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Mathematics in Undergraduate Study Programs: Challenges for Research and for the Dialogue between Mathematics and Didactics of Mathematics

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Abstract. The topic of undergraduate mathematics is of considerable concern for mathematicians in universities, but also for those teaching mathematics as part of undergraduate studies other than mathematics, for employers seeking to employ a mathematically skilled workforce, and for teacher education. Different countries have made and continue to make massive efforts to improve the quality of mathematics education across all age ranges, with most of the research undertaken particularly at the school level. A growing number of mathematicians and mathematics educators now see the need for undertaking interdisciplinary research and collaborative reflections around issues at the tertiary level. The conference aimed to share research results and experiences as a background to establishing a scientific community of mathematicians and mathematics educators whose concern is the theoretical reflection, the research-based empirical investigation, and the exchange of best-practice examples of mathematics education at the tertiary level. The focus of the conference was mathematics education for mathematics, engineering and economy majors and for future mathematics teachers.

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Introduction by the Organisers

The workshop *Mathematics in Undergraduate Study Programs: Challenges for Research and for the Dialogue between Mathematics and Didactics of Mathematics*, focused on *Mathematics education at the tertiary level* and organised by Rolf Biehler (Institut für Mathematik, Universität Paderborn), Reinhard Hochmuth (Institut für Mathematik und ihre Didaktik, Leibniz Universität Hannover), Dame Celia Hoyles (London Knowledge Lab, UCL Institute of Education, University of London) and Patrick W. Thompson (Dept. of Mathematics and Statistics, Arizona State University), took place at Oberwolfach, Dec 7-13th 2014. The topic of undergraduate mathematics is of considerable concern for mathematicians in universities, but also for those teaching mathematics as part of undergraduate studies other than mathematics, for employers seeking to employ a mathematically skilled workforce, and for teacher education. Different countries have made and continue to make massive efforts to improve the quality of mathematics education across all age ranges, with most of the research undertaken particularly at the school level. A growing number of mathematicians and mathematics educators now see the need for undertaking interdisciplinary research and collaborative reflections around issues at the tertiary level. The conference aimed to share research results and experiences as a background to establishing a scientific community of mathematicians and mathematics educators whose concern is the theoretical reflection, the research-based empirical investigation, and the exchange of best-practice examples of mathematics education at the tertiary level. The focus of the conference was mathematics education for mathematics, engineering and economy majors and for future mathematics teachers.

Aims were: (1) To create and sustain a *national and international interdisciplinary community of mathematicians and researchers in mathematics education* who are interested in tertiary level mathematics education. (2) To share and discuss a range of topics and concerns with a clear focus on *grounded approaches* in *theoretical and empirical aspects*, thus moving the field beyond the simple exchange of opinions and beliefs.

The workshop set out to share results and best practice around the following topics:

- The transition between school and university mathematics
- The content and goals of undergraduate and first year university mathematics education given new developments in mathematics at school level and at university level
- The design and evaluation of competence-oriented curricula elements including innovative methods for assessing students knowledgetaking into account needs of the specific study programmes (mathematics majors, teachers, engineers)
- Didactical analyses of mathematical content including general aspects such as formal representations, proving, and specific aspects concerning the learning of calculus and linear algebra

- Knowledge about subject specific learning obstacles and learning strategies, the development of motivation and beliefs, development of instruments that take the specificity of mathematics into account
- The design and evaluation of support systems for students including tutoring, mentoring and student support centres
- Difficulties that university mathematics departments face, both cognitive and cultural, in attempts to enhance the quality of students mathematical thinking and understanding

The conference was organised around the following sections: Learning mathematics at the undergraduate level, mathematical meaning making, transition between school and university, technology, reasoning and proof, maths in teacher education, maths as a service subject at university in particular in engineering studies, and common ground for research in undergraduate mathematics teaching. Each theme included keynote talks followed by small group discussions chaired by different participants, who made notes of the main issues raised and shared them with the whole group in a plenary session at the end of the conference. The conference brought together experts from eight countries, in particular from Great Britain, France, USA and Germany.

Reflections and next steps

Aim 1: There was much discussion on the nature of the school/university transition in general, with particular reference to topics that are known to present challenges for students such as understanding limits and real numbers. Several themes became very apparent and were taken up in small group discussions: how to promote the shift from intuitive to rigorous reasoning, from pragmatic to conceptual approaches, or from a naive to a scientific point of view. Discussion centred on how this might be managed in ways that were intellectually honest but also practical. Regarding intellectual honesty, one approach put forward was that didactics could usefully considered as ‘applied mathematics’, where mathematics is actually seen to be used and fundamental in the solution of problems or there is an *intellectual necessity* for a mathematical point of view.

Aim 2: Many presentations addressed methodological challenges for research in tertiary mathematics education, not least in terms of the adequacy of theoretical frameworks and the need for mathematicians and mathematics educators to work together on common projects. Two fruitful ways forward were suggested by some presentations: one related to analysing student responses to carefully constructed assessment questions and conjecturing about the routes of any misconceptions; another using interdisciplinary design research methodologies with the aim to address known didactical or epistemological obstacles through detailed task designs and evaluations while exploiting digital technologies in ways that enable students to explore the relevant mathematical structures, ways simply not possible with paper and pencil tools.

The workshop was remarkable in that it highlighted the diversity of undergraduate students mathematical experiences among countries. It seems there must be greater clarity of what is *meant* by mathematics teaching at university level before we can consider if and how we might change it.

There was an overriding consensus that it is the collective responsibility of mathematicians and mathematics educators to work together on common projects to make progress and take the field forward.

The workshop ended with real and shared motivation to engage further, and in particular work together in more collaborative research. There are projects to further the conference's presented work: (1) a special issue of IJRUME (International Journal of Research in Undergraduate Mathematics Education) with original research papers from the workshop will be published, and (2) shared ways to devise a coherent agenda of systematic research to produce results robust enough to transcend country divisions (e.g. the International Network for Didactic Research in University Mathematics initiative).

The challenges are not to be underestimated. To achieve these there is a need for methodological advances as well as further opportunities to share, perhaps at the khdm follow up conference "Didactics of Mathematics in Higher Education as a Scientific Discipline" to be held in December 2015 in Hannover at Schloss Herrenhausen.

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Workshop: Mathematics in Undergraduate Study Programs: Challenges for Research and for the Dialogue between Mathematics and Didactics of Mathematics

Table of Contents

Abstracts

Mathematical analysis at university: Some dilemmas and possibilities

Carl Winsløw

Most research on university level mathematics education concerns the teaching and learning of calculus, i.e. a highly didactified practical version of differential and integral calculus for functions of one or more variables. The high didactification refers here to the long tradition of textbooks, exams, syllabi and other practices which this domain of teaching has generated. Typical tasks for students begin with one or more concrete functions, given by a formula, and the students are then asked to compute limits, maxima, integrals, and so on. Most exercises in classical calculus can be done by computer algebra systems such as Maple or Mathematica, while new exercises can be developed to retain at least some necessity of student work. However, students do not work with theoretical justifications of the algorithms they (or their software) are based on, nor with conditions that ensure the existence of the objects they compute. In the scholarly counterpart of calculus, the theoretical superstructure relies on topological notions like completeness and continuity, which are not part of the usual calculus course.

In most European countries, calculus is taught from upper secondary school to introductory university level courses within a large variety of study programmes. The large volume of calculus teaching certainly justifies the intensive research activity that is devoted to clarify and solve some of the didactic challenges related to this domain. However, students who continue with mathematics will soon be required to work also with the formal theory of limits, derivatives and so on. They must, so to speak, acquire the complete and official version of classical analysis, as it is understood today. And even later in a mathematics programme, they will also encounter abstractions such as spaces of functions equipped with various topologies, operators on spaces of functions, and so on. In many universities, these two kinds of transition within the analysis sequence are experienced as very difficult by students and therefore by teachers. The dilemma of calculus, painfully realised by students as they work their way into real analysis, is that the mathematical meaning of its objects and indeed the foundations of their existence is not part of calculus itself, as they rest on topological notions which students do not know or learn in calculus courses. Moreover, the transistion problems just described can be aggravated by incomplete results from earlier mathematics teaching of algebra (e.g. the notion of function) and even basic arithmetic. This is usually accentuated by the informal and somewhat rapid treatment that is usually given to technical tools from logic and set theory.

The work reported on in my talk concerns efforts developed over several years at the University of Copenhagen, beginning around 2000, in view of dealing with the two transition problems just described. Some references to this work are given at the end of this note. In essence, our work consists in developing new forms and contents of assignments of work to students, with the aim of facilitating their access first to the theoretical level of calculus, then to the more abstract practice and knowledge of modern analysis based on function spaces, norms and operators. The following is a simple example of such an assignment - a thematic project which is worked on by students during the course and at the same time constituting one of several items for the final oral (and individual) exam of a course where measure theory is not yet available, but where students have already had a course which treats the fundamental theorem of analysis in detail [2].

Thematic project: The space of Riemann integrable functions
Goal: Work out properties of function spaces using key notions such as uniform continuity and completeness, and construct simple proofs based on inequalities and norms. Students may refer to literature from a previous course if using a result from there, without repeating the proof. They may to use (and refer to) other sources but here proofs must be included.
Let

- $\mathscr{R}$ = set of Riemann integrable functions on a closed interval I
- $\mathscr{I}$ = set of step functions on I
- $\mathscr{C}$ = set of continuous functions on I

(1) Prove $f \in \mathscr{R} \iff \forall \epsilon > 0 \exists u, o \in \mathscr{I} : u \leq f \leq o \wedge \int_I (o - u) \leq \epsilon$
(2) Prove that $\mathscr{C} \subseteq \mathscr{R}$ by using 1. [Hint: If $f \in \mathscr{C}$, in fact f is uniformly continuous. Use this together with the uniform partition of I and suitably defined step functions]
(3) Explain why $\mathscr{R}$ is a subspace of $L^\infty(I)$, and prove that when $\mathscr{R}$ is given the inherited topology, the map $f \mapsto \int_I f$ is continuous from $\mathscr{R}$ to $\mathbb{R}$.
(4) Is $(\mathscr{R}, \|\|_\infty)$ Banach? [Hint: $\|f - g\|_\infty \leq \epsilon \iff f - \epsilon 1_I \leq g \leq f + \epsilon 1_I$]

The students are thus supposed to investigate the fundamental theorem of analysis, and related properties, in a context of normed function spaces (cf the explicit aims at the beginning, always provided to students), in a relatively directed setting. The design and a posteriori analysis of data was based on the anthropological theory of the didactic [1, 3]. The main point here is that mathematical practice is modelled using the notion of praxeologies, which enables one to situate individual tasks with respect to a broader analysis of how the practice blocks they generate can be related through common theoretical blocks. Another point of this is that the types of transition referred to above may be described (much) more precisely.

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Conceptualisation of the continuum: An educational challenge for undergraduates

Viviane Durand-Guerrier

The concept of continuum is one of the most difficult to master for undergraduates. In French universities, in general, no construction of the set of real numbers is provided, so that many students identified real numbers with finite decimal approximations. In particular, the distinction between density and continuity, which plays a fundamental role in real analysis remains mainly at an implicit level.

1. A brief insight on numbers in the French curriculum. At Primary school, the syllabus covers: the elaboration of the concept of natural number relying on discrete collections and on the ordered sequence of numbers; the introduction of rational numbers (fractions) and finite decimals appears in the context of measurement of continuous magnitudes, along with an arithmetic treatment of these numbers. The numerical line plays an important role. At Middle school, students go on developing numerical competences with natural numbers, decimals numbers and fractions. They meet irrational numbers through the square root of natural numbers that are not perfect square of another natural number. At High school, students meet two other emblematic irrational numbers, namely π and e, that they learn to use via their representations. They deal with approximations, mainly via calculators. At University of Montpellier, as in many French universities, in first year, an axiomatic definition of the real numbers set is given, via the supremum property, without any explicit construction. The infinite decimal writings and the corresponding characterisation of the nature of numbers are also introduced explicitly, and improper writings, such a 0,9, are discussed with students. In 2013-2014, a questionnaire was addressed to first year students in the second semester in order to have an overview on their knowledge on numbers. The results have shown the fragility of their knowledge, so that we have decided to put more emphasis in our lectures on continuum.

2. Intuition and formalisation of the continuum in mathematics. Commenting H. Weyl view on continuum Longo [2] writes: “Our intuition about continuum is built from common and stable elements, from invariants which emerge from a plurality of acts of experience: the perception of time, of movement, of a line extended, of a trace of a pencil.” He adds that “This [The pencil on a sheet] is the most common experience of the continuum”, reminding that Cauchy himself “() in his first demonstration of the Theorem of the mean value (...) does not go further than the intuition of the continuum that comes from string and curves traced by a pencil and their crossing.” He indicates that the main thesis of his paper is that these difficulties are, so to say, the formal symptom of the inherent difficulties in the intuitions of the continuum., and states that “Cantor and Dedekind have proposed a precise mathematical formalisation of the intuition of continuum with at least three points of contacts with our intuitive demands: the invariance of scale, the absence of jump and of holes.” Indeed, Dedekind [1] proposes an extension of the rational number set through the notion of cuts, which appears as a formalisation of the intuition of continuum. He proves that the new set is order-complete with respect to the cut process. The construction of Dedekind is closely related to the concept of supremum and to the property of completeness of the real numbers set. The fact that the rational numbers set, although dense, is not order-complete is difficult to grasp, in particular because the holes in the rational line have no thickness, so that they are not "holes" in the common sense, and hence it is very difficult to show by a drawing the difference between the rational line and the real line.

3. Students encounter the question of the continuum. Pontille [3] reports on a research led in the frame of the "MATh-en-JEANS"[1] in 1993-1994. All along the school year, voluntary students accompanied by their teachers and by the mathematician who submitted the problem have worked on the following fixed-point problem: *Let us consider a map f of $\{1, 2,\ldots,n\}$ into $\{1, 2,\ldots,n\}$ where k is a non null natural number. f is supposed to be increasing ; show that exists an integer k such that $f(k) = k$; k is named a fixed point.* Then, study possible generalisations in the following cases, with f an increasing map.

(1) Is there a fixed distance between two finite decimals?
(2) Given any positive finite decimal, is it possible to find a smaller such number?
(3) Is there a relation between questions 1 and 2?

After having solved the case of natural numbers, students tried to adapt their proof, which relies strongly on the fact that every integer has a successor. This leads them to encounter significant epistemological questions:

(1) $f : \mathbb{D} \cap [0, 1] \to \mathbb{D} \cap [0, 1]$, where $\mathbb{D}$ is the set of finite decimal numbers
(2) $f : \mathbb{Q} \cap [0, 1] \to \mathbb{Q} \cap [0, 1]$, where $\mathbb{Q}$ is the set of rational numbers
(3) $f : [0, 1] \to [0, 1]$ or any other generalization[2]

Finally, the students concluded that the answer to both questions 1 and 2 were no. Then a new question appeared: "Is it possible that the graph of an increasing map of the finite decimal numbers set $\mathbb{D}$ in itself crosses the first bisector outside $\mathbb{D}$?" Through drawings and discussion, the students developed various representations of the numbers at stake and finally conclude that it is possible that the crossing be on a hole. Coming back to the initial question, they looked for a counterexample with simple and concrete maps. They finally found a counterexample for $\mathbb{D}$ and a counterexample for $\mathbb{Q}$. It is noticeable that this discovery only occurred following the students reconsidering, reorganising and adapting their knowledge about decimal and rational numbers.

The construction of Dedekind is an example of elaboration of a theoretical concept relying on an empirical intuition. The order-completeness of the set of real numbers is a fundamental property of this set and an essential part of the conceptualisation of mathematical continuum. The students work on the fixed-point

[1] http://www.mathenjeans.fr/

[2] In cases 1 and 2, there are counterexamples. In case 3, the statement is true, due to the supremum property

problem show that this problem is a good candidate to introduce secondary or undergraduate students, so that prospective or in service teachers, to the delicate concept of continuum, in perspective with the concept of density. It also emphasizes the necessity of giving time to students in order to allow a conscious reflection on the mathematical concepts at stake.

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The use of real numbers in secondary school and entering the university

Jürg Kramer

In our talk we aimed at highlighting the problem of the teaching of the real numbers at German schools on the lower and upper secondary level, proposed some desirable changes for the teaching of this subject at schools, and drew some consequences for the teacher education at universities.

1. Findings. In the nationwide educational standards for the secondary schools ("Bildungsstandards", see [1], [2]) established by the ministers for education of the federal states of Germany the teaching and building-up of the number system, in particular of the real numbers, finds only marginal mention. After the introduction of the rational numbers $\mathbb{Q}$ in the lower secondary level, the appearance of irrationalities is, in general, reduced to the demonstration of the existence of $\sqrt{2}$. This leads to the somewhat provocative perception that the set of real numbers at the lower secondary level $\mathbb{R}_{\mathrm{Sec\,I}}$ is reduced to the set $\mathbb{Q} \cup \{\sqrt{2}, \pi\}$.

Even though there is the guiding principle ("Leitidee") denoted "Algorithm and Number" in the educational standards for the upper secondary level, no substantial time is devoted to the teaching of the real numbers themselves. When it comes to the introduction of differentiation and integration, the completeness of the real numbers is taken for granted using a so-called "propedeutic" understanding of the notion of limit. As a consequence, the set of real numbers at the upper secondary level $\mathbb{R}_{\mathrm{Sec\,II}}$ continues de facto to be reduced to the set $\mathbb{Q} \cup \{\sqrt{2}, \pi\}$.

2. Desirable aims for the high school education. It is desirable that pupils at the lower secondary level understand and can well handle fractions and their representation as terminating and repeating decimals. Furthermore, by zooming into the ("naively-founded") real number line, they should be able to construct an arbitrarily fine grid consisting of terminating decimals on the real number line, as well as the first irrational numbers such as $\sqrt{2}$, $\sqrt{3}$, etc.

On the upper secondary level, high school students should be taught the following two models for the real numbers $\mathbb{R}$:

(1) $\mathbb{R}$ as the set of infinite decimals without recurring 9's,
(2) $\mathbb{R}$ as the real number line.

Using the grid consisting of terminating decimals constructed for the real number line, they should be able to approximate any real number on the number line by means of nested line segments whose end-points consist of terminating decimals.

3. Consequences for the teacher students' education. Of course, translating rigorously between the aspects (1) and (2) of the real numbers mentioned above is a difficult task and can only be partially explained at schools. However, we think that teachers of mathematics at the upper secondary level should be mathematically rigorously taught that crucial part of the construction of the number system in course of their university education.

A possible approach is to construct the real numbers using rational Cauchy sequences and then relating them to the set of infinite decimals without recurring 9's as well as the real number line with a view towards the teaching of these contents at school. We will briefly illustrate this concept more closely; for details we refer for example to chapter IV of the book [4].

We let $\mathbb{D}$ denote the set of infinite decimals (including the ones with recurring 9's); we observe that the fundamental operations of addition and multiplication cannot be carried out right away for this set. We introduce the set M of rational Cauchy sequences with componentwise addition $+$ and multiplication $\cdot$; one checks that $(M, +, \cdot)$ is a commutative ring. Two rational sequences are now called to be equivalent, if their difference is a zero-sequence. Passing to the set of equivalence classes leads to the set $\mathbb{R}_{\text{Cauchy}}$ of real numbers (following Cauchy). It turns out that the addition and multiplication on M give $\mathbb{R}_{\text{Cauchy}}$ the structure of a field which, moreover, is seen to be ordered. With these facts at hand, one is thus enabled to introduce the notion of a limit of a sequence of real numbers and define *real* Cauchy sequences. The challenge now consists in showing that such Cauchy sequences always possess a limit in $\mathbb{R}_{\text{Cauchy}}$. In particular, we can now prove that the (rational Cauchy) sequence $0.9, 0.99, 0.999, \ldots$ has limit 1.

Once these prerequisites are in place, one can relate $\mathbb{R}_{\text{Cauchy}}$ to the set $\mathbb{D}$. For this one considers the obvious map $\varphi\colon \mathbb{D} \to \mathbb{R}_{\text{Cauchy}}$ given by assigning to an infinite decimal the (equivalence class of the) rational Cauchy sequence obtained by successively adding the subsequent decimal. The map φ is easily seen to be surjective, but obviously not injective since the images of $0.\bar{9}$ and 1 are equal. In this respect one shows that two infinite decimals with the same image by φ are either equal or one of them is terminating, while the other has a tail of recurring 9's. Letting $\mathbb{R}_{\text{deci}}$ denote the subset of $\mathbb{D}$ of decimals without recurring 9's, we have thus established a bijection between the set $\mathbb{R}_{\text{deci}}$ and $\mathbb{R}_{\text{Cauchy}}$.

In order to finally relate $\mathbb{R}_{\text{Cauchy}}$ to the real number line $\mathbb{R}_{\text{line}}$, one first establishes the equivalence of the completeness of $\mathbb{R}_{\text{Cauchy}}$ with the validity of the principle of nested intervals. In order to be then able to define a map from $\mathbb{R}_{\text{Cauchy}}$ to $\mathbb{R}_{\text{line}}$, one has to elaborate on the real number line by postulating that every infinite set of nested line segments with rational end-points in $\mathbb{R}_{\text{line}}$ has to have a non-empty intersection, which then necessarily consists of exactly one element (this is equivalent to Dedekind's approach in [3]). With this axiom at hand, the map $\psi\colon \mathbb{R}_{\text{Cauchy}} \to \mathbb{R}_{\text{line}}$ is given by assigning to $\alpha \in \mathbb{R}_{\text{Cauchy}}$ determined by an infinite set of nested intervals determined by strictly ascending, resp. descending sequences of rational numbers, the intersection of the corresponding infinite set of nested line segments with rational end-points in $\mathbb{R}_{\text{line}}$. It is then routine to show that the map ψ constitutes the desired bijection between $\mathbb{R}_{\text{Cauchy}}$ and $\mathbb{R}_{\text{line}}$.

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Making the fundamental theorem of calculus fundamental to calculus

Pat Thompson

A long list of research studies in mathematics education documents students difficulties in understanding fundamental ideas of the calculus. In this presentation I explain how it is the very structure of the calculus that we commonly ask students to learn that makes many of their difficulties inevitable, and I share a project that restructures the calculus so that students learn it as a coherent body of ideas and methods.

In this restructured calculus, two problems illuminate what calculus is about and drive its development. The first is the problem of knowing how fast a quantity is changing and wanting to know how much of it there is. The second is the problem of knowing how much of a quantity there is and wanting to know how fast it is changing. Framing the calculus in these terms puts the Fundamental Theorem of Calculus at its center, from the very start. The first problem entails starting with a function that tells us how fast a quantity is changing at each moment of its change in relation to another quantity and deriving a function whose values say how much of it has accumulated in respect to a value of the other quantity. The second problem entails starting with a function that tells us how much of a quantity there is in relation to another quantity and deriving a function whose values say how fast it is changing in respect to a value of the other quantity. The interplay of these two fundamental problems of calculus crystalizes into a formal description of their relationship that appears as the Fundamental Theorem of Calculus.

The first part of the restructured calculus focuses on having students conceptualize accumulation from rate of change. This phase culminates in students conceptualizing the function F, defined as $F(x) = \int_a^x f(t)dt$, as a function that gives an exact accumulation of a quantity whose value changes at each value of x at a rate of $f(x)$. This understanding entails two things about the Fundamental Theorem: that $\int_a^x f(t)dt$ is a function and that $f(x)$ is its rate of change at each value of x.

The second part of the course focuses on having students conceptualize rate of change from accumulation. If values of a function g give an amount of a quantity, then we can think of that amount as having accumulated. Therefore, $g(x) = \int_a^x r(t)dt$ for some rate of change function r and some value a. This realization necessitates the development of a method to approximate r, g's rate of change at a moment, which culminates in students conceptualizing the function r as a limiting function defined, in effect, as $r(x) = \lim_{h\to 0}[g(x+h) - g(x)]/h$. This understanding entails two things about the Fundamental Theorem: That r is a function, and that $g(x)$, as r's accumulation function, is equal to $\int_a^x r(t)dt$ for some value of a and therefore that g is a closed form representation of $\int_a^x r(t)dt$.

My research team has conducted qualitative studies of students learning in the restructured calculus course with the aim of uncovering epistemological obstacles they meet and how the course might be adjusted to accommodate them. Next year we will begin conducting large scale comparisons of students learning in the restructured course and in courses that have the more common structure of limits, derivatives, integrals, and the Fundamental Theorem of Calculus.

Designing mathematics curricula to bridge a gap between students intuition and mathematical rigor: The case of advanced calculus

Kyeong Hah Roh

Introductory real analysis (IRA), or advanced calculus, is mandated for most university students majoring in mathematical sciences. Through the course, undergraduate students are expected to understand the theoretical foundation of calculus conceptually as well as rigorously. Unfortunately, many mathematics students point to IRA as one of their most difficult university courses and express their fears of IRA. Based on a project for designing and implementing a research-based curriculum for IRA, this presentation reports the results from four iterations of teaching experiments[4] conducted in a undergraduate IRA course at a large university in the United States. As an inquiry-based learning (IBL) course, a conventional lecture style instruction was minimized in the newly designed IRA course and instead students were often expected to contribute in knowledge construction by making and justifying conjectures for definitions or theorems. To support such student activities, a series of questions was only provided to students at the beginning of each activity. After the topics were explored in class, more detailed class-notes (e.g., a list of definitions and theorems) were provided to students; however, even in the case proofs were not offered by the instructor but students were often working in group to construct their groups proofs.

The IRA course was developed based on a theoretical standpoint toward mathematical intuition as cognitive capacity obtained by experiencing proper examples and lasting. Fischbein[1] classified intuition into primary intuition and secondary intuition via its origin. *Primary intuitive cognition* refers to “intuitive cognition that develops in individuals independently of any systematic instruction as an effect of their personal experience." All representations and interpretations naturally developed prior to formal instruction based on logic and rigorous proofs belong to primitive intuitive cognition. On the other hand, once a student is able to see directly a theorem, and the theorem becomes a belief and a self-evident conception so that the student directly conceives it without needing further justification, we then consider that a new, secondary intuition has been acquired by the student. In this classification of intuition, the point to be aware of is that secondary intuition can be obtained and developed by appropriate training.

The empirical data and findings from the IRA course indicates how the curricular materials bridge a gap between students intuition about calculus concepts and mathematical rigor. One of the examples is what is called the "ϵ-strip activity"[2, 3] implemented for the limit of a sequence. Students were first asked to make a conjecture to be applied to any sequence to determine its convergence. At this point, students were expected to use their primary intuition about limit prior to the formal definition of limit. After realizing problems of their conjectures and a need for a unifying idea to describe the limit of any sequence, ϵ-strips were introduced to students as strips with constant width and infinite length. These ϵ-strips were made of translucent paper so as to observe the graph of a sequence through the ϵ-strips. In addition, a red line was drawn in the center of each ϵ-strip so as to mark a value to be tested for the limit of a sequence. Students were then asked to evaluate the following statements A and B:

- A: L is a limit of a sequence when for any ϵ-strip, infinitely many points on the graph of the sequence are inside the ϵ-strip as long as the ϵ-strip is centered at L.
- B: L is a limit of a sequence when for any ϵ-strip, only finitely many points on the graph of the sequence are outside the ϵ-strip as long as the ϵ-strip is centered at L.

At the first glance most students did not accept statement B as a proper one to describe the limit of a sequence. However, in a process of seeking for counterexamples of statement B, students were rather able to find out that statement B is not only necessary but is also sufficient in determining the limit of any sequence. Furthermore, such an activity led students deeper semantic understanding of statement B: (1) the width of an ϵ-strip and the number of points outside the ϵ-strip are coordinated to one another; (2) ϵ is arbitrarily decreasing towards 0; and (3) the number of points outside an ϵ-strip is determined depending on the value of ϵ whereas ϵ is independent of N. Such deeper semantic understanding of statement B became part of students secondary intuition and remained as a powerful mental imagery that supported students understanding of the formal ϵ-N definition of the limit of a sequence.

This presentation illustrates how the ϵ-strip activity plays a role in developing students (secondary) intuition that is compatible with the formal definition of limit. The students were initially situated in such a way that they felt perplexed, frustrated, and confused due to the impreciseness of their conceptions of convergent sequences. Students primary intuition about convergence became a true intellectual problem to them. Students then suggested one idea after another, seeking a possible way to properly describe the convergence sequences, and tested their hypotheses by (mental) actions with ϵ-strips. It is also worth noting that after the ϵ-strip activity, the students continued reflecting on their new conception (secondary intuition) of convergence and utilized it to make sense of other properties of convergent sequences by using the ϵ-strips.

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Examining individual and collective level mathematical progress

Chris Rasmussen

Recent work in mathematics education research has sought to integrate different theoretical perspectives to develop a more comprehensive account of teaching and learning. The need for such integration arises from several challenges, including communicating more effectively among researchers, integrating empirical findings from different perspectives, and improving mathematics learning and teaching via more coherent research findings. The benefits of such integration include gaining explanatory and descriptive power, reducing the compartmentalization of theories, and fostering a discourse on theory development [1].

An early effort at integrating different theoretical perspectives is Cobb and Yackel's [2] emergent perspective and accompanying interpretive framework. The framework consists of three constructs that function as analytic tools for analyzing the collective (social norms, sociomathematical norms, and classroom mathematical practices) and three constructs for analyzing individuals (beliefs, mathematical beliefs, and individual conceptions and activity). The two sets of constructs are viewed as reflexively related, which means that individual beliefs and activities are constitutive of collective practices. Conversely, the joint activity of the collective gives shape and purpose to individuals goal-directed activities and conceptions. Hence, a primary assumption from this point of view is that mathematical development is a process of active individual construction and a process of mathematical enculturation.

In this research I expand the interpretive framework for coordinating social and individual perspectives by offering an additional set of constructs to examine the mathematical progress of both the collective and the individual. Prior work with the interpretative framework has raised my awareness of the opportunity (and need) to go beyond the constructs in the interpretative framework. In particular, prior research has treated the individual conceptions and activity construct as a single construct that addresses how individual students participate in classroom mathematical practices. In an effort to be more inclusive of a cognitive framing that would posit particular ways that students think about an idea, I split the individual conceptions and activity construct into two constructs, one for individual participation in mathematical activity and one for mathematical conceptions that individual students bring to bear in their mathematical work. To interpret student participation, I draw on Krummheuer's [3] ideas of production design and recipient design, which traces the roles individual speakers take regarding the originality of the content and form of the utterance. With these two constructs for individual progress we now can ask the following two questions: How do individual students contribute to mathematical progress that occurs across small group and whole class settings? What conceptions do individual students bring to bear in their mathematical work?

The construct of classroom mathematical practices, which refers to the normative ways of reasoning that emerge as learners solve problems, explain their thinking, represent their ideas, etc., is a way to conceptualize the collective mathematical progress of the local classroom community. Work at the undergraduate level has also highlighted the fact that, in comparison to K-12 students, university mathematics and science majors are more intensely and explicitly participating in the discipline of mathematics. However, the notion of a classroom mathematical practice was never intended to capture the ways in which the emergent, normative ways of reasoning relate to various disciplinary practices. In order to more fully account for what often occurs at the undergraduate level, I expand the interpretive framework to explicate how the classroom collective activity reflects and constitutes more general disciplinary practices, such as defining, algorithmatizing, symbolizing, modeling, and theoremizing. Thus, we can now answer the following two different questions about collective mathematical progress: What is the mathematical progress of the classroom community in terms of the disciplinary practices of mathematics? What are the normative ways of reasoning that emerge in a particular classroom?

In addition to using various combinations of the four constructs to more fully interpret students mathematical progress, there exist multiple ways in which coordination across the four constructs. For instance, one could choose an individual student within the classroom community and trace his/her utterances for the ways in which they contributed to the emergence of various normative ways of reasoning and/or disciplinary practices. Alternatively, when considering a normative way of reasoning, a researcher could investigate who the various individual students are that are offering the claims, data, warrants, and backing in the Toulmin analysis used to document normative ways of reasoning. How do those contributions coordinate with those students production design roles? For instance, does a student ever utilize an utterance that a different student authored as data for a new claim that he is authoring, and in what ways may that capture or be distinct from other students individual mathematical conceptions? One might also imagine ways to coordinate across the two individual constructs as well as across the two collective constructs. For example, how do patterns over time in how student participation in class sessions relate to growth in their mathematical conceptions? Are different participation patterns correlated with different mathematical growth trajectories? In what ways are particular classroom mathematical practices consistent (or even inconsistent) with various disciplinary practices? Finally, one could take up more directly the role of the teacher in relation to the four constructs and their coordination. Future work will more carefully delineate methodological steps needed to carry out the various ways in which analyses using the different combinations of the four constructs can be coordinated. Rasmussen, Wawro, and Zandieh [4] make a first step in this direction, using an example from linear algebra.

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Approaches to teaching mathematics and their relation to students' mathematical meaning making

Barbara Jaworski

Kilpatrick, Hoyles and Skovsmose [2] suggest that "Teachers of mathematics must deal with questions of meaning, sense making, and communication if their students are to be proficient learners and users of mathematics" (p. 8). They raise the questions:

- "How can meanings for teachers and didacticians be developed?" and
- "How can teachers best explore the meanings which students have constructed?" (p. 16).

Skovsmose writes further "...for students to ascribe meanings to concepts that have to be learned, it is essential to provide meaning to the educational situation in which the students are involved" (p. 85). I seek to redress the deficiency expressed by Speer, Smith and Horvath [4], who write "very little research has focused directly on teaching practicewhat teachers do and think daily, in class and out, as they perform their teaching work" (p. 99). The studies to which I refer below arise from an in-depth focus on specific examples of teaching practice with qualitative analyses which reveal aspects of teaching, capture the intentions and reflective thinking of the teacher and subject outcomes to critical scrutiny through rigorous analysis. They inform our more general understanding of teaching at this level and its relations to students meaning-making, raise and elaborate on issues that arise in the practice of teaching and offer insights that can be of relevance and significance more generally.

1. A study in the UK of 6 mathematicians tutorial teaching over one university term (8 weeks): observation of tutorials and interviews with tutors. Qualitative analyses led to a characterization of teaching approaches from the perspective of the tutor. All tutors recognized students difficulties and dealt with them in differing ways which were seen to fit into or between four characterizations: *Naive and Dismissive; Intuitive and Questioning; Reflective and Analytic; and Confident and Articulate*, the whole being characterized as *Spectrum of Pedagogical Awareness*[3].). A theoretical construct "The Teaching Triad" [1] consisting of 3 domains: *Management of Learning* (ML), *Sensitivity to Students* (SS), and *Mathematical Challenge* (MC), was used to analyze teaching episodes in some depth. Findings showed *largely* that ML involved teachers in *showing* and *explaining*; SS involved *ensuring the student was made aware of the correct mathematics*, while MC left it up to the student to go away and make sense of the mathematics presented to them.

2. An ongoing study in Greece of teaching in mathematics lectures involves observation of lectures and interviews in two universities with six lecturers who are research mathematicians. The *Teaching Triad* is used as an analytical frame to characterise teaching episodes. Analyses are showing "an uneasy balance" between *Sensitivity to Students* and *Mathematical Challenge*. This is reflected in a tension between a lecturers wish to include students thinking in the activity of a lecture while at the same time presenting mathematical meanings in a rigorous form. The ways in which this tension is addressed in teaching episodes can be seen to fit with differing positions on the *Spectrum of Pedagogical Awareness*.

3. A UK study of the teaching of linear algebra involved a small community of inquiry of two mathematics educators and one mathematician (the teacher) in which teaching was explored and characterised. The focus was centrally on the thinking and actions of the teacher with the three members engaging deeply with ideas from linear algebra, approaches to teaching and the engagement of students. The result was an in-depth characterization of teaching, achieved using an Activity Theory perspective. Central to this was the lecturers articulation of teaching goals and his realization of the goals in his day to day teaching. The study provides important insights into the teaching of linear algebra.

4. An ongoing UK study of teaching in small group tutorials, involving observation and interviews. Initially 26 tutorials were observed from 26 different tutors, including both mathematicians and mathematics educators. Subsequently tutorials of three tutors were studied in depth. The focus of analysis is on teachers knowledge for teaching in mathematical, didactical and pedagogic domains, drawing on a range of theoretical positions in the literature. One key concept emerging is tutors use of mathematical examples; extracts of dialogue between tutor and students provide insights into students meaning-making and tutors adaptation of teaching to students thinking as they see it.

The papers referenced in each case above provide details of findings with supporting evidence from teaching-learning dialogues. Overall, we see collaboration between mathematicians and mathematics educators, teachers and researchers, revealing actions and goals of teaching as related to students meaning making, and associated growth of knowledge in teaching. However, student verbalisation is rarely articulate enough to have certainty as to their meanings. We continue to explore the question **"How can we foster student expression of mathematical meanings in relation to the teaching experienced?"** Further references to papers associated with the four studies above can be obtained by contacting Barbara Jaworski on b.jaworski@lboro.ac.uk

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The emergence of quantification during guided reinvention of formal limit definitions

Michael Oehrtman
(joint work with Craig Swinyard, Jason Martin)

There is extensive research identifying conceptual barriers to students' understanding of limits and resulting misconceptions. For example, students are liely to view a sequence as becoming simply indestinguishable from its limit [1], believe the terms must always get closer to the limit [2], have significant difficulty with the three quantifiers in the definition [3], and conclude that ε should depend on N [4]. We conducted a study to generate insights into how students might leverage their intuitive understandings of sequence convergence to construct a formal definition. Our aim was not to avoid standard misconceptions, rather allow students to resolve challenges derived from their initial intuitive conceptions. We engaged six pairs of students, none of whom had previously seen a formal definition of sequence convergence, in a series of 90-minute sessions to collaboratively construct a definition using mathematical notation and quantification equivalent to the conventional definition. I outline several of the problems and solutions developed by one pair of students, Megan and Belinda, in developing their formal definition. I then discuss their individual recollections and reconstructions of their definition six months later and compare their work to the other pairs of students.

We adopted a developmental research design rooted in the theory of Realistic Mathematics Education "to design instructional activities that link up with the informal situated knowledge of the students, and enable them to develop more sophisticated, abstract, formal knowledge, while complying with the basic principle of intellectual autonomy" [5]. We first asked each pair of students to generate graphs of qualitatively distinct examples of sequences that converge to 5 and sequences that do not converge to 5 to serve as the foundation for articulating and testing their emerging definitions. Task design then followed the guided reinvention heuristic through a cyclic process of writing a definition to include all of their examples and exclude all of their non-examples, identifying problems and potential solutions, then returning to write a new definition.

For Megan and Belinda, this cyclic process resulted in 31 distinct written definitions for sequence convergence over three sessions of the teaching experiment. Their initial definitions consisted of informal dynamic characterizations of a sequence "approaching" its limit. Examining non-monotonic examples, they recognized this was not true for all convergent sequences and quickly introduced the idea that convergent sequences should approach their limit "after some point N." While struggling to express these ideas, the students introduced the quantity $|a_n - 5|$ as an "error" for each term in the sequence and the relationship $|a_{n+1} - 5| < |a_n - 5|$ to express a requirement of "decreasing errors," thus implicitly resolving the problematic non-quantitative nature of their descriptions. While these solutions introduced several new problematic issues, the students focused instead on the meaning of closeness and infinitesimal language. Through these discussions, they eventually reinterpreted closeness in terms of "getting within an acceptable range" or "error bound ε" and immediately transferred this requirement to the qualification of occurring "after some point N." The students recognized that a single choice of an error bound would allow for limits to have multiple values and include some sequences that did not converge, which they wrestled with for over an hour. As soon as the facilitators suggested a universal quantification of the error bound (an idea previously expressed by the students in a context unrelated to this problem), they immediately recognized it as an appropriate solution. Shortly afterwards, they also recognized a damped oscillating sequence should converge and that their universal quantification of ε rendered the need for decreasing errors unnecessary. Throughout the guided reinvention, the students also introduced several notational ambiguities, which they were able to quickly resolve once focused on them as problems.

The ideas developed by the students during the guided reinvention became useful to them and persistent in their reasoning to the extent that they were generated as solutions to cognitive challenges that they explicitly engaged and resolved. First, each component of the formal definition was derived as a solution to a problem made explicit to the students through their comparison of their current definition to the graphs of examples and counterexamples. Second, many components of the formal definition appeared in Megan's and Belinda's verbal expressions long before they were even considered as a potential part of their definition. Finally, six months later, Megan and Belinda did not recall the complete definition, but rather reengaged the same problems and employed the same solutions, albeit more rapidly, to reconstruct their definition.

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Mathematics support at English universities

Duncan Lawson

Throughout the 1990s there was growing disquiet amongst Professional Bodies, Learned Societies and University academics about the mathematical competence of new undergraduate students in science and engineering disciplines. A Government report in 2007 examined the success rates of students in higher education across all disciplines and found that many students of physical sciences and engineering require additional academic support in mathematics in order to succeed. Since then, the mathematical requirements of many other subjects, such as biological sciences and social sciences, have increased significantly and several prestigious reports have highlighted shortcomings in the mathematical confidence and ability of both students and staff in these disciplines.

A key report, by the Advisory Committee on Mathematics Education, stated that "*We estimate that of those entering Higher Education in any year, some 330,000 would benefit from recent experience of studying mathematics (including statistics) at a level beyond GCSE, but fewer than 125,000 have done so.*" [1] [Note: GCSE qualifications, typically taken at age 16, two years before entering university, represent the end of compulsory mathematics education in England.]

A common response by universities to the mismatch between the desired and actual mathematical competences of new undergraduate students has been the introduction of mathematics support. Mathematics support is a facility offered to students which is **in addition** to their normal programme of teaching. The manner in which mathematics support is provided differs across universities; however the most common model is the "drop-in" centre. In this model, students drop-in to the centre at a time of their own choosing to seek help from the centres tutors. Students must themselves elect to access the support, visiting the centre is not compulsory. The scale of mathematics support provided varies across institutions. At those institutions with large-scale provision, the drop-in centre has its own dedicated location where physical resources such as books and handouts are available, supplemented by online resources from an institutional website and/or from generic sites such as **math**centre (www.mathcentre.ac.uk).

Typically, mathematics support provision aims to provide an attractive and safe environment in which students may improve their mathematical skills. In this context, 'safe' means non-judgemental, students are not criticised for what they do not know and no question is regarded as too basic; furthermore the support provision is not involved in the assessment of students. A key aspect of mathematics support is to build students confidence and challenge the commonly held view that '*mathematics is something you can either do or you can't*', with most students placing themselves in the category of those who cannot.

In 2005, **sigma**, a collaboration in mathematics support between Loughborough and Coventry Universities, was designated as a Centre for Excellence in Teaching and Learning (CETL) by the Higher Education Funding Council for England (HEFCE). Working within the CETL programme (2005-2010) and the National HE STEM programme (2009-2012), **sigma** has promoted the development of mathematics support across the sector by a range of activities including providing funding to institutions wishing to establish their own provision, publishing guides for staff providing mathematics support, mentoring those establishing new provision, training mathematics support tutors and funding PhD research into mathematics support. A national survey [3] showed that in 2012, 85% of universities provided some kind of mathematics support (up from 65% in 2004 and 48% in 2000).

A major national study [2] evaluated mathematics support from the student perspective. This study reported that 22% of students on service mathematics courses (i.e. courses for students of disciplines other than mathematics such as engineers, scientists, economists, etc.) had considered dropping out because of difficulties with the mathematical elements of their course and of these, 63% reported that their decision not to drop out had been influenced by the mathematics support available. Currently **sigma** is undertaking a 3-year programme, funded by HEFCE, to establish a sustainable mathematics support community of practice across the sector. The **sigma** network, a voluntary association of mathematics support practitioners, has been established (www.sigma-network.ac.uk) as a first stage in this process.

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Learning mathematics at school and at university: Common features and fundamental differences

Lisa Hefendehl-Hebeker

Mathematics at school and mathematics at university have common roots and practices, but are domains of knowledge, which regularly refer to different stages of scientific development and to different levels of mathematical activity. Though learners meet these domains in different stages of their individual intellectual development, they feel a deep gap between them, which often seems to be unbridgeable. In the presentation these ideas were unfolded and clarified by examples.

To begin with a basic observation: Mathematics originates from elementary roots (numbers, simple spatial concepts) and developed towards a highly elaborated domain of research. Mathematics at school is situated close to these roots, mathematics at university on the contrary in a highly advanced stage of this process. In consequence there are fundamental differences with respect to the considered contents, the kind of representation and the accompanying standards, conventions and aims.

According to Freudenthal "our mathematical concepts, structures, ideas have been invented as tools to organise the phenomena of the physical, social and mental world." [3] The development of a mathematical discipline usually starts by organising phenomena of the real world. For this purpose mathematicians create mental objects, which can be considered as phenomena of the mental world. These phenomena again are organised by mental concept of higher order and so on. In this way a "praxeological progression" (Carl Winsløw) arises as an open process, conveyed by mental activities like organization, sophistication, formalization, deeper foundation, extension and generalization, restructuring and establishing a strong global architecture.

According to this way of speaking concepts and ideas of school mathematics mainly are tools to organize the physical and social world of our daily life. "Most concepts in school mathematics can be traced back to an origin in material physical activities of some sort or another (such as counting, measuring, drawing, constructing)." [1] This may happen in an intellectually demanding way, even with local deductions and rigorous reasoning, but on the whole the ontological connection to reality persists. Thus school mathematics in most cases does not exceed the conceptual level and the stage of knowledge of the 19th century.

Within the last 150 years the amount of mathematical knowledge experienced an almost exponential increase and the phenomenology changed in a fundamental way. Items like "axiomatic method" or "dissolving the ontological connection" indicate, that modern mathematical theories are in the first instance deductively structured mental worlds of their own, represented as abstract sets with specific structural features and build upon highly complex concepts like "vector" or "function". To grasp these concepts requires specific mental activities such as "encapsulation" [2], "objectivation" [4] or "reification" [5, 6].

These processes even can be iterated, e.g. when factor groups of factor groups are considered. They require not only new mental objects to be conceptualised, they also require a flexible change between the view of such an object as a complete entity and the dissolution into its individual components. Such a flexible change is necessary for example when special properties of a function such as linearity or monotony are to be shown or it has to be unfolded that the operation in a factor group is well defined (independency of the chosen representatives).

The development of mathematics towards a theory of abstract mental concepts has an exterior counterpart in the current tools of representation. The professional mathematical language is a highly developed artefact, usually related to the semantic of sets (which often means that other semantic implications or phenomenological roots are hidden and have to be revealed by the learners). Further it is highly conventionalised and it causes a high density of information, which requires a specific reading ability.

A general problem in mathematics teaching says that teachers often act within a long-range-perspective, which is not shared by the learners. This "problem of wide horizon" is even more serious in university mathematics than in school mathematics. Indicators are the high elaboration of concepts, the ideal of theoretical closure and the principle of optimal systemic fit. The latter entails for example that sometimes definitions are formulated in a way, which makes it easy to operate with the defined properties, but the original idea is hidden.

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Developing the mathematical reading skills of first year students

Joachim Hilgert
(joint work with Max Hoffmann and Anja Panse)

In the wake of a compulsory reorganization of teacher training in 2011 the department of mathematics at Paderborn University moved the traditional first-semester (4h) analysis course to the third semester and replaced it by a new (2h) course. In a first step we designed a course which, compared to the standard analysis course, has about one third of content and focusses on material chosen to bridge the well-known double discontinuity. The first half is devoted to developing basic mathematical strategies and concepts (in particular equivalence relations) starting from a concrete problem: find a workable test to check divisibility by 7. The second half has an overall goal: the stepwise construction of the real numbers from the natural numbers (via the integers and the rational numbers). This course was taught for the first time in the winter term 2012/13.

Student learning did not improve in the expected and desired way. We failed to activate the students according to the intended (traditional) three-step-process: attending lecture – reworking material – doing exercises. Most students did not do the reworking part. As none of these steps can be omitted without seriously jeopardizing student learning we worked out a new teaching concept for the 2013/14 edition of the course in order to improve the student activation potential by reorganizing those three steps. In parallel a student-teacher discussion group was founded to analyze the reasons of the high rate of failure in first-semester mathematics courses. The suggestions and wishes that were issued by this discussion group turned out to be highly compatible with our teaching concept, so we implemented it: Using the elementary text [1], originally designed for high school students looking for orientation, we taught the course in an *inverted classroom* approach, offering low threshold communication (anonymous via internet), detailed learning goals, and a variation of transparent examination modes (see [2, 3] for details).

While we got very positive student feedback during the course, in particular concerning the student-teacher communication, transparency, service and organization, student learning did not increase significantly. In this version the reading assignments were done at best in a perfunctory way, even by hard working students. This lead us to analyze the students' mathematical reading skills more closely. It seems that they are not prepared to deal successfully with existing mathematical texts. As we aim at intellectual autonomy of our students, our approach to deal with this problem is not to prepare special texts with reduced complexity and increased redundancy for them, but rather to reshape their reading skills. We focus on three aspects:

Deceleration: The adequate reading process for mathematical texts is a very slow process, in which the reader continuously evaluates his or her comprehension of the text. This is a feature that sets mathematical reading apart from many other forms of reading, especially forms that dominate most of the reading required in modern communication media. We need to develop an awareness for the fact that different forms of text require totally different reading approaches.

Attention to detail: Language is a very redundant system. Mathematics is, for very good reasons, not (repeating definition each time it is used does not facilitate the comprehension of overall arguments). In mathematics we need to be reading for detail not for gist. Skimming or scanning - useful skills in different contexts – are of little use for mathematical texts.

Decoding strategies: Even a decelerated, very attentive reading attitude will lead the individual to passages where immediate comprehension is not possible. Therefore every reader needs to acquire a toolbox of methods to identify such passages and to deal with them. As there is no algorithm to follow, we have to offer examples of strategies, such as "Revise definitions of terms used", "Generate simple examples", "Reformulate in simple special cases" or "Fill in omitted details".

In the winter term 2014/15 we added the concept of a *stumbling block* to our teaching concept in order to address these three key aspects of mathematical reading skills. A stumbling block by definition is a text passage a reader cannot immediately decode. This is a reader oriented concept. In principle every text passage can be a stumbling block to some reader. Different readers will identify different stumbling blocks. Training mathematical reading skills using stumbling blocks proceeds in two phases. In the first phase we give the students reading assignments in which certain text passages are marked, and examples of questions these passages may raise as well as possible strategies to deal with these questions. In a second phase we ask the students to produce their own stumbling blocks in new reading assignments, to formulate corresponding questions and to try out the strategies learnt in phase 1 in trying to answer them.

While we have not yet systematically implemented, let alone evaluated, the training of reading skills via stumbling blocks, interviews and questionaires on the first trials show that students welcome the marking of stumbling stones in reading assignments, but resent having to produce their own stumbling blocks. So we are thrown back to our central problem: how to achieve student activation in processes which require more than a mechanical effort?

We are reluctant to use repressive measures like yet more tests and assignments as a prerequisite for participation in the final exams as we feel this is counterproductive to our goal, the students' intellectual autonomy. On the other hand, we do not see a realistic chance to achieve our goal within the given time frame solely following the paradigm of inquiry based learning. It would be helpful to have an agreed on catalogue of concrete skills beginning/finishing students ought to have, so that a transparent schedule of learning for students to gauge their individual progress can be worked out. The hope is that students then see the need for and the reward of an intellectual effort on their own.

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Self-determination and interest development of first-year mathematics students

Michael Liebendörfer

Students' motivation is an important factor for their learning. Across various disciplines, interest as a specific person-object-relationship is a good predictor of students' effort, their use of deep learning strategies and good learning outcomes [2, 3, 4]. However, in Germany as in many other countries, university teachers often report a lack of students' intrinsic motivation of both mathematics majors and pre-service teachers. German drop-out rates range from 40 % to 80% [1]. The aim of this ongoing PhD-project is therefore to analyse and reconstruct the interest development of first-year mathematics students in a typical german university. This will hopefully give ideas for innovations in teaching but also for additional support.

In order to investigate students' interest development, the study was conducted at Kassel university, Germany. Mathematics major students and pre-service teachers of the higher secondary track (German Gymnasium) were contacted in a first-year course on real analysis which covered topics from real and complex numbers and their properties up to mutlivariable calculus based on definition and proof. For homework, the students had to hand-in their solutions for an exercise sheet each week. The solutions were marked and then returned and discussed in a weekly tutorial. Receiving on average 50 % of the maximum score was a precondition to take the exam. After 4 weeks of study, 21 students agreed to come to the interview and gave permission to use their data. For a second interview at the end of the first semester, 17 of them returned and 12 of them came for a third interview in their second semester. The guideline interviews asked for the students' study experience and learning behaviour, especially investigating their motivational experience.

The data were analysed using techniques from the Grounded Theory methodology [5]: coding for abstract phenomena, constant comparison of cases and situations and memo writing.

Although the results are preliminary, we present here a phenomenon which we call *interest shift.* To illustrate this phenomenon, we present the case of a student we call Matthew. Therefore, we summarise Matthew's view from each of the three interviews, mainly rephrasing his words.

First interview: Matthew wants to become a teacher. In school, he always liked mathematics, and in a preparatory course at university he liked it even more, especially proving. In the first lectures, he feels good. He wants to achieve the 50 % and also wants to understand the lecture. After the lectures, he re-writes his lecture notes and tries to understand every line using supplemental books. Afterwards, he starts solving the tasks. There is much work to do, but he can manage it. Sometimes it is stressful to have no idea what to do for a task.

Second Interview: Matthew now feels stressed. He still finds the new mathematics interesting, but he thinks for teachers, one might have it easier. He feels gaps in his knowledge and can't achieve the 50 % on his own. Receiving good grades is central for him now. However, Matthew likes calculation tasks and is still ambitious. He meanwhile reviews the lecture notes only as part of working on tasks. He works together with peers and started to sometimes copy their solutions.

Third Interview: Matthew wants to teach mathematics at school and therefore will not need proof. He feels very stressed and focusses on passing the exam by mastering calculations. He often doesn't understand the lectures as well as the proof tasks. Matthew doesn't find the new mathematics interesting anymore. He needs to have 50 % and to know the theorems, but not the proofs. He still has in mind to review the lecture notes, but he doesn't.

The case of Matthew illustrates the *interest shift* very distinctly as Matthew had a very positive attitude at the beginning which he then gradually lost. The *interest shift* could also be observed when looking at other students. Some of them started with a less positive attitude towards proof, others had already shifted their interest towards good marks (and not understanding) after four weeks.

The *interest shift* has some typical elements. First, there is a high stress level which can be attributed especially to pressure to get sufficiently good marks and high pressure of time. Thus, students' intrinsic motivation gets suppressed as they pursue performance goals rather than mastery goals. Their study behaviour shifts from learning in order to understand the contents (e.g. by reviewing the lecture) towards learning to get good marks (e.g. by copying). In the cases of pre-service teachers, a partial identification with their new goals and their altered learning behaviour appeared. While before the shift they highlight that teachers need to understand, after the shift they tend to argue that the university mathematics will not be needed in school. It is also worth noting that the *interest shift* has some self-reinforcing features. Students who did not (yet) experience the shift more often reported learning activities like reviewing the lecture notes and thus getting ideas for the tasks or more generally pursuing their own mathematical interest. Both activities were reported to be very motivating. This is also true for their identification with the mathematicians' community. Thus, while in the short run, intrinsically motivated learning is suppressed by the shift, in the long run students may run out of intrinsic motivation at all.

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Mathematics and Mathematics Education: A story of paths just crossing or of meeting at a vanishing point?

Elena Nardi

In this paper I draw on my experiences of collaborating with research mathematicians in order to tell a story of paths crossing – of mathematicians and mathematics educators intersecting at various points in *research, teaching, professional development* and engagement activities. In this story I position myself as a (non-research) mathematician who chose to become a researcher in mathematics education and has been involved in aforementioned activities with research mathematicians for more than 20 years.

The story has three parts.

First, I trace the relationship between research mathematicians and researchers in mathematics education in *research* and I offer a potted account of issues that have been helping this relationship grow, grow at a not-always satisfactory pace or occasionally stall. To this aim I draw on the findings from ([3]: Chapter 8) and elsewhere to discuss the perceived benefits, obstacles and desires regarding this relationship.

Secondly, I offer examples of initiatives that have been propelling the deepening of this relationship, from *research, teaching and professional development*. With regard to *research*, I draw on examples of studies dating from 1990s to to-date, to trace the evolution of collaborative research between mathematics educators and mathematicians from studies of university mathematics students' learning of particular mathematical topics (for example: [2]) to a progressively shifting focus on university mathematics teachers' perspectives on mathematics and mathematics teaching (for example: [1]; the research underpinning *Amongst Mathematicians*, [3]). With regard to *teaching*, within mathematics departments, I offer examples that trace two developments within university mathematics curricula: towards a more inclusive approach to mathematics education modules within university mathematics programmes; and, towards engaging mathematics education researchers with university mathematics teaching (as is the case in my institution as well as elsewhere in the UK, for example Loughborough University's Mathematics Education Centre). With regard to *professional development*, I offer examples that trace the evolution from generally unpopular, non-discipline specific training of new lecturers in mathematics to more appealing, mathematics-specific training formats. In these formats, that we have begun to see recently in several institutions, university mathematics education researchers are involved in the training of new mathematics lecturers, offering them opportunities to familiarize themselves with mathematics education research findings and to reflect on their teaching practice.

Thirdly, I offer examples of initiatives where the two communities have been working together towards the strengthening of another, very crucial, relationship: that of the public with mathematics. The examples I draw on are from *teaching* (mathematics or about mathematics to non-mathematics students) and *engagement* (activities that aim to draw non-mathematical audiences into the world of mathematics and into considering the possibility of mathematical studies). With regard to *teaching*, I put forward examples of: *teaching outside mathematics departments*-and particularly within the social sciences and the humanities-to illustrate the increasing interest in engagement with the world of mathematics of students hitherto resisting, or at best being skeptical about, such engagement (I particularly draw on the example of a *Changing public perceptions of mathematics* module that I teach to Year 3 BA Education students in my institution). With regard to *engagement activities*, I put forward examples such as MAUD, *Maths At Uni Days*, that I have been contributing to in my institution since 2006, to illustrate some of the ways in which departments of education, mathematics and Further Mathematics Centres in the UK have been coming together to organize events that showcase the importance and appeal of mathematics as well as its capacity to open windows to a wide range of professions.

Through these three tiers of examples, I aim to put forward the thought that this story of mathematicians and mathematics educators intersecting at various points in *research, teaching, professional development* and *engagement activities* can be re-imagined, not merely as a story of paths crossing-but as a story of paths meeting at a vanishing point, a point where the boundaries between the two communities may fade into insignificance, and even recede.

While this abstract uses examples of activity initiated by myself and my close collaborators, and the selection is also heavily UK-focused, international activity-such as the bulk of works discussed at the workshop in Oberwolfach in December 2014-offers much optimism and lends further validity to the claim I make here.

This presentation was based on a paper currently in preparation for *Mathematics Today*, the Special Issue *Windows on Advanced Mathematics*, edited by Dame Celia Hoyles and Richard Noss, due for publication in 2016.

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Mathematics and mathematics education: Searching for common ground

Tommy Dreyfus

A few years ago, Ted Eisenberg asked where the mathematics is in todays mathematics education. He felt that ethnographic research, semiotics, and other research directions drag mathematics education too far afield. Indeed, we may ask whether mathematics education has matured on the basis of its origins in mathematics or whether it has matured away from its origins and lost its way.

Focusing more specifically on research in mathematics education, one may ask how this research, using methodologies of the social sciences, can be relevant to learning and teaching mathematics, an exact science. The differences in research methodologies, as well as in the types of knowledge generated by research, are different. This may lead to communication gaps between mathematics education and mathematics. Communication may be further restricted by mathematics educators stressing what they see as the socio-political role of mathematics education, and by mathematicians refusing to see anything beyond content as important for mathematics education. Each group is then likely to accuse the other one of having a limited view while exhibiting a limited view themselves.

So a need arises to search for common ground between mathematics and mathematics education. In May 2012, a symposium in honor of Ted Eisenberg on this issue took place in Beer Sheva, Israel, and a book that reflects the work of the symposium was recently published [1]. An effort was made to include mathematicians, mathematics educators, and especially the few people who wear both hats by leading two parallel research careers, one in mathematics and one in mathematics education.

The following themes were addressed at the symposium and in the book:

- Mutual expectations between mathematics and mathematics education
- Problem solving in mathematics and in mathematics education
- Visualization in mathematics and in mathematics education
- Justification and proof in mathematics and in mathematics education
- Mathematical literacy - what is it?
- Mathematics education policy
- Collaboration between mathematics and mathematics education

The resulting insights include the following.

Research on problem solving has been based on a succession of models for problem solving and comparison between expert and novice problem solving behavior based on these models. It has led to successive refinement of theory and development of practice.

Visualization is one of several ways of thinking mathematically; however, visualization in mathematics teaching may fail due to lack of communication: Visual and logical thinking are not opposites but complementary and often a fluent interplay between analytical rigor and visual intuition is at stake such as in Rohs ϵstrips (this workshop). Hence, visual thinking must be taught, and is taught best by teachers with mixed and flexible personal preference.

"Proof is what distinguishes mathematics from other domains of knowledge" [2] and "Students cannot be said to have learned math, or even about math, unless they have learned what a proof is" [3]. Hence proof should be taught to all students. Addressing issues, shortcomings and potentials of teaching and learning proof is one of the choice issues on which the collaboration between mathematics and mathematics education is necessary and powerful. However, there is no agreement how. Recent research has shown that mathematicians often behave quite differently in how they seek conviction and read proofs. If mathematicians are not in agreement what constitutes a proof in elementary calculus [4], what are we to teach our students? Philosophers of mathematics have stressed the communicative and explanatory function for proof and this may well be its main function in mathematics education as well. However, this will only be meaningful for students, if there is an appropriate intellectual need, and this leads to questions how to design learning activity.

Being mathematically literate may have different interpretations; literacy may be defined in terms of mathematical habits of mind, or quite differently in terms of being able to use mathematics in the "real world"; moreover, "real world" may have various interpretations, ranging from minimal competencies like making sure one gets correct change, to sophisticated abilities like knowing when to approach an issue mathematically and how to do this, for example, by modeling.

While the nature of policy issues requires collaboration between mathematics and mathematics education, and while such collaboration has great potential, it is also fraught with potential friction because priorities may be different; for example, mathematical literacy may be in the foreground for some and preparation for university studies for others.

Several projects of genuine collaboration between mathematicians and mathematics educators have been presented. Further successful such collaborative ventures have been described at the present workshop by Nardi. While there are many ways and levels of collaboration, including between individuals, between organizations, and between individuals within a common organizational framework, the underlying theme in all of them is that no substantial and sustainable improvement of mathematics education is possible without mutual trust and respect, without common engagement in shared problems, and without a coordinated effort to build on the complementarity of expertise of mathematicians and mathematics educators.

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Preparing future mathematicians to transform undergraduate mathematics curriculum and instruction

Marilyn Carlson

The mathematics education research community has learned a considerable amount over the past 20 years about what is involved in learning and understanding key ideas in introductory university mathematics courses. This knowledge has not had a noticeable impact on the teaching of undergraduate mathematics. To complicate matters, future mathematics faculty (mathematics graduate teaching assistants) are not being introduced to the research on learning undergraduate mathematics, nor are they being prepared to use formative data and scientific methods to improve their teaching and curricula. As a result, few future mathematicians are developing knowledge to assume leading roles in evaluating and improving undergraduate mathematics curriculum and instruction.

The *Pathways to Transforming Undergraduate Mathematics Education* program (TUME) was designed to address this problem. *Pathways TUME* is a 2-year mathematics education certification program for PhD mathematics students. It is designed to prepare future mathematicians to leverage research on learning key ideas of a course and engage in processes of scientific inquiry to design and refine curriculum and instruction. *Pathways TUME* leverages past studies on knowing and learning key ideas of precalculus[3] and calculus that have produced research-developed curricula, instructor resources, and teacher professional development that is resulting in more conceptually focused teaching and significant gains in student learning.

PhD students apply to *Pathways TUME* and are selected based on their interest and potential to emerge as instructional leaders in undergraduate mathematics. *Pathways TUME Scholars* are assigned to teach a class of precalculus using Pathways Precalculus materials and instructional resources . Their introduction to the program is a weeklong summer workshop that engages them in mathematical tasks that perturb their understanding of what it means to understand and learn key ideas of precalculus. They are also introduced to *Pathways* instructor resources and student materials[2] that have been documented to be effective for advancing precalculus teachers mathematical knowledge for teaching. During the academic year the *Pathways TUME Scholars* attend a weekly 2-hour seminar. During the seminar they are engaged in activities to support them in: i) reflecting on what is involved in learning key ideas of upcoming lessons, ii) considering various pedagogical approaches for different phases of a lesson, iii) collecting and analyzing formative data of their students learning, and iv) reflecting on the degree to which their instructional choices are effective in supporting student learning.

At the end of their first year, *Pathways TUME Scholars* complete a project in which they investigate their teaching effectiveness based on criteria for effective instruction that emerged through their participation in the seminar. They create and administer written tasks that they designed to assess their students thinking and understandings, and they collect classroom video from 3 consecutive class sessions. They use rubrics developed during the seminars to score their student responses, and they use criteria for "effective teaching" to analyze their video data. The results of these analyses are reported in a manuscript that includes the results of their analyses and a plan for adapting their teaching during the upcoming year. During the second year in the program they repeat this experience in the context of teaching calculus or an introductory proof course, with similar mentoring in a research-developed curriculum and professional development seminar.

The results from the first 1.5 years of implementing the Pathways TUME Program reveal positive shifts in future mathematicians: i) views of what effective teaching entails, ii) mathematical knowledge for teaching key ideas of function and rate of change, iii) instructional effectiveness, iv) students learning of the courses key ideas as assessed by the Precalculus Concept Assessment (PCA) instrument[1], and v) ability to reflect on and adapt their instructional practices. To date the program has impacted 32 future mathematicians.

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Revisiting programming to enhance mathematics learning: an English perspective

Dame Celia Hoyles, Richard Noss

There was an explosion of research and development into the relationships and mutual influences of programming and mathematics in the nineteen eighties. After several decades when programming fell out of fashion, it has now returned at least in England - and the time is ripe to revisit this prior research as it still has much to tell us: much has changed but not everything.

For a variety of reasons (including a belated recognition of the role of programming and mathematics in the CGI/Animations industry), programming is now very much on the agenda. But little of the explosion of interest is focused on mathematics, despite the very substantial research base of the role of programming on mathematical thinking especially using Logo [4, 5, 7], together with substantial curricular contributions [1, 2, 3], each of which shows compellingly how programming is a medium for exploring mathematics [1].

In this presentation, we make the case for the non-empty intersection of computational and mathematical thinking, an intersection that can, we argue, be exploited to the benefit of both. We then show how programming opens new ways of expressing mathematics in some ways, to point to new mathematics; and especially one which is more democratic in who can learn it, and how. This does not happen by chance: past work showed clearly the importance of design and teacher mediation, which if not taken seriously might lead to the exclusion of different groups (for example, girls). We give some examples from learners who are younger than those considered by the workshop and with good reason: they are the university students of the future who are surrounded by mathematics embedded in many of the artefacts of their working and cultural lives. Yet in England at least so much of this mathematics is invisible and it is still acceptable to profess complete ignorance of mathematics, and even proclaim that the subject has no uses for the individual and for the society.

Our examples will involve novel representations of mathematical ideas; considering the programming of agent-based systems as an alternative to representing complex phenomena using differential equations; representing infinite decimals as tangible and infinitely precise programs; and some examples of Scratch, a relatively new descendant of Logo. Each of these examples illustrate the opportunity (and need) for mathematicians to consider what Seymour Papert has called the what of learning mathematics not just the how. We would also hope that with careful and collaborative iterative design, we can move nearer to the vision set out by Papert in 1972: Let the students learn mathematics as applied mathematics in the sense that mathematical knowledge is an instrument of power, making it possible to do things of independent worth that one could not otherwise do.

And we can respond to his more recent challenge to mathematicians and mathematics educators to rethink at least 10 percent of the mathematics we teach in the light of the presence of digital technologies: the new infrastructures they offer and the collaborative networks that they make possible [6].

We conclude with an outline of our latest project, ScratchMaths, a three-year research project, which will design and build a 2 year curriculum and associated professional development programme for teachers. The first year (for Grade 5) focuses on developing computational thinking from which experiences to explore mathematical thinking will be developed in Grade 6. Besides a team that will collect and analyse qualitative data, an independent evaluation will subject the work to a randomized control trial.

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Technology and resources use by university teachers

Ghislaine Gueudet

Introduction: from technology to resources

The use of technology at university has been studied for many years. In most research works, technology means software, like Computer Algebra Systems or programming technology, for example. But the available technology evolves: recently, studies appeared about the use of Virtual Learning Environments, in particular for distant learning. Studying the consequences of these evolutions for the teaching and learning practices requires to introduce a comprehensive concept of resource. A resource for the teacher is defined here as anything likely to re-source the teachers practice: technologies, but also traditional textbooks on paper, or even discussions with colleagues [1]. Teachers look for resources, transform them: we call this the documentation work of a teacher. Along this work, teachers develop documents, which associate resources and professional knowledge. The structured set of all these documents, developed along the years by a teacher is called his/her documentation system. Understanding the evolutions resulting from the use of technology requires to study the teachers documentation systems. We set up a study of these systems in the context of a university in France, investigating the work of six teachers with different profiles for their teaching in the first and second year of university with scientific students.

Results: in increasing, but still limited place of technology in teachers' documentation systems

The documentation systems of the teachers interviewed comprise several subsystems: documentation for lectures, for tutorials, for assessment, for communication.

The place of technology in 'usual' documentation systems: for communication only

For 4 out of the 6 teachers we met, only the sub-system for communication incorporates technology in some of the documents developed. For the preparation of lectures, the resources used are the notes of colleagues who previously taught this

course, when such notes exist (often under the form of a polycopie transmitted to the students from the beginning of the year as a .pdf file on the teachers webpage); or mathematics books otherwise. Most of the lectures are still traditionally given on the blackboard; only one of the six teachers tried for the first time in 2013-2014 to project slides. For the tutorials, the main resource is a list of exercises shared between the different teachers who intervene (and with the students). The teachers interviewed never search for resources on the Internet to prepare their teaching. Proposing the polycopie, the exercises sheets, the previous exam texts on his/her professional webpage is done by 3 of the 4 lecturers. The lecturers webpage is a central resource for the communication from the lecturer to his/her colleagues and his/her students.
A central place of software in particular documentation systems
Nevertheless the use of digital resources remains limited, compared for example with secondary school. As mentioned above, one of the colleagues interviewed teaches in a symbolic computation course. This course uses Maple in fact learning to use Maple is an objective of the course, where no new mathematics content is presented; the students learn to produce algorithms and programs linked with the mathematical content of the other courses, concerning matrices (Gauss algorithm), the resolution of linear systems, the search for prime numbers etc. The colleague teaching this course is herself researcher in formal computation. She has professional beliefs about the usefulness of programming, for the learning of mathematics. Maple is central in her resource system, since it intervenes in several documents she developed for different teaching or research objectives. A similar case is observed with a colleague using Scilab for his teaching of numerical analysis; Scilab intervenes in documents in all the subsystems except for assessment. This study remains naturally limited, since it only concerns six teachers, in a single university; enlarging this study in order to produce comparative analyses is one of our research perspectives.

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Designing design tools for visualizations in mathematics

Jürgen Richter-Gebert

[1] Mathematical structures live in an abstract world linked by abstract relationships. Nevertheless many mathematical structures have interpretations that can be expressed as pictures, diagrams, dynamic processes, spatial relations and other concrete metaphoric objects. Building concrete mathematical models has a long

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and far reaching history and tradition in the development of mathematics. Nevertheless not all mathematical effects that have a concrete haptic interpretation can be translated to a specific model that can be realized in a three-dimensional or two-dimensional scenario. Nowadays computer visualizations can be used to create models of mathematical objects or processes that go beyond what is physically possible. However, this freedom generates responsibility. A computer visualization has to "convince" the intended user that it actually is a model of an intended mathematical reality.

In what follows we want to give a few guidelines for the design of computer visualizations as well as for the design of tools for creating computer visualizations. These design criteria have proven to be useful in various concrete examples that demonstrate mathematical effects in educational contexts as well as in the development of such tools.

1. Design of Visualizations

Concentration on the Essence: Computer visualizations should highlight clearly one specific effect/theorem/phenomenon/technique. It should be made very clear what should be observed by a concrete visualization. Within its specific context a visualization should on the other be as generic as possible and cover (in a well defined way) the essence and the reach of a certain effect.

Clear Rules: A computer can perform many calculations that are not under the control of the user. As a consequence (in order to create trust) the implemented principles that underly a visualization should be made very clear. This can be achieved in two different ways. Either the principles should be so simple that interaction with the visualization immediately reveals them (think for instance of a construction in dynamic geometry, a visualized flow of a vector field, or a program that generates patterns that follow certain rules of symmetry [1]). If this is not possible then they should be clearly stated in an accompanying text or they should be transported by a second series of visualizations specifically designed for this purpose.

Simulations over Animations (and Annotations): Some visualizations follow a stimulus response pattern that demonstrate certain pre-programmed effects as a response of a certain user action. In such a scenario it is always necessary that the user trusts the expertise of the author of the animation. It turns out to be way more effective to use the full computational power of a device and create simulations rather than pre-programmed animations. In a simulation the author programs the underlying mechanisms rather than their effects. As a result in a simulation the effects that are visualized turn out to be emergent structures from (often) relatively simple rules. In a broad sense such simulations cover areas like drag mode in dynamic geometry, physics simulations, computer algebra systems, calculations in abstract groups and many more.

2. Design of Tools for the creation of visualizations

The above mentioned criteria have an influence on tools that are used to create such visualizations. On a very low level any suitably powerful computer programming language can be considered to be such a tool. However there are also scenarios that are closer to mathematics itself (for instance consider the computer algebra system Mathematica or the math visualization software Cinderella [2].) Here are a few reasonable requirements for the design of such tools.

Modularity and atomicality: A tool for design of mathematical visualizations should allow an author to create a mathematical scenario by the composition of relatively atomic units that cover a certain mathematical concept (think of routines that calculate the eigenvalues of a matrix, polynomial root finders, tools for performing geometric primitive operations, etc.). In fact the design of such modules may itself be a challenging part and require a lot of expert knowledge.

Consistentency: The tools should be designed in a way that maps a "mathematical reality" in a well defined way. Ultimately in a visualization there should be no effects that turn out to be artifacts induced by the insufficiency of the the underlying tool. This may sound easier than it actually is. It may turn out that the implemented primitives must be aware of their interrelations and history to create a reasonable mathematical model of a situation. A prominent example is, for instance, the fact that in order to achieve a continuous behavior of a dynamic geometry program one has to take into account the history of the movement. To avoid unmotivated jumping of dependent elements one often must embed the entire scenario of a construction in an ambient complex space and make path continuation on underlying Riemann surfaces. Ideally the designer of a tool for mathematical visualization provides built-in structures to take care of such effects.[2]

Ease of use: Despite of the intrinsic complexity of the underlying mathematics a visualization tool should be made as "easy-to-use" and ergonomic as possible. A mathematician creating visualizations with a tool should be relieved from as much technical parts as necessary for being able to concentrate on the essence and educational requirements of a certain visualization.

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DNR-based curricula: The case of complex numbers

Guershon Harel

My presentation discussed DNR-based curricula, with particular reference to a curricular unit on complex numbers. The design of the unit was inspired by and roughly follows the development of this subject in the history of mathematics. Consonant with DNR, the units instructional objectives are formulated in terms of both ways of understanding and ways of thinking, not only in terms of the former as traditionally is the case. The design of the unit factors in three major considerations: (a) the developmental interdependency between ways of understanding and ways of thinking, as dictated by the duality principle, (b) the intellectual needs of the students and the epistemological justifications suitable to their background knowledge and current mathematical abilities, as implied from the necessity principle, and (c) ways to facilitate internalization, organization, and retention of knowledge, as it is called by the repeated reasoning principle.

In accordance with the DNRs definition of learning and the instructional principle of intellectual need, i.e., the *necessity principle*, the unit was designed around alternating sequences of perturbations and resolutions of these perturbations. The development that leads up to complex numbers and the investigation into the meanings of complex numbers provide students with repeated opportunities for applying familiar ways of understanding and ways of thinking and acquiring new ones. Consistent with the *repeated reasoning principle*, the reoccurrence of these opportunities was by designto help students organize, internalize, and retain the knowledge they learn. To this end, each lesson concludes with a set of practice-of-reasoning problems, aiming at helping students internalize and organize the accumulated ways of understanding and ways of thinking they have learned at and up to that lesson. Some of these problems are rather demanding, as the reader may witness for herself or himself. The *duality principle* manifests itself throughout the unit in that students prior ways of thinking are taken into account, and those that targeted are developed through the solution of problems understood and appreciated as such by the students.

The unit is divided into three stages, corresponding to the historical development of complex numbers, which roughly proceeded in three stages: (1) the solution of the cubic equation, (2) the struggle to make sense of this solution, and (3) the emergence of complex numbers out of this struggle, and their power in solving mathematical problems. Accordingly, the units 12 lessons are organized around three stages: Stage 1 is comprised of Lessons 1-5; its aim is to delineate the ideas underlying the development of the cubic formula. Stage 2 is comprised of Lessons 6-8; its aim is to draw attention to the puzzling behaviors of the cubic formula. Stage 3 is comprised of Lesson 9-11; its aim is to resolve these puzzles by constructing a new set of numbers (the field of complex numbers), investigate their algebraic and geometric meanings, and articulate their remarkable value to understanding polynomial equations (i.e., the Fundamental Theorem of Algebra).

The questions we[1] faced in the process of translating the history of development of complex numbers into a curriculum are generalizable and relevant to the development of any curriculum. Specifically, the questions are:

(1) How should ideas underlying the historical development of the subject be represented and sequenced in the unit as to anchor them in students current knowledge, intellectually necessitate them, and provide opportunities for reasoning about them and with them repeatedly?
(2) What desirable ways of thinking are potentially afforded by this history?
(3) What is a typical background knowledge and cognitive ability of the student populations to whom the unit is intended (from high-school seniors to college freshmen and sophomores)?
(4) How much time can reasonably be allocated to this unit in the existing high school or college programs?
(5) How compatible the content of the unit with the content of current programs and national reforms?

Ways of thinking afforded by the unit include: (1) *structural reasoning*, with its various instantiations, manifested in two styles of reasoning: *theory building* and non-computational; (2) *deductive reasoning*, focusing on *definitional reasoning* and *reasoning in terms of quantifiers, connectives, and conditional statements*; and (3) *reflective reasoning*, with its two instantiations, *retrospective reflection* and *forward reflection*.

Research questions that follow from this study include:

(1) What are the learning trajectories of the ways of thinking claimed to be afforded by the unit and advanced among students through the DNR-based intervention?
(2) What is the extent and depth of the students acquisition of these ways of thinking?
(3) What is the impact of the instructional interventions reported here on retention of these ways of thinking and on student achievement in more advanced mathematical classes?

Using generic proofs as an element for developing proof competencies in the course "Introduction into the culture of mathematics"

Rolf Biehler, Leander Kempen

Reasoning and proving play a central role in mathematics and are known to be a major hurdle for many beginning university students (e.g., [3]). This discontinuity is supplemented by a second one for pre-service teacher students: They do not only have to deal with the increased demand for rigour at university, but also, to learn a repertoire of different kinds of reasoning and proofs they may use when teaching at school afterwards. Felix Klein called this the so-called double discontinuity for pre-service teachers.

[1]The unit was developed in collaboration with Evgenia Harel

In order to address the transition to advanced mathematics, the University of Paderborn offers the course “Introduction into the culture of mathematics”. It was held for the first time in 2011/2012 as a requirement for the first year secondary (non grammar schools) pre-service teachers, who had just passed their German “Abitur” before.

In the context of arguing and proving, four different kinds of proofs are used to engage students in the proving process and to address the double discontinuity mentioned above: The generic proof with numbers, the generic proof with figurate numbers, the proof with figurate numbers including “geometric variables” and the so-called formal proof.

As a generic proof, we consider the combination of operations on generic examples and a following valid narrative reasoning. In the concrete examples an argument has to be presented, why the statement is true in these specific cases. Afterwards, it has to be explained, why this argument also fits all possible cases and therefore is a valid general argument (see [1]). When the concrete examples are done by using arrays or patterns of dots, we call this a generic proof with figurate numbers.

For having the possibility of using variables also in the notation system of figurate numbers, we defined the symbol $\bullet \cdots \bullet$ as a “geometric variable” to represent an arbitrary number. So, it gets possible to construct proofs with geometric variables in the context of figurate numbers. The formal proof, which makes use of variables, is considered as the fourth kind of proof.

In our course, the students are to find conjectures in the context of natural numbers and figurate numbers and to prove them with all four kinds or proofs. Our aim is to make students understand the important borderline between purely empirical verifications and generally valid proofs, including several types of generic proofs. We see the benefits of generic proofs in the following: The exploration of a proposition by studying examples becomes a part of the proving process, it gets possible to develop ideas for a (formal) proof and the students get to know several valid forms of reasoning that can also be used at school. In the transition to formal proof, the functions and advantages of the mathematical symbolic language can be promoted in a meaningful way.

To get to know students proof competence, experience and attitudes, we conducted an entrance survey in winter term 2014/15. The results show students’ little prior experience with proofs, their serious problems when solving even simple argumentation tasks and their misinterpretation of generic proofs as purely empirical verifications (see also [2]).

We further analyzed students’ work in a task of the final exam of the course, where students were to prove the following statement with all four kinds of proofs mentioned above: “The sum of 6 consecutive natural numbers is always odd”. In the formal proof 79% of the students gave valid arguments, in the generic proof with numbers 57%, in the proof with geometric variables 28% and in the generic proof with figurate numbers 27%. These findings underline students’ problems with reasoning even after having passed our bridging course. Here, the narrative reasoning seems to be a special challenge, in the notation system of figurate numbers in particular. The two analyses are part of a larger empirical study that forms the core of the Ph.D. project of the second author.

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Proof scripts: a vehicle for assessing and enhancing proof comprehension

Rina Zazkis

"Proof scripts" refer to presenting a proof as a scripted dialogue that elaborates on various aspects of the proof. The idea of communicating and exploring ideas in a form of a dialogue has ancient roots. It was used by Socrates, Plato and Galileo, among others. In contemporary literature, in relation to mathematics or mathematics education, the dialogical method was used by Lakatos, and recently by Nardi and Sfard. In particular, the construct of proof scripts is influenced by "Lesson Plays" [5].

Lesson play refers to an imagined interaction between a teacher and students, presented in the form of a script for dialogue between teacher and students. It is a valuable tool for teachers, teacher-educators and researchers. Playwrights understanding of mathematics is revealed in their scripts. Proof scripts can be seen as a variation on this idea, where the interaction is around a particular proof. To date, proof scripts were used in several research studies attending to the following theorems and their proofs:

- Fermat Little theorem [2]
- Derivative of even function [3]
- Pythagorean theorem [4]
- Euclids Proposition 20 Book 7, known as "infinitude of primes" [6]

The task for students is to consider a given proof and to write a dialogue about the proof, focusing on its particular elements. The dialogue can be between two students trying to understand the proof or between a student and a teacher, who examines the students proof comprehension. In some task the choice of characters is imposed, in others the choice is left to the scriptwriter. While considering the proof scripts composed by students a researcher or a teacher can attend to and analyze the following:

- What aspect of the proof do the writers attend to?
- What do the writers consider as requiring explanation? What are points of difficulty?
- How is the perceived difficulty treated?

For example, in one popular proof of Fermat Little theorem the last step involves division in a congruence statement. The division is legal because the numbers involved in the statement are relatively prime. However, most script writers (prospective secondary teachers) either did not attend to the issue or provided some variation of the following explanation:

Teacher: Lets divide both sides by $(p-1)!$ and get $a^{p-1} \equiv 1 \mod p$.
Student: Is it allowed to divide like this?
Teacher: Yes. p is a prime number, so it is different from 1, therefore, $(p-1)$ is different from 0 and so it is possible to divide by it.

This dialogue provides a lens into the script writers knowledge as well as a starting point for a classroom discussion. In another study participants were presented a popular proof of the Pythagorean theorem. In this proof a point is chosen (points K, L, M, N) on each side of a given square $ABCD$, such that each side is divided into two segments of lengths a and b respectively (e.g., $AK = a$, $KB = b$). Then the area of $ABCD$ is calculated in two ways: by squaring its side of length $a + b$, and by adding the area of $KLMN$ and the 4 corner triangles. A short algebraic manipulation leads to the desired $a^2 + b^2 = c^2$. The scripts showed that in most cases participants were concerned with the algebraic manipulation rather than with the geometry of the situation, that is, why $KLMN$ is a square.

The proof script method provides an opportunity for a script writer to demonstrate personal mathematical and pedagogical strength, as well as to explore, reveal and overcome potential misconceptions. The following quote is applicable here: “What are the significant products of research in mathematics education?” I propose two simple answers:

(1) The most significant products are the transformations in the being of the researchers.
(2) The second most significant products are stimuli to other researchers and teachers to test out conjectures for themselves in their own context. [1]

As such, I invite colleagues to explore the following conjecture: *Proof Scripts provide an excellent vehicle for investigating and enhancing proof comprehension.*

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Mathematicians and educators: Divided by a common language

William McCallum

It is well-known that Winston Churchill once described England and the United States as "two nations divided by a common language." In preparing for this talk, I discovered that this well-known fact is false. The truth of the story is even better. According to [4]

> The British wanted to raise an urgent matter ... and told the Americans they wished to "table it" (that is, bring it to the table). But to the Americans, tabling something meant putting it aside. "A long and even acrimonious argument ensued," Churchill wrote, "before both parties realized that they were agreed on the merits and wanted the same thing."

In 2006 I participated in a similar discussion between mathematicians and mathematics educators about the algorithms for arithmetic operations that students learn in elementary school. It became clear from this discussion that there was no commonly accepted meaning within the group for such simple words as understanding, creativity, invent, and algorithm.

This led me to wonder: What are the fundamental intellectual differences between the study of mathematics and the study of mathematics education that might affect the interpretation of words? What are the structural differences determined by the subject matter itself? I am not interested in other differences here: political and social differences, struggles over authority and control, or psychological differences.

Perhaps the most fundamental difference is that mathematicians and mathematics education researchers have different ways of knowing. Knowledge about mathematics emerges from introspection and logical reasoning. Knowledge about mathematics education emerges from observation and scientific reasoning. Mathematicians sometimes make empirical claims about education as if they had deduced them logically. Mathematics education researchers sometimes treat pieces of mathematics as mere raw material for empirical study, in a way that is insensitive to mathematical norms.

In addition to these different ways of knowing, there are also some differences in the nature of the subject matter, which I will try to characterize here. First, mathematics is structurally rigid. A characteristic intellectual satisfaction of mathematics is the click of a piece falling into place. By contrast, mathematics educators must take account of "the role played by a certain cognitive flexibility in improving mathematical work" [1]. A characteristic intellectual satisfaction of mathematics education is seeing the growth of student understanding (including misunderstandings or partial understandings).

Secondly, mathematics is very tightly connected. The graph of mathematical knowledge has very few loose ends. Those that exist are either forgotten (e.g., general topology) or eventually connected back to the main body. The traces of discovery are tidied away. On the other hand, the growth of mathematical knowledge in a student's mind is full of dead ends and backtrackings. Areas of exploration that are of minor mathematical interest can be of major pedagogical usefulness (e.g. special strategies for arithmetic operations). In mathematics education, the traces of discovery are valued.

Thirdly, although all scholars value the truth, there are differences in what sorts of truths are valued. In mathematics, in the words of my undergraduate mentor Alf van der Poorten, "the important thing isn't getting it right—it's not getting it wrong." Mathematicians are trained to resist the siren song of the almost right; endless reformulation is a nightmare. On the other hand, in mathematics education, incorrect answers and imperfect understandings are valid pieces of evidence. Some of the most enlightening pieces of evidence are the almost right answers. Reformulation is useful in learning.

I'd like to mention one final difference between the two disciplines. In addition to the epistemological and structural differences I have mentioned, there is a difference in what the practitioners of the disciplines are inclined to care deeply about. It is a consequence of the fundamental difficulty in mathematics education: effective mathematics teaching must be faithful to the mathematics, attend to student thinking, and find a way of connecting the two. There is a strong temptation to put one of these to the background, and focus on the other.

During my career as a mathematician I have searched for activities that break through this foreground/background problem. The most powerful activity I have found is the common examination of student work by teachers, education researchers and mathematicians. There is a nice example of this in a case study about grading undergraduate calculus homework in [2], and I describe another example in [3]. Examining student work brings into play the expertise of each group—mathematicians and mathematics educators—in a way that is mutually recognized. Mathematicians see mathematical depths that educators might not see, and that might be responsible for some of the difficulties; educators understand what is going on in a student's mind in a way that might be opaque to mathematicians. Thus, examining student work provides one possible way to bridge the language differences between mathematicians and educators.

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KLIMAGS: Researching and improving mathematics courses for future primary teachers

Werner Blum, Rolf Biehler, Reinhard Hochmuth

The KLIMAGS Project, embedded in the khdm Centre, investigates the mathematics courses for first year primary school students at the universities of Kassel and Paderborn. KLIMAGS started in October 2010, directed by P. Bender, R. Biehler, W. Blum and R. Hochmuth, and aimed at knowing more about the knowledge, competencies, beliefs, interest, and strategies which beginning primary education students have in arithmetic and geometry, how that knowledge etc. develops in the first year of university studies, and what effects targeted innovations in university courses in arithmetic and geometry have on this development. The research design of KLIMAGS was as follows. The student cohort 2011/12 was the Control Group (CG), both in Kassel and in Paderborn, with courses in arithmetic and geometry as taught in the years before; there were four points of measurement in the first two semesters. The student cohort 2012/13 was the Experimental Group (EG), with certain innovations in these courses, and the same points of measurement. For these measurements, special achievement tests were developed, with 52 items in arithmetic and 26 items in geometry. All tests were IRT scaled, with EAP/PV reliabilities between .7 and .8 and item parameters ranging from 2.6 to 3.6.

Our presentation concentrated on the courses in *arithmetic* in *Kassel*. In winter semester 2011/12, the beginners (CG) were investigated in a pretest/posttest design, and in winter semester 2012/13 the corresponding cohort (EG) in the same way. The *innovation* in the EG lecture and the accompanying written homework consisted of treating all modes of representation (enactive, iconic, symbolic) for divisibility rules based on position systems, an explicit change between these representations as well as a meta-cognitive explication of connections and a reflection on the relevance for the students learning, whereas in the CG lecture only iconic and symbolic representations were treated, with deliberately no meta-cognitive elements. The innovation in the EG tutorials was a professionalization of the tutors, in particular through tutor training in diagnosis, feedback and learning support. All other aspects of the two courses (especially the lecturer) were as identical as possible. Treatment control took place by analyses of lecture scripts, videographed lectures and students written exercises. Our *research questions* were whether this intervention actually results in a clearly better understanding and a significantly higher achievement progress for EG students compared to CG students in the content areas of position systems and divisibility rules, and whether the performance of both student groups develops equally in all other content areas. Our *sample* consisted of 322 students (mean age M = 22.6 years, SD = 4.8; 80.6% female), mainly in their first year of study (91.6%). The sub-sample relevant for the evaluation consisted of the 131 students who have taken part in both pre- and post-tests (M = 21.9, SD = 4.4, 85.5% female; 95% first year), 69 in CG, 62 in EG. On a 5% level, there was no significant difference between the total sample and the sub-sample of 131. We had a rotation test *design* where each test consists of two sub-tests: items with a direct relation to the intervention (11 items, dimension 1) and items without such a direct relation (41 items, dimension 2).

The main *quantitative results* are given in the following table (obtained by variance analyses with repeated measurements over latent person abilities).

Table 1. Pre and Post Scores With and Without Intervention

	Dim. 1 (Pre)	Dim. 2 (Pre)	Dim. 1 (Post)	Dim. 2 (Post)
CG	1.76(1.20)	-1.45(0.80)	-0.98(1.29)	-0.06(1.09)
EG	-1.91(1.21)	-1.24(0.75)	-0.31(1.50)	0.07(0.89)

Dimension 1: items not related to intervention
Dimension 2: items not related to intervention

Thus, the essential results are:

- The pre-test results of CG and EG are not significantly different (dimension 1: $t(129) = 0.702; p = 0.484$, dimension 2: $t(129) = 1.547; p = 0.124$)
- Both groups show a big achievement progress (so both courses were efficient)
- The achievement progress of the EG in dimension 1 is significantly higher ($F(1, 131) = 8.764; p < .01; \eta^2 = 0.064$)
- In dimension 2 there is equal progress for both groups ($F(1, 131) = 0.199$; $p = .66; \eta^2 = 0.002$)

So, the results show clear advantages of the Experimental Group. This is also revealed by *qualitative analyses* of students solutions. DIF analyses show a significant difference between CG and EG in the post-test for 6 items, all in favour of the EG. For instance, for an item where the students had to bundle balls in the position system with base 5, the solution rates were 6% in the CG and 39% in the EG. Of course, all these results are still disillusioning from a normative point of view.

For the courses in *geometry* in Kassel we obtained similar results. The intervention was analogous: multiple representations, change between them and meta-cognitive explication for the content area of congruence mappings. Here, we found significant advantages for the EG even on the whole test. So an obvious conclusion is that lecture innovations as implemented in Kassel seem promising. However, the aim ought to be to have much more transfer to other content areas, what perhaps can be reached by more meta-cognitive explications and reflections and a stronger connection of the mathematics courses with the corresponding didactics courses. And there have, of course, to be reinforced efforts to raise primary students interest in mathematics, to change their beliefs, to advance strategies, and thus to contribute to a further improvement of their achievement progress.

Engineering students' recognition of university mathematics and their conceptualization of the role of mathematics for their future career

Eva Jablonka, Christer Bergsten

Our presentation reported from a larger study on the transition from upper secondary to tertiary mathematics education in Sweden, motivated by a discussion about decreasing pre-knowledge and pass rates in courses where mathematics is a service subject (manly in engineering programmes). In this study of students enculturation into a disciplinary practice different from school mathematics, embedded in the wider institutional practice of higher education, our interest concerned mathematical, didactical and social issues, with a focus on students perspectives. From the literature we had identified a range of critical issues connected to the transition: Pass rates and participation; alignment of curriculum; differences in teaching formats and assessment; change in expected study habits; differences in atmosphere and sense of belonging; differences in pedagogical awareness of teachers; changes in level of formalisation and abstraction; and an unclear role of mathematics for the students career paths. Our presentation discussed findings related to the last two issues.

The study is a theoretically informed ethnographic investigation based on a set of data comprising individual interviews with 60 students enrolled in different engineering programmes (mechanical engineering, computer technology, physics and electric engineering, industrial economy, and energy and environment) at two universities at three occasions during their first year of study, complemented with their results from a diagnostic mathematics test and exams; a focus group interview with lecturers; observations of lectures and tutorials; and documents (e.g. teaching materials, exam papers). In the generation and analysis of the data we draw on theoretical constructs from the sociology of knowledge, including key notions from Basil Bernstein such as strong/weak knowledge classification, singulars vs regions, and horizontal and vertical knowledge structures, and recognition rules. We assumed that in order to be successful, students need to understand the principles for distinguishing between contexts and recognise the specificity of the discourse, in which they engage.

To investigate students awareness of what counts as legitimate mathematical knowledge in the new institution (university), we were interested in the relation between their recognition of the knowledge criteria reflected in the mathematical discourse at this institution and their success in the mathematics studies during the first year. At the second interview (after their first exams), students were asked to compare excerpts from four different calculus textbooks (A, B, C, D) as to which of these they perceive as being more or less mathematical and why. For the operationalization of the classification and recognition rules, the concept of register from Functional Linguistics was employed. The mathematical register at university level can be characterised by technicality, grammatical metaphor, definitions and proof, impersonal style, etc., giving the ranking CBAD of the four texts with C as the most mathematical. This ranking was also obtained in a group interview with eight of the students lecturers. The student interviews showed that among the low-achieving students, less than half made this ranking, while more than two thirds of the high-achieving students did so. These two groups of students also differed in the type of arguments on which they based their rankings. In contrast to most of the low achieving students, the arguments from high-achieving students were generally not focused on their affective position as a reader (as was the case with the lower achievers), but rather on the identification of what the texts were about and how this was accomplished.

In the third interview, the students were asked about their rationales for choosing university studies in general, for their choice of the particular study programme, and how they viewed the role of mathematics in their future career. Our interest concerned differences in how they perceived the role of mathematics for practice and knowledge development in their fields of engineering and for their imagined future professional practice. We expected different perceptions linked to the more theoretical or more practical nature of the programmes. While their arguments for the choice of study direction pointed to some differences between programmes, what they said about the role of mathematics was, contrary to our expectations, not related to their particular programme choices. Less successful students, however, focussed on a change of an essence in their thinking afforded by their mathematics studies, while the usefulness of mathematics was more emphasised amongst more successful students. All students comments reflected a view of mathematics as a closed subject, unlinked to the social context of its use.

The findings concerning the relation between the students' understanding of the principles for knowledge classification (recognition rules) and their achievement, point to the significance of further investigations into how they acquire these principles and why some students acquire these principles in a relatively short time and some do not. The reasons seem to be related to how students perceive the role of mathematics for their future careers.

About the use of mathematics in signal analysis: Practices in an advanced electrical engineering course

Reinhard Hochmuth, Stephan Schreiber

Mathematics plays an important role in engineering studies and there is a general consensus that difficulties with mathematics are a major reason for high dropout rates, and thus, the teaching and learning of mathematics in engineering studies should be improved.

The project KoM@ING[1] (www.kom-at-ing.de) addresses this situation by exploring the following question by a specific combination of quantitative IRT-models and more qualitative process-oriented studies: Which topics, concepts, heuristic strategies and competences are relevant for being successful in basic theoretical courses like “Technical Mechanics” or “Theoretical Basics in Electrical Engineering” and in more advanced courses like “Signals and Systems” or specific lab courses? In the subproject located in Hannover we focus on a qualitative perspective and suggest approaches that allow to describe and to analyze students’ use of mathematics in advanced engineering contexts, exemparily in courses on Signal Anlysis and System Theory (SST).

Students in engineering courses face mathematics in at least three different institutional contexts: They have to pass courses in higher mathematics (HM), they must apply mathematics in their basic engineering courses and they are confronted with more advanced mathematical concepts like the delta-distribution in advanced courses, e.g. courses on SST. It is not clear how students are able to integrate these variations of mathematics.

In SST courses and text books the objective is to describe and analyze specific electrotechnical phenomena. Thereby mathematics is used to apprehend aspects of this phenomena by laws. To be applied to those phenomena mathematics needs an electrotechnical discourse, which is to some extend necessarily dualistic and contradictory. The indicated contradictory relations between different uses and meanings of mathematical concepts are finally realized within discourses related to different institutional contexts. This kind of epistemological-philosophical consideration contributes to a deeper understanding of differences and contradictions between different mathematical practices.

In view of the different contexts in which mathematics is learned and used in engineering studies, this presentation focuses on epistemic relations between mathematics in higher mathematics courses and mathematics in engineering courses and how these relationships can be reflected or reconstructed by applying the well-established approach of Anthropological Theory of Didactics (ATD) (see [1] and references therein).

ATD in general aims at a precise description of knowledge and its epistemic constitution within different institutional contexts. A basic concept of ATD are praxeologies consisting of a practical (know how, “doing math”) and a theoretical block (knowledge block, discourse necessary for interpreting and justifying

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the practical block, "spoken surround"). The practical block includes a type of task and the relevant solving techniques. The theoretical block covers the technology explaining and justifying the used technique and the theory justifying the underlying technology.

From ATD analyses of widespread SST-textbooks and lecture notes of cooperating lecturers we found on an institutional level that in SST courses an *internal* subject-specific mathematical practice is developed, which is context imbedded and context specific, self-consistent and has specific reasoning patterns that are in particular different from HM-practices. The specific relations of HM- and the SST-related practices can be described as follows: Mathematics-related praxeologies in SST are mainly determined by the electrical engineering context and by HM-praxeologies that are embedded in this context. Furthermore, on the level of routines or calculation rules practical blocks of HM-praxeologies directly appear in practical blocks of SST-praxeologies.

The acquisition of this "pragmatic" mathematical practice with its subject-specific shortcuts and simplifications turns out to be necessary for the ability to act efficiently with respect to tasks posed in this advanced courses. Furthermore, context specific selections of competing solution techniques for given type of tasks are required. Moreover, mathematical symbols represent electrotechnical quantities, that allow and require specific (partly mathematically inconsistent) manipulations.

ATD analyses of videorecorded task processing studies of students solving typical SST-tasks show switchings between HM- and SST-type techniques and corresponding logos-parts during the solution process. For analysing possible causes of the specific selection processes between discourse possibilities we extend the ATD approach by selected concepts from Bernstein's discourse analysis (classification, framing, recognition and realization rules). First analyses support that the SST-discourse is very heterogeneous related to considered subdimensions of classification in contrast to the HM-discourse, which is strongly classified.

First analyses of conducted interview studies with students show that concept images are strongly connected to technical notions and relative to idealized constructs contradictory. Successful students have ideas about characteristics of the different subject-specific discourses but have difficulties to recognize the appropriate discourse in the view of tasks.

Summarizing, we carried out basic subject-specific reconstructions of mathematical practices from an institutional point of view, which give first hints for helpful curricula innovations. In addition, we established connections to concepts for individual learning and task solving processes from an subject-scientific point of view (see [2]).

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Exploring the significance of engineering students' problem solving competencies for task performances

Bettina Roesken-Winter
(joint work with Malte Lehmann)

Mathematics is a key aspect to develop competences in engineering education. One essential competency students need to acquire is mathematical problem solving[3]: "This competency includes on the one hand the ability to identify and specify mathematical problems [...] and on the other hand the ability to solve mathematical problems [...]. What really constitutes a problem is not well defined and it depends on personal capabilities whether or not a question is considered as a problem". What remains unclear is whether problem solving is a general or domain-specific strategy, meaning that one has to distinguish engineering students' problem solving competencies in physics and mathematics. For investigating students' *mathematical* problem solving skills one can focus on the inner structure of processes considering heuristics and beliefs, or on the outer structure considering timing and organization of processes. We chose to focus on heuristics and timing by drawing on Polya's [2] phases of the problem solving process, and Bruder and Collet's [1] work on heuristics which places the methods in the center and explains the process of problem solving by heuristic tools and heuristic strategies. In order to capture students' *physical* problem solving skills we additionally elaborate on the *Epistemic Games*–model provided by Tuminaro and Redish [4], but these results are not presented in this abstract.

All data was collected within the scope of the project KoM@ING which aims at investigating competencies of engineering students. By quantitative studies, our project partners have gained IRT-scaled tests on mathematical and physical-technical skills. In two qualitative studies we investigate students' difficulties when working on these tasks in terms of their problem solving behavior. We respect to the theory framework mentioned above we developed a category system to capture students' problem solving. Our research questions are:

- How can phases and heuristics be classified in terms of their popularity, their universality and their potential?
- In how far can the category system help us to describe and understand the barriers of the mathematics and physics tasks that students encounter in terms of their problem solving skills?
- When working on task variations in Technical Mechanics that imply increasing difficulty, do students succeed by using more and different heuristics?

Participants in study one were 37 engineering students. They worked in small groups on the five easiest and most difficult tasks from the IRT-test. Data was gathered by applying the thinking aloud method and analyzed by using the developed category system. Participants in study two were eight students who worked on selected task variations from Technical Mechanics. In sum 64 tasks processes were selected and thoroughly analyzed

Results show that Polya's phases can be found in all students' work, but that they are differently relevant, as only 34% of all phases could be observed in the easy tasks. When solving the difficult tasks students usually pass through all phases. All students frequently used heuristics tools, but only up to 22% did so when working on the easy tasks. The same applies for the use of heuristic strategies which occurred in up to 30% when students worked on those tasks. While working forward is the main strategy used in in the easy tasks, many different strategies, like conclusion of analogy or principle of decomposition, were found when students solved the difficult tasks. While the incomplete use of Polya's phases and the restricted choice of heuristics have no negative effect on success in the easy tasks, the use of different strategies and passing through all phases increases the success in difficult tasks.

In sum, one can conclude that a reduced number of problem solving phases or underutilization of heuristics have no negative effect on success in easy tasks, but using different strategies and utilizing all problem solving phases increases the probability of success in difficult tasks. In addition, those students were successful who already applied various heuristics while solving easy tasks and kept their behavior when being confronted with difficult tasks. As students' problem solving competencies proved to play a key role for success, implications for mathematics and physics lectures could be to implement an extensive strategy training. Future data analysis will be dedicated to exploring in detail the interplay of physics and mathematics problem solving and how interventions to support students could adequately be designed.

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Short Contributions

A reading course on Galois theory

Hans-Georg Rück

Subject: Algebra, Galois Theory (3rd year math majors)

Text: "A Field Guide to Algebra", by Antoine Chambert-Loir (Springer)

Homework:

- Reading Text (4 - 6 pages per sesssion)
- Preparing at least 2 Questions for other Students

"Lecture":

- Discussing the Text
- Answering Questions
- Further Applications

Results:

- 2 Rounds (2013 with 8 students, 2014 with 7 students)
- Completely Different Performance

Problems:

- Incompleteness (Teacher)
- Completely Different **Reading Skills**

Where Do Students Learn How to Read Mathematics?

Mathematics plus language equals mathematical language?

Maria Specovius-Neugebauer

As it is well known most of the first year students have large difficulties in understanding and formulating mathematics, especially the particulars of the mathematical language. This does not only show up when the task is to formulate a simple proof or verify a definition but also when students are asked just to reproduce a definition. My guess is that these difficulties are not only related to the unfamiliar vocabulary and the (at least in German) extensive use of the subjunctive but also to the fact that at least some of the students are oblivious to the subtleties of language in general.

In this context two case studies were started. The first one is related the first year's Analysis course. The (conventional) lectures were accompanied by a Moodle course which included transparencies with all definitions and results. Furthermore, at the end of each section a special chapter titled "Things you should know" was included. Mostly I formulated questions there in order to encourage the students to reflect wether they really got the definitions, were able to reproduce them, to apply them to examples or to procure counterexamples and wether they understood the relations between various concepts. In addition some exercises dealt especially with the phenomena appearing for example while interchanging the order of quantifiers. The students knew in advance that the final written examination would contain questions which were already formulated in the chapters "Things you should know". Yet a significant part of the students is not able to correctly reproduce the main definitions of an Analysis I course let alone the relations between them. A more detailed analysis exploring the extensive data collected during several courses is still to be done.

The other study concerns the observation of seminar talks, especially for a proseminar (first elementary seminar) or a seminar for teacher students. The difficulty for the lecturer here is that the students do not yet have enough mathematical background to read advanced texts. Using elementary German textbooks often causes the students just to repeat the text. They do not dare to use language in their own way. While confronted with English textbooks students tend to translate the text word by word without grasping the meaning of it. In particular this can be observed with texts like [1], a book which is written in a more informal way leaving several details to the reader. In both cases it struck me that the students did not even realize that quite often they did not understand the argumentation. In order to understand this phenomenon I started to require a written reflection from the students. In these reflections they should formulate how they dealt with the text, in particular which parts they experienced as difficult and why. However, it turned out that only students with a good understanding of the texts also wrote an elaborate reflection.

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Static and emergent shape thinking: Implications for multivariable calculus

Eric Weber

The focus of this poster was a construct called *shape thinking* that characterizes students meanings for graphical representations of functions. While the construct was used by Patrick Thompson of Arizona State University for years in studying teachers' and students' thinking about functions, it has only recently been formally presented through collaborations and discussions between Patrick Thompson (Arizona State University), Kevin Moore (University of Georgia) and Eric Weber (Oregon State University).Briefly, a student engages in *static shape thinking* when she is thinking of a graph as being essentially a solid object, like a piece of wire. The graph may be associated with a particular function, but that association is no more than associating a face with a name. A student engages in *emergent shape thinking* when she sees a graph as a record of variables having varied togetherthe record being of their co-occurring valueswithin some set of conventions (e.g., a rectangular or polar coordinate system). If the student envisions the variables having covaried smoothly and continuously then she sees the graph as connected by virtue of its variables continuous variation.As Thompson

notes in this volume, the calculus is about two fundamental problems: “The first is the problem of knowing how fast a quantity is changing and wanting to know how much of it there is. The second is the problem of knowing how much of a quantity there is and wanting to know how fast it is changing.” Those students who are emergent shape thinkers are well positioned to make sense of a calculus in which these two problems are central issues. In contrast, those students who are static shape thinkers have no sense of a graph as a representation of an invariant relationship between quantities, and consequently, a rate of change as a measure of that relationship between quantities. In my own work, I have observed the consequences of static and emergent shape thinking not only at the introductory calculus level, but also the multivariable calculus level.Here, I briefly consider an

example of rate of change that illuminates some of the issues that arise for students in multivariable calculus. A static shape thinker tends to think about rate of change as the slant of a graph, or the tangent line as a solely geometric object that conveys no sense of variation or covariation. In three dimensions, the static shape thinker then focuses on finding the slant of a graph, or a geometric object which touches the graph at a single point. Very often this does not involve any sense of measuring quantities to determine the rate of change, often avoiding any complexities like measuring rate of change in a direction. In contrast, an emergent shape thinker tends to think about rate of change as a measure of how fast one quantity changes with respect to another and develops measurement processes to determine this covariation. In three dimensions, the student focuses on measuring how fast one quantity changes with respect to two independent quantities, often leading naturally to discussion of how three quantities can covary and how one might measure that relationship. While this is one example amongst many, like integration and graphing, it communicates the importance of attending to shape thinking and supporting emergent shape thinking for students in calculus.

Learning strategies of first year university students

Robin Göller

Introduction & Research Questions. According to Biggs and Tang [1] good teaching focuses in what the student does. In the past years much research has been done to investigate the difficulties accompanying the secondary-tertiary transition (e.g. [2]) but still little is known about what students actually do to cope with these difficulties. Therefore the presented study aims

(1) to describe the learning strategies during the first year.
(2) to expose preconditions affecting these learning strategies.

Methodology. To investigate these questions interviews are being conducted with pre-service teachers and undergraduates with a major in mathematics four times during their first year (two weeks before the first semester, around the forth week of the first semester, last weeks of the first semester, middle of the second semester). The interviews are structured by the theoretical considerations given below, but enable and support the interviewee to present his own thoughts and opinions. The analysis of data is based on the Grounded Theory Method.

Learning Strategies & Investigated Preconditions. Learning strategies may differ depending on the particular learning context. We distinguish the day-to-day learning during the semester and the preparation for exams. In both contexts students need to decide to what extend they focus on tasks or recapitulate the lectures, work in groups or alone and which material resources (lecture notes, books, internet) they use. Additionally, the interviews aim to figure out students' explanations and evaluation of their reported learning strategies.

There is evidence to expect that learning strategies and problem solving strategies are highly based on established experience [3]. On the other hand the secondary-tertiary transition is accompanied by several ruptures [2]. Therefore the interviews aim to identify students' belief systems (mathematical world views) including habits and routines how mathematics is learned and taught based on experiences made in school and how they manage to adapt them to the requirements at university.

Finally, goal-setting and self-reflection are crucial for the regulation of learning strategies. A goal can refer to longtime objectives as well as to a single task or problem. Surely, the exams are an external occasion that provokes a goal-setting and reflection process. Additionally, it is of interest to investigate students' conclusions of this reflection process for their future learning.

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Developing new mathematical results through preparing capstone course materials for secondary teacher candidates

Sergei Abramovich

This presentation is a reflection on the author's work in preparing capstone course materials for prospective secondary mathematics teachers. It started with a spreadsheet-based exploration of the difference equation $f_{n+1} = af_n + bf_{n-1}$ with real coefficients and initial data. In the case of Fibonacci sequence, the notion of Fibonacci sieve of order k was introduced by first deleting every second term of the sequence, then every second term of the remaining sequence, and so on. Repeating this operation k times yields what may be called the Fibonacci sieve of order k described by the above equation with a being the 2^k th Lucas number and $b = -1$. Developed computationally by trial and error, this result can be proved by using Binet's formulas for Fibonacci and Lucas numbers.

Considering the ratios of two consecutive terms of a Fibonacci sieve, the notion of the Golden Ratio (GR) can be extended to include Lucas numbers as generalized GRs. This prompts exploring the equation for other values of parameters using a spreadsheet and various computer algebra systems. From such explorations, two distinct cases emerge: $a^2 + 4b \geq 0$ and $a^2 + 4b < 0$. In the latter case, the polynomials $P_n(x) = x^{mod(n,2)}P_{n-1}(x) + P_{n-2}(x), P_0(x) = 1, P_1(x) = x + 1$ were (eventually) introduced. The roots of these Fibonacci-like polynomials (not to be confused with Catalan's Fibonacci polynomials and Jacobsthal polynomials, two other classes of recursively-defined polynomials associated with Fibonacci numbers) are responsible for the cyclic behavior of the ratios f_{n+1}/f_n, something that demonstrates how generalized GRs can form the strings of numbers of any given length. Permutations with rises turned out being pertinent means for describing the behavior of cycles generated by the smallest root of a Fibonacci-like polynomial regardless of the cycles length. More specifically, it can be proved that a p-cycle generated by such a root can always be associated with one permutation with two rises, one permutation with p rises and $p - 2$ permutations with $p - 1$ rises. An open problem about the polynomials $P_n(x)$ not having complex roots was formulated and then verified numerically for all $n \leq 200$ by using Maple. Whereas this property of the Fibonacci-like polynomials was discovered within a complex context of difference equations, Binet's formulas, and continued fractions, an important didactical aspect of this discovery is that it can be easily presented even at the secondary level by simply using Pascal's triangle as a classic frame of reference. It appears that within this frame a dialogue between mathematics and didactics of mathematics can be mutually beneficial.

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M@thWithApps: More cognitive activation within a mathematics lecture by using social apps for learning?! Results of an empirical study

Rita Borromeo-Ferri

In past years a lot of promising concepts for teaching mathematics in the undergraduate level were introduced. One example for getting a stronger and more lively communication between students and docents in a large lecture is using voting systems ("Clickers"). However research studies regarding this kind of method in the field of mathematics are at the beginning. In the winter term 2012/2013 with the lecture Applied Mathematics the pilot-study of my project M@thWithApps started. Goal of the project was to involve students (becoming primary teachers) more cognitively within the lecture by using the Webapp ARSnova, which was never used before in a mathematics lecture. All of the 140 students received Tablet-PCs from the Service Center for Teaching of the University of Kassel for the whole semester period. Starting point for this project were several thoughts of the docent for improving the teaching in mathematics lectures: "I like that my students get a better view in their own processes of understanding" or "I need real-time feedback to agree immediately on the problems of the students." A central research question of the pilot-study was: Are the students more cognitively activated throughout the lectures when using the App? The App was continuously used during the semester and the exercises. A pre- and post-test design was used to test the cognitive activation. Central results of the pilot study were: The item "While using the App ARSnova I was involved during the lecture" showed, that the sample of the students was divided in two equal groups of acceptance. The interest of the students concerning mathematics increased significantly (Likert-scale from 1-5, 3 items, Cronbachs-Alpha >.64, Pre-test mean 2.27, Post-test mean 2.94, effect: d = 0.96); no control group. Based on these results a second study was conducted in the winter term 2013/2014. Intervention-Study with a new cohort of the lecture "Applied Mathematics"; sample 145 students was developed concerning a specific content for the students: conditional probability (Pre- and Post-Test Design; "Tablet-group" and control group; Pre-Test (20min) - Intervention (70min) - Post-Test (25min)). A central research question was: Which difference in the performance of the students can be stated concerning the introduction of conditional probability with and without using the App? The results showed that the performance of the "Tablet-group" using the App is higher than in the control group. Future prospects of the study are that beneath quantitative testing the learning process of the students shall be reconstructed by interviews or learning diaries. Summarized it can be stated that a stronger cognitive activation happened, because in a "regular" lecture not all 140 students raise their hands on a question the docent is asking. It seems that this way of anonymity is helpful for many students and at the same time make them think about the contents of the lecture.

Pictures and proofs: Which come first?

Deborah Hughes-Hallett

Mathematicians use graphs often, both in their own work and in teaching. This article suggests our pedagogy would benefit from a better understanding of the role of pictures.

One view is that graphs and pictures are merely illustrations—an extra, perhaps even a crutch—while the "real" mathematics is symbolic. From this vantage point, graphs and pictures may be useful, but they are never central.

However, two other roles for pictures are illustrated by the following examples:

- **A graph may be the culmination of analysis**
  - *Example 1*: A continuous function of one variable with constant slope has a graph that is a line.
    The justification depends on the fact that all triangles with endpoints on the graph and sides parallel to the axes are similar. An extension of this argument suggests that a function whose derivative is continuous and increasing "bends up," while one whose derivative is continuous and decreasing "bends down."
  - *Example 2*: The distribution of the sampling distribution of means becomes more "scrunched" as the sample size increases. This conclusion follows from an analysis of the variance of sums of independent random variables.
- **A graph may provide crucial insight that suggests a strategy for solving a problem or for doing a proof.**
  - *Example 3*: What can we conclude if the graph of the logarithm of prices against time is a line? The shape of this graph allows us to conclude that prices are increasing exponentially and suggests the proof.
  - *Example 4*: Consider a definition which may not be familiar to mathematicians: quasiconcavity of a function.[1] The definition is:

    $$f(\alpha x_1 + (1-\alpha)x_2) \geq \min f(x_1), f(x_2).$$

    For most people, an attempt to understand an unfamiliar definition starts with picturing it. Only then can can they move on to seeing its implications. A graph is a central tool in realizing the power of this definition.

Given the importance of graphical reasoning as technology advances, it is time for a much greater understanding of the features of illustrations that aid reasoning.

The concept of marginal cost - one name for two different mathematical objects in mathematics and economics

Frank Feudel

The notion of marginal cost is a very important application of the concept of the derivative in the field of economics and is often used in cost theory. The problem is that it is defined differently in mathematics and in economics. Therefore in the following two different symbols are used (MC_m and MC_e). In mathematics marginal cost (MC_m) is defined as the derivative of a cost function $C : [0;\infty[\to [0;\infty[$ (the independent variable is the output, the dependent variable are the costs): $MC_m(x) := C'(x) = \lim_{h\to 0} \frac{C(x+h)-C(x)}{h}$ (if the limit exists). The corresponding unit would be "Euro per unit of quantity" if the unit of the cost function is Euro.

[1] This property is important to economists.

In economics the notion of marginal cost (MC_e) is understood and sometimes even defined as the following: "Marginal cost is the additional cost, which arises, when the output is increased by one unit." In mathematical terms, this could be described with the following definition: $MC_e(x) := C(x+1) - C(x)$. The corresponding unit would be "Euro" if the unit of the cost function is Euro. But the marginal cost is then often calculated with the derivative.

From a mathematical point of view, the two definitions define different mathematical objects. The marginal cost $MC_m(x)$ is the rate of change of a cost function at a certain output x. The marginal cost $MC_e(x)$ is the value of absolute change of the costs when the output x is increased by one unit. It is no rate. The two objects $MC_e(x)$ and $MC_m(x)$ are often mixed up, even in professional literature of economics. In an exercise book to Wöhe [1], a standard German textbook for economics, one even finds statements like $C'(1)+C'(2)+C'(3)+C'(4) = C(4)$ for every cost function with $C(0) = 0$ meaning that the sum of costs for the first four units in the sense of MC_e is equal to the total cost for the four units. But the calculation of the marginal cost with the derivative leads to this wrong equation.

A main goal of a lecture "Mathematics for students of economics" concerning the concept of marginal cost should be that the students are aware, that $MC_m(x)$ and $MC_e(x)$ define different mathematical objects whose numerical values may also differ. But the two different objects are connected via the approximation formula $C(x + h) - C(x) \approx C'(x) \cdot h$ for h close to 0. This formula is valid because of the connection between the difference quotient and the derivative via the limiting process. Because $h = 1$ can be often considered as small in economics the numerical values of $MC_m(x)$ and $MC_e(x)$ are often pretty close. It is therefore from an economic point of view reasonable to identify both objects.

In an interview study after the mathematical course it should be found out to what extent the students are able to connect the definitions of marginal cost taught in their mathematical and their economical course and to what extent they possess the relevant knowledge about the derivative to justify that connection.

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Our CAT speaks maths

Hans Dietz

Often mathematics courses end up with disappointing results in contrast to the students' perceived own effort. To our experience a main reason for this is a frequent inaptitude to read an understand mathematical texts properly. As a remedy, we introduced "CAT", an in-teaching methodological support for business students at the University of Paderborn. Its main concern is to support students in building valid mental concepts of given mathematical texts. Accordingly, its main procedure *"check-list 'reading"* guides the reading process, in particular when dealing with new concepts. These are its stages:

- *Translating stage:* By carefully "spelling" a given phrase the individual meaning of all its signs, letters, etc. is identified precisely and unambiguously. In this process both the student's written vocabulary and a conscious knowledge management play a key role. In addition, this stage provides a fluent "read out" in order to support communication and mental processing.
- *Conceptual stage:* This stage provides examples, non-examples, and visualizations, as far as appropriate. It should be finished by giving a talk including discussion, or at least being able to do so.
- *Embedding stage:* This stage aims at forming a network of the different concepts under consideration, e.g. by providing important statements, applications, etc.

We recommend the students to use a *concept base*, i.e. a type of form that extends the vocabulary by the results of the forementioned stages (for details see, e.g., [2] and [1]). Our experiences made so far indicate that CAT matches indeed a students' need. The extent of its exploitation and the effects on the students' results are subject to ongoing research.

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Aligning teaching and learning mathematics in university service courses

Simon Goodchild

Teachers of mathematics at university level face a number of challenges, for example: Very large classes; Students motivated to study their programme (engineering, science, etc.), but not necessarily mathematics, which comes as an essential element of the programme; Students from many programmes grouped together for shared core mathematics courses; Inadequate classroom accommodation; Demanding syllabuses; Students inadequate prior knowledge and understanding of mathematics; Competition for students study time (e.g. other courses, part-time employment); Students unprepared for the level of independence in learning expected at university; Internal debate about the purpose of mathematics education to prepare for the workplace, research institute, or academic institution.

These challenges are experienced through: Students poor performance in assessments; Poor attendance at lectures, seminars and problem classes; Unacceptable failure and drop-out rates.

Many teachers are working to address these challenges using digital technologies and other approaches to support students learning, including: Video streamed (web-cast) lectures, video tutorial production, flipped classroom approaches and blended learning techniques; Digital simulations and visualization developing software and program templates that students can manipulate without developing programming expertise; Digital assessment for formative and summative purposes; Mathematical modelling to augment courses with authentic situations.

It seems that much of the innovation misses the desired target (or does not have the desired effect) because it is premised on a false model of the student. The teaching approaches presume students to be accomplished versatile learners, who are able to exercise metacognitive processes and regulate their own learning. Students course evaluation feedback and course results are not sufficient to expose learning approaches. One goal for partnership between mathematics teachers and mathematics education researchers is to achieve a better alignment of innovative teaching and students learning. The mathematics education researcher can be a partner in mathematics teachers learning communities and engage in research into students learning experiences and the effectiveness of learning support. Students can also be offered instruction about how to learn mathematics effectively especially using the digital technologies available.

My concern is to develop a research agenda (or perhaps agendas) that will serve to align better colleagues imaginative and innovative teaching with students learning.

Determinants of math performance of first-year business administration and economics students

Rainer Vosskamp
(joint work with Angela Laging)

Many mathematics educators are confronted with inadequate first-year students' basic mathematical skills. In particular in study programs where mathematics has the role as an auxiliary science tremendous deficits are reported. While there are only few studies our project deals with three research questions focusing on first-year business administration and economics students: (a) What determines math performance? (b) What determines changes in math performance? (c) What inventions foster math performance?

To find answers to the questions in the winter term 2011 first-year business administration and economics students enrolled in a course on "Mathematics for business administration and economics" at University of Kassel were asked to participate in two voluntary tests and two surveys at time $T1$ (week 1 of 14) and $T2$ (week 9 of 14). This allows us to define scales representing the test performance of the students (π^{T1}, respectively π^{T2}) socio-economic and biographic variables (S_j), educational und psychological scales (P_j), scales representing learning strategies (L_j), work habits (W_j) as well as the use of support services (e. g. tutorials, weekly tests with feedback, an open learning environment ("MatheTreff") (U_j).

The theoretical fundament of our analysis is twofold: We follow the concept of education production functions which was developed in economics of education and is often applied to analyze PISA data (see e. g. [1]). And, we use results from theories from different research fields (e. g. education science, psychology, economics of education) in order to motivate predictors used in our model.

First results are received by applying hierarchical block regression techniques. Successively every block of variables is integrated in the model if the determination coefficient R^2 increases significantly. It turns out that the students' performance in $T1$, π^{T1}, is determined by all three blocks of variables S, P and L. In more detail, the participation in a math preparation course (S_3), the kind of graduation (S_7), the grade of graduation (S_{11}), the math grade achieved at school (S_{12}), the students math self-efficacy (P_1) and interest in math (P_3) are examples for variables with statistically significant positive impact. The results for π^{T2} are driven by the impact of π^{T1} on π^{T2}. The variables of block S still account for more variance, again with S_7 and S_{11} as significant predictors. Moreover, the scales representing learning strategies (L) and the use of support services (U_j) show a very limited impact. Further results will be published in [2].

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