Mathematisches Forschungsinstitut Oberwolfach

Report No. 15/2015

DOI: 10.4171/OWR/2015/15

Algebraic Geometry

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15 March – 21 March 2015

Abstract. The workshop covered a broad variety of areas in algebraic geometry and was the occasion to report on recent advances and works in progress. Special emphasis was put on the role of derived categories and various stability concepts for sheaves, varieties, complexes, etc. The mix of people working in areas like classification theory, mirror symmetry, derived categories, moduli spaces, p-adic geometry, characteristic p methods, singularity theory led to stimulating discussions.

Mathematics Subject Classification (2010): 14Cxx, 14Dxx, 14Exx, 14Fxx, 14Gxx, 14Jxx, 32Qxx, 32Sxx.

Introduction by the Organisers

Algebraic geometry is a vast and thriving subject with a countless number of researchers in the field worldwide. While most conferences focus on more specialized topics, this workshop was designed to give a broader view on various aspects of algebraic geometry with the aim to spread ideas across subfields. To make this really happen we targeted researchers with a broad range of interests, working on topics that usually require a mix of different techniques. The result was an intense exchange of ideas, with a very attentive and lively audience throughout the 21 talks of 50 minutes each, and continuing with productive discussions in the lunch breaks and after dinner, often late into the evening. The schedule was sufficiently relaxed to permit free time and to consent to recover energy for discussions. This worked out perfectly well and the atmosphere was generally felt to be extremely stimulating and productive. In fact, many participants expressed their satisfaction about the design of the workshop and the new mix of topics and people. The fact that many young participants got the chance to present their work was generally very appreciated.

The big success of the workshop was due to the high quality of the participants, with a large number of prime players together with many young but already very visible participants. The level of the workshop is also illustrated by the fact that half of the participants came from overseas.

A larger number of talks was devoted to derived categories of coherent sheaves, addressing questions of semi-orthogonal decompositions for a particularly interesting class of Fano manifolds (Alexander Kuznetsov), non-commutative enhancements and deformations related to rational curves in Calabi-Yau varieties (Will Donovan), spaces of stability conditions on abelian and Calabi-Yau varieties (Arend Bayer), a conceptually new approach to stability of objects (Daniel Halpern-Leistner), derived categories of moduli spaces under wall crossing (Matthew Ballard) and moduli spaces of stable objects in derived categories (Yukinobu Toda).

New results on moduli spaces of sheaves on surfaces were presented in talks by Giulia Saccà. In a joint work with Arbarello, she describes the singularities of the moduli space of sheaves on K3 surfaces in strictly semistable points in terms of quiver varieties, which is important for the understanding of wall crossing phenomena. Aspects of mirror symmetry were highlighted in the talks by Helge Ruddat (mirror symmetry for conifold transitions) and Alessio Corti (classification of Fano surfaces via their mirror Landau-Ginzburg potentials).

Talks by János Kollár (Numerical flatness and stability criteria), Zsolt Patakfalvi (Projectivity of moduli spaces of KSBA stable pairs) and Chenyang Xu (Degeneration of Fano Kähler-Einstein manifolds) concentrated more on foundational problems related to moduli theory of algebraic varieties.

In her talk, Enrica Floris explained a recent result with Paolo Cascini addressing deformation invariance of plurigenera for foliations of surfaces. This is in analogy to Siu's result, one of the central results in classification theory, but the case of foliations turns out to be considerably more involved.

The talk by Bhargav Bhatt reported on ongoing work with Peter Scholze that transports a well-known result on the structure of the affine Grassmannian as an ind-projective scheme in characteristic zero to the p-adic case. Besides its fundamental importance, the role of Keel's criteria for basepoint freeness in positive characteristic made this of particular interest to participants working in classification theory.

Rationality questions have always been of special interest to algebraic geometers. In his talk Burt Totaro reported on very recent results on hypersurfaces of not too small degree not being stably rational, which strengthens earlier results by Kollár proving non-rationality.

François Charles explained his new approach to the Tate conjecture for K3 surfaces which is based on a version of Zarhin's trick for K3 surfaces via moduli spaces of stable sheaves and which uses boundedness results for birational equivalence classes of hyperkähler manifolds. The method eventually shows finiteness of (unpolarized) K3 surfaces over finite fields which had been shown to imply the Tate conjecture.

Although not giving talks themselves, the presence of more senior participants like Paolo Cascini, Gerard van der Geer, Ludmil Katzarkov, Jun Li, Mircea Mustata, Mihnea Popa, and Karl Schwede was important for the success of the workshop. There was a lively exchange of ideas between the generations which was appreciated by all.

The Mathematische Forschungsinstitut Oberwolfach provided an excellent environment and inspiring atmosphere for this workshop and we are grateful for its hospitality.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, "US Junior Oberwolfach Fellows".

Workshop: Algebraic Geometry

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Abstracts

Numerical flatness and stability criteria
János Kollár

The lecture reported on some new numerical flatness and stability criteria.

Recall the projective case of a theorem of Hironaka [Hir58].

Theorem 1. *Let T be a regular, 1-dimensional scheme and $X \subset \mathbb{P}^N_T$ a closed subscheme, flat over T. Then*

(1) $t \mapsto \deg(\operatorname{red} X_t)$ *is a lower semicontinuous function on T.*
(2) *If the reduced fibers* $\operatorname{red} X_t$ *are normal then the following are equivalent.*
  (a) $t \mapsto \deg(\operatorname{red} X_t)$ *is locally constant on T,*
  (b) $t \mapsto \chi\big(\operatorname{red} X_t, \mathcal{O}_{\operatorname{red} X_t}(m)\big)$ *is locally constant for every m and*
  (c) *the fibers X_t are reduced.*

We are looking for theorems of this type. The first part should be a general assertion that some invariants related to Hilbert functions are lower or upper semicontinuous on the base. Then, under some geometric assumptions, we aim to show that constancy of the leading coefficient implies constancy of the whole Hilbert function, hence flatness.

Simultaneous canonical models.

The following two results will be treated in [Kol15].

Theorem 2 (Simultaneous canonical models I)**.** *Let S be a seminormal scheme of* $\operatorname{char} 0$ *and $f : X \to S$ a morphism of pure relative dimension n. For $s \in S$ let X^r_s be any resolution of the fiber X_s. Then*

(1) $s \mapsto \operatorname{vol}(K_{X^r_s})$ *is a lower semicontinuous function on S and*
(2) *the canonical models of the X^r_s form a flat family iff this function is locally constant (and positive).*

Part (1) was first observed and proved in [Nak86].

The following is a similar result for normal lc pairs, but the lower semicontinuity of Theorem 2 changes to upper semicontinuity.

Theorem 3 (Simultaneous canonical models II)**.** *Let S be a seminormal scheme of* $\operatorname{char} 0$ *and $f : (X, \Delta) \to S$ a flat morphism with log canonical fibers (X_s, Δ_s). Then*

(1) $s \mapsto \operatorname{vol}(K_{X_s} + \Delta_s)$ *is an upper semicontinuous function on S and*
(2) *the canonical models of the fibers form a flat and stable family iff this function is locally constant.*

See [Kol13b] for the definition and explanation of the stability condition.

Families of Cartier divisors.

Example 4. *Consider the family of quadric surfaces*

$$X := \big(x_1^2 - x_2^2 + x_3^2 - t^2x_0^2 = 0\big) \subset \mathbb{P}^3_{\mathbf{x}} \times \mathbb{A}^1_t.$$

The fiber X_0 is a cone, the other fibers are smooth. Consider the Weil divisors

$$D := \big(x_1 - x_2 = x_3 - tx_0 = 0\big) \quad \textit{and} \quad E := \big(x_1 + x_2 = x_3 - tx_0 = 0\big).$$

The fibers D_t, E_t form a pair of intersecting lines on X_t for every t. It is easy to compute that

(1) $(aD_0 + bE_0)^2 = \frac{1}{2}(a+b)^2 \geq 2ab = (aD_t + bE_t)^2$ *and*
(2) *equality holds* $\Leftrightarrow$ $a = b$ $\Leftrightarrow$ $aD + bE$ *is Cartier.*

We aim to prove that this example is quite typical, as far as intersection numbers are concerned. The following result was conjectured in [Kol13a] and proved there for log canonical fibers. The extension to normal fibers is done in [BdJ14]. Non-normal versions are proved in [Kol14].

Theorem 5 (Relative Cartier criterion)**.** *Let C be a smooth curve and $f : X \to C$ a proper, flat family of normal varieties of dimension n. Let D be a Weil divisor on X such that its restriction D_c is an ample Cartier divisor for every c. Then*

(1) $c \mapsto \big(D_c^n\big)$ *is an upper semicontinuous function on C and*
(2) *D is a Cartier divisor on X iff the above function is constant.*

Ampleness is needed for $n \geq 3$, the main reason is that $\big((-D)^n\big) = (-1)^n\big(D^n\big)$. Thus, on a 3–fold, ample divisors behave anti-symmetrically while divisors pulled-back form a surface behave symmetrically.

Grothendieck–Lefschetz theorems for the local Picard group.

Let us recall the form given in [Gro68].

Theorem 6 (Grothendieck–Lefschetz)**.** [Gro68, XIII.2.1] *Let $(x \in X)$ be an excellent local scheme, $x \in D \subset X$ a Cartier divisor. Set $U := X \setminus \{x\}$, $U_D := D \setminus \{x\}$ and let L be a line bundle on U such that $L|_{U_D} \cong \mathcal{O}_{U_D}$.*

Then $L \cong \mathcal{O}_U$, provided $\operatorname{depth}_x \mathcal{O}_D \geq 3$.

We would like to apply this to families of varieties over a smooth curve $f : X \to C$ with D being a fiber. In this context assuming that the fibers are S_2 is natural but S_3 is not. The following strengthening was conjectured in [Kol13a] and proved there for log canonical fibers. The extension to normal fibers is done in [BdJ14] and the form below is established in [Kol14].

Theorem 7. *Let $(x \in X)$ be a local scheme that is essentially of finite type over a field and $x \in D \subset X$ a Cartier divisor. Set $U := X \setminus \{x\}$, $U_D := D \setminus \{x\}$ and let L be a line bundle on U such that $L|_{U_D} \cong \mathcal{O}_{U_D}$.*

Then $L \cong \mathcal{O}_U$, provided $\operatorname{depth}_x \mathcal{O}_D \geq 2$ *and* $\dim_x D \geq 3$.

Variation of $\mathbb{R}$-divisors.

Let X be a proper, normal algebraic variety of dimension n over a field K and D an $\mathbb{R}$-divisor on X. Set $h^0(D) := \dim_K H^0(X, \mathcal{O}_X(\lfloor D \rfloor))$ and define the *volume* of D as $\mathrm{vol}(D) := \limsup h^0(mD)/(m^n/n!)$. The volume is preserved by $\mathbb{R}$-linear equivalence, but the individual $h^0(mD)$ are not; see Example 9. We claim that, although the volume does not determine $h^0(mD)$, the only way to change it by subtracting or adding an effective divisor is to change the volume.

Theorem 8. [FKL15] *Let X be a proper, normal algebraic variety over a perfect field, D a big $\mathbb{R}$-divisor on X and E an effective $\mathbb{R}$-divisor on X. Then*

(Subtraction version.) *The following are equivalent.*

(1^-) $\mathrm{vol}(D - E) = \mathrm{vol}(D)$.
(2^-) $h^0(mD - mE) = h^0(mD)$ *for all* $m > 0$.
(3^-) $E \leq N_\sigma(D)$; *the negative part of the Zariski–Nakayama decomposition.*

(Addition version.) *The following are equivalent.*

(1^+) $\mathrm{vol}(D + E) = \mathrm{vol}(D)$.
(2^+) $h^0(mD + mE) = h^0(mD)$ *for all* $m > 0$.
(3^+) $\mathrm{Supp}(E) \subseteq \mathbf{B}_+^{\mathrm{div}}(D)$, *the divisorial part of the augmented base locus of* D.

Example 9. *Let $S \to \mathbb{P}^1$ be a minimal ruled surface with a negative section $E \subset S$ and a positive section $C \subset S$ that is disjoint from E. Let $F_1, \dots, F_4$ be distinct fibers. Then $C \sim_{\mathbb{R}} C + (F_1 - F_2) + \sqrt{2}(F_3 - F_4)$.*

Note that $\lfloor mC + m(F_1 - F_2) + m\sqrt{2}(F_3 - F_4) \rfloor$ has negative intersection with E for all real $m > 0$. This implies that, for every $m > 0$ we have

$$h^0\big(mC + m(F_1 - F_2) + m\sqrt{2}(F_3 - F_4)\big) < h^0\big(mC\big).$$

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Moduli of Bridgeland semistable objects on 3-folds and Donaldson-Thomas invariants

Yukinobu Toda
(joint work with Dulip Piyaratne)

1. Introduction

Let X be a smooth projective variety over $\mathbb{C}$. In [Bri07], Bridgeland introduced the complex manifold

$$\mathrm{Stab}(X) \tag{1}$$

called the *space of stability conditions* on $D^b\,\mathrm{Coh}(X)$. Roughly speaking, a point of $\mathrm{Stab}(X)$ is given by data $(Z, \mathcal{A})$, where $Z\colon K(X) \to \mathbb{C}$ is a group homomorphism, $\mathcal{A} \subset D^b\,\mathrm{Coh}(X)$ is the heart of a bounded t-structure, satisfying some axioms. In particular, it gives the notion of Z-semistable objects in $\mathcal{A}$. The space (1) is important in connection with mirror symmetry, birational geometry, counting invariants, etc. However in general, the space (1) is a difficult object to study. At least we need to settle the following issues for the applications:

- We need to prove $\mathrm{Stab}(X) \neq \emptyset$.
- We need to show the existence of nice moduli stacks of semistable objects.

The above issues are settled for $\dim X \leq 2$, but open in $\dim X = 3$. When $\dim X = 3$, the first issue was addressed in [BMT14], and reduced to proving Bogomolov-Gieseker (BG for short) type inequality conjecture among Chern characters of certain two term complexes. The purpose of this study is to solve the second issue for 3-folds satisfying the BG inequality conjecture in [BMT14].

2. BG type inequality conjecture

Let X be a smooth projective 3-fold over $\mathbb{C}$, and take $B, \omega \in \mathrm{NS}(X)_{\mathbb{Q}}$ such that ω is an ample class. In [BMT14], we constructed the heart of a bounded t-structure

$$\mathcal{B}_{\omega,B} \subset D^b\,\mathrm{Coh}(X)$$

given as a tilting of $\mathrm{Coh}(X)$ determined by the ω-slope stability on it. Furthermore, we constructed a *tilt* slope function

$$\nu_{\omega,B}(E) := \frac{\mathrm{ch}_2^B(E)\omega - \mathrm{ch}_0^B(E)\omega^3/6}{\mathrm{ch}_1^B(E)\omega^2} \tag{2}$$

on $\mathcal{B}_{\omega,B}$. Here $\mathrm{ch}^B(E) := e^{-B}\,\mathrm{ch}(E)$. Note that the numerator of the above slope function is the imaginary part of the central charge

$$Z_{\omega,B}(E) := -\int_X e^{-i\omega}\,\mathrm{ch}^B(E).$$

The slope function (2) defines the *tilt semistable objects* on $\mathcal{B}_{\omega,B}$.

Conjecture 1. ([BMT14]) *For any tilt semistable object $E \in \mathcal{B}_{\omega,B}$ with $\nu_{\omega,B}(E) = 0$, we have the inequality*

$$\operatorname{ch}_3^B(E) \leq \frac{1}{18} \operatorname{ch}_1^B(E)\omega^2.$$

The above conjecture is known to be true in the following cases: $X = \mathbb{P}^3$ (cf. [Mac14]), X is a smooth quadric (cf. [Sch]), and X is an etale quotient of an abelian 3-fold (cf. [MP], [BMS]). One of the important observations on Conjecture 1 is that it is equivalent to the following conjecture:

Conjecture 2. ([BMS], [PT]) *For any tilt semistable object $E \in \mathcal{B}_{\omega,B}$, we have the inequality*

$$\begin{aligned}(\operatorname{ch}_1^B(E)\omega^2)^2 - 2\operatorname{ch}_0^B(E)\omega^3 \operatorname{ch}_2^B(E)\omega \\ + 12(\operatorname{ch}_2^B(E)\omega)^2 - 18\operatorname{ch}_1^B(E)\omega^2 \operatorname{ch}_3^B(E) \geq 0.\end{aligned} \tag{3}$$

Conjecture 1 is proved to be equivalent to Conjecture 2 in [BMS] when B and ω are proportional, and in [PT] in general. The advantage of the inequality (3) is that it also ensures the support property after tilting. More precisely, let

$$\mathcal{A}_{\omega,B} \subset D^b \operatorname{Coh}(X)$$

be the further tilting of $\mathcal{B}_{\omega,B}$ determined by the tilt stability constructed in [BMT14]. Let $\operatorname{Stab}_{\omega,B}(X)$ be the space of stability conditions on $D^b \operatorname{Coh}(X)$ whose central charges factor through the map

$$K(X) \ni E \mapsto (\operatorname{ch}_0^B(E)\omega^3, \operatorname{ch}_1^B(E)\omega^2, \operatorname{ch}_2^B(E)\omega, \operatorname{ch}_3^B(E)) \in \mathbb{Q}^4.$$

The inequality (3) shows that

$$\sigma_{\omega,B} := (Z_{\omega,B}, \mathcal{A}_{\omega,B}) \in \operatorname{Stab}_{\omega,B}(X). \tag{4}$$

Below we denote by $\operatorname{Stab}^{\circ}_{\omega,B}(X)$ the connected component of $\operatorname{Stab}_{\omega,B}(X)$ which contains $\sigma_{\omega,B}$.

3. Results

Let X be a smooth projective variety over $\mathbb{C}$. By the result of Lieblich [Lie06], there is an algebraic stack $\mathcal{M}$ locally of finite type which parametrizes objects $E \in D^b \operatorname{Coh}(X)$ satisfying

$$\operatorname{Ext}^{<0}(E, E) = 0.$$

Suppose that there is a stability condition $\sigma = (Z, \mathcal{A})$ on $D^b \operatorname{Coh}(X)$. Then for any $v \in H^*(X, \mathbb{Q})$, we have an abstract substack

$$\mathcal{M}_\sigma(v) \subset \mathcal{M} \tag{5}$$

which parametrizes Z-semistable objects $E \in \mathcal{A}$ with $\operatorname{ch}(E) = v$. A priori, it is not obvious whether $\mathcal{M}_\sigma(v)$ is an algebraic stack nor it is of finite type. Indeed, the inequality (3) is used to show the following:

Theorem 3. ([PT]) *Let X be a smooth projective 3-fold satisfying Conjecture 1. Then for any $\sigma \in \mathrm{Stab}^{\circ}_{\omega,B}(X)$, the stack $\mathcal{M}_{\sigma}(v)$ is a proper algebraic stack of finite type over $\mathbb{C}$, such that the embedding (5) is an open immersion.*

Remark 4. *A similar statement was proved in* [Tod08] *for K3 surfaces, and the same argument is applied to any surface. The result of Theorem 3 is a 3-fold generalization of these works.*

The result of Theorem 3 is used to define the Donaldson-Thomas invariants counting Bridgeland semistable objects on Calabi-Yau 3-folds, as predicted in [KS]:

Theorem 5. ([PT]) *Let X be a smooth projective Calabi-Yau 3-fold satisfying Conjecture 1. Then for any $v \in H^*(X, \mathbb{Q})$, there is a map*

$$\mathrm{DT}_*(v) \colon \mathrm{Stab}^{\circ}_{\omega,B}(X) \to \mathbb{Q} \tag{6}$$

such that $\mathrm{DT}_{\sigma}(v)$ virtually counts σ-semistable objects $E \in D^b \operatorname{Coh}(X)$ with $\mathrm{ch}(E) = v$.

Remark 6. *So far, the only known Calabi-Yau 3-folds satisfying Conjecture 1 are A-type Calabi-Yau 3-folds, that are given by etale quotients of abelian 3-folds* [BMS].

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Hypersurfaces that are not stably rational

Burt Totaro

A central problem of algebraic geometry is to determine which varieties are rational. In particular, we want to know which smooth hypersurfaces in projective space are rational.

An easy case is that smooth hypersurfaces X of degree at least $n+2$ in $\mathbf{P}^{n+1}$ have nonzero sections of the canonical bundle $K_X = \Omega^n_X$ and hence are not rational. There is no known example of a smooth hypersurface of degree at least 4 in any dimension which is rational.

For all $d \geq 2\lceil (n+3)/3 \rceil$, Kollár showed that a very general complex hypersurface of degree d in $\mathbf{P}^{n+1}$ is not ruled and therefore not rational [4], [5, Theorem 5.14]. Very little is known about rationality in lower degrees, except for cubic 3-folds and quintic 4-folds [1], [6, Chapter 3]. Kollár's proof is based on degenerations of hypersurfaces to singular Fano varieties Y in characteristic 2 which are not separably uniruled. In particular, Y is not ruled. It follows that the hypersurfaces in characteristic 0 which degenerate to Y are not ruled and hence are not rational.

Most methods for proving non-rationality give no information about stable rationality. By definition, a variety is *stably rational* if some product of the variety with projective space is rational. Nonetheless, Voisin showed in 2013 that a very general quartic double solid (a double cover of $\mathbf{P}^3$ ramified along a quartic surface) is not stably rational [8]. Her method was to show that these varieties have Chow group of zero-cycles which is not *universally trivial*; that is, the Chow group can increase when the base field is increased.

Colliot-Thélène and Pirutka simplified and generalized Voisin's method. They deduced that very general quartic 3-folds are not stably rational [2]. This was impressive, in that non-rationality of smooth quartic 3-folds was the original triumph of Iskovskikh-Manin's work on birational rigidity, while stable rationality of these varieties was unknown [3].

We show that for all $n \geq 3$ and all $d \geq 2\lceil (n+2)/3 \rceil$, a very general complex hypersurface of degree d in $\mathbf{P}^{n+1}$ is not stably rational [7]. This covers all the degrees in which Kollár showed that these hypersurfaces are non-rational, and a bit more. In particular, very general quartic 4-folds are not stably rational, whereas it was not even known whether these varieties are rational.

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Construction of line bundles on p-adic analogs of the affine Grassmannian

Bhargav Bhatt
(joint work with Peter Scholze)

The affine Grassmannian (associated to the group $G := \mathrm{GL}_n$ over $\mathbf{C}$) is the set $\mathrm{Gr}(\mathbf{C}) := G(\mathbf{C}(\!(t)\!))/G(\mathbf{C}[\![t]\!])$. This set has a natural algebro-geometric structure as an ind-variety, and plays a fundamental role in geometric representation theory. This talk discussed some geometric features of a p-adic analog recently constructed by Zhu [3] by replacing the field $\mathbf{C}$ with $\mathbf{F}_p$, and the discrete valuation ring $\mathbf{C}[\![t]\!]$ with the p-adic integers $\mathbf{Z}_p$.

1. The affine Grassmannian

As the construction of the p-adic analog is motivated by the classical picture, we recall the latter first. Observe that the set $\mathrm{Gr}(\mathbf{C})$ parametrizes $\mathbf{C}[\![t]\!]$-lattices in $\mathbf{C}(\!(t)\!)^{\oplus n}$. Using this interpretation, one defines an algebro-geometric structure on this set as follows:

Definition 1. *Let* Gr *be the functor on* $\mathbf{C}$*-algebras defined by setting* $\mathrm{Gr}(R)$ *to be the set of isomorphism classes of pairs* (E, ϕ)*, where* E *is a finite projective* $R[\![t]\!]$*-module, and* $\phi : E[\frac{1}{t}] \to R(\!(t)\!)^{\oplus n}$ *is an isomorphism.*

It turns out that with these definitions, one has a good algebro-geometric properties:

Theorem 1. *The functor* Gr *is represented by an ind-projective variety.*

We briefly explain the source of projectivity. Consider the subfunctor $\mathrm{Gr}_+ \subset \mathrm{Gr}$ defined by setting $\mathrm{Gr}_+(R)$ to be those $(E, \phi) \in \mathrm{Gr}(R)$ such that $\phi(E)$ is contained in the standard lattice $R[\![t]\!]^{\oplus n} \subset R(\!(t)\!)^{\oplus n}$. Using group actions, it is fairly easy to reduce the projectivity question to that of Gr_+. On Gr_+, there is a universal map $\Phi : \mathcal{E} \to \mathcal{O}_{\mathrm{Gr}_+}[\![t]\!]^{\oplus n}$. The cokernel $\mathcal{Q}$ of this map may be viewed as a $\mathcal{O}_{\mathrm{Gr}_+}$-module via the canonical embedding $\mathcal{O}_{\mathrm{Gr}_+} \to \mathcal{O}_{\mathrm{Gr}_+}[\![t]\!]$; viewed as such, $\mathcal{Q}$ is a vector bundle on Gr_+ (which has finite rank after pullback along any $\mathrm{Spec}(R) \to \mathrm{Gr}_+$), and thus has a well-defined determinant $\det(\mathcal{Q})$; using embeddings into Grassmannians, one checks that this line bundle is ample on Gr_+, proving projectivity.

Remark 1. *One of the most fundamental features of the geometry of* Gr *is the so-called geometric Satake isomorphism between representations of the Langlands dual copy of* GL_n *and certain equivariant perverse sheaves on* Gr*. More generally, a similar picture exists once we replace the group* GL_n *with any reductive group* G *(although the corresponding affine Grassmannian is no longer projective).*

2. The p-adic analog

One of the main ingredients necessary in endowing the set $\mathrm{Gr}(\mathbf{C})$ with an algebro-geometric structure was the functor $R \mapsto R[\![t]\!]$ which associates a flat $\mathbf{C}[\![t]\!]$-algebra to any R-algebra. In order to replace $\mathbf{C}$ with $\mathbf{F}_p$ and $\mathbf{C}[\![t]\!]$ with $\mathbf{Z}_p$ in this story, one must thus have functorial p-adic deformations for $\mathbf{F}_p$-algebras. As it is not usually possible to find a *single* p-adic deformation of a given finite type $\mathbf{F}_p$-algebra, we restrict to the following setting:

Definition 2. *An $\mathbf{F}_p$-algebra R is* perfect *if Frobenius is an isomorphism; likewise, an $\mathbf{F}_p$-scheme X is perfect if $\mathcal{O}_X(U)$ is perfect for each affine $U \subset X$. Let* Perf *be the category of perfect $\mathbf{F}_p$-algebras.*

Each $\mathbf{F}_p$-scheme X admits a perfection X_{perf} characterized by $X_{\mathrm{perf}}(R) = X(R)$ for any $R \in \mathrm{Perf}$; explicitly, the cover $X_{\mathrm{perf}} \to X$ is purely inseparable, and constructed by extracting all p-power roots of all (local) functions on X. Any perfect ring R admits a unique (up to unique isomorphism) p-adic deformation $W(R)$ given by the Witt vector construction; this association globalizes to attach a flat p-adic formal scheme $W(X)$ to any perfect scheme X. Moreover, one can do algebraic geometry in the world of perfect schemes: there exist robust notions of finitely presented maps, proper maps, closed immersions, projective maps, ample line bundles, algebraic spaces, etc; to a crude approximation, each notion may be defined by applying the functor $X \mapsto X_{\mathrm{perf}}$ to the corresponding notion in usual algebraic geometry (though intrinsic definitions exist). In this world, Zhu defines:

Definition 3. *Let* Gr *be the functor on* Perf *defined by setting* $\mathrm{Gr}(R)$ *to be isomorphism classes of pairs (E, ϕ) where E is a finite projective $W(R)$-module, and $\phi : E[\frac{1}{p}] \to W(R)[\frac{1}{p}]^{\oplus n}$ is an isomorphism.*

It makes sense to ask if the analog of Theorem 1 is true in this setting. Zhu was not quite able to prove this, but, using a quotient construction and some stack theory, he showed:

Theorem 2 (Zhu)**.** *The functor* Gr *is representable by an ind-proper (in the perfect sense) algebraic space.*

Remark 2. *Zhu has proved an analog of the geometric Satake isomorphism in the p-adic setting; curiously, his proof uses geometric Satake in equicharacteristic p!*

Zhu conjectured that Gr is ind-projective. The main obstruction here is the lack of a ring homomorphism $R \to W(R)$ analogous to the map $R \to R[\![t]\!]$ used in §1. Indeed, in defining the line bundle $\det(\mathcal{Q})$ in §1, we relied crucially on the ability to regard an $R[\![t]\!]$-module as an R-module via restriction of scalars along $R \to R[\![t]\!]$. This is not possible in the p-adic case, so one is naturally confronted with the following question:

Question 1. *Given a perfect ring R and a map $f : M \to N$ of finite projective $W(R)$-modules that $f[\frac{1}{p}]$ is an isomorphism, the cokernel Q is a finite $W(R)$-module of projective dimension* 1*. Is there a well-defined determinant $\widetilde{\det}(Q) \in$* $\mathrm{Pic}(R)$ *which coincides with the obvious definition when Q is killed by p?*

One of our main results is that the preceding question has a positive answer. In order to explain why, it is convenient to work in a more general framework. Hence, we introduce the following notation:

Notation 1. *For any perfect scheme X, let $D_{\mathrm{perf}}(X)$ be the derived category of perfect complexes on X, and similarly for the formal scheme $D_{\mathrm{perf}}(W(X))$; let $D_{\mathrm{perf},X}(W(X)) \subset D_{\mathrm{perf}}(W(X))$ be the full subcategory spanned by those K that become acyclic after inverting p. Let $K(X)$ be the K-theory space of $D_{\mathrm{perf}}(X)$, and write $K_X(W(X))$ for the K-theory space of $D_{\mathrm{perf},X}(W(X))$. The association $K \mapsto \det(K)$, as defined by Knudsen-Mumford* [2]*, extends naturally to give an additive map $\det : K(X) \to \mathrm{Pic}(X)$ of spaces*[1]*; here $\mathrm{Pic}(X)$ is viewed as a groupoid or, equivalently, a 1-truncated space.*

In the notation of Question 1, the map f defines an object of $D_{R,\mathrm{perf}}(W(R))$. Note that there is an obvious functor $D_{\mathrm{perf}}(R) \to D_{\mathrm{perf},R}(W(R))$, which induces an additive map $K(R) \to K_R(W(R))$ of spaces. Hence, to answer the previous question, it is enough to extend det across $K(R) \to K_R(W(R))$. This is indeed the case:

Theorem 3. *There is a natural additive map $\widetilde{\det} : K_X(W(X)) \to \mathrm{Pic}(X)$ extending $\det : K(X) \to \mathrm{Pic}(X)$.*

To understand this, consider a special case: $X = \mathrm{Spec}(R)$ is affine, and R is the perfection of a finitely presented $\mathbf{F}_p$-algebra R_0. Then any $K \in D_{\mathrm{perf},R}(W(R))$ admits a finite filtration with graded pieces being (shifted) R-modules M_i. If R is smooth (so R_0 is classically smooth), then, one can find such a filtration with the M_i's themselves being R-perfect. This implies that $D_{\mathrm{perf}}(R) \to D_{\mathrm{perf},R}(W(R))$ induces an isomorphism on K-theory, which immediately constructs $\widetilde{\det}$ in this case; explicitly, one sets $\widetilde{\det}(K) := \otimes_i \det(M_i) \in \mathrm{Pic}(R)$. If R_0 is not smooth, this recipe does not work. Instead, our proof of Theorem 3 reduces to the smooth case using de Jong's alterations, and the following non-flat descent result for glueing line bundles along alterations:

Theorem 4. *Vector bundles satisfy effective descent for the h-topology on* Perf.

Here the h-topology on Perf is the Grothendieck topology where the covers are generated by perfections of proper surjective finitely presented maps and fppf covers. The analog of Theorem 4 is completely false in the classical setting: if X is any scheme, then $X_{red} \to X$ is an isomorphism after h-sheafification, yet vector bundles on X and X_{red} are typically inequivalent. With Theorem 4 in hand, the following result is not unreasonable:

Theorem 5. *The functor* Gr *is a represented by an ind-projective scheme.*

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[1]This map is monoidal (i.e., one has natural isomorphisms $\det(K \oplus L) \simeq \det(K) \otimes \det(L)$ for $K, L \in D_{\mathrm{perf}}(X)$), but *not* symmetric monoidal (i.e., we cannot switch K and L functorially); this issue can be fixed by replacing $\mathrm{Pic}(X)$ with the groupoid of $\mathbf{Z}$-graded line bundles, and will be ignored in this report.

This verifies Zhu's conjecture, and our proof is independent of Theorem 2. For the proof, note that Theorem 3 furnishes the line bundle on Gr, whose equicharacteristic analog gives projectivity via an embedding into Grassmannian, as indicated in §1. In our case, however, we cannot reduce to classical Grassmanians to verify the ampleness of this line bundle; instead, our proof[2] is a direct inductive analysis of Gr in terms of the natural stratification (obtained by measuring relative positions of lattices associated to points in Gr with respect to a standard lattice), together with Keel's theorem [1], which facilitates movements between strata.

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Degeneration of Fano Kähler-Einstein manifolds

Chenyang Xu

(joint work with Chi Li, Xiaowei Wang)

Constructing moduli spaces for higher dimensional algebraic varieties is a fundamental problem in algebraic geometry. For Fano varieties, it is a difficult question in algebraic geometry what kind of general Fano varieties we should parametrize in order for us to obtain a nicely behaved moduli space, especially if we aim to find a compact one, and how to construct it. Nevertheless, the recent breakthrough in Kähler-Einstein problem, namely the solution to the Yau-Tian-Donaldson Conjecture ([1, 2, 3, 6]) is a major step forward, especially for understanding those Fano manifolds with Kähler-Einstein metrics. Furthermore, it implies that the right limits of smooth Kähler-Einstein manifolds form a bounded family.

In [4], we use the analytic results they established to investigate the geometry of the compact space of orbits which is the closure of the space parametrizing smooth Fano varieties. The first question is about the uniqueness of the degeneration, which in general fails for Fano varieties. However, if we post the Kähler-Einstein metric condition, the uniqueness of the degeneration holds.

More precisely, let $\mathcal{X} \to C$ be a flat family over a pointed smooth curve $(C, 0)$ with $0 \in C$. Suppose

(1) $-K_{\mathcal{X}/C}$ is relatively ample;
(2) for any $t \in C^\circ := C \setminus \{0\}$, $\mathcal{X}_t$ is smooth and $\mathcal{X}_0$ is klt;
(3) $\mathcal{X}_0$ is K-polystable.

[2] In fact, one may prove Theorem 5 directly along these lines, avoiding Theorem 3 and K-theory, provided one proves and employs certain more refined descent results in perfect algebraic geometry.

Then

(1) after a possible shrinking of C around 0, we can conclude that $\mathcal{X}_t$ is K-semistable for all $t \in C^\circ$ and K-stable if we assume further $\mathcal{X}_0$ has a discrete automorphism group;
(2) for any other flat projective family $\mathcal{X}' \to C$ satisfying (1)-(3) as above and

$$\mathcal{X}' \times_C C^\circ \cong \mathcal{X} \times_C C^\circ,$$

we can conclude $\mathcal{X}'_0 \cong \mathcal{X}_0$;
(3) $\mathcal{X}_0$ admits a weak Kähler-Einstein metric. If we assume further that $\mathcal{X}_t$ is K-polystable, then $\mathcal{X}_0$ is the Gromov-Hausdorff limit of $\mathcal{X}_t$ endowed with the Kähler-Einstein metric for any $t \to 0$.

With all this knowledge, then we can show that there is a well-behaved orbit space for smoothable K-semistable Fano varieties. We show that for $N \gg 0$, let Z^* be the semi-normalization of the open set of $\mathrm{Chow}(\mathbb{P}^N)$ parametrizing all smoothable K-semistable Fano varieties in $\mathbb{P}^N$. Then the algebraic stack $[Z^*/\mathrm{SL}(N+1)]$ admits a proper *good moduli space* $\mathcal{KF}_N$. Furthermore, for sufficiently large N, $\mathcal{KF}_N$ does not depend on N.

In [5], we explore the projectivity of the CM line bundle Λ_{CM}, which can be showed to descend on $\mathcal{KF}_N$. It has a continuous metric coming from the Deligne pairing which is the extension of the Weil-Petersson metric on the open locus $\mathcal{KF}_N^\circ$ parametrizing smooth Kähler-Einstein manifolds. For this reason, we expect Λ_{CM} is ample on $\mathcal{KF}_N$. However, our current method only yields that it is big and nef, and its restriction to $\mathcal{KF}_N^\circ$ is ample.

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On some Küchle fourfolds

Alexander Kuznetsov

In the classification of Fano threefolds the main role is played by threefolds X with $\mathrm{Pic}(X) = \mathbb{Z} \cdot K_X$ (called prime Fano threefolds). In this class there are 10 deformation families, distinguished by their anticanonical degree $d_X = (-K_X)^3$, which is even, does not exceed 22, and is not equal to 20. The bottom part of the list, i.e., X with $d_X \leq 8$, is comprised by complete intersections in weighted projective spaces, while the top part, i.e., X with $d_X \geq 10$, consists of zero loci of regular sections of equivariant vector bundles on Grassmannians.

In 1995 Oliver Küchle has classified in [2] all prime Fano fourfolds which can be realized as zero loci of regular sections of equivariant vector bundles on Grassmannians. His list consists of 21 examples among which 6 are not really prime fourfolds (three of them have larger Picard group and three have divisible canonical class). Geometry of some of these varieties was discussed in [3] and [1]. Moreover, in [4] it was shown that two families of prime Küchle fourfolds actually coincide.

In the talk we discuss all these varieties with a special stress on the structure of their derived categories. In particular, basing on results of Casagrande we showed that the variety (b9) (the Hilbert scheme of lines on the intersection of two 5-dimensional quadrics) has a full exceptional collection of length 48. The case of the other fourfold which could have an exceptional collection (variety (c3)) is not so clear.

We also discussed varieties which could have a K3 category as a semiorthogonal component. In the original list there are three such varieties. We showed that the variety (d3) is isomorphic to the blowup of $(\mathbb{P}^1)^4$ along a K3 surface and that the variety (c7) is isomorphic to the blowup of a cubic fourfold along a Veronese surface. The main result of the talk is a birational description of the most interesting of such varieties, i.e., variety (c5). We show that such X is a half-anticanonical section of a fivefold Y which fits into a diagram

$$\begin{array}{ccccccc} & E & \longrightarrow & \tilde{Y} & \longleftarrow & E' & \\ \swarrow & & \pi \swarrow & & \searrow \pi' & & \searrow \\ F & \longrightarrow & Y & & Y' & \longleftarrow & Z \end{array}$$

where $F = \mathrm{Fl}(1, 2; 3)$ is the flag variety, Z is a $\mathbb{P}^1$-fibration over a del Pezzo surface of degree 6, Y' is a hyperplane section of the symplectic Lagrangian Grassmannian $\mathrm{SGr}(3, 6)$, and π and π' are the blowups with centers F and Z and exceptional divisors E and E'. We conjecture that Y has a full exceptional collection of length 12 with a rectangular Lefschetz structure. This would imply existence of a K3 category in the derived category of X. Finally, we conjecture that the Hilbert scheme of rational twisted cubic curves in X is birational to a hyperkähler fourfold.

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Projectivity of moduli spaces of KSBA stable pairs and applications

Zsolt Patakfalvi
(joint work with Sándor Kovács)

In this talk, we will work over an algebraically closed base field k of characteristic 0. Stable (log-)varieties are higher dimensional generalizations of stable ((weighted) pointed) curves, where the latter were introduced by Mayer [May69] and Mumford [Mum64], Knudsen [Knu83a, Knu83b] and Hassett [Has03]. The moduli space of stable log varieties contains an open part parametrizing log-canonical models. Hence it can be regarded as a compactification of a moduli space parametrizing "birational equivalence classes", where the quotes are warning that one should be careful about what birational equivalence means for pairs.

Let us recall now carefully the definitions. Recall that a *demi-normal* variety is an equidimensional S_2 reduced scheme over k which has nodes in codimension one.

Definition 1. *A* pair (Z, Γ) *consist of an equidimensional demi-normal variety* Z *and an effective* $\mathbb{Q}$*-divisor* $\Gamma \subset Z$*. A* stable log-variety (Z, Γ) *is a pair such that*

(1) Z *is proper,*
(2) (Z, Γ) *has* slc singularities*, and*
(3) *the* $\mathbb{Q}$*-Cartier* $\mathbb{Q}$*-divisor* $K_Z + \Gamma$ *is* ample*.*

For the definition of slc singularities the reader is referred to [Kol13, 5.10]

Definition 2. *A* family of stable log-varieties*,* $f : (X, D) \to Y$ *over a normal variety consists of a pair* (X, D) *and a flat proper surjective morphism* $f : X \to Y$ *such that*

(1) D *avoids the generic and codimension* 1 *singular points of every fiber,*
(2) $K_{X/Y} + D$ *is* $\mathbb{Q}$*-Cartier, and*
(3) (X_y, D_y) *is a connected stable log-variety for all* $y \in Y$*.*

After having made the necessary definitions, we turn to the main statement. Since there are multiple suggestions for the moduli functor at this point and more our hoped, we make a flexible statement prescribing only the value of a functor over normal test schemes. To do this precisely we need the next definitions, the first of which has technical reasons: without a condition as there, the considered functors would not be proper.

Definition 3. *A set* $I \subseteq [0, 1]$ *of coefficients is said to be* closed under addition*, if for every integer* $s > 0$ *and every* $x_1, \ldots, x_s \in I$ *such that* $\sum_{i=1}^{s} x_i \leq 1$ *it holds that* $\sum_{i=1}^{s} x_i \in I$*.*

Definition 4. *Fix* $0 < v \in \mathbb{Q}$, $0 < n \in \mathbb{Z}$ *and a finite set of coefficients* $I \subseteq [0,1]$ *closed under addition. A functor* $\mathcal{M} : \mathfrak{Sch}_k \to \mathfrak{Sets}$ *(or to groupoids) is* a moduli functor of stable log-varieties of dimension n, volume v and coefficient set I, *if for each normal* Y,

(1)

$$\mathcal{M}(Y) = \left\{ \begin{array}{c|l} (X,D) & \text{(1) } f \text{ is a flat morphism,} \\ \big\downarrow f & \text{(2) } D \text{ is a Weil-divisor on } X \text{ avoiding the generic and the codimension 1 singular points of } X_y \text{ for all } y \in Y, \\ Y & \text{(3) for each } y \in Y,\ (X_y, D_y) \text{ is a stable log-variety of dimension } n, \text{ such that the coefficients of } D_y \text{ are in } I, \text{ and } (K_{X_y} + D_y)^n = v, \text{ and} \\ & \text{(4) } K_{X/Y} + D \text{ is } \mathbb{Q}\text{-Cartier.} \end{array} \right\},$$

and the line bundle $Y \mapsto \det f_* \mathcal{O}_X(r(K_{X/Y} + D))$ *associated to every family as above extends to a functorial line bundle on the entire (pseudo-)functor for every divisible enough integer* $r > 0$.

Having made the necessary definitions we state our main theorem.

Theorem 1. *Any algebraic space that is the coarse moduli space of a moduli functor of stable log-varieties with fixed volume, dimension and coefficient set (as defined above) is a projective variety over* k.

The above theorem has been known for varieties (without log-, so without boundary divisor) by the work of Kollár [Kol90] which was extended by Fujino [Fuj12]. We also note that there are functors as above that yield a coarse moduli space which is an algebraic space (e.g., [KP15, Sec 5]).

The above main theorem can be applied to prove a new version of the logarithmic subadditivity of Kodaira dimension conjecture. Earlier results on this conjecture are [Fuj14a, Fuj15, Fuj14b, Nak04]. Recall also that the Kodaira dimension of an arbitrary (so not necessarily projective) algebraic variety X is defined via finding a resolution X_0' of X with a projective compactification X' such that $D' := (X' \backslash X_0')_{\mathrm{red}}$ is simple normal crossing, and then setting $\kappa(X) := \kappa(K_{X'} + D')$. Our theorem is roughly speaking the logarithmic version of [Kol87] and it is as follows.

Theorem 2.
(1) *If* $f : (X,D) \to (Y,E)$ *is a surjective map of log-smooth projective pairs with coefficients at most* 1, *such that* $D \geq f^*E$ *and* $K_{X_\eta} + D_\eta$ *is big, where* η *is the generic point of* Y, *then*

$$\kappa(K_X + D) \geq \kappa\left(K_{X_\eta} + D_\eta\right) + \kappa(K_Y + E).$$

(2) *Let* $f : X \to Y$ *be a dominant map of (not necessarily proper) algebraic varieties such that the generic fiber has maximal Kodaira dimension. Then*

$$\kappa(X) \geq \kappa\left(X_\eta\right) + \kappa(Y).$$

Another application is a joint work with Chenyang Xu concerning the CM line bundle. This line bundle was originally defined in differential geometry and it is connected to the existence of Kähler-Einstein metrics. In particular, its degree over any (modular) curve yields the Donaldson-Futaki invariant of the corresponding family.

Theorem 3. *(joint with Chenyang Xu* [PX15]*) The CM line bundle on the moduli space of stable log-varieties is ample.*

Earlier, the above result was proven only for the locus parametrizing canonically polarized manifolds [Sch12].

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Birational geometry of cluster varieties

Mark Gross
(joint work with Paul Hacking, Sean Keel)

The talk discusses aspects of the paper [2].

Cluster algebras were introduced by Fomin and Zelevinsky in [4]. Fock and Goncharov introduced a more geometric point of view in [3], introducing the $\mathcal{A}$ and $\mathcal{X}$ cluster varieties constructed by gluing together "seed tori" via birational maps known as cluster transformations.

In this talk, motivated by our study of log Calabi-Yau varieties initiated in the two-dimensional case in [1], we give a simple alternate explanation of basic constructions in the theory of cluster algebras in terms of blowups of toric varieties. Each seed roughly gives a description of the $\mathcal{A}$ or $\mathcal{X}$ cluster variety as a blowup of a toric variety, and a mutation of the seed corresponds to changing the blowup description by an elementary transformation of a $\mathbb{P}^1$-bundle. Certain global features of the cluster variety not obvious from the expression as a union of tori are easily seen from this construction. For example, it gives a simple geometric explanation for the Laurent phenomenon (originally proved in [5]). From the blowup picture it is clear that the Fock-Goncharov dual basis conjecture, particularly the statement that tropical points of the Langlands dual $\mathcal{A}$ parameterize a natural basis of regular functions on $\mathcal{X}$, can fail frequently.

In the talk, we explained the basic philosophical point of view demonstrating how a study of log Calabi-Yau varieties can naturally lead to the basic notions of cluster algebras. We then describe how cluster transformations, which a priori are birational maps between algebraic tori, can be viewed naturally as isomorphisms between blowups of certain associated toric varieties. In this manner, cluster transformations can be interpreted as elementary transformations, a standard procedure for modifying $\mathbb{P}^1$-bundles in algebraic geometry. This procedure blows up a codimension two center in a $\mathbb{P}^1$-bundle meeting any $\mathbb{P}^1$ fibre in at most one point, and blows down the proper transform of the union of $\mathbb{P}^1$ fibres meeting the center. The key result is a precise description of the $\mathcal{X}$, principal $\mathcal{A}$ cluster varieties and $\mathcal{A}$ cluster varieties with general coefficients up to codimension two in terms of a blowup of a toric variety. The toric variety and the center of the blowup is specified very directly by the seed data determining the cluster variety.

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Noncommutative enhancements of families of rational curves

Will Donovan
(joint work with Michael Wemyss)

Pairs of varieties with equivalent derived categories of coherent sheaves may now be obtained by a range of different constructions. For a given such pair, characterising the set of equivalences, taken up to isomorphism, is a deep and difficult problem. This set is dependent on the automorphism groups of the varieties, and also on phenomena intrinsic to the derived category, about which much remains to be discovered. I report on new results in this area, for pairs of complex 3-folds obtained by flops, making use of noncommutative deformation theory.

I begin by explaining the relevance of noncommutative deformations for simple flops [DW1], before describing work in progress for general flops [DW2, DW3].

Simple flops. To illustrate how the non-uniqueness of equivalences can be interesting in a well-known example, we consider a flop $\phi\colon X \dashrightarrow X'$ of smooth projective 3-folds, with indeterminacy locus $C \cong \mathbb{P}^1$. In this case there exists a pair of canonical derived equivalences

$$\Phi\colon \mathrm{D}(X) \overset{\sim}{\longleftrightarrow} \mathrm{D}(X') : \Phi'$$

given by Fourier–Mukai transforms associated to the graph of ϕ [Bri02]. These equivalences are not, however, mutually inverse. The following theorem characterises the difference between Φ and $(\Phi')^{-1}$ in terms of noncommutative deformations.

Theorem 1. [DW1] *For a flopping $C \cong \mathbb{P}^1$ as above, write $\mathcal{E}_C$ for the universal noncommutative deformation of $E_C = \mathcal{O}_C(-1) \in \mathrm{Coh}(X)$. Then there exists a distinguished triangle of Fourier–Mukai transforms*

$$\mathrm{FM}(\mathcal{E}_C^\vee \overset{\mathbb{L}}{\underset{A}{\boxtimes}} \mathcal{E}_C) \longrightarrow \mathrm{Id}_{\mathrm{D}(X)} \longrightarrow (\Phi' \circ \Phi)^{-1} \longrightarrow \tag{1}$$

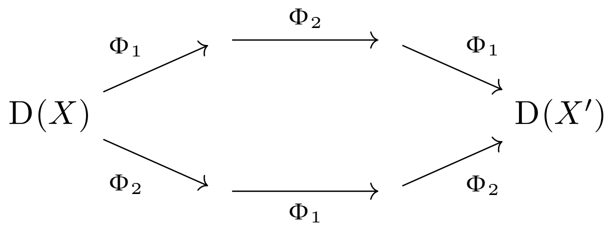
acting on $\mathrm{D}(X)$, where A is a $\mathbb{C}$-algebra representing an appropriate functor of noncommutative deformations.

The use of deformations to understand the flop–flop functor $\Phi' \circ \Phi$ was pioneered by Toda in the cases where C is a $(-2,0)$-curve or $(-1,-1)$-curve [Tod07]. In these case, the sheaf E_C has a 1-parameter deformation, or trivial deformations, respectively. In the further case where C is a $(-3,1)$-curve, we find that the deformation algebra A in Theorem 1 has 2 generators, and is *always* noncommutative [DW1, §3.4], showing the necessity of noncommutative deformations for proving this general result.

Theorem 1 may be compactly reformulated in the language of spherical functors, with the universal deformation $\mathcal{E}_C$ inducing a spherical functor $S\colon \mathrm{D}(A) \to \mathrm{D}(X)$ with twist T_S [ST01, Ann07, AL13]. We then have $T_S^{-1} \cong \Phi' \circ \Phi$.

General flops. There also exist 3-fold flops with indeterminacy locus a nodal curve C, given by a tree of $\mathbb{P}^1$s. As above, there is an equivalence Φ associated to such a flop. Furthermore, the connected components C_i of the (reduced) curve C may flop individually, each yielding an equivalence Φ_i. We give a simple example, before indicating results in the general case.

Example 1. *Take a nodal curve $C \subset X$ with two irreducible components C_i, each flopping individually, and having no infinitesimal deformations. In this case the corresponding flop functors Φ_i satisfy the braid relation expressed in the commutative diagram below. Here X' is obtained by flopping the nodal curve C.*



The unlabelled vertices of the hexagon correspond to 3-folds obtained by flops of the C_i, and their respective transforms.

Braiding. In the general case, where the curves C_i may have infinitesimal noncommutative deformations, we establish braid relations as follows in [DW3].

$$\underbrace{\Phi_1 \circ \Phi_2 \circ \Phi_1 \circ \cdots}_{d} \cong \underbrace{\Phi_2 \circ \Phi_1 \circ \Phi_2 \circ \cdots}_{d} \quad (d \geq 3) \tag{2}$$

The degree d of the relation is determined by the chamber structure of a certain space of GIT stability conditions associated with a neighbourhood of the curve C, as investigated in [W, §§5–7]. In the example above, this chamber structure comes from the familiar A_2 Coxeter hyperplane arrangement. In general, it comes from a simplicial hyperplane arrangement which is not of Coxeter type.

Using the relations (2), we construct actions of certain generalised pure braid groups on $\mathrm{D}(X)$, associated to the topology of the simplicial hyperplane arrangement.

Deformations for nodal curves. The generalisation of Theorem 1 to a flop of a nodal curve C, with n components C_i, proceeds as follows. We take

$$E_C = \bigoplus \mathcal{O}_{C_i}(-1) \in \mathrm{Coh}(X).$$

The direct sum decomposition of E_C means that it naturally has a noncommutative deformation over the semisimple base ring $\mathbb{C}^n$, with the corresponding universal deformation $\mathcal{E}_C$ being a module over a certain augmented $\mathbb{C}^n$-algebra A. The appropriate noncommutative deformation technology will be treated in [DW2], following for instance [Lau02, Eri07, Seg08, ELO09]. Then the flop equivalences Φ and Φ' fit into a distinguished triangle as in (1) above, with the tensor product now taken over the $\mathbb{C}^n$-algebra A. We expect that the spherical functor formulation also generalises to this setting, giving a spherical functor $S\colon \mathrm{D}(A) \to \mathrm{D}(X)$.

Noncommutative enhancements. We think of the $\mathbb{C}^n$-algebra A as a noncommutative enhancement of the disjoint union of n points corresponding to the semisimple base ring $\mathbb{C}^n$, arising from the geometry of the discrete family of curves C_i. This enhancement is key for the construction of a spherical functor S in the setting above, and it is hoped that such enhancements will play an interesting role in other situations.

Continuous families. In 4-folds, and in higher dimensions, there exist flops of continuous families of rational curves. In this setting we expect a noncommutative enhancement of the base of the family to yield a spherical functor. Such an enhancement, albeit for the case of a contraction of a divisor in dimension 3, is studied in [W, §4.4].

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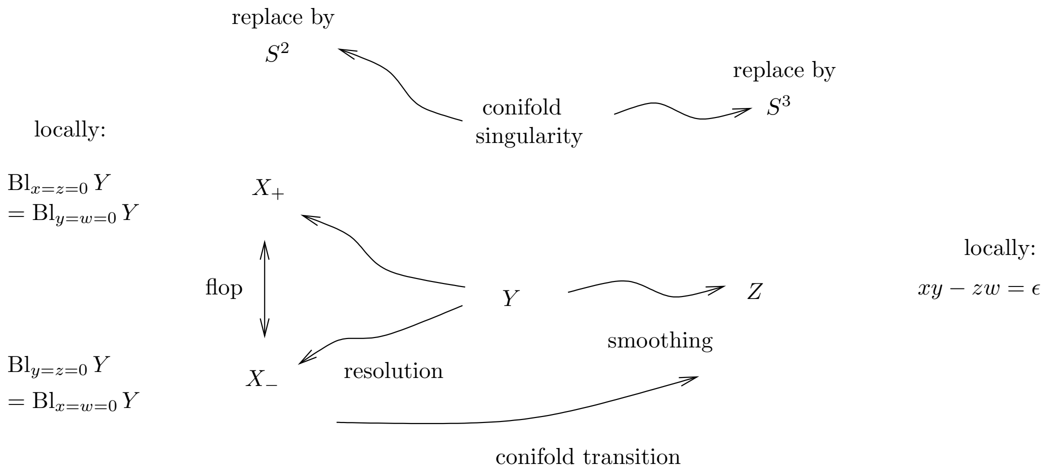
A proof of Morrison's Conjecture

Helge Ruddat

(joint work with Bernd Siebert)

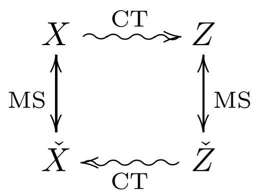
Introduction. A *conifold* is a three-dimensional Calabi-Yau variety Y which is smooth outside of a set of ordinary double points $(xy - zw = 0)$. We consider two ways to associate to Y a smooth Calabi-Yau manifold, one via a small resolution which we call X, the other by smoothing we denote Z.



Resolutions exist analytically but not necessarily symplectically. Smoothings exist symplectically but not necessarily analytically. A *conifold transition* $X \rightsquigarrow Z$ is the process of traversing from the left to the right in the above diagram. (Note that there are two choices for the small resolution related by a flop.)

The significance of conifold transitions derives in part from the *web conjecture* by Miles Reid [Re87]: *Any two Calabi-Yau threefolds are connected by a sequence of conifold transitions and their inverses.*

The motivation for us has been to prove the following conjecture by Morrison [Mo97]: *For $X \rightsquigarrow Z$ a conifold transition and $\check{X}, \check{Z}$ mirror symmetry duals of X, Z respectively, one finds $\check{X}, \check{Z}$ are related by a conifold transition $\check{Z} \rightsquigarrow \check{X}$.*



Obstructions. Let $C_1, ..., C_p$ denote the S^2s replacing the conifold points in a small analytic resolution X of Y. Let $[C_i] \in H_2(X, \mathbb{Z})$ denote the homology class of C_i. Friedman [Fr86] and Tian [Ti92] proved that

$$\exists \text{ a smoothing } Z \text{ of } Y \text{ analytically} \quad \Leftrightarrow \quad \exists \lambda_i \neq 0 : \lambda_1[C_1] + ... + \lambda_p[C_p] = 0.$$

Mirror symmetry interchanges complex and symplectic moduli. Morrison's conjecture inspired Smith-Thomas-Yau [STY02] to consider the S^3s that replace the conifold points in a symplectic smoothing Z of Y, denoting these $L_1, ..., L_p$ and their classes in $H_3(Z, \mathbb{Z})$ by $[L_i]$. They prove

$$\exists \text{ a resolution } X \text{ of } Y \text{ symplectically} \quad \Leftrightarrow \quad \exists \lambda_i \neq 0 : \lambda_1[L_1] + ... + \lambda_p[L_p] = 0.$$

Considering mirror symmetry with $\check{Z} \rightsquigarrow \check{X}$ the conjectural mirror transition to $X \rightsquigarrow Z$, we expect C_i to correspond to L_i under mirror symmetry and this can be explained geometrically with the Strominger-Yau-Zaslow picture and has been suggested according to this by Gross and Ruan.

Mirror symmetry for conifolds. Mirror symmetry is about maximal degenerations, so in order to even make sense of Morrison's conjecture, we need to study conifold transitions in maximally degenerating families. The most versatile construction for mirror symmetry duals has been given by Gross-Siebert [GS11], so we prove that their construction extends to the mirror symmetry of conifolds $Y \leftrightarrow \check{Y}$. This actually means that we start with a maximally degenerating family $\mathcal{Y} \to \operatorname{Spf} \mathbb{C}[\![t]\!]$ where the generic fibre is a projective conifold Y. We obtain a mirror family $\check{\mathcal{Y}} \to \operatorname{Spf} \mathbb{C}[\![t]\!]$ and obtain the mirror dual of Y as the generic fibre $\check{Y}$. The lifting of these families to analytic families is ongoing work.

The proof of Morrison's Conjecture. Our main result is the following. Let $S, \check{S}$ denote the set of conifold points in $S, \check{S}$. Equivalent are

(1) there is an analytic smoothing $\mathcal{Z}$ of $\mathcal{Y}$,
(2) $\exists \alpha \in H^1(Y \setminus S, \Omega^2_{Y\setminus S})$ that maps non-trivially to each summand in $H^2_S(Y, \Omega^2_Y) \cong \mathbb{C}^S$,
(3) $\exists \alpha \in H^1(\check{Y} \setminus \check{S}, \Omega_{\check{Y}\setminus \check{S}})$ that maps non-trivially to each summand in $H^2_{\check{S}}(\check{Y}, \Omega_{\check{Y}}) \cong \mathbb{C}^{\check{S}}$,
(4) there is a projective resolution $\check{\mathcal{X}}$ of $\check{\mathcal{Y}}$.

What we mean by (1) and (4) is that the families $\mathcal{Z}$ and $\check{\mathcal{X}}$ are over $\operatorname{Spf} \mathbb{C}[\![t]\!]$ again and their generic fibres are a smoothing and resolution of Y and $\check{Y}$ respectively.

It should be noted how this result identifies the Friedman-Tian obstruction with the Smith-Thomas-Yau one under mirror symmetry. Given a conifold transition $\mathcal{X} \rightsquigarrow \mathcal{Z}$ with both ends projective, we thus obtain the mirror dual one $\check{\mathcal{Z}} \rightsquigarrow \check{\mathcal{X}}$ proving Morrison's conjecture.

Sketch of Proof. Let Θ denote the tangent sheaf. The proof is based on identifying all of the following maps.

$$H^1(Y \setminus S, \Omega^2_{Y\setminus S}) \to H^2_S(Y, \Omega^2_Y)$$

$$H^1(Y \setminus S, \Theta_{Y\setminus S}) \to H^2_S(Y, \Theta_Y)$$

$$H^1(B \setminus S, i_*\Lambda) \to H^2_S(B, i_*\Lambda)$$

$$H^1(Y \setminus S, \Omega^1_{\check{Y}\setminus \check{S}}) \to H^2_S(\check{Y}, \Omega^1_{\check{Y}})$$

The first identification follows from Y being a Calabi-Yau threefold, so $\Omega^2_Y \cong \Theta_Y$. The second is derived from Hodge theory of toric degenerations. Here, B denotes the dual intersection complex of the central fibre of $\mathcal{Y}$ which is an affine manifold with singularities in codimension two. The inclusion of its smooth locus is denoted $i : B_{\mathrm{sm}} \to B$, Λ is the local system of integral tangent vectors on B_{sm} and $S \subset B$ denotes the set of four-valent points in $B_{\mathrm{sm}} \setminus B$. The third line controls the smoothings $\mathcal{Z}$ of $\mathcal{Y}$ in the Gross-Siebert program linking (1) and (2). The identification with the last row above is again Hodge theory of toric degenerations so we find equivalence with (3) in the theorem. For (4) one checks that the classes in (3) give Weil divisors that one can blow up for a small resolution and any small resolution comes from such a Weil divisor.

□

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Θ-reductive moduli problems, stratifications, and applications

Daniel Halpern-Leistner

For most moduli problems in algebraic geometry, the existence of a quasi-projective fine moduli space fails in myriad ways. The language of stacks is necessary to deal with the issue that objects can have (finite or infinite) automorphism groups, but even then the moduli problem can be "too big."

Example 1. *Let $\mathscr{M}_{R,D}$ be the moduli of vector bundles over a smooth curve C or rank R and degree D. This algebraic stack is locally finite type but not quasi-compact: indeed the quantity $\dim H^0(C,E)$ is semicontinuous and obtains arbitrarily large values, so the moduli functor can not receive a surjective map from a quasi-projective scheme.*

The solution to this problem is the Harder-Narasimhan (HN) filtration: Every unstable bundle admits a unique filtration whose associated graded pieces are semistable with slopes arranged in decreasing order. This leads to the Shatz stratification

$$\mathscr{M}_{R,D} = \mathscr{M}^{ss}_{R,D} \cup \bigcup \mathcal{S}_\alpha \tag{1}$$

where $\mathscr{M}^{ss}_{R,D}$ admits a projective good moduli space, and $\mathcal{S}_\alpha$ denotes the moduli of vector bundles of a fixed HN type (indexed by the rank and degrees of the associated graded pieces $\alpha = (r_1,\ldots,r_p;d_1,\ldots;d_p)$). Assigning a bundle to its associated graded defines a map $\mathcal{S}_\alpha \to \mathcal{Z}_\alpha := \mathscr{M}^{ss}_{r_1,d_1} \times \ldots \times \mathscr{M}^{ss}_{r_p,d_p}$ whose fibers are affine spaces.

We present a program for "solving" other moduli problems in this manner, by introducing a type of stratification which we call a Θ-stratification.

0.1. **Θ-reductive stacks.** Our main character is the algebraic stack $\Theta := \mathbb{A}^1/\mathbb{G}_m$. A vector bundle on Θ is the same as a vector space with a weighted descending filtration. This leads to the observation that the mapping stack $\underline{\mathrm{Map}}(\Theta, B\mathrm{GL}_n)$ is algebraic – it is an infinite disjoint union of quotients of partial flag varieties by GL_n. In fact, this is a special case of a more general result

Theorem 1. *Let $\mathcal{X}$ be a (derived) locally finite type algebraic stack with quasi-affine diagonal over a field. Then $\underline{\mathrm{Map}}(\Theta, \mathcal{X})$ is a locally finite type algebraic stack with quasi-affine diagonal.*

Example 2. *In the example of $\mathscr{M}_{R,D}$, a T-point of $\underline{\mathrm{Map}}(\Theta, \mathscr{M}_{R,D})$ a vector bundle E on $C \times T$ along with a flat family of weighted descending filtrations, i.e. sequence $\cdots E_{w+1} \subset E_w \subset \cdots \subset E$ of vector bundles which stabilizes to E on the right and 0 on the left, and such that $\mathrm{gr}_w E_\bullet = E_w/E_{w+1}$ is a vector bundle for all w.*

Evaluation of a map $\Theta \to \mathcal{X}$ at the point $1 \in \mathbb{A}^1$ defines a map of algebraic stacks $\mathrm{ev}_1 : \underline{\mathrm{Map}}(\Theta, \mathcal{X}) \to \mathcal{X}$ which corresponds to forgetting the data of the filtration in the previous example.

Definition 1. [7] *Let $\mathcal{X}$ be a locally finite type stack with quasi-affine diagonal over a field. Then $\mathcal{X}$ is* Θ-reductive *if for any finite type k-scheme T over $\mathcal{X}$, the connected components of $T \times_{\mathcal{X}} \underline{\mathrm{Map}}(\Theta, \mathcal{X})$ are proper over T.*

Example 3. *If X is affine and G is a reductive group acting on X, then $\mathcal{X} = X/G$ is Θ-reductive. However, this fails for more general quasi-projective X.*

Example 4. *Let X be a projective scheme, and fix a t-structure on $D^b\,\mathrm{Coh}(X)$ satisfying certain properties (Noetherian, generic flatness, boundedness of generalized Quot-spaces; see* [7]*). Then the moduli stack of flat families of objects in $D^b\,\mathrm{Coh}(X)^\heartsuit$ is Θ-reductive. In particular, the usual moduli stack of flat families of coherent sheaves is Θ-reductive.*

The stack $\mathscr{M}_{R,D}$ is not Θ-reductive, but it is an open substack of the Θ-reductive stack $\underline{\mathrm{Coh}}(C)$, the moduli of flat families of coherent sheaves on C. We hope to study many more moduli problems by finding natural enlargements which are Θ-reductive and then constructing Θ-stratifications as follows.

0.2. **Θ-stratifications.** It turns out that among all weighted descending filtrations of an unstable vector bundle the numerical invariant

$$\mu(f : \Theta \to \mathscr{M}_{R,D}) = \frac{\sum_w w\,(R\deg(\mathrm{gr}_w E_\bullet) - D\mathrm{rk}(\mathrm{gr}_w E_\bullet))}{\sqrt{\sum_w w^2 \mathrm{rk}(\mathrm{gr}_w E_\bullet)}}$$

is maximized by a unique (up to simultaneous rescaling) choice of weights on the Harder-Narasimhan filtration, which lets us canonically identify points on $\mathcal{S}_\alpha \subset \mathscr{M}_{R,D}$ with points on the mapping stack. On an arbitrary stack $\mathcal{X}$, one can construct a function generalizing the function μ from a pair of cohomology classes in $H^2(\mathcal{X};\mathbb{Q})$ and $H^4(\mathcal{X};\mathbb{Q})$.

In general, we define a Θ-stratification to be an open substack $\mathcal{S} \subset \underline{\mathrm{Map}}(\Theta, \mathcal{X})$ such that $\mathrm{ev}_1 : \mathcal{S} \to \mathcal{X}$ is a locally closed immersion (satisfying some additional nice properties). Note that in general $\underline{\mathrm{Map}}(\Theta, \mathcal{X})$ will have many more connected components than $\mathcal{X}$, and $\mathcal{S}$ plays the role of the disjoint union of Shatz strata.

Theorem 2. [3] *Let $\mathcal{X}$ be a Θ-reductive stack. Then any locally convex, bounded numerical invariant defines a Θ-stratification of $\mathcal{X}$.*

The notion of a Θ-stratification is a simultaneous generalization of the Shatz stratification as well as the canonical stratification of the unstable locus in GIT. Our theorem leads to new examples of Θ-stratifications, not known to be related to GIT, such as a stratification of the stack of flat families of objects in the heart of a Bridgeland stability condition on the derived category of a $K3$ surface.

0.3. **Some applications.** Kirwan's surjectivity theorem [8] says that for a smooth (local) quotient stack with Θ-stratification the restriction $H^*(\mathcal{X}; \mathbb{Q}) \to H^*(\mathcal{X}^{ss}; \mathbb{Q})$ is surjective. This leads to beautifully explicit formulas [9] expressing the difference in the Poincare polynomial of $\mathcal{X}$ and $\mathcal{X}^{ss}$ as a sum of contributions from each stratum. Recently these results have been categorified to a structure theorem [2, 1] for the derived category $D^b(\mathcal{X})$, where a direct sum decomposition of $H^*(\mathcal{X})$ is categorified by an infinite semiorthogonal decomposition of $D^b(\mathcal{X})$.

Using the modular interpretation as a mapping stack allows one to generalize this result beyond the smooth global quotient situation.

Theorem 3. [5, 6] *Let $\mathcal{X}$ be a locally finite type derived*[1] *algebraic stack with a quasi-affine diagonal. If $\mathcal{X}$ has a derived Θ-stratification, then there is an infinite semiorthogonal decomposition*

$$D^- \operatorname{Coh}(\mathcal{X}) = \langle \ldots, D^- \operatorname{Coh}(\mathcal{X}^{ss}), \mathcal{A}_\alpha^{w_\alpha}, \mathcal{A}_\alpha^{w_\alpha+1}, \ldots, \mathcal{A}_{\alpha'}^{w_{\alpha'}}, \ldots \rangle$$

where $\mathcal{A}_\alpha^w \simeq D^- \operatorname{Coh}(\mathcal{Z}_\alpha)^w$. When $\mathcal{X}$ is quasi-smooth, then a version of this theorem holds with $D^b \operatorname{Coh}$ instead of $D^- \operatorname{Coh}$.

Here $\mathcal{Z}_\alpha$ are the "centers" of the strata (generalizing the example of the Shatz stratification above), the category $D^- \operatorname{Coh}(\mathcal{Z}_\alpha)^w$ is the full subcategory with weight w with respect to a canonical generic $\mathbb{G}_m$-stabilizer in $\mathcal{Z}_\alpha$. Algebraic symplectic stacks always satisfy the $D^b \operatorname{Coh}$ version of the theorem, so we have

Corollary 3.1. *Let $\mathcal{X}$ be an algebraic symplectic stack with a Θ-stratification. Then $K(D^b \operatorname{Coh}(\mathcal{X})) \to K(D^b \operatorname{Coh}(\mathcal{X}^{ss}))$ is a split surjection.*

Specializing to global quotients over the ground field $\mathbb{C}$, it is possible to recover the Atiyah-Segal equivariant topological K-theory from $D^b \operatorname{Coh}(X/G)$ [4]. This leads to some surprising implications for the topology of singular stacks:

Corollary 3.2. *Let $\mu : X \to \mathfrak{g}^*$ be an algebraic moment map for a Hamiltonian action of a reductive group G on a projective-over-affine algebraic symplectic variety X, all over $\mathbb{C}$. Let $X_0 = \mu^{-1}(0)$, and let $G_c \subset G$ be a maximal compact subgroup. Then $K_{G_c}^{top}(X_0) \to K_{G_c}^{top}(X_0^{ss})$ is a split surjection.*

[1]Even if $\mathcal{X}$ is a classical (non-derived) stack, the derived structure on the strata $\mathcal{S}_\alpha \subset \underline{\mathrm{Map}}(\Theta, \mathcal{X})$ will differ from the naive structure as a classical locally closed substack.

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Wall crossing for derived categories of moduli spaces of sheaves on rational surfaces

Matthew Robert Ballard

A central question in the theory of derived categories is the following: given a smooth, projective variety X, how does one find interesting semi-orthogonal decompositions of its derived category, $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X)$? Historically, two different parts of algebraic geometry have fed this question: birational geometry and moduli theory.

This talk focuses on the intersection of birational geometry and moduli theory. Namely, given some moduli problem equipped with a notion of stability, variation of the stability condition often leads to birational moduli spaces. As such, it is natural to compare the derived categories in this situation. Let us consider the well-understood situation of torsion-free rank two semi-stable sheaves on rational surfaces [EG95, FQ95, MW97]. The flipping of unstable strata under change of polarization was investigated to understand the change in the Donaldson invariants. It provides the input for the following result, which can be viewed as a categorification of the wall-crossing formula for Donaldson invariants.

Theorem 1. *Let S be a smooth projective rational surface over $\mathbb{C}$ with effective anti-canonical bundle and let L_- and L_+ be ample lines bundles on S separated by a single wall defined by unique divisor ξ satisfying*

$$L_- \cdot \xi < 0 < L_+ \cdot \xi$$
$$0 \leq \omega_S^{-1} \cdot \xi.$$

Let $\mathcal{M}_{L_\pm}(c_1, c_2)$ be the $\mathbb{G}_m$-rigidified moduli stack of Gieseker $L_\pm$-semi-stable torsion-free sheaves of rank 2 with first Chern class c_1 and second Chern class c_2.

There is a semi-orthogonal decomposition

$$\mathrm{D^b}(\mathrm{coh}\,\mathcal{M}_{L_+}(c_1,c_2)) = \Big\langle \underbrace{\mathrm{D^b}(\mathrm{coh}\,H^{l_\xi}),\ldots,\mathrm{D^b}(\mathrm{coh}\,H^{l_\xi})}_{\mu_\xi},\ldots$$
$$\underbrace{\mathrm{D^b}(\mathrm{coh}\,H^{0}),\ldots,\mathrm{D^b}(\mathrm{coh}\,H^{0})}_{\mu_\xi},\mathrm{D^b}(\mathrm{coh}\,\mathcal{M}_{L_-}(c_1,c_2))\Big\rangle$$

where

$$\begin{aligned} l_\xi &:= (4c_2 - c_1^2 + \xi^2)/4 \\ H^l &:= \mathrm{Hilb}^l(S) \times \mathrm{Hilb}^{l_\xi - l}(S) \\ \mu_\xi &:= \omega_S^{-1}\cdot\xi \end{aligned}$$

with the convention that $\mathrm{Hilb}^0(S) := \mathrm{Spec}\,\mathbb{C}$.

Theorem 1 follows from the general technology that goes under the heading of windows in derived categories. Windows provide a framework for addressing the central question put forth above; they are a machine for manufacturing interesting semi-orthogonal decompositions of $\mathrm{D^b}(\mathrm{coh}\,X)$. They have a rich history with contributions by many mathematicians and physicists. Here we build off the ideas of [H-L15, BFK12] to extend the semi-orthogonal decompositions of [BFK12] in the setting of smooth Artin stacks using an appropriate type of groupoid in Białynicki-Birula strata.

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Invariance of plurigenera for foliations on surfaces

Enrica Floris
(joint work with Paolo Cascini)

Let X be a smooth algebraic surface. A foliation $\mathcal{F}$ on X is, roughly speaking, a locally free coherent subsheaf $T_{\mathcal{F}}$ of the tangent bundle of X. The dual of $T_{\mathcal{F}}$ is called the canonical bundle of the foliation $K_{\mathcal{F}}$. In the last few years birational methods have been successfully used in order to study foliations. More precisely, geometric properties of the foliation are translated into properties of the canonical bundle of the foliation. One of the most important invariants describing the properties of a line bundle L is its Kodaira dimension $\kappa(L)$, which measures the growth of the global sections of L and its tensor powers. The Kodaira dimension of a foliation $\mathcal{F}$ is defined as the Kodaira dimension of its canonical bundle $\kappa(K_{\mathcal{F}})$. In their fundamental works, Brunella and McQuillan give a classfication of foliations on surfaces on the model of Enriques-Kodaira classification of surfaces.

The next step is the study of the behaviour of families of foliations. Brunella proves that, for a family of foliations $(X_t, \mathcal{F}_t)$ of dimension one on surfaces, satisfying certain hypotheses of regularity, the Kodaira dimension of the foliation does not depend on t.

By analogy with Siu's Invariance of Plurigenera, it is natural to ask whether for a family of foliations $(X_t, \mathcal{F}_t)$ the dimensions of global sections of the canonical bundle and its powers depend on t. During the talk we discussed to which extent an Invariance of Plurigenera for foliations is true and under which hypotheses on the family of foliations it holds. After giving basic definitions and some examples, we presented the following result

Theorem 1 (Cascini, Floris). *Let $(X_t, \mathcal{F}_t)_{t\in\Delta}$ be a family of foliations with reduced singularities. Then, for any sufficiently large positive integer m, the dimension $h^0(X_t, \mathcal{O}_{X_t}(mK_{\mathcal{F}_t}))$ is constant for all $t \in \Delta$.*

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(Uni)-rationality of Ueno-type manifolds and complex dynamics.

Fabrizio Catanese
(joint work with Keiji Oguiso and Alessandro Verra)

1. Ueno type varieties

Let k be any field of characteristic $\neq 2, 3$ containing a primitive third root of unity ω, respectively a primitive fourth root of unity i.

Let E be either the anharmonic elliptic curve E_4, with affine equation

$$E_4 = \{(x, y) | y^2 = x(x^2 - 1)\},$$

which admits the following order 4 automorphism:

$$g_4 : (x, y) \mapsto (-x, iy),$$

(the field of g_4-invariant rational functions is the field $k(x^2)$).

Or let E be the equianharmonic elliptic curve E_6, with affine equation

$$E_6 = \{(x, y)|y^2 = x^3 - 1\},$$

birational to the curve with affine equation

$$E'_6 = \{(x, y)|y^6 = x^2(x - 1)\},$$

which admits the following order 6 automorphism:

$$g_6 : (x, y) \mapsto (x, -\omega y)$$

(such that the field of g_6-invariant rational functions is the field $k(x)$).

The **Ueno-type manifolds** are the minimal resolutions of singularities $X_{n,m}$ of the quotient of E_m^n by the diagonal action of g_m.

It is classical that these manifolds are rational for $n \leq 2$, and the arguments of Ueno ([10])show that

- the Kodaira dimension of $X_{n,6}$ is 0 if $n \geq 6$ and $-\infty$ if $n \leq 5$,
- the Kodaira dimension of $X_{n,4}$ is 0 if $n \geq 4$ and $-\infty$ if $n \leq 3$.

Later Kollár showed a more general result: if Z has trivial canonical bundle and a finite group G acts on Z, either the quotient Z/G has Kodaira dimension 0, or it is uniruled.

Ueno asked about separable unirationality of the manifold $X_{3,4}$, and Oguiso asked the similar question for $X_{n,6}$, $3 \leq n \leq 5$.

Interest for these open questions was revived by Campana, who showed that $X_{3,4}$ is rationally connected.

The rebirth of interest in the rationality of these manifolds stems also from complex dynamics and entropy, since these manifolds admit an action by $GL(n, \mathbb{Z})$ (and indeed by $GL(n, R_m)$, where R_m is the cyclotomic ring $\mathbb{Z}[i]$, resp. $\mathbb{Z}[\omega]$).

In fact, $GL(n, \mathbb{Z})$ and $GL(n, R_m)$ act on the product E_m^n and since we divide by a central automorphism the action descends to the quotient, and then extends biregularly to $X_{n,m}$ since the resolution is just obtained by blowing up the singular points of the quotient.

2. Recent results and questions

In the case of the Ueno manifolds, Oguiso and Truong proved in [4] that $X_{3,6}$ is rational. They not only proved the rationality of $X_{3,6}$, but also showed that in this way one gets a rational variety with a primitive automorphism of positive entropy. Here, according to a concept introduced by De-Qi-Zhang, an automorphism $f : X \to X$ is said to be birationally inprimitive if there is a nontrivial rational fibration $\pi : X \to Y$, and a birational automorphism ϕ of Y such that $\pi \circ f = \phi \circ \pi$. De-Qi-Zhang showed that if a threefold X admits a primitive birational automorphism of positive entropy, then either X is a torus, or it is a $\mathbb{Q}$-Calabi-Yau manifold, or X is rationally connected.

Campana proved that $X_{3,4}$ is rationally connected, unirationality was proven by Catanese, Oguiso and Truong in [1], and later Colliot-Thélène proved rationality in [2] using the conic bundle description of [1].

Unirationality of $X_{3,4}$ was proven in a joint work with Oguiso and Truong, later Colliot Thelene showed, using our conic bundle realization, that $X_{3,4}$ is indeed rational (even if the conic bundle is a non trivial element of the Brauer group).

Together with Oguiso, I proved recently the following

Theorem 1. $X_{4,6}$ *is unirational.*

Open questions are:

Question 1. Is $X_{4,6}$ rational ?
Question 2. Is $X_{5,6}$ unirational ?

3. Methods of proof

In my talk I explained the methods of proof for the cases of $X_{3,4}$ and $X_{4,6}$.

The first step is computational, and consists in finding a minimal system of generators for the field of invariant rational functions on E_m^n: for instance, in the case of $X_{3,6}$ one finds three generators, hence these three elements are algebraically independent and the variety is k-rational.

In the case of $X_{3,4}$ one finds 4 generators t_1, t_2, u_1, u_2 and one equation, which can be written as a diagonal quadratic form of the form

$$u_1^2 - A(t_1, t_2)u_2^2 - B(t_1, t_2) = 0.$$

We get, birationally, a conic bundle over the projective plane, and the method of [1] consisted in showing that the conic bundle has a bisection Z which is rational: then the pull back of the conic bundle to Z is a conic bundle with a section hence it is rational.

Colliot-Thélène proved that the conic bundle does not have a section: in fact, if K is the function field of the plane, $A, B \in K$ and to such a diagonal conic over K one associates a central algebra over K, $M_{A,B}$, generated by $1, i, j, ij = -ji$ and defined by $i^2 = A, j^2 = B$.

By a general theorem the algebra is a division algebra if and only if the conic does not have any K-rational point (in the contrary case $M_{A,B} \cong M(2, 2, K)$). Moreover, two such conics are K-isomorphic if and only if the corresponding algebras are isomorphic (they yield the same element of the Brauer group).

Colliot-Thélène proved also that in this case the conic is isomorphic to one of the form

$$u_1^2 + t_1 u_2^2 + t_2 = 1,$$

hence the function field is generated by t_1, u_1, u_2 and $X_{3,4}$ is rational.

We proved the unirationality of $X_{4,6}$, showing that it is birational to a diagonal cubic surface S over the function field $K := k(t_1, t_2)$

$$A(t_1, t_2)(u_1^3 - 1) + B(t_1, t_2)(u_2^3 - 1) + C(t_1, t_2)(u_3^3 - 1) = 0.$$

The surface S admits 27 rational points (just let u_j be a cubic root of 1).

Then, by a theorem of B. Segre, it follows that S is unirational, and we observe that the unirational degree is at most 6.

Using other classical results of B. Segre, Swinnerton-Dyer and Colliot-Thélène on cubic surfaces and on diagonal cubic surfaces we show that the surface S is K-unirational, but it is not K- rational.

Is it possible, like it was done for the conic bundle case, to change the cubic surface birationally and prove the rationality of $X_{4,6}$?

Sandro Verra observed that the coefficients $A(t_1, t_2), B(t_1, t_2), C(t_1, t_2)$ correspond to a very special system of plane cubics, yielding the Del Pezzo surface of degree 2 which is the double cover of $\mathbb{P}^2$ branched over a complete quadrilateral.

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Curve counting on abelian surfaces and threefolds

Qizheng Yin

(joint work with Jim Bryan, Georg Oberdieck, Rahul Pandharipande)

In the past 20 years, much progress has been made in the study of curve counting invariants on $K3$ surfaces, and their connections to modular forms. This includes the Yau-Zaslow formula for genus 0 invariants, proven by Klemm-Maulik-Pandharipande-Scheidegger [2], and the Katz-Klemm-Vafa formula governing invariants in all genera, recently proven by Pandharipande-Thomas [4]. More recently, the work of Oberdieck-Pandharipande [3] reveals a beautiful (conjectural) link between curve counting on $K3 \times E$ and the Igusa cusp form χ_{10}.

Our joint project [1] deals with analogous problems for abelian surfaces and threefolds. As in the $K3$ case, reduced theories are required in the presence of holomorphic 2-forms. Some additional difficulty is brought by the translation action of the abelian variety: one needs to define and compute invariants that count curves up to translation.

For abelian surfaces, we prove the analogue of the KKV formula evaluating certain reduced up-to-translation Gromov-Witten invariants in primitive classes. We also conjecture a multiple cover formula expressing imprimitive invariants in terms of primitive ones (analogue of the full KKV formula), and prove it in genus 2 (analogue of the full Yau-Zaslow formula). Other results include the (quasi-)modularity of all descendent invariants in primitive classes, and the counts of hyperelliptic curves in those classes.

For abelian threefolds, based on the GW/Pairs correspondence, the multiple cover formula, and stable pairs calculations in base curve classes, we give a conjectural formula for the reduced up-to-translation Gromov-Witten invariants in all primitive classes. Crucial evidence is provided by stable pairs calculations beyond base classes. We also discuss connections between the surface and threefold theories.

The circle of ideas involves reduced theories (especially the Kiem-Li cosection), degeneration and localization, the counts of polarized isogenies, the link between abelian surfaces and Kummer $K3$'s, the GW/Pairs correspondence, and motivic and toric stable pairs calculations.

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Generic vanishing and compact Kähler manifolds

Christian Schnell

(joint work with Giuseppe Pareschi, Mihnea Popa)

The term "generic vanishing" refers to a collection of results about the cohomology of line bundles with trivial first Chern class. The first results of this type were obtained by Green and Lazarsfeld [3, 4]; they were proved using Hodge theory and are therefore valid on arbitrary compact Kähler manifolds. About 10 years ago, Hacon [5] found a more algebraic approach, using vanishing theorems and the Fourier-Mukai transform, that has led to many additional results in the projective case. In joint work with Giuseppe Pareschi and Mihnea Popa, we show that the newer results are in fact also valid on arbitrary compact Kähler manifolds.

In my talk on Thursday morning, I first explained how generic vanishing can be used to prove the following theorem by Chen and Hacon [1].

Theorem. *Let X be a smooth projective algebraic variety over the complex numbers. Then X is birational to an abelian variety if and only if* $\dim H^1(X, \mathbb{C}) = 2\dim X$ *and* $P_1(X) = P_2(X) = 1$.

The two most important ingredients in the proof are:

(1) Results by Green and Lazarsfeld [3, 4] about the structure of the set
$$S^0(X, \omega_X) = \left\{ L \in \operatorname{Pic}^0(X) \;\middle|\; H^0(X, \omega_X) \neq 0 \right\},$$
in particular that every irreducible component is a translate of a subtorus.

(2) Results by Hacon and Chen and Jiang about the direct image $\operatorname{alb}_* \omega_X$ of the canonical bundle under the Albanese mapping $\operatorname{alb}\colon X \to \operatorname{Alb}(X)$. More precisely, Hacon [5] showed that $\operatorname{alb}_* \omega_X$ is a *GV-sheaf* ("nef"), and Chen and Jiang [2] showed that it is a direct sum of pullbacks of *M-regular* sheaves ("semi-ample").

Then I explained how one can generalize the theorem of Chen and Jiang to higher direct image sheaves of the form $R^j f_* \omega_X$, where $f\colon X \to T$ is any morphism from a compact Kähler manifold to a compact complex torus. One application of this new result is that the theorem of Chen and Hacon remains true in the larger class of compact Kähler manifolds.

A brief outline of the proof goes as follows. Using Saito's version of the decomposition theorem, one can reformulate the problem in terms of polarizable real Hodge modules on the torus T. By a theorem of Deligne, the underlying regular holonomic $\mathscr{D}$-module of a polarizable real Hodge module is semi-simple; we then show that each simple factor is, up to tensoring by a flat line bundle, the pullback of a simple regular holonomic $\mathscr{D}$-module with positive Euler characteristic on a projective subvariety of some quotient torus. (The observation that Hodge modules on compact tori come from abelian varieties is due to Botong Wang.) The above decomposition into simple factors is compatible with the Hodge filtration, and therefore induces the desired decomposition
$$R^j f_* \omega_X \simeq \bigoplus_{k=1}^{n} \left(q_k^* \mathscr{F}_k \otimes L_k \right),$$
where $q_k\colon T \to T_k$ is surjective with connected fibers, $\mathscr{F}_k$ is M-regular and supported on a projective subvariety of the torus T_k, and $L_k \in \operatorname{Pic}^0(T)$ is a holomorphic line bundle of finite order.

After the talk, Kollár raised the question of whether it is really necessary to use Hodge modules in this context. As far as we know, there is no elementary proof (without Hodge modules) of the above result; moreover, what is known about the structure of compact Kähler manifolds of Kodaira dimension 0 is not enough to reduce the problem directly to the projective case. Instead, the point of our approach is that one can compute the cohomology of X in terms of the cohomology of subvarieties of T, with polarizable real Hodge modules as coefficients. In this context, it is now possible to reduce to the case of abelian varieties, basically because every subvariety of T is a torus bundle over a projective variety.

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Zarhin's trick for $K3$ surfaces

François Charles

It is well-known that the moduli space of polarized $K3$ surfaces is not of finite type: indeed, the degree of a polarization can take any even positive integer value, and thus gives rise to infinitely many connected components of the moduli space. In particular, working over a given finite field k, this moduli space might have infinitely many points – though each of its component only has finitely many.

In contrast with this geometric argument, it has been shown by Lieblich-Maulik-Snowden in [3] that, assuming finiteness of the Brauer group of $K3$ surfaces over finite fields, there exist only finitely many $K3$ surfaces over any given finite field, up to isomorphism. Since then, finiteness of the Brauer group has been proven for $K3$ surfaces over finite fields of characteristic at least 3 in [5, 2, 4].

A similar situation holds for abelian varieties, and – as it is usually called – Zarhin's trick gives a simple geometric explanation. Zarhin indeed showed that if A is an abelian variety over an arbitrary field, then the abelian variety $(A \times A^\vee)^4$ admits a principal polarization. From this fact, it can be proved that there exist only finitely many abelian varieties of given dimension over a given fight field.

Zarhin's trick for $K3$ surfaces refers to the fact that there exists a similar construction for $K3$ surfaces. Indeed, we show in [1] that if X is a projective $K3$ surface over a field, then there exists a moduli space of stable sheaves on X of low dimension with a big line bundle of low degree, even if the degree of X is large – precise estimates can be given in terms of lattice theory. Furthermore, over a field of characteristic zero or a finite field, we prove boundedness results for the binational equivalence class of holomorphic symplectic varieties that occur from such a construction. Such binational boundedness results for unbounded families of $K3$ surfaces do in turn make it possible to prove finiteness of Brauer groups without using abelian varieties.

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Symplectic singularities of moduli spaces of sheaves and quiver varieties

Giulia Saccà

(joint work with E. Arbarello)

A number of examples of symplectic varieties and symplectic resolutions come from representation theory or from moduli spaces of sheaves on K3 or abelian surfaces. In the first category, we find the nilpotent cone of a complex semisimple Lie algebra and its Springer resolution, quotients of $\mathbb{C}^2$ by a finite group of symplectic automorphism and their minimal resolution, and Nakajima quiver varieties. As for the case of moduli spaces, the singularities arise in two circumstances, when the Mukai vector is not primitive, or when the polarization (more generally, the stability condition) is not general. The Hilbert–Chow morphism, from the Hilbert scheme of points on a holomorphic symplectic surface to the symmetric product of the surface itself, is an example of extremely fruitful interaction between the two worlds.

Two instances of singularities due to a non primitive Mukai vector were studied by O'Grady [5], [6], who found two new examples of irreducible holomorphic symplectic manifolds by exhibiting symplectic resolutions of two singular moduli spaces on a K3 and on an abelian surface, respectively. Subsequently, Kaledin, Lehn, and Sorger showed in their inspiring paper [2] that in the remaining cases with non primitive Mukai vector there is no symplectic resolution. The aim of [1] is to continue their investigation, and study the case when the singularities of a moduli space of sheaves arise from the choice of a non generic polarization. In certain cases, moving slightly the polarization to a general one induces a symplectic resolution the singular moduli space; to be more specific, our aim is to find a local analytic model of these singularities, as well as of these natural symplectic resolutions.

The case we will be considering is the one of pure dimension one sheaves on a K3 surface S. By definition, these are sheaves whose support, as well as that of any non trivial sub sheaf, has dimension one. Yoshioka showed that, letting v be the Mukai vector of a pure dimension one sheaf on S, the ample cone $\mathrm{Amp}(S)$ admits a finite wall and chamber structure relative to v: for polarizations lying outside of the walls (i.e., in a chamber) the moduli space $M_H(v)$ is smooth, while if the polarization H_0 is contained in a wall, then the corresponding moduli space $M_{H_0}(v)$ is singular. We consider the case of pure dimension sheaves because, if H is in a camber containg H_0 in its closure, then there is natural *regular* morphism $h : M_H(v) \to M_{H_0}(v)$, which is a symplectic resolution. Given a singular point $x \in M_{H_0}(v)$, we use quiver varieties to describe the structure of $h : h^{-1}(U) \to U$, where $U \subset M_{H_0}(v)$ is an analytic neighborhood of x.

A quiver, denoted by Q, is an oriented graph. Let $I = \{1, 2, \dots, s\}$ be the set of vertices and denote by E the set of edges. For an edge $e \in E$, let $s(e), t(e) \in I$ be the source and target of e, respectively. For a dimension vector $\mathbf{n} = (n_1, \dots, n_s) \in \mathbb{Z}^s_{\geq 0}$, let V_i be an n_i–dimensional complex vector space and let

$$\mathrm{Rep}(\mathbf{n}) = \bigoplus_{e \in E} \mathrm{Hom}(V_{s(e)}, V_{t(e)}) \oplus \mathrm{Hom}(V_{t(e)}, V_{t(e)})$$

be the space of $\mathbf{n}$–dimensional representions of Q. This vector space has a natural symplectic form, preserved by the obvious action of $G := GL(\mathbf{n}) = \prod GL(V_i)$; in this context there is a moment map with values in the Lie algebra $\mathfrak{g}$ of G

$$\mu : \mathrm{Rep}(\mathbf{n}) \to \mathfrak{g}, \qquad \sum (x_e, y_e) \mapsto \sum [x_e, y_e]$$

which allows one to consider sympectic reduction. Roughly speaking, this means that the smooth locus of the quiver variety $\mathfrak{M}_0 := \mu^{-1}(0) /\!\!/ G$ has a holomorphic symplectic form. Let $\chi \in \mathrm{Hom}(G, \mathbb{Z})$ be a rational character of G. If $\mathbf{n}$ is primitive, by considering the variation of GIT quotient $\mathfrak{M}_\chi := \mu^{-1}(0) /\!\!/_\chi G$ we get a projective morphism

$$\xi : \mathfrak{M}_\chi \to \mathfrak{M}_0. \qquad (1)$$

As showed by Nakajima [4], there is a wall and chamber decomposition of $\mathrm{Hom}(G, \mathbb{Z}) \otimes \mathbb{Q}$, so that if χ is chosen in a chamber then (1) is a symplectic resolution.

Recall that given a singular point $x \in M_{H_0}$ there is a unique up to isomorphism polystable sheaf F in the S–equivalence class represented by x. We can now state the main theorem.

Theorem 1. *Let v be a primitive Mukai vector, and let $x \in M_{H_0}(v)$ be a singular point corresponding to a polystable sheaf F. There exists a quiver Q and a dimension vector $\mathbf{n}$ such that*

i) *If the differential graded Lie algebra (dgla) $R\,\mathrm{Hom}(F, F)$ is formal, then, here is a local (analytic) isomorphism $\psi : (\mathfrak{M}_0, 0) \cong (M_{H_0}(v), x)$;*
ii) *If, in addition, F is pure of dimension one. Then for every polarization H lying in a chamber containing H_0 in its closure, there is a character χ lying in a chamber of $\mathrm{Hom}(G, \mathbb{Z}) \otimes \mathbb{Q}$ such that the symplectic resolutions*

$$\xi : \mathfrak{M}_\chi(\mathbf{n}) \to \mathfrak{M}_0(\mathbf{n}), \quad \text{and} \quad h : M_H(v) \to M_{H_0}(v),$$

correspond to each other via ψ.

A few remarks are in order. First of all, the statement of *ii*) holds, more generally, whenever the map $M_H(v) \to M_{H_0}(v)$ is a regular morphism. In higher rank, this is not always the case, and one looks instead at resolutions arising from Matsuki–Wentorth twisted stability or from Bridgeland stability conditions. This is work in progress (in the case of ideal sheaves we recover Nakajima's description of the Hilbert–Chow morphism).

If $F = \oplus_{i=1}^s F_i^{n_i}$, with the sheaves F_i pairwise non isomorphic and H_0–stable, then the quiver Q has s vertices, and for every $i < j$, it has $\dim \mathrm{Ext}^1(F_i, F_j)$ vertices from i to j, and if $i = j$ it has $\dim \mathrm{Ext}^1(F_i, F_i)/2$ loops at the vertex i. This can be defined for arbitrary polystable sheaf, but if F is pure of dimension one, then Q is "essentially" the dual graph of its support. Notice that $\mathrm{Aut}(F) = G$.

By definition, the dgla $R\,\mathrm{Hom}(F, F)$ is called formal if it is quasi–isomorphic to its cohomology algebra $\mathrm{Ext}^*(F, F)$. From our point of view, what this means is that deformation space Def_F is isomorphic to a complete intersection of quadrics in $\mathrm{Ext}^1(F, F)$. A subtle point in the proof of *i*) is that this isomorphism can be chosen to be $\mathrm{Aut}(F)$–equivariant. With our choice of Q and $\mathbf{n}$, we see that this intersection of quadrics is isomorphic to $\mu^{-1}(0) \subset \mathrm{Rep}(\mathbf{n})$ and we conclude with a G–equivariant version of Artin's approximation due to Bierstone and Milman. It is conjectured by Kaledin and Lehn [3] that any polystable sheaf $F = \oplus_{i=1}^s F_i^{n_i}$ on a K3 surface satisfies this formality condition. In the case when all the F_i are locally free sheaves, the conjecture was proved by Zhang [7], using results by Kaledin and Verbitsky. The idea is therefore to use a Fourier–Mukai equivalence in order to send the torsion sheaves F_i to locally free sheaves M_i, so that in the cases where the M_i are stable one can apply Zhang's theorem to the polystable sheaf $\oplus M_i^{n_i}$. For example, when the equivalence is the spherical twist associated to $\mathcal{O}_S$, then M_i is the dual of the Lazarsfeld–Mukai bundle of F_i. We prove the stability of the M_i in a number of cases, thereby proving the conjecture of Lehn and Zhang for the corresponding pure dimension one sheaves.

The proof of *ii*) uses the geometry of the Quot scheme, of an étale slice around a point corresponding to F, and natural determinant line bundles. In view of this, the assignment $H \mapsto \chi$, though not unique (as the isomorphism classes of the moduli spaces are constant within each chamber!), can be chosen to be given by the following formula

$$G \ni (g_1, \dots, g_s) \mapsto \chi(g_1, \dots, g_s) = \prod_{i=1}^{s} \det(g_i)^{(D_i \cdot H - D_i \cdot H_0)}, \text{ where } D_i := c_1(F_i).$$

Moreover, this assignment preserves the wall and chamber structure of both sides, in the sense that it maps the walls in $\mathrm{Amp}(S)$ to the walls of $\mathrm{Hom}(G, \mathbb{Z})$.

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Stability conditions on abelian threefolds, and some Calabi-Yau threefolds

Arend Bayer
(joint work with Emanuele Macrí, Paolo Stellari)

In this talk, I described by the main result of [1]: a description of a connected component on the space of Bridgeland stability conditions on abelian threefolds, and on Calabi-Yau threefolds obtained as (crepant resolutions of) quotients of abelian threefolds by a finite group action.

The existence of stability conditions on three-dimensional varieties in general, and more specifically on Calabi-Yau threefolds, is an important open problem in the theory of Bridgeland stability conditions. Until recent work by Maciocia and Piyaratne [4, 5], they were only known to exist on threefolds whose derived category admits a full exceptional collection.

Our approach is based on [3]. The construction is based on the auxiliary notion of *tilt-stability* for two-term complexes, and a conjectural Bogomolov-Gieseker type inequality for the third Chern character of tilt-stable objects. It was shown in [3] that this conjecture would imply the existence of Bridgeland stability conditions, and, in the companion paper [2], a Reider-type theorem, including a version of Fujita's conjecture, on very ampleness of adjoint line bundles on threefolds.

Conjectural inequality and main result: Our first result is a better understanding of the conjectural inequality proposed in [3]. Let (X, H) be a polarized threefold, and $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$. We define the twisted slope by

$$\mu_\beta(E) = \frac{H^2 \mathrm{ch}_1^\beta(E)}{H^3 \mathrm{ch}_0(E)}$$

where ch_1^β is defined via the twisted Chern character $\mathrm{ch}^\beta(E) = e^{-\beta H}(E)$. The main technical construction is the definition of the following subcategory of $D(X) = D^b(\mathrm{Coh} X)$:

$$\mathrm{Coh}^\beta(X) = \left\{ E^{-1} \to^d E^0 \in D(X) \colon \begin{cases} \mu_\beta(F) > 0 & \text{for all quotients } F \text{ of } \mathrm{cok}\ d \\ \mu_\beta(G) \leq 0 & \text{for all subobjects } F \text{ of } \ker d \end{cases} \right\}$$

This is an abelian subcategory. We use the following function to define a slope-like notion of stability on $\mathrm{Coh}^{\beta}(X)$, which we call *tilt-stability*:

$$\nu_{\alpha,\beta} = \frac{H\mathrm{ch}_2^{\beta} - \frac{1}{2}\alpha^2 H^3\mathrm{ch}_0^{\beta}}{H^2\mathrm{ch}_1^{\beta}}.$$

We propose the following conjecture:

Conjecture 1. *Let (X, H) be a smooth polarized threefold. Assume that $E \in \mathrm{Coh}^{\beta}(X)$ is $\nu_{\alpha,\beta}$-semistable. Then*

(1)
$$\alpha^2\left(\left(H^2\mathrm{ch}_1^{\beta}(E)\right)^2 - 2H^3\mathrm{ch}_0(E)H\mathrm{ch}_2^{\beta}(E)\right) + 4\left(H\mathrm{ch}_2^{\beta}(E)\right)^2 - 6H^2\mathrm{ch}_1^{\beta}(E)\mathrm{ch}_3^{\beta}(E) \geq 0.$$

We prove that this conjecture is equivalent to a seemingly more special conjecture proposed in [3], which only treated objects with $\nu_{\alpha,\beta} = 0$.

Our main result is the following:

Theorem 2. *Conjecture 1 holds when (X, H) is an abelian threefold, or a smooth quotient of an abelian threefold.*

The case of abelian varieties of Picard rank one was proved previously in [4, 5]. The case of étale quotients includes some Calabi-Yau threefolds with infinite fundamental group and $H_1 = 0$, called *Calabi-Yau threefolds of abelian type*.

Main Application: Space of stability conditions. Let X be an abelian threefold, of a Calabi-Yau threefold of abelian type, or a *Calabi-Yau threefold of Kummer type* (a crepant resolution of a singular quotient Y/G of an abelian threefold Y). The main application of Theorem 2 is a description of a connected component of the space of stability conditions on X. More precisely, we consider stability conditions whose central charge factors via the map

(2) $$v_H \colon K(X) \to \mathbb{Q}^4, \quad E \mapsto \left(H^3\mathrm{ch}_0(E), H^2\mathrm{ch}_1(E), H\mathrm{ch}_2(E), \mathrm{ch}_3(E)\right).$$

(In the case of Kummer threefolds, we apply the BKR-equivalence before taking the Chern character.)

Inside the space $\mathrm{Hom}(\mathbb{Q}^4, \mathbb{C})$, consider the open subset $\mathfrak{V}$ of linear maps Z whose kernel does not intersect the (real) twisted cubic $\mathfrak{C} \subset \mathbb{P}^3(\mathbb{R})$ parameterized by $(x^3, x^2y, \frac{1}{2}xy^2, \frac{1}{6}y^3)$; it is the complement of a real hypersurface. Such a linear map Z induces a morphism $\mathbb{P}^1(\mathbb{R}) \cong \mathfrak{C} \to \mathbb{C}^*/\mathbb{R}^* = \mathbb{P}^1(\mathbb{R})$; we define $\mathfrak{P}$ be the component of $\mathfrak{V}$ for which this map is an unramified cover of topological degree $+3$ with respect to the natural orientations. Let $\widetilde{\mathfrak{P}}$ be its universal cover.

We let $\mathrm{Stab}_H(X)$ be the space of stability conditions for which the central charge factors via the map v_H in (2), and which satisfy the support property.

Theorem 3. *Let X be an abelian threefold, or a Calabi-Yau threefold of abelian or Kummer type. Then $\mathrm{Stab}_H(X)$ contains $\widetilde{\mathfrak{P}}$ as a connected component.*

These stability conditions are obtained by deforming tilt-stability via a contribution of ch_3^{β} that is "small" compared to the contributions of $\mathrm{ch}_{\leq 2}^{\beta}$; this can be made precise due to inequality (1).

Other applications. The above-mentioned Reider-type theorems are based on the following cohomology vanishing statement:

Corollary 4 ([2])**.** *Let X be a Calabi-Yau threefold of abelian type. Given $a \in \mathbb{Z}_{>0}$, let L be an ample line bundle on X satisfying*

- $L^3 > 49a$,
- $L^2D \geq 7a$ *for every integral divisor class D with $L^2D > 0$ and $LD^2 < a$, and*
- $L.C \geq 3a$ *for every curve $C \subset X$.*

Then $H^1(L \otimes I_Z) = 0$ for every 0-dimensional subscheme $Z \subset X$ of length a.

In addition, if $L = A^{\otimes 5}$ for an ample line bundle A, then L is very ample.

As a special case of Conjecture 1, we also get new inequalities for Chern classes of slope-stable vector bundles:

Corollary 5. *Let X be one of the following threefolds: projective space, the quadric in $\mathbb{P}^4$, an abelian threefold, or a Calabi-Yau threefold of abelian type. Let H be a polarization, and let $c \in \mathbb{Z}_{>0}$ be the minimum positive value of H^2D for integral divisor classes D. If E is a sheaf that is slope-stable with respect to H, and with $H^2c_1(E) = c$, then*

$$3c\mathrm{ch}_3(E) \leq 2\left(H\mathrm{ch}_2(E)\right)^2.$$

The assumptions hold when $\mathrm{NS}(X)$ is generated by H, and $c_1(E) = H$. Even for vector bundles on $\mathbb{P}^3$, this result is new for rank bigger than three.

Both Corollaries would hold similarly for any threefold on which Conjecture 1 can be proved.

Proof strategy. The proof of Theorem 2 for abelian threefolds starts by a reducing the conjecture to the following statement:

(*) If E is $\nu_{\alpha,\beta}$-stable for (α, β) near $(0, 0)$, and if E satisfies $H\mathrm{ch}_2(E) = 0$, then it also satisfies $\mathrm{ch}_3(E) \leq 0$.

This reduction uses wall-crossing for tilt-stability as (α, β) vary, and that tilt-stability is preserved under pull-back by $\underline{n}\colon X \to X$, the multiplication by n.

We then prove statement (*) via very classical methods: further pull-backs under $\underline{n}$ as $n \gg 0$, stability of line bundles (and the associated Hom-vanishing), restriction of global sections to divisors, and finally bounds on the cohomology of sheaves on surfaces given by the Grauert-Mülich theorem.

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Mirror Symmetry and Classification of Orbifold del Pezzo Surfaces

Alessio Corti
(joint work with Mohammad Akhtar, Tom Coates, Liana Heuberger, Alexander Kasprzyk, Alessandro Oneto, Andrea Petracci, Thomas Prince, Ketil Tveiten)

I stated the four conjectures that follow. More detail can be found in [1]

Conjecture 1. *There is a one-to-one correspondence between:*

- *the set $\mathfrak{P}$ of mutation equivalence classes of Fano polygons; and*
- *the set $\mathfrak{F}$ of qG-deformation equivalence classes of locally qG-rigid class TG del Pezzo surfaces with cyclic quotient singularities.*

The correspondence sends P to a (any) generic qG-deformation of the toric surface X_P.

Conjecture 2. *Let P be a Fano polygon and let X be a generic qG-deformation of the toric surface X_P. Let L_P^T denote the affine space of maximally-mutable Laurent polynomials with Newton polygon P and T-binomial edge coefficients, and let $H_X^{ts} \subset H_X$ denote the twisted sectors of age less than 1:*

$$H_X^{ts} = \bigoplus_{i=1}^{r} \mathbb{C}\mathbf{u}_i$$

There is an affine-linear isomorphism $\varphi\colon L_P^T \to H_X^{ts}$, the mirror map, such that the regularized quantum period $\widehat{G}_{\mathfrak{X}}$ of $\mathfrak{X}$ and the classical period π_P of P satisfy[1] *$\widehat{G}_{\mathfrak{X}} \circ \varphi = \pi_P$.*

Conjecture 3. *Let P_1 and P_2 be Fano polygons with the same singularity content. Suppose that there is an affine-linear isomorphism $\varphi\colon L_{P_1}^T \to L_{P_2}^T$ such that $\pi_{P_1}(a,t) = \pi_{P_2}(\varphi(a),t)$. Then P_2 is obtained from P_1 by a chain of mutations.*

Conjecture 4. *Let X_1 and X_2 be del Pezzo surfaces of class TG with the same set of qG-rigid cyclic quotient singularities, and let $\varphi\colon H_{X_1}^{ts} \to H_{X_2}^{ts}$ be the obvious identification. Suppose that $\widehat{G}_{\mathfrak{X}_1} = \widehat{G}_{\mathfrak{X}_2} \circ \varphi$. Then X_1 and X_2 are qG-deformation equivalent.*

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[1] We think of $\widehat{G}_{\mathfrak{X}}$ and π_P as functions from H_X^{ts} and L_P^T to $\mathbb{C}[\![t]\!]$.

The Nash problem on families of arcs

Tommaso de Fernex
(joint work with Roi Docampo)

Let X be a complex algebraic variety. The arc space X_∞ of X is the scheme parametrizing formal arcs on X; the K-valued points of X_∞ are K-valued arcs

$$\alpha\colon \mathrm{Spec} K[[t]] \to X.$$

The natural projection $\pi_X\colon X_\infty \to X$ is defined by setting $t = 0$. Note that α defines a valuation

$$\mathrm{val}_\alpha\colon \mathcal{O}_{X,\pi_X(\alpha)} \to K[[t]] \to \mathbb{Z} \cup \{\infty\}$$

which extends to the function field of X if and only if α maps the generic point of $\mathrm{Spec} K[[t]]$ to the generic point of X.

If X is a smooth variety, then X_∞ is the inverse limit of a tower of affine bundles, and thus it is irreducible. The following theorem of Kolchin shows that the same conclusion holds even if X is singular.

Theorem 1 (Kolchin [4])**.** *For any variety X, the arc space X_∞ is irreducible.*

In [5], Nash proposed to look at the set of arcs through the singularities of X

$$\pi_X^{-1}(\mathrm{Sing}\, X) = \{\alpha \in X_\infty \mid \pi_X(\alpha) \in \mathrm{Sing}\, X\}.$$

Theorem 2 (Nash [5])**.** *There is a finite decomposition*

$$\pi_X^{-1}(\mathrm{Sing}\, X) = \bigcup_{i=1}^{s} C_i$$

into irreducible components C_i.

The proof of this theorem uses resolution of singularities. It shows, in particular, that if $f\colon Y \to X$ is a high enough resolution and $f_\infty\colon Y_\infty \to X_\infty$ is the map induced by f, then there are prime divisors $E_1, \dots, E_s$ on Y such that

$$C_i = \overline{f_\infty(\pi_Y^{-1}(E_i))}.$$

This implies that if $\alpha_i \in C_i$ is the generic point, then val_{α_i} is the divisorial valuation associated to E_i. We call the valuations $\mathrm{val}_{\alpha_1}, \dots, \mathrm{val}_{\alpha_s}$ the *Nash valuations* over X.

Nash Problem. *Describe the Nash valuations in terms of resolutions of singularities.*

Nash observed that every Nash valuation over X is *essential*, which means that its center on any resolution of singularities $g\colon X' \to X$ is an irreducible component of $g^{-1}(\mathrm{Sing}\, X)$. This yields the definition of the *Nash map*, which associate an essential valuation to any irreducible component of $\pi_X^{-1}(\mathrm{Sing}\, X)$.

Our main theorem provides a sufficient condition for a valuation to be a Nash valuation. We say that a divisorial valuation val_E centered in X is a *terminal* *valuation* over X (in the sense of the minimal model program) if E can be realized as an exceptional divisor on a minimal model $f\colon Y \to X$.

Theorem 3 (de Fernex–Docampo [1]). *Every terminal valuation over X is a Nash valuation.*

If $\dim X \leq 2$, then terminal valuations are the same as essential valuations, and therefore the theorem implies that the Nash map is surjective. This is an elementary fact in dimension one. In dimension two, it gives a new proof of the following theorem.

Theorem 4 (Fernandez de Bobadilla–Pe Pereia [2]). *Nash valuations over a surface are the same as essential valuations (and thus the Nash map is surjective).*

Another sufficient condition to be a Nash valuation is to be minimal (in the valuative sense) among divisorial valuations centered in the singular locus of X. By showing that every essential valuation on a toric variety is a minimal valuation, Ishii and Kollár proved that the Nash map is surjective for toric varieties.

Theorem 5 (Ishii–Kollár [3]). *Nash valuations over a toric variety are the same as essential valuations (and thus the Nash map is surjective).*

Apart from curves, surfaces, and toric varieties, only very few other examples are known where the Nash map is surjective, and there are examples in all dimensions greater than two where the Nash map is not surjective.

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