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Subfactors and Conformal Field Theory

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Abstract. Connections between subfactor theory and conformal field theory have been expected since the early days of the former in 1980's, and recently we see more and more evidence for deeper relations. It was our aim to attract experts from a wide range of topics related to subfactors and CFT. Many of the participants met for the first time at Oberwolfach, and there were numerous very fruitful interactions.

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Introduction by the Organisers

Subfactor theory in today's form was initiated by Jones in the early 1980's. It has revolutionized the theory of operator algebras and through it, many surprising connections to low-dimensional topology, quantum groups, statistical mechanics and quantum field theory were discovered. Two-dimensional conformal field theory has been well studied during the last 30 years, and it has also been connected to vast ranges of subjects in mathematics and physics. Formal similarities between subfactor theory and conformal field theory were apparent since the early days, but we have recently seen more and more explicit connections. The workshop gathered mathematicians and physicists covering a wide range of topics in these and related areas.

(1) Subfactors and fusion categories

The Haagerup subfactor was found as an exceptional ("exotic") subfactor in 1990's. Its siblings and generalizations have been studied by many researchers, but their real meaning has not been understood. The Haagerup subfactor has been believed to be connected to conformal field theory and vertex operator algebras, particularly through the work of Evans-Gannon, but still many important details have to be clarified. Also the representation theoretic aspects of subfactor theory have recently caught much attention.

We had talks of V. Jones, Evans, Morrison, Snyder, C. Jones, Vaes, Izumi, Grossman, Penneys, Brothier, Liu and Shlyakhtenko on these topics. Haagerup was among the participants. We also had an informal talk of Gannon on an approach from subfactors to non-unitary fusion categories on Thursday evening.

(2) Algebraic quantum field theory

In algebraic quantum field theory, we study nets of observable algebras parameterized by spacetime regions. This approach has found deep connections to subfactor theory in the 1980s through works of Longo and Fredenhagen-Rehren-Schroer. Two-dimensional conformal field theory has been extensively studied in this context and the associated mathematical object is called a local conformal net. Its connection to the theory of vertex operator algebras, algebraic axiomatizations of chiral conformal field theory was not well-understood beyond many apparent formal similarities, but Carpi gave a talk on connecting the two theories directly for the first time. Longo, Tanimoto, Rehren, Müger and Bischoff also gave talks on these topics.

(3) Vertex operator algebras

A vertex operator algebra first appeared in studying Monstrous Moonshine in 1980's through works of Borcherds and Frenkel-Lepowsky-Meurman. VOAs were well developed to a sophisticated theory over many years. Though it was started independently from algebraic quantum field theory and has had deep connections to algebra such as finite sporadic simple groups and modular functions, it has become clear that VOAs must be closely related to algebraic quantum field theory. There are many similarities between the two theories. Mason, Duncan and Lam gave talks on these topics.

(4) Conformal field theory and tensor categories

Various aspects of conformal field theory have been studied in the context of modular tensor categories. Unitary tensor categories have been extensively studied, but non-unitary ones also appear, particularly in connection to logarithmic conformal field theory. Schweigert, Runkel, Fuchs, Creutzig and Schommer-Pries gave talks on these topics.

(5) Other topics

Tener gave a talk on an example of a Segal type conformal field theory. Teichner talked about moduli spaces of field theories, and Henriques gave a talk on Stolz-Teichner cocycles. Wassermann's talk was about analysis on trinions. Ogata explained some of her work on gapped Hamiltonians in quantum statistical mechanics.

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Workshop: Subfactors and Conformal Field Theory

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Abstracts

Does every subfactor arise in conformal field theory?

Vaughan Jones

We describe an ongoing effort to construct a conformal field theory for every finite index subfactor in such a way that the standard invariant of the subfactor, or at least its quantum double, can be recovered from the CFT. There is no doubt that interesting subfactors arise in CFT nor that in some cases the numerical data of the subfactor appears as numerical data in the CFT. But there are supposedly "exotic" subfactors for which no CFT is known to exist, the first of which was constructed by Haagerup in a tour de force in [14],[1]. In the last few years ideas of Evans and Gannon (see [8]) have made it seem plausible that CFT's exist for the Haagerup and other exotic subfactors constructed in the Haagerup line (see [20]). This has revived the author's interest in giving a construction of a CFT from subfactor data.

The most orthodox way to do this would be to extract form the subfactor the Boltzmann weights of a critical two-dimensional lattice model then construct a quantum field theory from the scaling limit of the n-point functions. Looking at the monodromy representations of the braid group one would then construct a subfactor as in the very first constructions of [15]. This "royal road" is paved with many mathematical difficulties and it is probably impossible to complete with current technology except in the very simplest examples.

There are alternatives, however, to using the scaling limit of the n-point functions. The algebraic (Haag-Kastler [13],[7],[12]) approach has been quite successful in understanding some aspects of conformal field theory-[10],[23],[9]. After splitting the CFT into two chiral halves, this approach predicts the existence of "conformal nets"- von Neumann algebras $\mathcal{A}(I)$ on the Hilbert space $\mathcal{H}$, associated to closed intervals $I \subset S^1$, and a continuous projective unitary representation $\alpha \mapsto u_\alpha$ of DiffS^1, on $\mathcal{H}$, satisfying four axioms:

$$
\begin{aligned}
&(i) && \mathcal{A}(I) \subseteq \mathcal{A}(J) \text{ if } I \subseteq J \\
&(ii) && \mathcal{A}(I) \subseteq \mathcal{A}(J)' \text{ if } I \cap J = \emptyset \\
&(iii) && u_\alpha \mathcal{A}(I) u_\alpha^{-1} = \mathcal{A}(\alpha(I)) \\
&(iv) && \sigma(Rot(S^1)) \subset \mathbb{Z}^+ \cup \{0\}
\end{aligned}
$$

Here by $Rot(S^1)$ we mean the subgroup of rotations in Diff(S^1). $Rot(S^1)$ may be supposed to act as an honest representation which can therefore be decomposed into eigenspaces (Fourier modes). The eigenvalues, elements of $\hat{S^1} = \mathbb{Z}$, are the spectrum σ of the representation.

There may or may not be a vacuum vector Ω in $\mathcal{H}$ which would be fixed by the linear fractional transformations in Diff(S^1), and would be cyclic and separating for all the $\mathcal{A}(I)$.

The $\mathcal{A}(I)$ can be shown to be type III_1 factors so subfactors appear by axiom (ii) as $\mathcal{A}(I') \subseteq \mathcal{A}(I)'$ where I' is the closure of the complement of I.

Non-trivial examples of such conformal nets were constructed in [29],[27] by the analysis of unitary loop group representations ([25]). These examples can be exploited to construct many more.

On an apparently completely different front, the study of subfactors for their own sake led to the development of "planar algebras" which in their strictest form ([16]) are an axiomatization of the standard invariant of a subfactor but by changing the axioms slightly they yield an axiomatization of correspondences (bimodules) in the sense of Connes ([5]), and systems of such. The most significant ingredient of a planar algebra is the existence of a positive definite inner product which is interpreted diagrammatically. More precisely a planar algebra is a graded vector space $\mathcal{P} = (P_n)$ of vector spaces where n is supposed to count the number of boundary points on a disc into which the elements of P_n can be "inserted". Given a planar tangle - a finite collection of discs inside a big (output) disc, all discs having boundary points and all boundary points being connected by non-crossing curves called strings, the insertion of elements of $\mathcal{P}$ into the internal discs produces an output element in P_n, n being the number of boundary points on the output disc.

The idea of obtaining a "continuum limit" by letting the number of boundary points on the discs fill out the circle has been around for over 20 years but this paper is the first one to take a concrete, though by no means big enough, step in that direction. Planar algebra is an abstraction of the notion of (planar) manipulations of the tensor powers of a given finite dimensional Hilbert space (thus in some sense a planar version of [24]), and our constructions below of limit Hilbert spaces are really versions, aimed at a scaling rather than a thermodynamic limit, of von Neumann's original infinite tensor product-[28]. Background for this point of view is detailed in [17].

Given the difficulty of following the royal road using the scaling limit, we are trying to construct the local algebras $\mathcal{A}(I)$ directly from a planar algebra. A well known idea in physics is the block spin renormalization procedure ([4]). Here one groups the spins in a block on one scale and replaces the blocks by spins of the same kind on a coarser scale. Hamiltonians (interactions) between the spins and blocks of spins are chosen so that the physics on the block spin scale resembles the physics on the original scale. This procedure is tricky to implement but we shall use the idea. For, however one plays it, in constructing a continuum limit one must relate the Hilbert space on one scale to the Hilbert space on a finer scale. It is this relation that we are trying to produce using structures suggested by planar algebra.

More precisely, given a planar algebra $\mathcal{P}$, for a choice of n points on S^1, called B_n, we will associate the Hilbert space $\mathcal{H}_n = P_n$, and for an inclusion $B_n \subset B_m$ we will use planar algebra data to group boundary points into blocks and construct a projection from $\mathcal{H}_m$ onto $\mathcal{H}_n$. Alternatively we are defining isometries of $\mathcal{B}_n$ into $\mathcal{B}_m$ and the Hilbert space $\mathcal{H}$ of the theory will then be the direct limit of the $\mathcal{H}_n$.

With this idea we have been led to unitary (projective) representations of Thompson's groups of PL homeomorphisms of S^1 and $[0, 1]$ which play the role of

Diff(S^1)in our not yet continuum limit. The idea is to use an element of $R \in P_n$ for $n > 2$ to embed Hilbert spaces associated with finitely many points into each other by grouping together $n-1$ "spins" on one scale into a single spin on a more coarse scale. This is just what is done in block spin renormalisation.

Although this block spin idea does not introduce dynamics, we will see that it does produce interesting unitary representations of T and F. In particular these representations do depend on the planar algebra data used to construct them. A perhaps surprising byproduct arises if one uses "crossings" from the Conway knot-theoretic skein theory and knot polynomial theory. For then the coefficient of the "vacuum vector" in the representation is an unoriented link. One of our main results is that all unoriented links arise in this way.

As we have said, the block spin approach is purely kinematic. In order to introduce physics into the picture we should construct a Hamiltonian on the limit Hilbert space. In a first step in this direction we have constructed a transfer matrix $T(\lambda)$ depending on a spectral parameter λ. The condition that the transfer matrix, initially defined only on the finite dimensional approximates, passes to the limit Hilbert space, turns out to be the Yang Baxter equation. Once an initial T operator is chosen on a certain scale, all T's on finer scale are determined by the Yang Baxter equation. Unfortunately these T operators do not commute among each other and the usual way of constructing a local Hamiltonian as the logarithmic derivative of the transfer matrix with respect to λ does not seem to work.

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Standard Subspaces and the Localisation of Particles

Roberto Longo

(joint work with G. Lechner and with V. Morinelli, K.-H. Rehren)

This talk concerns the particle localisation properties in Quantum Field Theory. We analyse nets of standard subspaces on a Hilbert space (see [4]), then one can soon get the corresponding results for nets of von Neumann algebra on the exponential (Fock) space obtained via the second quantisation procedure. We answer two long standing, natural questions.

Nets with minimal length [3]. We construct natural local nets of real closed linear subspaces $H_V(I)$ of a complex Hilbert space $\mathcal{H}$, associated with intervals I of the real line such that

$$H_V(I) \text{ is cyclic if } \ell(I) > r \quad \text{and} \quad H_V(I) = \{0\} \text{ if } \ell(I) < r$$

for any given $r \in [0, \infty]$, where $\ell(I)$ is the length of I. We say that r is the minimal length of H_V. We have similar results for nets on the Minkowski plane $\mathbb{R}^2$.

We start with a free field one-particle net H and we put $H_V(a, \infty) = VH(a, \infty)$, $H_V(-\infty, b) = H(-\infty, b)$, for every $a, b \in \mathbb{R}$, where V is an endomorphism of $H(0, \infty)$ constructed in [6], i.e. V is a unitary on $\mathcal{H}$, $VH(0, \infty) \subset H(0, \infty)$ and V commutes with translations. We define the net H_V by

$$H_V(a, b) \equiv H_V(a, \infty) \cap H_V(-\infty, b), \quad a < b .$$

Now V is associated with an inner function φ on the upper half-plane which is symmetric ($\bar{\varphi}(z) = \varphi(-\bar{z})$) [6], namely $V = \varphi(P)$ where P is the translation one-parameter unitary group generator. Let's consider a symmetric, infinite Blasckhe product φ (with no translation factor to avoid trivial constructions). The growth of the zeros of φ has effect on the minimal length r of H_V. We have:

For every $r > 0$ there exists φ s.t. the minimal length of H_V is equal to r

where φ is as above and $V = \varphi(P)$.

We notice that H_V is local, translation covariant with positive energy, but not Lorentz covariant, and duality for wedge regions is not satisfied (unless $H_V = H$).

Problem. Does there exist a natural local net of standard subspaces on double cones of the Minkowski spacetime $\mathbb{R}^4$ with positive minimal length?

Particles with infinite spin [5]. In Wigner classification of unitary, positive energy, irreducible representations of the Poincaré group [9], massless representations fit in two classes, the ones with finite spin (helicity) and the ones with infinite spin, according to the representations of the "little group", the Euclidean group of the plane $E(2)$. Particles with infinite spin have been long disregarded, but for a result of Yngvason [8] that they cannot appear in a Wightman theory [8].

The procedure in [2] gave however a canonical construction of a local net H^U of closed, real linear subspaces on the Minkowski spacetime $\mathbb{R}^4$ associated with any unitary, positive energy, representation U of the Poincaré group. Let U_0 be a massless representation with infinite spin; the space $H^{U_0}(\mathcal{O})$ was shown to be standard (cyclic) for certain unbounded regions $\mathcal{O}$ (space-like cones) but it remained open whether there are non-zero vectors localised in bounded regions. Generalised (string-like) Wightman fields associated with U_0 were later constructed [7], but the above localisation problem remained unsettled.

We show in [5] that $H^{U_0}(\mathcal{O})$ is trivial if $\mathcal{O}$ is bounded, say $\mathcal{O}$ a double cone, namely

$$H^{U_0}(\mathcal{O}) \equiv \bigcap_{\mathcal{O} \subset W} H^{U_0}(W) = \{0\}$$

where W runs on all wedge regions containing $\mathcal{O}$.

As a consequence, if A is a (Fermi-)local net of von Neumann algebras on a Hilbert space, covariant under a unitary, positive energy, irreducible representation U of the Poincaré group, with the vacuum Reeh-Schlieder cyclicity property on double cones, then no infinite spin representation can appear in the irreducible direct integral decomposition of U (up to measure zero), provided that A satisfies the fundamental Bisognano-Wichmann property [1] (KMS condition for the boost one-parameter groups on wedge von Neumann algebras).

Problem. Given a local net of standard subspaces $H(\mathcal{O})$ of a Hilbert space $\mathcal{H}$ on double cones $\mathcal{O}$ of the Minkowski spacetime $\mathbb{R}^4$, covariant under a unitary, positive energy representation U of the Poincaré group, does there exist a unitary, positive energy representation U' of the Poincaré group on $\mathcal{H}$ such that H is U'–covariant and H satisfies the Bisognano-Wichmann property with respect to U'?

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RCFT correlators and surface defects in three-dimensional topological field theory

Christoph Schweigert

(joint work with Jürgen Fuchs)

1. Some categorical structures in 2d conformal field theory

In the first part of the talk, we have reviewed some categorical structures in conformal field theory.

Monoidal categories. A fusion category $\mathcal{A}$ is a finitely semisimple, $\mathbb{C}$-linear monoidal category such that the monoidal unit is absolutely simple. Chiral conformal field theory (with certain finiteness conditions) provides many examples of *braided* fusion categories. On the other hand, the Drinfeld center $Z(\mathcal{A})$ of any fusion category $\mathcal{A}$ provides a class of examples of braided fusion categories. A braided fusion category is called *non-degenerate* if the natural monoidal functor

$$\mathcal{C} \boxtimes \mathcal{C}^{rev} \xrightarrow{\simeq} Z(\mathcal{C})$$

obtained from the braiding and the reversed braiding is a braided equivalence. A non-degenerate category is called *modular* if it is a ribbon category. Modular tensor categories encode the “Moore-Seiberg-data” or modular data of a rational conformal field theory.

Module categories. About 15 years ago, it became clear that boundary conditions and boundary fields for a given full, local conformal field theory with chiral data encoded in a modular tensor category $\mathcal{C}$ are described by a module category over $\mathcal{C}$. For simplicity, we restricted ourselves in this talk to (bi)module categories that are finitely semisimple.

If A is an associative algebra in $\mathcal{A}$, then the category A-mod$_{\mathcal{A}}$ of left A-modules internal in $\mathcal{A}$ is a right module category. In applications to conformal field theory, A has the additional structure of a (special symmetric) Frobenius algebra; this induces additional structure on the category A-mod, called a *module trace* [14].

If $\mathcal{C}$ is a *braided* fusion category and $\mathcal{M}$ a right module category over $\mathcal{C}$, then right multiplication $M \mapsto M \otimes U$ by any $U \in \mathcal{C}$ provides a $\mathbb{C}$-linear endofunctor of $\mathcal{M}$. These functors can be endowed, via the braiding and the opposite braiding, respectively, with two different structures of a module functor. This provides two monoidal functors, called *braided induction* (or α-induction) [11, 2, 13]

$$\alpha^{\pm}_{\mathcal{M}} : \quad \mathcal{C} \to \mathcal{C}^*_{\mathcal{M}} ,$$

with $\mathcal{C}^*_{\mathcal{M}}$ the fusion category of module endofunctors of $\mathcal{M}$.

Suppose that $\mathcal{C}$ is modular, with a set I of representatives of simple objects, and that $\mathcal{M} \simeq A$-mod$_{\mathcal{C}}$ for a special symmetric Frobenius algebra $A \in \mathcal{C}$. Let

$$Z_{ij}(\mathcal{M}) := \dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{C}^*_{\mathcal{M}}}(\alpha^{+}_{\mathcal{M}}(U_i), \alpha^{-}_{\mathcal{M}}(U_j)) \;\in \mathbb{Z}_{\geq 0}$$

for $i, j \in I$. Then the matrix $Z = (Z_{ij})$ defines a modular invariant for the modular data of $\mathcal{C}$ [2]. The TFT construction of RCFT correlators (see the next section) implies that it is even the modular invariant giving the torus partition function of a consistent full conformal field theory.

In the specific case of a Drinfeld center, $\mathcal{C} = Z(\mathcal{A})$, $\mathcal{C}$-module categories are equivalent, as a bicategory, to $\mathcal{A}$-bimodule categories. For an $\mathcal{A}$-bimodule category, one obtains from left and right multiplication two monoidal functors

$$(*) \qquad Z(\mathcal{A}) \xrightarrow{L} \mathrm{End}_{\mathcal{A}}(\mathcal{B}) \xleftarrow{R} Z(\mathcal{A}) .$$

If the bimodule category is invertible, this induces [4] a braided auto-equivalence of $Z(\mathcal{A})$, and thus a bijection between the Brauer-Picard group of $\mathcal{A}$ and the group of isomorphism classes of braided auto-equivalences of its Drinfeld center $Z(\mathcal{A})$.

2. Topological field theory with defects

The aim of the second part of the talk was to show that three-dimensional topological field theories with defects provide a natural framework not only to understand the relation between the Brauer-Picard group of $\mathcal{A}$ and braided auto-equivalences of $Z(\mathcal{A})$ (and other aspects of (categorified) representation theory), but also of the TFT construction of RCFT correlators.

In a review of the TFT construction (see e.g. [15]), we emphasized two aspects: First, the triangulation of the world sheet that enters in this construction is an ad hoc idea that is not implemented in terms of natural field theoretic structures. And second, different Frobenius algebras can give equivalent sets of correlators; Frobenius algebras are thus redundant decoration data.

Defects in three-dimensional TFTs provide a natural way to solve this problem. An extended topological field theory is, for our purposes, a symmetric monoidal 2-functor

$$TFT: \quad \mathrm{cob}_{3,2,1} \to \text{2-vect}\,.$$

For TFT with defects, one works with the larger bicategory $cob^{\partial}_{3,2,1}$ that contains also manifolds with singularities, or stratified manifolds, or manifolds with defects (see e.g. [12, Sect. 4.3] and [1] for definitions of such categories).

Although a complete construction of such theories is not available to date, non-trivial results have been obtained. In particular a natural class of topological ("gapped") indecomposable surface defects separating topological field theories of Reshetikhin-Turaev type based on $\mathcal{C}_1$ and $\mathcal{C}_2$ has been identified [8]. They are described by a pair consisting of a fusion category $\mathcal{W}$ and a braided equivalence

$$\mathcal{C}_1 \boxtimes \mathcal{C}_2^{rev} \xrightarrow{\simeq} Z(\mathcal{W})\,.$$

As a consequence, there is an obstruction to the existence of surface defects which takes values in the Witt group [3] of modular tensor categories.

The braided equivalence (∗) should then be related to the functor associated in a TFT of Turaev-Viro type to the cylinder with a circular defect labeled by an invertible $\mathcal{A}$-bimodule category. This has been shown [9] for a subclass of Turaev-Viro theories, so-called Dijkgraaf-Witten theories, which can be obtained by linearizing categories of principal bundles. Defects correspond to a variant of relative bundles, and one indeed finds the functor (∗).

Generalizing a paradigm from two-dimensional rational conformal field theory [5] to three-dimensional field theories, one can realize symmetries of three-dimensional TFTs in terms of invertible surface defects. The case of Dijkgraaf-Witten theories based on an abelian group A and vanishing 3-cocycle on A has been explicitly analyzed [7].

Finally, a category valued trace for bimodule categories has been developed [6] which provides natural candidates for those categories that the extended TFT assigns to circles with defects. These results are steps towards a version of the TFT construction of RCFT correlators that implements a suggestion of [10]: replace the triangulation of the world sheet by a surface defect. Indeed, it is understood [8, Sect. 6] that a surface defect $B \in \mathcal{C}$-bimod and a generalized Wilson line W separating the transparent defect $T_{\mathcal{C}}$ from the defect B give rise to a special symmetric Frobenius algebra in $\mathcal{C}$.

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From Segal CFT to conformal nets

James Tener

In the first part of my talk, I presented a version of Graeme Segal's geometric formalism for a (spin) chiral conformal field theory. A Segal CFT assigns Hilbert spaces to labelled, parametrized circles and finite-dimensional vector spaces of trace class maps to complex cobordisms of these circles. To give an example, I introduced the free fermion spin Segal CFT, and discussed its construction in arbitrary genus via fermionic second quantization.

There is a straightforward dictionary between vertex operator algebras and the restriction of a Segal CFT to genus zero surfaces with all boundary components labelled by the vacuum. A concise version of this statement for the free fermion is the following.

Theorem. *Let Σ be the Riemann surface obtained by removing from the unit disk the disk of radius of radius r about 0 and the disk of radius s about z. Let T be the operator assigned to this surface by the free fermion Segal CFT, normalized to satisfy $T(\Omega \otimes \Omega) = \Omega$. Then $T(\xi \otimes \eta) = Y_{\mathcal{F}}(r^{L_0}\xi, z)s^{L_0}\eta$, where $Y_{\mathcal{F}}$ is the free fermion state-field correspondence.*

In the second half of the talk, I discussed a method for constructing conformal nets from genus zero Segal CFT, and relating these conformal nets to the vertex operator algebra corresponding to the pairs of pants in the above theorem. Given a chiral Segal CFT in genus zero, one can consider limits of pairs of pants as one of the incoming circles expands to overlap with the outgoing circle on an interval. If the operators assigned to these pairs of pants converge in that limit, then one can construct a conformal net from the limit operator in a straightforward way.

I then discussed whether or not the free fermion had the property that one can assign a bounded operator to degenerate pairs of pants, and presented partial results to this effect.

Vertex rings and von Neumann regular rings

Geoffrey Mason

A theme of this Oberwolfach conference was the search for connections between subfactor theory and vertex (operator) algebras. My talk was directly related to this possibility, and arose out of ideas concerning *vertex rings* (henceforth, VRs).

The general idea is to find *functorial* relations between VRs and *von Neumann regular rings* rather than subfactors *per se*. Von Neumann originally introduced his regular rings in order to coordinate von Neumann algebras, but here only the rings will be relevant. Moreover, we only deal with *commutative* von Neumann rings – mainly because VRs are, despite their outward appearance, commutative objects. The extension of our work to the noncommutative case remains open. Finally, we have to deal with *sheaves* of VRs, which nowadays may be regarded as natural.

Although the axioms for a VR are generally stated in terms of vector spaces over a base field (usually the complex numbers) they make complete sense over any unital, commutative, associative base ring k, and we need to work at this level of generality in order to state and prove our main results. Indeed, we do not even have to make k explicit; a VR is an abelian group V equipped with biadditive 'products'

$$V \times V \to V, (u, v) \mapsto u(n)v, \quad (n \in \mathbb{Z})$$

together with a distinguished element (vacuum) $\mathbf{1}$. The following axioms are imposed:

(a) $u(n)v = 0 \;\; \forall n \gg n_0(u,v)$

(b) $u(-1)\mathbf{1} = u, \;\; u(n)\mathbf{1} = 0 \; (n \geq 0) \;\; \forall u \in V$

(c) $$\sum_{i\geq 0}\binom{r}{i}(u(t+i)v)(r+s-i)w =$$

$$\sum_{i\geq 0}(-1)^i\binom{t}{i}\left\{u(r+t-i)v(s+i) - (-1)^t v(s+t-i)u(r+i)w\right\}$$

The *center* of V is defined as $C(V) := \{u \in V | u(n) = 0 \; (n \neq -1)\}$. This is a commutative, associative ring with respect to the -1^{th} product, and $\mathbf{1}$ is the identity. We may take $k := C(V)$ as the base ring of V inasmuch as all products $u(n)v$ are k-bilinear. This construction defines an insertion of categories

$$\mathbf{C} \hookrightarrow \mathbf{V}$$

where $\mathbf{C}, \mathbf{V}$ are the categories of commutative rings and vertex rings respectively. This is a particularly good functor because of

Theorem. $\mathbf{C}$ is a *coreflective* subcategory of $\mathbf{V}$, i.e., $\mathbf{C} \hookrightarrow \mathbf{V}$ has a *right adjoint*, given by the center construction $V \mapsto C(V)$.

For VOAs defined over $\mathbb{C}$, say, this is an easy result. It is less obvious in the generality stated here. Informally, the Theorem says that $\mathbf{V}$ is a *natural* extension $\mathbf{C}$. It is well-known that $\mathbf{C}$ is equivalent to the category $\mathbf{A}$ of *affine schemes*, and we might ask if there is an analogous category $\mathbf{V}$? of what we might call *affine vertex schemes* that fits into a functor diagram

$$\begin{array}{ccc} \mathbf{V} & \longleftrightarrow & \mathbf{V}? \\ \uparrow & & \uparrow \\ \mathbf{C} & \longleftrightarrow & \mathbf{A} \end{array}$$

Following ideas of Pierce in the commutative ring case, we consider a qualitatively weaker question. Pierce showed that there is an equivalence between $\mathbf{C}$ and a category **redA** of *reduced sheaves* of commutative rings. Here, the objects are sheaves of commutative rings over a base which is a *Stone space* (compact, Hausdorff, totally disconnected), moreover each fiber has only the two trivial idempotents $0, 1$. Every commutative ring R gives rise to such a reduced sheaf by constructing an *etalé* space whose base is $Spec(B(R))$, the Stone space of the set $B(R)$ of idempotents of R made into a Boolean ring in a standard manner. For a general ring this is a rather crude variant of the affine scheme of R, but if R is a commutative von Neumann regular ring then it is *equivalent* to the affine scheme. Moreover in this case, Pierce showed that the reduced sheaf defined by R is a sheaf of *fields*.

We are able to similarly construct a category of *reduced sheaves of vertex rings* **redV** whose objects are etalé spaces over a Stone space whose fibers are vertex rings with a certain idempotent property. These may be constructed internally from any vertex ring over $k = C(V)$ as a sheaf over $Spec(B(C(V)))$.

$$\begin{array}{ccccc} \mathbf{redV} & \longleftrightarrow & \mathbf{V} & \longleftrightarrow & \mathbf{V}? \\ \uparrow & & \uparrow & & \uparrow \\ \mathbf{redA} & \longleftrightarrow & \mathbf{C} & \longleftrightarrow & \mathbf{A} \end{array}$$

If $k = C(V)$ is a von Neumann regular ring then the corresponding sheaf of vertex rings has fibers which are vertex rings defined over a *field*. Conversely, the vertex ring of global sections of a reduced sheaf of vertex rings defined over a field is a vertex ring defined over a von Neumann regular ring. In this way, we obtain analog of Pierce's construction, namely an equivalence of categories between reduced sheaves of vertex rings defined over a field and vertex rings whose center $k = C(V)$ is von Neumann regular.

Subfactors, twisted equivariant K-theory and conformal field theory

David E. Evans

I focused on the Haagerup subfactor and whether its quantum double/Drinfeld center or asymptotic inclusion gives rise to or arises from a conformal field theory described by a conformal net of factors or a unitary vertex operator algebra and whether it can be built up from more unexceptional group constructions involving their deformations as quantum groups or loop groups.

The Haagerup subfactor [13] is the first finite depth subfactor $N \subset M$ beyond index 4. It has index $(5 + \sqrt{13})/2$ and the even Haagerup N-N system is a quadratic system with irreducible sectors (in the type III endomorphism picture) $\Delta = \{\mathrm{id}, u, u^2, \rho, \rho u, \rho u^2\}$. The fusions (product of sectors) are given by: $[u]^3 = [\mathrm{id}]$, $[u][\rho] = [\rho][u]^2$, $[\rho]^2 = [\mathrm{id}] + [\rho] + [\rho u] + [\rho u^2]$. These have statistical dimensions $\dim(u) = 1$ and $\dim(\rho) = \delta = (3 + \sqrt{13})/2$, since $\delta^2 = 1 + 3\delta$ from the fusion rules. These fusion rules were generalised in [15] generated with an irreducible ρ and a group G with fusion rules $[g][\rho] = [\rho][g^{-1}]$ for all $g \in G$ and $[\rho]^2 = 1 + \sum_{g\in G}[\rho g]$, with $\dim(\rho) = (|G| + \sqrt{|G|^2 + 4})/2$ In particular, the Δ or N-N system of the the Haagerup subfactor corresponds to $G = \mathbb{Z}_3$.

The existence of this subfactor was first established by Haagerup [13] who constructed basically the 6j-symbols or Boltzmann weights. Izumi later showed the existence of this subfactor by constructing endomorphisms on Cuntz algebras satisfying these fusion rules [15]. More recently [17] found the Haagerup subfactor by constructing the planar algebra or relative commutants. Izumi [15] put the Haagerup in a potential series of subfactors for the graphs 33...3 ($2n+1$ arms) and an abelian group of order $2n + 1$, and established existence and uniqueness for $\mathbb{Z}_3$ and $\mathbb{Z}_5$. We showed [4] by solving Izumi's polynomial equations, that there are (respectively) $1, 2$ subfactors of Izumi type $\mathbb{Z}_7$, and $\mathbb{Z}_9$, and found strong numerical evidence for at least $2, 1, 1, 1, 2$ subfactors of Izumi type $\mathbb{Z}_{11}, \mathbb{Z}_{13}, \mathbb{Z}_{15}, \mathbb{Z}_{17}, \mathbb{Z}_{19}$. The cases for $\mathbb{Z}_{11}$ and $\mathbb{Z}_{19}$ when there are further solutions are when the discriminants $11^2 + 4 = 5^3$ and $19^2 + 4 = 5 \times 73$ are composite. The extra complications regarding even groups were addressed in [6] and [16]. We [6] generalised Izumi's framework, weakening his equations and allowing solutions for even order abelian groups. In particular, we constructed new subfactors at indices $3+\sqrt{5}$ and $4+\sqrt{10}$ corresponding to the groups $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_6$ and with graphs 3333 and 333333.

Further quadratic systems are the near group ones. Take again G to be a finite abelian group and consider the fusion rules $[g\rho] = [\rho] = [\rho g]$, and $[\rho^2] = \oplus_g [g] \oplus n'[\rho]$ for some $n' \in \mathbb{Z}_{\geq 0}$. It was independently shown in [7] and [16] that n' is either $|G|-1$ or is a multiple of the order of the group. In particular, $n'=0$ is the case of the Tambara-Yamagami categories [18]. In the case when $n' = |G|$, we [7] found new solutions for groups up to order 13. The cases for $G = \mathbb{Z}_n$ $(n \leq 4)$, $\mathbb{Z}_2 \times \mathbb{Z}_2$ and one solution for $\mathbb{Z}_5$ were constructed in [15]. We analysed the modular data of their doubles in terms of quadratic forms on abelian groups.

More generally, there are mixed systems proposed in [7]. Let G be a finite group (not necessarily abelian) and suppose $\overline{[\rho]} = [\rho][g_\rho]$ for some $g_\rho \in G$. Let N be any subgroup of G: we require $[g][\rho] = [\rho]$ if and only if $g \in N$. Then $[\rho][g] = [\rho]$ iff $g \in g_\rho N g_\rho^{-1} =: N'$. The simple objects in this category are $[g]$ for $g \in G$ as well as $[g_i][\rho]$ for representatives g_i of cosets G/N. Let ϕ be any isomorphism $G/N \to G/N'$; we require $[g][\rho] = [\rho][g']$ if and only if $g' \in \phi(gN)$. Then $[\rho]^2 = \sum_{g \in N} [g] + \sum_i n'_i [g_i][\rho]$. We require ϕ to satisfy $g_\rho^{-1}\phi(\phi(g))g_\rho = g$ for all $g \in G$. The near-group categories [15, 7] correspond to the choice $N = G$ and $g_\rho = 1$; the Haagerup series [15, 4] corresponds to $N = 1$, $\phi(g) = -g$, $n'_i = 1$. The first such mixed system was found in [12] starting with the conformal inclusion $(G_2)_4 \to (D_7)_1$. There is a non-trivial simple current α of order 4 coming from $(D_7)_1$. Here G is $\mathbb{Z}_4$ with subgroup $N = \mathbb{Z}_2$. The fusion rules are $[\alpha][\rho] = [\rho][\alpha] = [\rho\alpha] \neq [\rho]$, $\quad [\alpha^2][\rho] = [\rho] = [\rho][\alpha^2]$, $[\rho]^2 = 2[\rho] \oplus 2[\rho\alpha] \oplus [\mathrm{id}] \oplus [\alpha^2]$.

I described the picture of the Haagerup modular data found in joint work with Terry Gannon [4] as arising from the double of the dihedral group D_3 and $SO(13)$ at level 2 by a notion of grafting. The modular data of the double of the Haagerup was determined by Izumi [15] using tube algebra computations and half braidings on the original system Δ. This was used to determine the fusion rules in [10] using the Verlinde formula $N_{i,j}^k = \sum_l (S_{i,l}/S_{0,l}) S_{j,l} S_{k,l}^*$. Fed back into the relation $S_{i,j} = \overline{T}_{i,i}\overline{T}_{j,j}T_{0,0}\sum_k T_{k,k}S_{k,0}N_{i,j}^k$ gives a simpler form for the S matrix as basically cyclotomic integers [11]. The S matrix was further considerably simplified in [4], and led to discovering characters which transformed according this representation of $SL(2,\mathbb{Z})$ and a description of the Haagerup modular data as arising from the double of the dihedral group D_3 and $SO(13)$ at level 2 by a notion of grafting. By level rank duality, one could alternatively start by considering $O(2)$ at level 13. Both parts, the finite dihedral and the infinite dihedral $O(2)$ can be described via $Z_3 \times Z_3$ and Z_{13} respectively and their orbifolds, which one can also picture as lattice theories. The fusion rules of the double are particularly elegant in this language and was presented in the lecture.

This formulation generalises to the the doubles of the Haagerup series, and to the doubles of the near group systems at least when the groups are of odd order, and then presumably to the mixed quadratic systems. For the doubles of odd near group systems, the modular data is built up from a G fold of modular data of the grafting of orbifolds of G with that of G' where G' is an associated abelian group of order $|G| + 4$, at least when the order of G is ≤ 13 [7].

In [4] we produced the modular data to the second solution of the generalized Haagerup system for Z_{11}, where the discriminant $11^2 = 4 = 5^3$ is composite. The first solution produces modular data corresponds to he grafting of the orbifolds arising from $Z_{11} \times Z_{11}$ and Z_{125} whilst the second solution can now be understood as being grafted from $Z_{11} \times Z_{11}$ and $Z_5 \times Z_{25}$. This new understanding was obtained during an analysis [9] of non-unitary systems and a more penetrating analysis of Izumi's equations for identifying the half-braidings which were incomplete even in the unitary setting. We could realise [9] non-unitary fusion categories using subfactor-like methods, and compute their quantum doubles and modular data.

Lattice theories were described [3, 5, 8] through twisted equivariant K-theory of a torus acting trivially on itself. This suggests a way producing the modular tensor category of the double via these torus models alone. The Haagerup system and subfactor can be recovered from a knowledge of the module categories of the double – particularly by an understanding of the monomial modular invariants through alpha-induction. Modular invariants for an even lattice system L have been understood in related work [8] as subgroups H (with a parity constraint) of L^*/L with some 2-cohomology $H^2(H, \mathbb{T})$. Alternatively as pairs of even lattices D_+ and D_- such that $L \subset D_\pm \subset L^*$, such that $D_+^*/D_+ \simeq D_-^*/D_-$ [5]. It is then natural to ask if the modular invariants of the grafting of what are basically orbifolded lattices can also be simply described or at least those which produce monomial modular invariants.

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Operator-algebraic construction of integrable QFT

Yoh Tanimoto

1. Half-sided modular inclusions

Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion of von Neumann algebras on a Hilbert space $\mathcal{H}$. Assume that Ω is cyclic for $\mathcal{M}$ and $\mathcal{N}$, separating for $\mathcal{M}$. Let σ^t be the modular automorphisms group for $\mathcal{M}$ with respect to Ω. We say that this inclusion $\mathcal{N} \subset \mathcal{M}$ is a **half-sided modular inclusion** if $\sigma^t(\mathcal{N}) \subset \mathcal{N}$ holds for $t \geq 0$.

Differently from the study of subfactors, the situation where the relative commutant $\mathcal{M} \cap \mathcal{N}'$ is large is of a great interest. We say that a half-sided modular inclusion is **standard** if Ω is cyclic for $\mathcal{M} \cap \mathcal{N}'$.

A remarkable result is that there is a one-to-one correspondence between standard half-sided modular inclusions and a class of quantum field theories, namely strongly additive Möbius covariant nets on S^1 [8] as we explain below.

This is the recurring theme of this article. In many cases, a few operator algebras with some additional information are enough to construct the full quantum field theory.

2. Quantum field theory and Haag-Kastler nets

Quantum field theory is a theoretical framework for elementary particle physics. While classical field theory is concerned with functions (possibly in a extended sense) on the spacetime $\mathbb{R}^d$, Streater and Wightman pointed out that quantum fields should be operator-valued distributions $\phi(x), x \in \mathbb{R}^d$ [17]. Haag proposed then to consider the net of von Neumann algebras $\mathcal{A}(O) = \{e^{i\phi(f)} : \mathrm{supp} f \subset O\}''$ generated by quantum field ϕ, where $O \subset \mathbb{R}^d$ is a spacetime region [9]. Such a net $\{\mathcal{A}(O)\}$, together with the representation U of the spacetime symmetry and the vacuum vector Ω, turns out to contain most of information of physical interest, e.g. the scattering amplitudes of particles, therefore, can be regarded as a mathematical framework of quantum field theory, and called **Haag-Kastler net**.

A Haag-Kastler net $(\mathcal{A}, U, \Omega)$ by definition should satisfy various axioms (see a recent review on conformal field theories [15]). An important axiom is **locality**, namely, if O_1 and O_2 are spacelike separated, then the corresponding von Neumann algebras $\mathcal{A}(O_1)$ and $\mathcal{A}(O_2)$ should commute. The action of U should be consistent with $\mathcal{A}$, namely $U(g)\mathcal{A}(O)U(g)^* = \mathcal{A}(gO)$ (**covariance**). When one considers one-dimensional spacetime $d = 1$, spacetime regions are intervals and spacelike separation of I_1, I_2 is simply replaced by disjointness $I_1 \cap I_2 = \emptyset$.

If one has a standard half-sided modular inclusion $(\mathcal{N} \subset \mathcal{M}, \Omega)$, one can construct a net on $\mathbb{R}$ by setting $\mathcal{A}(\mathbb{R}_+) = \mathcal{M}, \mathcal{A}(\mathbb{R}_+ + 1) = \mathcal{N}, \mathcal{A}(0, 1) = \mathcal{M} \cap \mathcal{N}'$. The crucial point is that one can obtain the spacetime symmetry group, in this case the Möbius group, from the modular operators for these von Neumann algebras, and indeed, one can extend the net to the circle S^1 [8].

3. Examples of Haag-Kastler nets

The most important problem in mathematical quantum field theory is the scarcity of examples in higher dimensions. Although there are many conformal field theories for $d = 1, 2$ [6] and even certain classification results are available [10], and under the program of Constructive Quantum Field Theory several interacting models have been obtained for $d = 2, 3$ [7], currently the only examples of Haag-Kastler nets in $d \geq 4$ are the (generalized) free fields [16].

Therefore, it is significant to study methods and techniques to produce examples of nets. We explain below some recent results for $d = 2$. We also mention several attempts to higher dimensions [4, 12] and new ideas which go through de Sitter spacetime [2].

4. Longo-Witten endomorphisms

Let us see that small additional information to a one-dimensional net is sufficient in order to construct a class of nets on the two-dimensional half-plane $\mathbb{R}^2_+ = \{(t, x) \in \mathbb{R}^2 : x > 0\}$. Let $(\mathcal{A}, U, \Omega)$ be a net on $\mathbb{R}$, where U is a unitary representation of the translation group $\mathbb{R}$ (we are not considering Möbius or conformal covariance here). We say that a unitary operator V implements a **Longo-Witten endomorphism** of the net $(\mathcal{A}, U, \Omega)$ if V commutes with U and $\mathrm{Ad}V(\mathcal{A}(\mathbb{R}_+)) \subset \mathcal{A}(\mathbb{R}_+)$. For two intervals $I_1 < I_2$ on the time axis $x = 0$, one associates a diamond $D = \{(x, t) \in \mathbb{R}^2_+ : t - x \in I_1, t + x \in I_2\}$. Conversely, any diamond in $\mathbb{R}^2_+$ is of this form. If we define $\mathcal{A}_V(D) = \mathcal{A}(I_1) \vee \mathrm{Ad}V(\mathcal{A}(I_2))$, then $\mathcal{A}_V$ is a local net on $\mathbb{R}^2_+$, time-translation covariant (with respect to U) [14].

Examples of Longo-Witten endomorphisms are given on the U(1)-current net by the second quantization operators [14]. Further examples which are not second quantization were obtained by the so-called boson-fermion correspondence [3].

5. Integrable QFT

By considering more specific examples, one can do more. Namely, we can construct a two-dimensional net on the full spacetime $\mathbb{R}^2$. Let $(\mathcal{A}, U, \Omega)$ be the massive complex free field net on $\mathbb{R}^2$. This can be identified with the the tensor product of two copies of the massive real free field net, the simplest two-dimensional net. For the wedge-shaped region $W_{\mathrm{R}} := \{(t,x) \in \mathbb{R}^2 : x > |t|\}$, the inclusion $\mathcal{A}(W_{\mathrm{R}} + (1,1)) \subset \mathcal{A}(W_{\mathrm{R}})$ is a standard half-sided modular inclusion with respect to Ω. The associated one-dimensional net is the tensor product of two copies of the U(1)-current net.

There is a one-parameter 2π-periodic Longo-Witten automorphisms acting on this one-dimensional net. Let us denote the unitary operators by $V(s)$. We can take its generator Q and write them as $V(s) = e^{isQ}, s \in \mathbb{R}$. Now the new net is defined on the tensor product Hilbert space. Let us introduce $\widetilde{V}(s) = e^{isQ\otimes Q}$. Define the new net first for the wedge by $\widetilde{\mathcal{A}}_s(W_{\mathrm{R}}) := \mathcal{A}(W_{\mathrm{R}}) \otimes \mathbb{C}I \vee \mathrm{Ad}\widetilde{V}(s)(\mathbb{C}I \otimes \mathcal{A}(W_{\mathrm{R}}))$. The other elements are simply the tensor products $\widetilde{U} = U \otimes U, \widetilde{\Omega} = \Omega \otimes \Omega$. The von Neumann algebras $\widetilde{\mathcal{A}}(O)$ for general region O is defined by locality (one assigns the commutant $\mathcal{A}(W_{\mathrm{R}})'$ to the reflected wedge W_{L}) and covariance with respect to $\widetilde{U}$. Then the triple $(\widetilde{\mathcal{A}}, \widetilde{U}, \widetilde{\Omega})$ is a two-dimensional Haag-Kastler net [18]. One can compute the so-called S-matrix, an invariant of a net, which is different from the identity operator and hence one says that this net is interacting.

With more complicated Longo-Witten endomorphisms, one can construct more examples. The resulting nets have S-matrices with a particularly simple structure, called factorizing, and some of them are believed to be related to the quantization of classical integrable Lagrangians. Another class of nets with factorizing S-matrices has been constructed by a different technique [11]. Further examples are currently under investigation [13, 1, 5].

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From vertex operator algebras to conformal nets and back

Sebastiano Carpi
(joint work with Yasuyuki Kawahigashi, Roberto Longo, Mihály Weiner)

We have two different mathematical formulations of chiral two-dimensional CFT (CFT on $S^1 \equiv$ compactified light-ray): vertex operator algebras (VOAs) and conformal nets on S^1. The VOA approach is mainly algebraic [6, 7, 11, 15]. A VOA (over $\mathbb{C}$) is a complex vector space V together with a linear map (the *state field correspondence*) $V \ni a \mapsto Y(a, z)$ satisfying certain physically motivated assumptions: locality, vacuum, conformal covariance, ... Here, for any $a \in V$, the *vertex oparator* $Y(a, z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$ is a formal power series with coefficients in $\mathrm{End}(V)$ or, equivalently, an operator-valued formal distribution. On the other hand, the conformal net approach is mainly operator algebraic (hence functional analytic). It corresponds to the chiral CFT version of the so called algebraic quantum field theory (AQFT) [9], see [12] for a recent review on the AQFT approach to two-dimensional CFT. A conformal net $\mathcal{A}$ on S^1 is a map $I \mapsto \mathcal{A}(I)$ from the set of intervals of S^1 into the family of von Neumann algebras acting on a fixed Hilbert space (the *vacuum Hilbert space*) satisfying (again) certain physically motivated assumptions: isotony, locality, vacuum, conformal covariance, ... Despite their significant differences from a mathematical point of view, these two formulations show their common "physical root" through many structural similarities. Moreover, various interesting chiral CFT models can be considered from both point of view with similar outputs.

In a recent paper [2] we studied, for the first time, the mathematical correspondence between VOAs and conformal nets from a general point of view. We start with a unitary simple VOA V [2, 3] assumed to satisfy certain nice estimates (energy bounds). Then we follow the traditional approach to the construction of local nets of von Neumann algebras starting from quantum fields (i.e. operator valued distributions in the sense of Wightman [17]), see [9]. For every smooth function

$f \in C^\infty(S^1)$, the operator valued distribution $Y(a,z)$ gives rise to a *smeared vertex operator* $Y(a,f)$ acting on the Hilbert space completion $\mathcal{H}_V$ of V. Then, for every open interval $I \subset S^1$ we consider the von Neumann algebra

$$\mathcal{A}_V(I) \equiv W^*(\{Y(a,f) : a \in V, f \in C_c^\infty(I)\})$$

generated by all the vertex operators smeared with test functions with support in I. Since the smeared vertex operators are in general unbounded it is not *a priori* clear that the locality axiom for the VOA V implies that the the map $I \mapsto \mathcal{A}_V(I)$ will satisfy locality i.e. that $[\mathcal{A}_V(I_1), \mathcal{A}_V(I_2)] = \{0\}$ whenever $I_1 \cap I_2 = \emptyset$. We say that V is *strongly local* if this is actually the case. We then prove that if V is a strongly local VOA then the map $I \mapsto \mathcal{A}_V(I)$ defines an irreducible conformal net on S^1.

The class of strongly local VOAs is closed under taking tensor products and unitary subVOAs. Moreover, for every strongly local VOA V, the map $W \mapsto \mathcal{A}_W$ gives a one-to-one correspondence between the unitary subVOAs W of V and the covariant subnets of $\mathcal{A}_V$.

Many known examples of unitary VOAs such as the unitary Virasoro VOAs, the unitary affine Lie algebras VOAs, the known $c = 1$ unitary VOAs, the moonshine VOA $V^\natural$, together with their coset and orbifold subVOAs, turn out to be strongly local. The corresponding conformal nets coincide with those previously constructed by different methods: the Virasoro nets [1, 13, 16], the loop groups nets [8, 18, 19], the coset conformal nets [20], the $c = 1$ conformal nets [21], the moonshine conformal net $\mathcal{A}^\natural$ [14] ....

The even shorter moonshine vertex operator algebra constructed by Höhn [10] also turns out to be strongly local being a subVOA of $V^\natural$. Moreover, the automorphism group of the corresponding conformal net coincides the VOA automorphism group which is known to be the baby monster group $\mathbb{B}$.

Note also that the (still hypothetical) Haagerup VOA with $c = 8$ considered by Evans and Gannon in [4] has been suggested to be a unitary subVOA of a unitary affine Lie algebra VOA and hence it should be strongly local.

Furthermore, a construction of Fredenhagen and Jörß [5] gives back the strongly local VOA V from the irreducible conformal net $\mathcal{A}_V$. More generally, in [2] we give conditions on an irreducible conformal net $\mathcal{A}$ implying that $\mathcal{A} = \mathcal{A}_V$ for some strongly local vertex operator algebra V.

We conjecture that every unitary VOA is strongly local and that every irreducible conformal net comes from a unitary VOA in the way described above.

The representation theory aspects of the correspondence $V \mapsto \mathcal{A}_V$ will be considered in future research.

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Small examples: subfactors and fusion categories

Scott Morrison

My talk covered three essential points:

(1) Fusion categories and modular tensor categories provide a notion of symmetry going beyond groups of symmetries. They are observed in topological phases of matter, and can be formally realised as the symmetries of Levin-Wen models.
(2) While the overall classification of fusion categories, of modular tensor categories, or of subfactors is likely to be intractable, there are tantalising hints of order, and of sparsity, in the classifications which have been established so far.
(3) The classification of small index subfactor planar algebras now extends up to index $5\frac{1}{4}$ unconditionally, and up to index $6\frac{1}{5}$ (leaving aside index exactly 6) with the added assumption of 1-supertransitivity [LMP15].

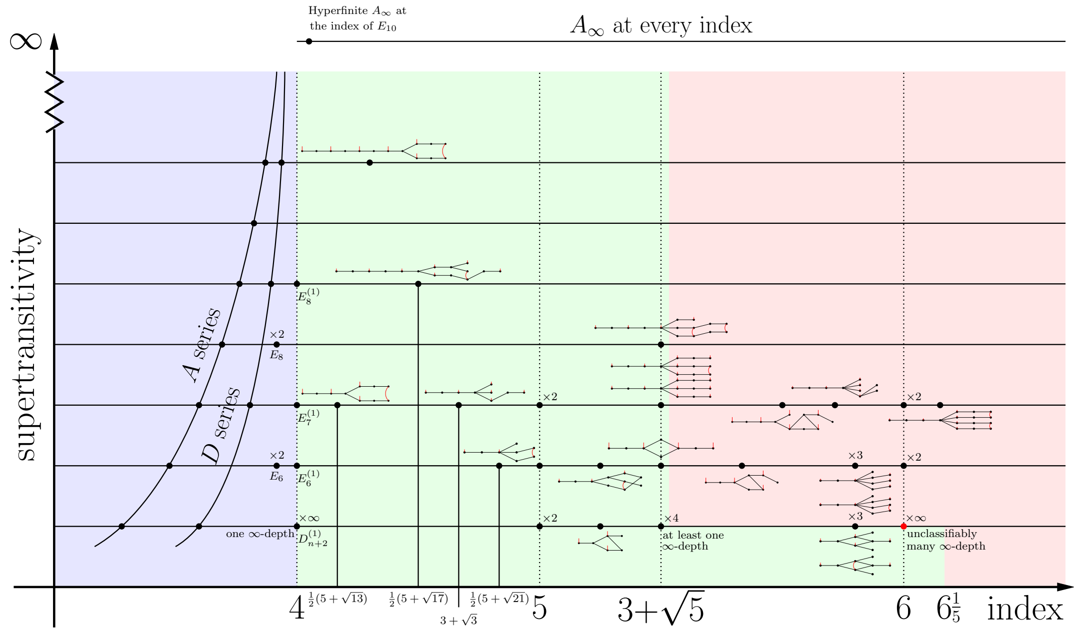
(The talk had a significant defect, in my opinion, in that I did not clearly explain the relationship between classifications of fusion categories and classifications of small index subfactors. One way to see a connection is that any object V in a unitary fusion category gives a subfactor planar algebra via $P_{n,+} = \mathrm{Inv}((V \otimes V^*)^{\otimes n})$, and the index is then $|\dim V|^2$, and so a classification of small index subfactor planar algebras gives control over fusion categories containing an object with small categorical dimension.)

The great surprise of the successive improvements to the classification of small index subfactors is that there have been so few new examples. The further suprise of recent years is that some of the apparently exotic examples, particularly the Haagerup subfactor and the Asaeda-Haagerup subfactor, are now seen to be parts of, or related to, better understood families [Izu01, EG14, GIS15]. We now have the situation that all concretely described fusion categories appear to fall into one of four classes:

(i) categories constructed from finite groups,
(ii) categories constructed from quantum groups at roots of unity,
(iii) quadratic categories, and
(iv) the fusion categories coming from the extended Haagerup subfactor [BMPS12].

Recent progress of the classification incorporates work of Penneys (on quadratic tangles style obstructions) [Pen15], Bigelow-Penneys (on principal graph stability) [BP14], Afzaly (on fast enumeration methods), and Calegari-Guo (on new number theoretic methods for obstructing cyclotomicity of square graph norms) [CG15]. We hope that the paper combining these ideas to complete the classification up to index $5\frac{1}{4}$ will soon be finished.

The latest 'map of subfactors' appears below.



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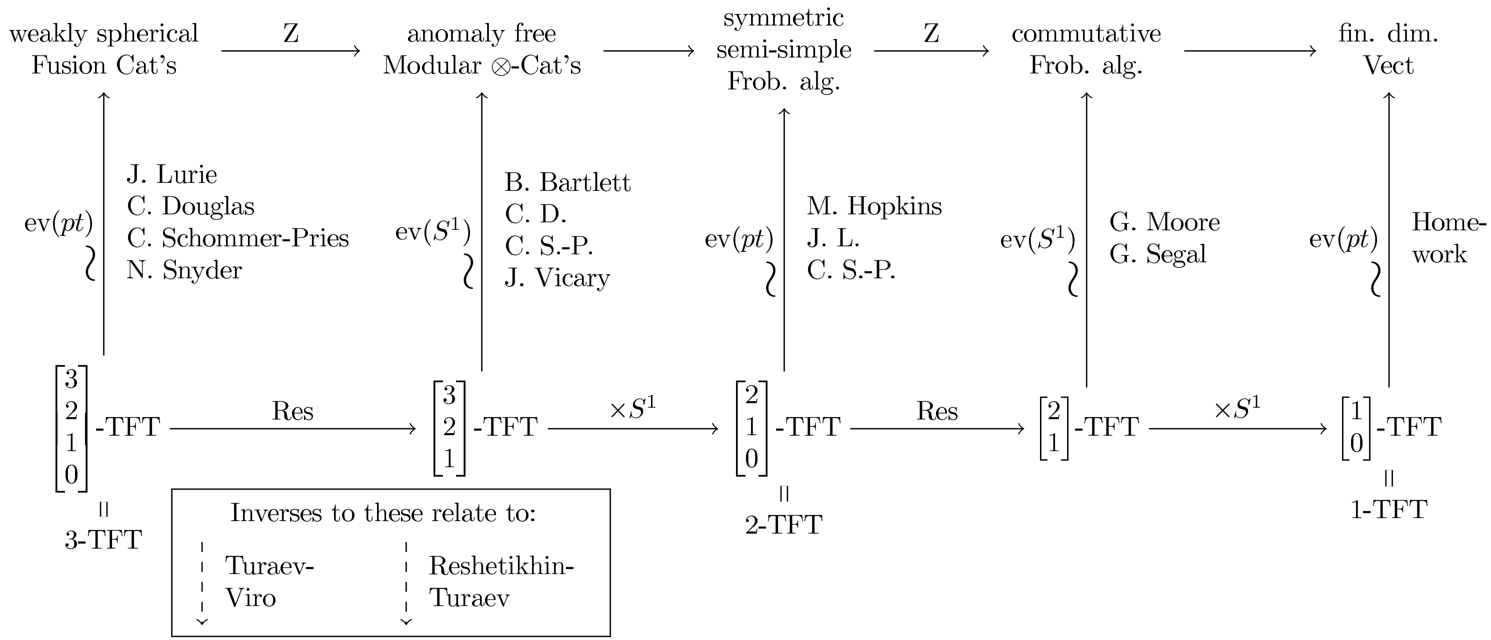
Moduli spaces of field theories

Peter Teichner



Thm: The above evaluation maps are equivalences

of:	3-groupoids	2-groupoids	2-groupoids	groupoids	groupoids
date:	2016	2014	2010	1990's	folklore
π_0:	?	?	$\coprod_{n\geq 0}(\mathbb{C}^\times)^n$	?	$\mathbb{N}_0$
π_1:	$\pi_1(\mathcal{A}) = \mathrm{Br}(\mathrm{Pic}(\mathcal{A}))$	?	$\pi_1(\lambda_1,\dots,\lambda_n) = \mathrm{Stab}(\vec{\lambda}) \leq S_n$	?	$\pi_1(n) = \mathrm{GL}_n$

Examples of Stolz–Teichner cocycles

André Henriques
(joint work with Christopher L. Douglas)

Disclaimer: The ideas and statements in this note are informal and speculative. We present an exploration of what might, one day, become interesting examples of Stolz–Teichner cocycles.

According to the Stolz–Teichner conjecture [5, 6], the moduli space of $(0,1)$-supersymmetric, $\mathit{Fer}(n)$-twisted, extended, 2-dimensional euclidian field theories has the homotopy type of the $(-n)^{\text{th}}$ space of the spectrum of topological modular forms:

$$TMF_{-n} \;\approx\; \left\{ \begin{matrix} \text{Moduli space of } (0,1)\text{-susy} \\ \mathit{Fer}(n)\text{-twisted extended } 2d \text{ EFTs} \end{matrix} \right\}. \tag{1}$$

Given a space X, a "degree $-n$ Stolz–Teichner cocycle" over X is, roughly speaking, a map from X to the right hand side of (1). Such a thing should represent a class in $TMF^{-n}(X)$. In the special case when X is a point, this is the same thing as an element of $\pi_n(TMF)$, the n^{th} homotopy group of the spectrum of topological modular forms.

We now explain, at a very rough level, the terms that appear in the right hand side of (1). First of all, in Stolz–Teichner's language, the term "*euclidian field theory*" means essentially the same thing as "quantum field theory on flat spacetime". A theory is *extended* if it is defined not on Minkowski/Euclidean space, but on compact (flat) 2-manifolds with boundary, on 1-manifolds with boundary (the theory assigns Hilbert spaces to 1-manifolds), and on points (the theory assigns von Neumann algebras to points – see [2] for an explanation of this concept in the context of CFTs). A theory is $(0,1)$-*supersymmetric* if the infinitesimal generator $\bar{L}_0$ of anti-holomorphic translations (on the Hilbert space associated to a circle, say) has an odd square root.

There is an obvious forgetful map from CFTs to EFTs: if a theory knows how to assign values to conformal surfaces, then in particular it knows how to assign values to flat surfaces. We shall be interested in the following question: how much of the right hand side of (1) can one see using only conformal field theories? Can one get interesting elements of $\pi_n(TMF)$ that way?

Let us first go back to (1) and discuss the meaning of "$\mathit{Fer}(n)$-twisted". We concentrate on the case $n \geq 0$. Here, $\mathit{Fer}(n)$ refers to the conformal field theory of n real chiral free fermions (with central charge $c = n/2$). We'll say that a conformal field theory is "$\mathit{Fer}(n)$-twisted" if it contains a copy of $\mathit{Fer}(n)$. (If n is negative, then the theory should instead contain $|n|$ antichiral fermions.) Given a conformal field theory Z, let us write $\chi(Z)$ for the maximal chiral sub-theory of Z, and $\bar{\chi}(Z)$ for the maximal anti-chiral sub-theory. Thus, we say that Z is "$\mathit{Fer}(n)$-twisted" if it comes equipped with an embedding $\mathit{Fer}(n) \hookrightarrow \chi(Z)$.

Finally, Z is $(0,1)$-supersymmetric if the canonical Virasoro $\overline{Vir} \subset \bar{\chi}(Z)$ comes equipped with an extension to an $N = 1$ super-Virasoro:

$$\begin{array}{ccc} \overline{sVir} & & \\ \cup & \searrow & \\ \overline{Vir} & \hookrightarrow & \bar{\chi}(Z) \end{array}$$

Recapitulating, we are looking at full CFTs with an embedding of $Fer(n)$ in the chiral part, and an $N = 1$ supersymmetric structure on the anti-chiral part. The easiest full CFT that satisfies the first requirement is $Fer(n) \otimes \overline{Fer(n)}$. We now need a supersymmetric extension of the standard Virasoro of $\overline{Fer(n)}$

$$\bar{T}(\bar{z}) = \tfrac{1}{2} \sum_{i=1}^{n} : \partial\bar{\psi}_i(\bar{z})\bar{\psi}_i(\bar{z}) :$$

This is the datum of a primary field $\bar{G}(\bar{z})$ of conformal dimension $\frac{3}{2}$, satisfying certain commutation relations. Fields of dimension $\frac{3}{2}$ are linear combinations of $: \bar{\psi}_i\bar{\psi}_j\bar{\psi}_k :$ and of $\partial\bar{\psi}_i$. The latter cannot occur in $\bar{G}(\bar{z})$ since it otherwise wouldn't be primary, so the most general form is the following:

$$\bar{G}(\bar{z}) = \tfrac{1}{6} \sum f^{ijk} : \bar{\psi}_i(\bar{z})\bar{\psi}_j(\bar{z})\bar{\psi}_k(\bar{z}) :$$

It turns out that the above field satisfies the required commutation relations if and only if f^{ijk} are the structure constants of a Lie algebra on $\mathbb{R}^n$ whose Killing form agrees with the standard inner product on $\mathbb{R}^n$ [1, 4]. In other words, there are exactly as many $N = 1$ supersymmetric structures on $\overline{Fer(n)}$ as there are n-dimensional semi-simple Lie algebras of compact type. All in all, this provides (a sketch of) a construction

$$(2) \qquad \left\{ \begin{array}{c} n\text{-dimensional semi-simple} \\ \text{Lie algebras of compact type} \end{array} \right\} \rightarrow \left\{ \begin{array}{c} \text{degree } -n \text{ Stolz–Teichner} \\ \text{cocycles (over a point)} \end{array} \right\}$$

At this point, the natural question is: are there elements in $\pi_n(TMF)$ that are naturally parametrised by n-dimensional semi-simple Lie algebras of compact type, and that one could reasonably conjecture are the images of the above construction? The answer turns out to be yes. The simply connected Lie group integrating a Lie algebra is a framed manifold, when equipped with its left invariant framing. It therefore represents a class in the n-dimensional framed bordism group. The latter is isomorphic to the n^{th} stable homotopy group of spheres $\pi_n^{stable}(S^0) = \pi_n(\mathbb{S})$ by the Pontrjagin-Thom isomorphism (here $\mathbb{S}$ denotes the sphere spectrum). Being a ring spectrum, TMF admits a unit map from the sphere spectrum, which induces a map $\pi_n(\mathbb{S}) \to \pi_n(TMF)$ at the level of homotopy groups. As mentioned in [3], many interesting classes in the homotopy groups of TMF are represented by compact Lie groups with their left invariant framing, and we conjecture that the construction (2) hits exactly those classes.

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Introduction to logarithmic conformal field theory

Ingo Runkel

1. Basic idea

Logarithmic two-dimensional conformal field theories first appeared in [1, 2]; a recent collection of articles on the topic can be found in the special issue [3].

A two-dimensional chiral conformal field theory is constructed out of modules of the Virasoro algebra. One natural class of modules to consider are unitary lowest weight modules, that is, lowest weight modules equipped with an inner product satisfying $(L_m x, y) = (x, L_{-m} y)$ for the generators L_m of the Virasoro algebra. It follows that L_0 is hermitian and therefore diagonalisable.

If one drops the unitarity requirement, there is no reason why L_0 should stay diagonalisable. This leads to logarithmic conformal field theories.

2. Why non-unitary?

There are several good reasons to look beyond unitarity. On the physical side, two-dimensional conformal field theories describe universality classes of two-dimensional critical statistical systems, and these need not be unitary. The most famous example is probably critical percolation, as first analysed via conformal field theory in [4] and whose conformal symmetry was proved in [5].

On the mathematical side, the best understood conformal field theories are build from vertex operator algebras whose representation categories are finitely semisimple. These theories should be thought of as very exceptional points in a fictitious "moduli space of two-dimensional conformal field theories". In going beyond these special points, one can drop the finiteness requirement or the semisimplicity requirement, or both. The simplest class of logarithmic conformal field theories arises if one keeps finiteness but not semisimpicity.

Vertex operator algebras whose representation category is modular (so in particular finitely semisimple) produce – via the Reshetikhin-Turaev construction – invariants of three manifolds with embedded ribbon graphs. This important application has been developed to some extend also for the non-semisimple case. In particular, one obtains projective actions of mapping class groups of surfaces with marked points [6]. However, much work remains to be done: neither is it understood (beyond examples) which vertex operator algebras give rise to such non-semisimple modular categories, nor how to construct a fully-fledged three-dimensional topological field theory out of these. The vertex operator algebra should certainly be C_2-cofinite (to have only a finite number of irreducible modules), but additional requirements may well be necessary.

3. Logarithms

Logarithmic conformal field theories earned their qualifier through the appearance of logarithms in their correlation functions, as opposed to just displaying power-law behaviour. This is most easily demonstrated by looking at a two-point correlator on the complex plane: The space of fields F of a conformal field theory carries an action of two copies of the Virasoro algebra and we denote their generators as L_m and $\overline{L}_m$. The two-point correlator (with one insertion fixed at zero) is a function $C : F \otimes F \times \mathbb{C}^\times \to \mathbb{C}$, and the standard notation is to write $\langle \phi(z)\psi(0)\rangle$ instead of $C(\phi \otimes \psi, z)$. Conformal covariance imposes

$$-z\tfrac{d}{dz}C(\phi \otimes \psi, z) \,=\, C(L_0\phi \otimes \psi, z) + C(\phi \otimes L_0\psi, z) \ ,$$

together with a corresponding condition for the antiholomorphic derivative and $\overline{L}_0$. Suppose now that two elements $\alpha, \beta \in F$ form a rank-two Jordan cell for L_0 and $\overline{L}_0$, say $L_0\alpha = \overline{L}_0\alpha = h\alpha$ and $L_0\beta = \overline{L}_0\beta = h\beta + \alpha$. A short calculation then shows that

$$\langle \beta(z)\beta(0)\rangle = |z|^{-4h}(C_1 + C_2 \log |z|)$$

for suitable constants C_1, C_2. For an L_0 and $\overline{L}_0$ eigenvector, the log-term would be absent and one would be left with only the power-law part.

4. Symplectic fermions

The most studied family of logarithmic conformal field theories is that of symplectic fermions [7]. A more systematic name for symplectic fermions would be "purely odd free superbosons". To construct free superbosons (see e.g. [8]), one starts from a finite-dimensional super-vector space $\mathfrak{h}$ and endows it with a super-symmetric pairing $(-, -)$. From this one builds the affine Lie super-algebra $\widehat{\mathfrak{h}} := \mathfrak{h} \otimes \mathbb{C}[t^{\pm 1}] \oplus \mathbb{C}K$ with (super-)bracket

$$[a_m, b_n] \,=\, m\,(a, b)\,\delta_{m+n,0}\,K \ ,$$

where a_m stands for $a \otimes t^m$ and K is an even central element. There is also a "twisted" version of this Lie super-algebra, $\widehat{\mathfrak{h}}_{\mathrm{tw}}$, where instead of $m, n \in \mathbb{Z}$ one takes $m, n \in \mathbb{Z} + \frac{1}{2}$.

For symplectic fermions one chooses $\mathfrak{h}$ to be purely odd, so that – forgetting the super-structure – one has a symplectic vector space and the above super-bracket becomes an anti-commutation relation (hence the name "symplectic fermions"). Let us think of $\mathfrak{h}$ as an abelian Lie super-algebra. Then it is not hard to see that the category of $\widehat{\mathfrak{h}}$-modules (resp. $\widehat{\mathfrak{h}}_{\mathrm{tw}}$-modules) which are bounded below and where K acts as 1 is equivalent to that of $\mathfrak{h}$-modules (resp. to the category of complex super-vector spaces) [9]. Define the category

$$\mathcal{SF}(\mathfrak{h}) := \mathcal{SF}_0(\mathfrak{h}) \oplus \mathcal{SF}_1(\mathfrak{h}) \ , \quad \mathcal{SF}_0(\mathfrak{h}) = \mathrm{rep}(\mathfrak{h}) \ , \quad \mathcal{SF}_1(\mathfrak{h}) = \mathrm{svect} \ ,$$

where the latter are the categories of finite-dimensional $\mathfrak{h}$-modules (in super-vector spaces) and of finite-dimensional super vector-spaces, respectively. In particular, $\mathcal{SF}_1$ is semisimple, while $\mathcal{SF}_0$ is not. By construction, $\mathcal{SF}(\mathfrak{h})$ is finite: it has finitely many simple objects up to isomorphism (namely four), all objects have finite composition series, and all hom-spaces are finite dimensional.

Via a conformal block calculation one can determine explicitly a braided monoidal structure on $\mathcal{SF}(\mathfrak{h})$, see [9, 10]. For example, let $T = \mathbb{C}^{1|0}$ denote one of the two simple objects in $\mathcal{SF}_1$. The tensor product of T with itself is $S(\mathfrak{h}) \in \mathcal{SF}_0$, the symmetric algebra of $\mathfrak{h}$ (in super-vector spaces, i.e. the exterior algebra when forgetting the super-structure). The latter is indecomposable but reducible as an $\mathfrak{h}$-module.

One can construct a C_2-cofinite vertex operator algebra out of symplectic fermions [11], and conjecturally its representation category is monoidally equivalent to $\mathcal{SF}(\mathfrak{h})$.

5. What else?

Apart from the family of symplectic fermions, parametrised by $d \in 2\mathbb{N}$, the dimension of $\mathfrak{h}$, another well-studied class of models are the W_p-triplet models, which are equally C_2-cofinite. The two families have one point in common, namely the (even part of) symplectic fermions at $d = 2$ is the triplet model at $p = 2$. Other examples of C_2-cofinite vertex operator algebras are the $W_{q,p}$-models which reduce to the W_p-models for $q = 1$. Then one has orbifolds and superconformal versions of the models mentioned so far, but after this we are already running out of examples.

That is to say, there are at this point relatively few classes of examples, which are moreover not too distinct from each other, and more variety would be desirable.

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Factorization and modular invariance beyond RCFT

Jürgen Fuchs
(joint work with Christoph Schweigert)

Mapping class group representations from finite ribbon categories. Let $\mathcal{C}$ be a factorizable finite ribbon category. It is known [6, 7] that one can assign to any punctured surface Σ, with each puncture labeled by an object of $\mathcal{C}$, a finite-dimensional vector space $Bl_{\mathcal{C}}(\Sigma)$ that carries a representation $\pi_{\mathcal{C}}$ of the mapping class group Map(Σ) of Σ. The construction of [6, 7] is compatible with the operation of sewing of surfaces. In case the category $\mathcal{C}$ is semisimple, and thus is a modular tensor category, the so obtained representations of mapping class groups coincide with those furnished by the state spaces of the three-dimensional topological field theory of Reshetikhin-Turaev type – or, equivalently, by the monodromy data of the conformal blocks of a chiral rational conformal field theory (RCFT) – based on the category $\mathcal{C}$. However, remarkably, semisimplicity of $\mathcal{C}$ is not required for the construction, whereby it is of potential relevance also for non-rational ("logarithmic") chiral conformal field theories.

It should be appreciated that in the first place the spaces $Bl_{\mathcal{C}}(\Sigma)$ are not associated with a (punctured) surface, but with a *marked surface* $\widehat{\Sigma}$, consisting of a surface Σ together with some combinatorial data, which can e.g. be expressed with the help of a cut system [8, 1] that furnishes a pair-of-pants decomposition of Σ. However, through the universal properties of coends, the construction in [6, 7] also provides unique isomorphisms between the vector spaces associated with different markings of one and the same surface Σ.

From chiral to full conformal field theory. In the RCFT case it is well known (see e.g. [2]) how to construct *full* local conformal field theories with chiral data described by the category $\mathcal{C}$. In contrast, for non-semisimple $\mathcal{C}$ it is an open problem whether such local theories exist, even though partial results are available for specific models, some of them [5] already for a long time.

When addressing the construction of full local conformal field theories in the framework of [6, 7], the following problems need to be solved:

(1) Find an object F in the enveloping category $\mathcal{D} := \mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}$ that gives the space of bulk fields of a full conformal field theory.

(2) To any punctured surface Σ with all punctures labeled by the so obtained object F of $\mathcal{D}$, assign a vector $v_F(\Sigma)$ in the space $Bl_{\mathcal{D}}(\Sigma)$ that is invariant under

the action $\pi_{\mathcal{D}}$ of the mapping class group $\mathrm{Map}(\Sigma)$, in such a way that the linear isomorphism $Bl_{\mathcal{D}}(\Sigma) \to Bl_{\mathcal{D}}(\#\Sigma)$ implementing a sewing $\Sigma \mapsto \#\Sigma$ of a surface maps the vector $v_F(\Sigma)$ to $v_F(\#\Sigma)$.

In the same way as in RCFT, the vectors $v_F(\Sigma)$ then play the role of the correlators of bulk fields in the full conformal field theory will bulk state space F.

For a subclass of factorizable finite ribbon categories the task of constructing spaces of bulk fields and mapping class group invariants has been achieved:

Theorem [4]**.** Suppose that $\mathcal{C}$ is the category H-mod of finite-dimensional modules over a finite-dimensional factorizable ribbon Hopf algebra H and ω is a ribbon automorphism of H. Then the following holds:
(i) The coend

$$F_\omega := \int^{U\in H\text{-mod}} U \boxtimes U^\vee \ \in H\text{-mod} \boxtimes H\text{-mod}^{\mathrm{rev}} \simeq H\text{-bimod}\,,$$

endowed with the co-regular left H-action and ω-twisted co-regular right H-action, carries a natural structure of an 'S-invariant' commutative cocommutative symmetric Frobenius algebra in H-bimod, as well as a natural structure of a module over Lyubashenko's coend

$$K := \int^{X\in H\text{-bimod}} X \otimes X^\vee \ \in H\text{-bimod}\,.$$

(ii) By suitably combining the structural morphisms of F_ω as a Frobenius algebra and as a K-module, one can assign to each surface Σ with all punctures labeled by F_ω a vector $v_F(\Sigma)$ in $Bl_{H\text{-bimod}}(\Sigma)$ that is invariant under the mapping class group action $\pi_{H\text{-bimod}}$.

In conformal field theory, the correlator assigned to the zero-holed torus, which is invariant under action of the the mapping class group $\mathrm{SL}(2,\mathbb{Z})$ of the torus, is known as the modular invariant bulk partition function. When the bulk state space is given by F_ω of the form above, this is also called a modular invariant of automorphism type.

It is worth pointing out that the appearance of coends, both for the bulk state space F_ω and for the 'glueing object' K, should not come as a surprise. Indeed, coends formalize the physical idea of summing over all states, and thus specifically, of combining all states of one chiral half of the theory with those of the other chiral half in the case of F_ω, respectively of describing sewing in terms of summing over all intermediate states in the case of K. In precise terms, what is meant is to take a sum over all objects of the relevant category, modulo all relations among those objects. This is exactly what a coend achieves. In the semisimple case, the prescription amounts to the familiar summations over (representatives of) the isomorphism classes of simple objects.

Mapping class group invariants from finite ribbon categories. It is natural to hope that the results of [4] survive when the assumption that $\mathcal{C}$ is equivalent to H-mod is dropped, i.e. for coends of the form $\int^{U\in\mathcal{C}} U \boxtimes \varpi(U)^\vee \in \mathcal{D} = \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$ with

$\mathcal{C}$ an arbitrary factorizable finite ribbon category and ϖ a ribbon automorphism of the identity functor $Id_{\mathcal{C}}$.

The methods of [4] cannot be extended directly to this general situation. In particular, the coalgebra structure of $F_\omega \in H$-bimod involves [4, Eq. (3.4)] the integral and cointegral of the Hopf algebra H, for which there is no direct substitute in the general case.

Nevertheless a generalization turns out to be possible, upon a suitable adjustment of the perspective, and compatibility with sewing can be achieved as well. Namely, based entirely on structure that already comes with the construction of the spaces $Bl_{\mathcal{D}}(\Sigma)$, a prescription can be obtained [3] that, given any object F in $\mathcal{D}$, selects for any surface Σ for which all punctures are labeled by F, a vector $v_F(\Sigma)$ in $Bl_{\mathcal{D}}(\Sigma)$. We conjecture that the so obtained vectors are invariant under the actions $\pi_{\mathcal{D}}$ of the mapping class groups and are mapped to each other by the linear isomorphisms that implement the sewing of surfaces, *if and only if* the object F carries a structure of an S-invariant commutative cocommutative symmetric Frobenius algebra in $\mathcal{D}$.

In the special case that $\mathcal{C}$ and $F = F_\omega$ are as in the Theorem above, the prescription of [3] reproduces the mapping class group invariants obtained in [4].

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Does the Asaeda-Haagerup subfactor come from a CFT?

Noah Snyder

One of the motivating questions of this conference is whether subfactors always have an associated conformal field theory, i.e. when are they CFT-realizable. So far, the finite depth finite index subfactors we know about come in three general families plus a few sporadic examples. The families are subfactors which come from finite groups, which come from loop groups, and the Izumi quadratic subfactors realized as automorphisms of the Cuntz algebra. Here "come from" is vaguely defined, but any such construction should be strong enough that you can transfer

CFT-realizability across it. There are two remaining sporadic subfactors outside these three families: the Asaeda-Haagerup and extended Haagerup subfactors. The goal of this talk is to explain recent joint work with P. Grossman and M. Izumi where we give a new construction of the Asaeda-Haagerup subfactor and show that it comes from an Izumi subfactor. The main technique is to study the Brauer-Picard groupoid or maximal atlas of the Asaeda-Haagerup subfactor following our earlier work with P. Grossman. That is, instead of studying subfactors one at a time, we study all overfactors at once which can be built from a fixed collection of bimodules. This has a richer combinatorial structure which incorporates all the information of all intermediate subfactors. By studying this larger structure we are able to identify a potential new subfactor which is Morita equivalent to the Asaeda-Haagerup subfactor but which is easier to construct and study. This subfactor can be constructed as the automorphisms of a Cuntz algebra, and then the Asaeda-Haagerup subfactor itself can be recovered by a simple planar algebraic skein theoretic argument. This new construction has many applications to the study of the Asaeda-Haagerup subfactor: computing the Drinfel'd center or asymptotic inclusion, studying graded extensions, and finding lattices of intermediate subfactors. Now that we know that the Asaeda-Haagerup subfactor comes from the Izumi series, only the extended Haagerup subfactor remains sporadic and mysterious. The Brauer-Picard groupoid approach does not lead to any insight into the extended Haagerup subfactor and other approaches will be needed.

Non-commutative analysis on trinions

Antony Wassermann

The holomorphic picture of conformal field theory, as outlined by G. Segal and Neretin for fermions, associates Hilbert spaces to the incoming and outgoing boundary circles of a bordered Riemann surface and a Hilbert-Schmidt operator between them corresponding to the surface. These operators can be described for trinions, the building blocks for the surfaces, and can be constructed analytically using singular integral operators for the Szegő projections. On the other hand in string theory this theory was described by physicists, including the two Verlindes, and Eguchi & Ooguri, who all used the Szegő kernel to compute string partition functions.

In this talk we described the techniques for developing complex function theory on a trinion, ie a disc with two circular holes. The approach uses the Schottky group Γ generated by inversions in the 3 circles and its index 2 subgroup of holomorphic Möbius transformations, Γ_0. Functions or differentials are constructed using Poincaré sums or products over Γ_0. The Szegő kernel corresponds to a sum $\sum \frac{g'(z)^{1/2}}{g(z)-w} = S(z,w)$, which requires the convergence of $\sum |g'(z)|^{1/2}$. As HF Baker noted, this can be written in terms of the Schottky-Klein prime form and the Riemann theta function, an analytic power series in the automorphic factors of the prime form. Schottky defined these as products over Γ_0, e.g.

$$E(x,y) = (x-y)\prod \frac{(x-g(y))(y-g(x))}{(x-g(x))(y-g(y))}$$

for the prime form. The convergence of $\sum |g'|^{s/2}$ is governed by the limit set Λ of Γ and Γ_0. Its Hausdorff dimension s_0 determines the convergence or divergence of the sums - always $0 < s_0 < 1$ so the formula for $S(z,w)$ applies only if $s_0 < \frac{1}{2}$ while the formula using $E(z,w)$ always works.

We explain why the Hausdorff s_0-measure defines a finite measure on Λ ergodic for Γ_0. Λ can be defined in terms of infinite words in the generators of Γ and this measure recovered using Ruelle's generalisation of the Perron-Frobenius theory for symbolic dynamics. It can be interpreted as a KMS-state for the Cuntz-Krieger algebra and an action of $\mathbb{R}$ determined by a ceiling function. The existence and uniqueness of the measure follows by applying Krein-Rutman's theory for compact operators leaving invariant a closed convex cone. Following D. Mayer, Pollicott and Zworski, this can be deduced by considering a transfer operator formed from composition operators between the Bergman spaces of the 3 discs associated with the trinion.

C^1-classification of gapped Hamiltonians

Yoshiko Ogata

I talked about the classification of gapped ground state phases in quantum spin systems. Let us consider ν-dimensional quantum spin systems. We consider Hamiltonians given by translation invariant finite range interactions. Given such an interaction Φ, the local Hamiltonian on a ν-dimensional cube Λ is given by $H_{\Lambda,\Phi} = \sum_{X\subset\Lambda} \Phi(X)$. We say that a Hamiltonian $H := (H_\Lambda)_\Lambda$ is *gapped* if there exists $\gamma > 0$ and $N_0 \in \mathbb{N}$ such that the difference between the smallest and the next-smallest eigenvalue of H_Λ, is bounded below by γ, for all cubes $\Lambda \subset \mathbb{Z}$ with $|\Lambda| \geq N_0$. What we would like to do is to classify gapped Hamiltonians given by translation invariant interactions.

In the context of quantum spin systems, a widely accepted criterion for the classification of gapped Hamiltonians is as follows: two gapped Hamiltonians are equivalent if and only if they are connected by a continuous path of uniformly gapped Hamiltonians. It is known that the "ground state structure" is an invariant of a bit stronger version of this, a C^1-*equivalence* [1]. We say two gapped Hamiltonians are C^1- equivalent if and only if they are connected by a continuous *and piecewise* C^1-path of uniformly gapped Hamiltonians. We call the classification of gapped Hamiltonians with respect to this equivalence relation, the C^1-*classification of gapped Hamiltonians.*

The need to prove the existence of a uniform spectral gap however makes the construction of relevant examples a hard problem, in particular in higher dimensions. In one dimension, the martingale method has been successfully applied to a large class of models, namely to systems with frustration free, finitely correlated ground states [3]. They are simple, yet correlated states, and [3] gives a general recipe to construct gapped Hamiltonians which have a finitely correlated ground states, with a simple control of the spectral gap above the ground state energy. The C^1-classification of the Hamiltonians given by this recipe was studied in [2].

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Subfactors, algebraic quantum field theory, and boundary conditions

Karl-Henning Rehren

(joint work with Marcel Bischoff, Yasuyuki Kawahigashi, Roberto Longo)

The representation theory of quantum field theory is in the algebraic framework controlled by the braided C* tensor category DHR(A) of the net of local algebras A [1]. We assume that DHR(A) has only finitely many inequivalent irreducible objects, and each object has finite dimension. In completely rational chiral conformal QFT, this is automatic, and DHR(A) is even modular. In much the same way as Q-systems $Q = (\theta, w, x)$ (Frobenius algebras) in $\mathrm{End}_0(N)$ (where N is a type *III* von Neumann factor) characterize finite-index subfactors and inclusions $N \subset M$, Q-systems in DHR(A) characterize relative local extensions $A \subset B$, i.e., nets of subfactors $A(O) \subset B(O)$ indexed by spacetime regions O, with suitable covariance properties such that $B(O_1)$ commutes with $A(O_2)$ at spacelike distance. B is local iff the Q-system is commutative. Thus, one can construct new QFT models, by classifying and computing Q-systems.

On the other hand, the category approach allows to "import" notions like modules and bimodules, centre, full centre and braided product of Frobenius algebras from category theory into QFT. In a recent review [2], we have clarified how these notions correspond to simple algebraic notions and operations with local nets. In particular, the centres correspond to maximal intermediate local extensions that can be obtained as relative commutants of algebras $B(W)$ associated with wedges, and the braided product yields a joint extension in which two given extensions are embedded in a "left-local" position.

The latter property is precisely a "boundary scenario" [3] in which two local quantum field theories are separated by a timelike boundary that is transparent for energy and momentum densities (i.e., energy and momentum are conserved at the boundary). Such scenarios model, e.g., phase transitions of relativistic quantum systems. Left locality is precisely the algebraic implementation of Einstein's principle of causality in the presence of the boundary.

"Boundary conditions" are solutions to the constraints that are imposed by this principle. They can be cast into the form of algebraic relations between the generating quantum fields of the two QFTs. In the case of full two-dimensional conformal QFTs, they roughly correspond to the "defects" studied in [4] in a different setup, but they are less general than the "defects" studied in [5].

In [2, 3], we have characterized boundary conditions as minimal projections in the algebraic centre of the braided product of two local extensions, that in turn correspond to certain bimodules between the two commutative Q-systems $Q^i = (\Theta^i, W^i, X^i)$ $(i = 1, 2)$ in DHR(A). The centre is isomorphic to Hom(Θ^1, Θ^2) equipped with a commutative convolution product coming from the braided product of the commutative Q-systems. In some cases, including those studied by [4], complete classifications are available, and can be used to explicitly compute the boundary conditions in terms of generalized Verlinde matrices [3].

As for the "fusion" of boundary conditions (work in progress), there are several mathematical options to define it: Relative tensor products of wedge algebras, fusion of correspondences and bimodule fusion are presumably all equivalent; they give rise to a fusion category, which has the boundary conditions as generating objects, but in general cases does not close among them. Another notion of fusion that corresponds to the natural composition of the spaces Hom(Θ^i, Θ^j), which is dual to the convolution product [6]. It closes among boundary conditions, but in general gives rise only to a fusion algebra without a tensor category structure.

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Representation theory of subfactors and C*-tensor categories.

Stefaan Vaes

(joint work with Sorin Popa)

A subfactor $N \subset M$ of finite Jones index can roughly be encoded by a group like object $\mathcal{G}$ and an "action" of $\mathcal{G}$ on M. This group like object has several incarnations. It can be viewed as the lattice of relative commutants $M_i' \cap M_j$, where $N \subset M \subset M_1 \subset \cdots$ is the Jones tower of the subfactor, which can be axiomatized as a λ-lattice (in the sense of Popa) or a planar algebra (in the sense of Jones).

The group like structure $\mathcal{G}$ can also be viewed as the category of M-bimodules arising inside ${}_M L^2(M_n)_M$, $n \geq 1$, and be axiomatized as a rigid C^*-tensor category $\mathcal{C}$. In this picture, an "action" of $\mathcal{C}$ on M corresponds to realizing $\mathcal{C}$ as a category of finite index M-bimodules.

In this talk, I present a recent joint work [8] with Sorin Popa in which we propose a unitary representation theory for the above group-like structures, determine it for the Temperley-Lieb-Jones (TLJ) λ-lattice, and use our new approach to give a systematic account of several geometric group theory properties for subfactors, as the Haagerup property, property (T), and (weak) amenability.

When Γ is a countable group and $\Gamma \curvearrowright^\alpha T$ is an outer action on the II_1 factor T, the representation theory of Γ can be encoded as follows by the crossed product inclusion $T \subset S = T \rtimes \Gamma$: there is a natural 1-to-1 correspondence between unitary representations of Γ and Hilbert S-bimodules $\mathcal{H}$ that are generated by T-central vectors. This correspondence associates to a unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{K})$ the S-bimodule $\mathcal{H}$ given by

$$\mathcal{H} = L^2(S) \otimes \mathcal{K} \quad \text{and} \quad au_g \cdot (x \otimes \xi) \cdot y = au_g xy \otimes \pi(g)\xi$$

for all $a \in T, g \in \Gamma, x, y \in S, \xi \in \mathcal{K}$. Note that $\mathcal{H}$ is generated as an S-bimodule by the T-central vectors $1 \otimes \xi$, $\xi \in \mathcal{K}$.

In [7], Popa associated to every finite index subfactor $N \subset M$ a canonical "crossed product type" inclusion $T \subset S$, where $T = M \overline{\otimes} M^{\mathrm{op}}$ and S is the symmetric enveloping algebra $M \boxtimes_{e_N} M^{\mathrm{op}}$ generated by $M \overline{\otimes} M^{\mathrm{op}}$ and a single projection e_N that serves as the Jones projection for both $N \subset M$ and $N^{\mathrm{op}} \subset M^{\mathrm{op}}$.

We then define in [8] an *SE-correspondence* of the subfactor $N \subset M$ as being a Hilbert S-module that is generated by T-central vectors. When $\pi : \Gamma \to \mathcal{U}(\mathcal{K})$ is a unitary representation and $\xi \in \mathcal{K}$, the function $\varphi_\xi : \Gamma \to \mathbb{C}$ given by $\varphi_\xi(g) = \langle \pi(g)\xi, \xi \rangle$ is a function of positive type. Similarly, we call *completely positive (cp) SE-multiplier* of $N \subset M$, every normal T-bimodular cp map $\varphi : S \to S$. They all arise as $\tau(\varphi(x)y) = \langle x\xi y, \xi \rangle$, where ξ is a T-central vector in an SE-correspondence.

By [7], the symmetric enveloping algebra S can be decomposed as

$$L^2(S) = \bigoplus_{\alpha \in \mathrm{Irr}(\mathcal{C})} H_\alpha \otimes \overline{H_\alpha}$$

where $\mathcal{C}$ is the category of M-bimodules that appear somewhere in the Jones tower, and $\mathrm{Irr}(\mathcal{C})$ is the set of irreducible objects in $\mathcal{C}$ (up to isomorphism). Note that $H_\alpha \otimes \overline{H_\alpha}$ naturally is a T-bimodule and that the above formula gives a decomposition of $L^2(S)$ into irreducible T-subbimodules, each appearing with multiplicity one. Therefore, a cp SE-multiplier $\varphi : S \to S$ is necessarily given by multiplication with a scalar $\varphi(\alpha)$ on each $H_\alpha \otimes \overline{H_\alpha}$.

In [8], we then give an intrinsic (in terms of the C^*-tensor category $\mathcal{C}$) characterization of which functions $\varphi : \mathrm{Irr}(\mathcal{C}) \to \mathbb{C}$ correspond to cp SE-multipliers. Arbitrary functions $\varphi : \mathrm{Irr}(\mathcal{C}) \to \mathbb{C}$ are in 1-to-1 correspondence with systems of linear maps

$$\theta^{\varphi}_{\alpha,\beta} : \mathrm{End}(\alpha \otimes \beta) \to \mathrm{End}(\alpha \otimes \beta)$$

that are $\mathrm{End}(\alpha) \otimes \mathrm{End}(\beta)$-bimodular and satisfy the obvious compatibility/naturality conditions in $\alpha, \beta \in \mathcal{C}$. We prove in [8] that the correct positivity condition amounts to requiring that all maps $\theta^{\varphi}_{\alpha,\beta}$ are completely positive on the multimatrix algebra $\mathrm{End}(\alpha \otimes \beta)$. We call such maps *cp-multipliers* on $\mathcal{C}$.

To every cp-multiplier on $\mathcal{C}$ corresponds a state on the fusion algebra $\mathbb{C}[\mathcal{C}]$ and a Hilbert space representation of this fusion algebra. This leads to the notion of an *admissible representation* of the fusion algebra. It turns out the fusion algebra admits a universal admissible representation, defining the universal C*-algebra $C_u(\mathcal{C})$. Note that it is crucial to restrict to admissible representations: the fusion algebra need not have a universal enveloping C*-algebra, because for the TLJ λ-lattice, it is the algebra of polynomials $\mathbb{C}[X]$.

Finally in [8], we relate cp-multipliers on the representation category $\mathrm{Rep}(\mathbb{G})$ of a compact quantum group $\mathbb{G}$ (in the sense of Woronowicz) to the notion of a central state (defined in [2]) on the $*$-algebra of polynomials on $\mathbb{G}$. As such, the approximation and rigidity properties for the categories $\mathrm{Rep}(\mathbb{G})$ turn out to be equivalent to the corresponding central approximation and rigidity properties for the discrete quantum groups $\widehat{\mathbb{G}}$. Using this connection and the main results of [2], we find all admissible representations for the TLJ λ-lattices and prove that they have the Haagerup property, as well as the complete metric approximation property. Using in turn the main result of [1], we find that the category $\mathrm{Rep}(\mathrm{SU}_q(3))$ has property (T). This then provides us the first examples of property (T) subfactors that are not defined using property (T) groups.

Very recently in [6], Neshveyev and Yamashita have given a fully C*-tensor categorical description of our representation theory. Indeed, on the side of subfactors, we defined SE-correspondences and these are indeed the “unitary representations” of the standard invariant of the subfactor. On the side of C*-tensor categories, we defined in [8] only the notion of a positive type function (called cp-multiplier) and the corresponding universal C*-algebra $C_u(\mathcal{C})$.

In [6], Neshveyev and Yamashita define a unitary representation of a rigid C*-tensor category $\mathcal{C}$ as a *unitary half braiding* σ on an ind-object X for $\mathcal{C}$. This means the following: such an ind-object can be seen as an infinite direct sum of objects in $\mathcal{C}$ and a unitary half braiding is a system of unitary morphisms $\sigma_\alpha : \alpha \otimes X \to X \otimes \alpha$ that is natural in α and satisfies

$$\sigma_{\alpha\otimes\beta} = (\sigma_\alpha \otimes 1)(1 \otimes \sigma_\beta) \; .$$

Note that when $\mathcal{C}$ has only finitely many irreducible objects, then the unitary half braidings on objects in $\mathcal{C}$ form the *Drinfeld center* of $\mathcal{C}$.

Neshveyev and Yamashita then prove in [6] that in the case where $\mathcal{C}$ is a category of finite index M-bimodules H_α with associated symmetric enveloping inclusion

$T \subset S$, there is a 1-to-1 correspondence between unitary half braidings on ind-objects and S-bimodules $\mathcal{H}$ with the property that $\mathcal{H}$, as a T-bimodule, can be decomposed as a direct sum of T-bimodules of the form $H_\alpha \otimes \overline{H_\beta}$, $\alpha, \beta \in \mathcal{C}$.

Note that this condition is weaker than $\mathcal{H}$ being generated by T-central vectors. Indeed, the SE-correspondences (i.e. the S-bimodules generated by T-central vectors) are the "weight 0" representations, which on the C*-tensor category side means that X contains (a multiple of) the trivial object and is generated by this trivial object under application of all σ_α.

Even more recently in [3], Ghosh and C. Jones presented a third point of view on the above representation theories, in the language of planar algebras and Ocneanu's *tube algebra* $\mathcal{A}$ that can be associated to any rigid C*-tensor category. The tube algebra $\mathcal{A}$ is a $*$-algebra. Ghosh and C. Jones prove in [3] that there is a 1-to-1 correspondence between $*$-representations of $\mathcal{A}$ and S-bimodules of the above type when $\mathcal{C}$ is a category of M-bimodules. I proved that for arbitrary rigid C*-tensor categories, there is a 1-to-1 correspondence between $*$-representations of $\mathcal{A}$ and unitary half braidings on ind-objects for $\mathcal{C}$. This proof can be found in [3] and generalizes the equivalence of Rep($\mathcal{A}$) and the Drinfeld center of $\mathcal{C}$, see [5].

This third point of view was presented in another talk by C. Jones. In the planar algebra picture, the representations of the TLJ planar algebra were determined by V. Jones in [4] and the above thus provides another approach to determining all unitary representations of the TLJ λ-lattice.

Finally note that in the three pictures, there is a natural tensor product of representations, in particular given by the Connes tensor product $\mathcal{H} \otimes_T \mathcal{H}'$ in the picture of S-bimodules. The above 1-to-1 correspondences preserve tensor products, i.e. define monoidal equivalences of categories.

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The tube algebra and representation theory of categories

Corey Jones
(joint work with Shamindra Ghosh)

The tube algebra $\mathcal{A}$ of a rigid C^*-tensor category $\mathcal{C}$ was introduced by Ocneanu in his study of paragroups. In the case $|Irr(\mathcal{C})| < \infty$, the tube algebra is a finite dimensional C^*-algebra. Using the tube algebra data, Ocneanu showed how to define a 2+1 TQFT from a subfactor. Later, it was realized that this fit into a broader picture. The category of finite dimensional representations of the tube algebra is in fact equivalent to the Drinfeld center of the category, $Z(\mathcal{C})$ [2], [4]. This category is always a modular tensor category hence defines a 2+1 TQFT in its own right through the now standard *RT* construction. Modular categories are also of interest since they are precisely the categories that arise as representations in 2-dimensional rational chiral conformal field theories. The tube algebra of a category provides a concrete way to determine the data of the Drinfeld center, and this has been exploited to determine the modular data of $Z(\mathcal{C})$ for many categories, see [3].

In the case where the number of isomorphism classes of simple objects is infinite, the tube algebra is an infinite dimensional $*$-algebra with no natural norm in sight. One can consider $*$-representations of $\mathcal{A}$ in $B(H)$, and in analogy with groups, one can define a universal C^* algebra $C^*(\mathcal{A})$ such that continuous representations of these C^* algebra are in $1-1$ correspondence with $*$ representations of $\mathcal{A}$ in $B(H)$ for some Hilbert space H. The category of $*$-representations $Rep(\mathcal{A})$ forms a braided monoidal category (though it is not rigid), and it was realized by Vaes that this category is contravariantly equivalent to $Z(\text{ind-}\mathcal{C})$, the Drinfeld center of the ind category of $\mathcal{C}$ studied by Neshveyev and Yamashita, see [1], [5].

There has recently been a great deal of interest in approximation and rigidity properties for subfactors, categories, and quantum groups. Popa and Vaes have introduced definitions of properties such as the Haagerup property and property (T) in the categorical setting that generalize the existing definitions for subfactors [7]. Their definitions can be formulated in terms of certain classes of admissible representations of the fusion algebra of the category. This class of admissible representations can be seen as defining a "unitary representation theory" for $\mathcal{C}$. Incidentally the fusion algebra is a corner of $\mathcal{A}$, and we show that the admissible representations of Popa and Vaes are exactly the representations that induce representations of $\mathcal{A}$. Said another way, admissible representations are precisely the representations of the fusion algebra that arise as restrictions of representations of $\mathcal{A}$ [1]. This suggests that the $Rep(\mathcal{A})$ is the appropriate setting for a categorical representation theory, and allows the properties defined by Popa and Vaes to be rephrased in the context of the tube algebra.

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Near-group categories and (de)-equivariantization

Masaki Izumi

A near-group category $\mathcal{C}$, introduced in [12], is a fusion category with only one non-invertible simple object. By definition, the set of equivalence classes of simple objects of $\mathcal{C}$ is of the form $\mathrm{Irr}(\mathcal{C}) = G \cup \{\rho\}$ with a finite group G. Possible fusion rules are

$$g \otimes h \cong gh, \quad g, h \in G,$$
$$g \otimes \rho \cong \rho \cong \rho \otimes g, \quad g \in G,$$
$$\rho \otimes \rho \cong \bigoplus_{g \in G} g \oplus \overbrace{\rho \oplus \rho \oplus \cdots \oplus \rho}^{m}.$$

The non-negative integer m is the only parameter other than G in the level of fusion rules. We call $\mathcal{C}$ a near-group category for G with multiplicity m.

Near-group categories with $m = 0$ are completely classified by Tambara-Yamagami in [13], and are called Tambara-Yamagami categories.

We concentrate on C^*-fusion categories because we freely use the fact that such a category can uniquely embed into the category of unital endomorphisms $\mathrm{End}(M)$ of an injective type III_1 factor. In this case, an invertible object corresponds to an automorphism of M. Thus for a C^*-near-group category $\mathcal{C}$ embedded into $\mathrm{End}(M)$, there exists a map $\alpha : G \to \mathrm{Aut}(M)$ inducing an injective homomorphism from G into $\mathrm{Out}(M)$ such that $\mathrm{Irr}(\mathcal{C}) = \{\alpha_g\}_{g \in G} \cup \{\rho\}$.

Theorem 1. *Let $\mathcal{C}$ be a C^*-near-group category for a finite group G with multiplicity $m \neq 0$. Let $n = |G|$, and let $d = (m + \sqrt{m^2 + 4n})/2$, which is the dimension of ρ.*

(1) *If d is rational, then either $m = n - 1$ or $n = 2^{2s+1}$, $m = 2^s$ for a natural number s.*
  (i) *If $m = n - 1$, then $n = p^t - 1$ for a prime number p and a natural number t, and G is a cyclic group.*

(ii) *If* $m = 2^s$ *and* $n = 2^{2s+1}$*, then* G *is a extra-special 2-group. For each extra-special 2-group of order* 2^{2s+1}*, there exist exactly 3 near-group categories with* $m = 2^s$*.*

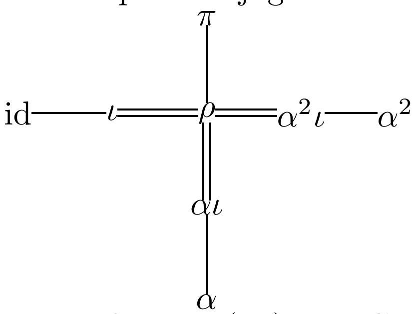
(2) *If* d *is irrational,* m *is a multiple of* n *and* G *is abelian.*

This theorem was obtained by the author around 2008 using a Cuntz algebra method developed in [6] and [7], though (i) had been obtained in [12]. In combination with [1], (i) implies the complete classification in this case. The proof of (2) can be found in [3] and [11]. The polynomial equations classifying the C^*-near-group categories with $m = n$ were obtained in [7]. For each finite group of order less than 14 except for $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, there exists at least one solution (see [3]). The proof of (ii), the polynomial equations classifying C^*-near-group categories in the general case of (2), and a detailed account of (de-)equivariantization of C^*-near-group categories will be given in [8].

The following was essentially shown in [6]:

Lemma 2. *There is a one-to-one correspondence between the equivalence classes of* C^**-near-group categories for* G *with multiplicity* $m = ln$ *and the isomorphism classes of* $2^G_l 1$ *subfactors.*

It is likely that for each finite abelian group G, there is a bound of l for the existence of near-group categories. For example, we have $l \leq 1$ for $G = \{e\}, \mathbb{Z}_2$ (see [10], [11]), and $l \leq 2$ for $G = \mathbb{Z}_3$ (see [9]). Existence of s near-group category with $l = 2$ was recently observed by Noah Snyder and Zhengwei Liu: the case of $G = \mathbb{Z}_3$ with $l = 2$ can be found in a list in [4, page 14]. It turns our that there are exactly two solutions of the polynomial equations in this case, and they are complex conjugate to each other. Thus there are exactly two $2^{\mathbb{Z}_3}_2 1$ subfactors:



. These subfactors are self-dual.

Let $\mathcal{C} \subset \mathrm{End}(M)$ be a C^*-near-group category with $\mathrm{Irr}(\mathcal{C}) = \{\alpha_g\} \cup \{\rho\}$. Choosing an appropriate representative of each α_g from its equivalence class, we can always assume that the equation $\alpha_g \cdot \rho = \rho$ holds for any $g \in G$, and in consequence α is a group action. Since $\rho \cdot \alpha_g$ is equivalent to ρ, there exists a unitary representation $\{U(g)\}_{g \in G}$ of G in M satisfying $\rho \cdot \alpha_g = \mathrm{Ad}\, U(g) \cdot \rho$. Now $N = \rho(M) \vee \{U(g)\}''$ is regarded as the crossed product $\rho(M) \rtimes G$, which is the corresponding $2^G_l 1$ subfactor. One can show that there exits a unique symmetric non-degenerate bicharacter $\langle \cdot, \cdot \rangle : G \times G \to \mathbb{T}$ satisfying $\alpha_g(U(h)) = \overline{\langle g, h \rangle} U(h)$. In particular, the subfactor N is preserved by the G-action α, and we can obtain a new subfactor $M \rtimes_\alpha H \supset N \rtimes_\alpha H$ for each subgroup H.

Victor Ostrik is the first to observe that a near-group category for $\mathbb{Z}_3 \times \mathbb{Z}_3$ with $m = 9$ produces the Haagerup category by de-equivariantization. We will give a systematic account of this kind of phenomena. Part of the information of the polynomial equations in the case of $m = n$ is given by a quadratic form of the group G, that is a function $a : G \to \mathbb{T}$ satisfying $a(g) = a(-g)$, $a(g)a(h) = \langle g, h\rangle a(g+h)$. We set $H^\perp = \{g \in G;\ \langle g, h\rangle = 1,\ \forall h \in H\}$. The crossed product $M \rtimes_\alpha H$ is the factor generated by M and a unitary representation $\{\lambda_h\}_{h\in H}$ satisfying $\lambda_h x = \alpha_h(x)\lambda_h$ for any $x \in M$ and $h \in H$. We can extend ρ and α to $M \rtimes_\alpha H$ by setting $\tilde{\rho}(\lambda_h) = a(h)U(h)\lambda_h$ and $\tilde{\alpha}_g(\lambda_h) = \langle g, h\rangle\lambda_h$. When $H \subset H^\perp$, the restriction of a to H is a character, and there exists $g_a \in G$ satisfying $a(h) = \langle h, g_a\rangle$ for any $h \in H$.

Theorem 3. *Assume $H \subset H^\perp$. Then $\tilde{\alpha}_g$ is inner if and only if $g \in H$, and $\tilde{\rho}$ has the following irreducible decomposition with the fusion rules of the irreducible components:*

$$[\tilde{\rho}] = \bigoplus_{g\in G/H^\perp} [\tilde{\alpha}_g][\sigma],$$

$$[\sigma][\sigma] = \bigoplus_{k\in H^\perp/H} [\tilde{\alpha}_{k-g_a}] \oplus |H^\perp/H| \bigoplus_{g\in G/H^\perp} [\tilde{\alpha}_g\sigma],$$

$$[\tilde{\alpha}_g][\sigma] = [\sigma][\tilde{\alpha}_{-g}].$$

Note that since $[\tilde{\alpha}_{g+h}] = [\tilde{\alpha}_g]$ for any $h \in H$ and $[\tilde{\alpha}_k\sigma] = [\sigma]$ for any $k \in H^\perp$, the above expression makes sense.

Theorem 4. *If H is a Lagrangian, that is, $H = H^\perp$ and the restriction of $a(g)$ to H is 1, then*

$$[\sigma][\sigma] = [\mathrm{id}] \oplus \bigoplus_{g\in G/H} [\tilde{\alpha}_g\sigma],$$

$$[\tilde{\alpha}_g][\sigma] = [\sigma][\tilde{\alpha}_{-g}].$$

Evans-Gannon [3] showed that there exists a unique solution of the polynomial equations for $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ with $m = 9$. For this solution, there exists two Lagrangian subgroups $\{(0,0),(1,1),(2,2)\}$ and $\{(0,0),(1,2),(2,1)\}$. These two subgroups correspond to the Haagerup category and Grossman-Snyder category found in [5]. For $\mathbb{Z}_9$ with $m = 9$, there are two solutions and $< 3 >$ is a unique Lagrangian in the both cases, which give two additional fusion categories with the same fusion rules as the Haagerup category. These two categories have non-trivial associators for the group part $\mathbb{Z}_3$ (cf. [2]).

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Some examples of fusion categories associated to small-index subfactors

Pinhas Grossman

To any finite depth, finte index subfactor $N \subseteq M$, there is associated a pair of fusion categories, called the even parts, and a Morita equivalence(=invertible bimodule category) between them. Given a particular subfactor, it is natural to ask: what are all of the fusion ctageories in the Morita equivalence class of the even parts, and what are all of the invertible bimodule categories between them?

For the Haagerup subfactor, the corresponding Morita equivalence class consists of three distinct fusion categories, with a unique, up to equivalence, invertible bimodule category between any pair of these. In particular, the Brauer-Picard group of Morita autoequivalences is trivial [1].

In this talk, we consider the self-dual 4442 subfactor, constructed in [3]. It was shown by Izumi [2] that the even parts of this subfactor are Morita equivalent to an equivariantization of the even part of the generalized Haagerup subfactor for the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ by a $\mathbb{Z}/3\mathbb{Z}$ action. We express this fact by writing the even part of the 4442 subfactor as $\mathcal{C}^{\mathbb{Z}/3\mathbb{Z}}$, where $\mathcal{C}$ is the even part of the generalized Haagerup subfactor.

In particular, $\mathcal{C}^{\mathbb{Z}/3\mathbb{Z}}$ contains as a subcategory the tensor category $\mathrm{Rep}(A_4)$, the category of representations of the alternating group on 4 letters, which can also be expressed as $A_4 = (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \rtimes \mathbb{Z}/3\mathbb{Z}$.

The 4442 subfactor gives a non-trivial Morita autoequivalence of $\mathcal{C}^{\mathbb{Z}/3\mathbb{Z}}$ and it is clear that there are at least three distinct fusion categories in the Morita equivalence class, corresponding to the three distinct fusion categories in the Morita equivalence class of $\mathrm{Rep}(A_4)$. We have the following classification result.

Theorem 1. *There are exactly* 4 *distinct fusion categories in the Morita equivalence class* $\mathcal{C}^{\mathbb{Z}/3\mathbb{Z}}$*, and the Brauer-Picard group is* S_3*.*

The Brauer-Picard group is generated by the autoequivalence coming from the 4442 subfactor, which has order 2, and an autoequivalence coming from a 4-dimensional algebra in $\mathrm{Rep}(A_4)$, which has order 3.

To find the fourth fusion category, which does not come from $\mathrm{Rep}(A_4)$, we look at $\mathcal{C} \rtimes \mathbb{Z}/3\mathbb{Z}$, which is a $\mathbb{Z}/3\mathbb{Z}$-graded extension of $\mathcal{C}$ in the same Morita equivalence class as $\mathcal{C}^{\mathbb{Z}/3\mathbb{Z}}$. There are four different algebras in $\mathcal{C}$ which give the generalized Haagerup subfactor for $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, corresponding to the four elements of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. While all of these give Morita autoequivalences of $\mathcal{C}$, only the one corresponding to the trivial group element gives an autoequivalence of $\mathcal{C} \rtimes \mathbb{Z}/3\mathbb{Z}$; the other group elements give algebras with module categories whose dual category is something different.

Once we have found all of the module categories over $\mathcal{C}^{\mathbb{Z}/3\mathbb{Z}}$ and $\mathcal{C} \rtimes \mathbb{Z}/3\mathbb{Z}$, we can turn our attention to $\mathcal{C}$ itself.

Theorem 2. *There are* 30 *simple module categories over* $\mathcal{C}$*, which all give Morita autoequivalences. The outer automorphism group of* $\mathcal{C}$ *has order* 12*, and the Brauer-Picard group has order* 360*.*

The large symmetry group for $\mathcal{C}$, which is the even part of the generalized Haagerup subfactor for the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, stands in stark contrast to the generalized Haagerup subfactors for $\mathbb{Z}/3\mathbb{Z}$, which has trivial Brauer-Picard group, and $\mathbb{Z}/4\mathbb{Z}$, which has Brauer-Picard group of order 2.

While it may seem strange that we study $\mathcal{C}^{\mathbb{Z}/3\mathbb{Z}}$ before studying $\mathcal{C}$, it is precisely the lack of symmetry of $\mathcal{C}^{\mathbb{Z}/3\mathbb{Z}}$ which makes classifying the module categories over the equivariantization easier, and then we can use the information we have about the equivariantization to deal with the underlying category $\mathcal{C}$.

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Umbral Moonshine

John F. R. Duncan

(joint work with Miranda C. N. Cheng and Jeffrey A. Harvey)

Following the extraordinary precedent [7] of Conway–Norton, a tradition has developed whereby the term *moonshine* is used to refer to an association of distinguished modular objects to the conjugacy classes of a finite group. In the case of Mathieu moonshine [13], and umbral moonshine more generally [3, 4, 5], the modular objects are vector-valued mock modular forms of weight 1/2, and the finite groups are automorphism groups of Niemeier lattices (i.e. self-dual even positive-definite lattices of rank 24, cf. e.g. [8]).

The precise nature of the mechanism which relates these two classes of objects remains obscure, but it is clear by now that there are many interesting areas of mathematics and mathematical physics involved in the correspondence. For example, K3 surface geometry, and the non-linear K3 sigma models of mathematical physics have been implicated since the first Mathieu moonshine observations [13] of Eguchi–Ooguri–Tachikawa. See [6] and [12] for two different, but related analyses, demonstrating connections between K3 surface geometry and umbral moonshine. See [10] for a recent review of moonshine.

Until recently, concrete connections to vertex algebra and conformal field theory have been missing from umbral moonshine, but that situation has changed now, with the construction in [11] of the umbral moonshine analogue of the Frenkel–Lepowsky–Meurman moonshine module [14, 15, 16], for the case of umbral moonshine corresponding to the self-dual even lattice of rank 24 with root system $X = E_8^3$. We next describe the modular objects which arise in this case.

The first *mock theta functions* were introduced by Ramanujan in his last letter to Hardy (cf. [17, 18]) in 1920. Amongst these are the two examples

$$\chi_0(q) := 1 + \frac{q}{1-q^2} + \frac{q^2}{(1-q^3)(1-q^4)} + \frac{q^3}{(1-q^4)(1-q^6)(1-q^6)} + \cdots, \tag{1}$$

$$\chi_1(q) := \frac{1}{1-q} + \frac{q}{(1-q^2)(1-q^3)} + \frac{q^2}{(1-q^3)(1-q^4)(1-q^5)} + \cdots, \tag{2}$$

which Ramanujan identified as having *order* 5. Ramanujan did not explain what he meant by order, and indeed an appropriate mathematical framework for analyzing the mock theta functions was missing for decades, until Zwegers' ground-breaking doctoral thesis [20], the contemporaneous work [2] of Bruinier–Funke on harmonic Maass forms, and the subsequent refinements and elaborations on the resulting theory of mock modular forms, due to Bringmann–Ono [1], and Zagier [19], and by now many others. (See [9] for a review.)

In the light of umbral moonshine, we can say that χ_0 and χ_1 are two components of a vector-valued mock modular form of weight 1/2, whose modularity is controlled by the E_8 root system. More precisely, if we define unary theta series

$$S_1^X := 3(S_{30,1} + S_{30,11} + S_{30,19} + S_{30,29}), \tag{3}$$

$$S_7^X := 3(S_{30,7} + S_{30,13} + S_{30,17} + S_{30,23}), \tag{4}$$

where $S_{m,r}(\tau) := \sum_{k=r \mod 2m} kq^{k^2/4m}$ for $q = e^{2\pi i\tau}$, and if we set

$$H_1^X(\tau) := 2(\chi_0(q) - 2)q^{-1/120}, \tag{5}$$

$$H_7^X(\tau) := 2\chi_1(q)q^{49/120}, \tag{6}$$

then $\check{H}^X := (H_1^X, H_7^X)$ is a *mock modular form* of weight 1/2 for $\mathrm{SL}_2(\mathbb{Z})$ with *shadow* $\check{S}^X := (S_1^X, S_7^X)$.

In concrete terms, this means that the function

$$\widehat{\check{H}^X}(\tau) := \check{H}^X(\tau) + \frac{e^{-\pi i/4}}{\sqrt{60}} \int_{-\bar{\tau}}^{\infty} \frac{\check{S}^X(z)}{\sqrt{z+\tau}} \mathrm{d}z, \tag{7}$$

called the *completion* of $\check{H}^X$, is a (non-holomorphic) modular form of weight 1/2 for $\mathrm{SL}_2(\mathbb{Z})$, with multiplier system inverse to that of the (holomorphic) cusp form $\check{S}^X$.

Observe that the indices defining $\check{S}^X$ are just the Coxeter number, and Coxeter exponents of E_8. Analogous formulas hold for each of the other 22 root systems arising from Niemeier lattices. Cf. [4].

In [11] a super vertex operator algebra V^X is constructed, for $X = E_8^3$, together with an action by automorphisms of the group $G^X \simeq S_3$, such that the graded-trace functions attached to the action of G^X on certain canonically-twisted modules $V^X_{\mathrm{tw},a}$ for V^X recover the vector-valued mock modular forms $\check{H}^X_g = (H^X_{g,1}, H^X_{g,7})$ associated to elements $g \in G^X$ in [4].

Theorem ([11])**.** *Set $X = E_8^3$ and let $H^X_{g,r}$ be as described in* [4] *for $g \in G^X$ and $r \in \{1,7\}$. Then we have*

$$H^X_{g,r}(\tau) = 2\mathrm{tr}_{V^X_{\mathrm{tw},r}} g g_{\rho/2} p(0) q^{L(0)-c/24} \tag{8}$$

for $r = 1$ and $r = 7$, where $q = e^{2\pi i \tau}$.

We refer to [11] for the precise definition of the right hand side of (8). The construction of V^X is achieved via an adaptation of the lattice vertex algebra method to suitable pairs of cones in lattices of indefinite signature. The correct choice of cone for $X = E_8^3$ is inspired by Zwegers' work [21].

The generalization of the construction of V^X to the remaining 22 cases of umbral moonshine, and the illumination of the role of V^X in conformal field theory, remain important problems for future work.

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Finite orbifolds in rational chiral CFTs

Michael Müger

The aim of my talk mainly was to draw the attention of the vertex operator algebraists in the audience to some fairly old work of mine on conformal orbifold theories, done in the operator algebraic setting for quantum field theory. I hope that they will prove similar results for orbifolds of VOAs.

I explained how a quantum field theory A living on the line and having a group G of inner symmetries gives rise to a category $G\text{–Loc}A$ of twisted representations. This category is a braided crossed G-category, as defined by Turaev in 2000. (Published only as a preprint at the time and ultimately subsumed in a book to which I contributed a short appendix [5].) Its degree zero subcategory is braided and equivalent to the usual representation category $\text{Rep}\, A$. The latter is known to be a modular category if A is completely rational, cf. [1].

Then I described the relation between $G-\text{Loc}A$ and the braided (in the usual sense) representation category $\text{Rep}\, A^G$ of the orbifold theory A^G, under the assumptions that that A is completely rational and G is finite. The main result is the existence of an equivalence $\text{Rep}\, A^G \simeq (G-\text{Loc}A)^G$ of braided tensor categories, which is a rigorous implementation of the insight that one needs to take the twisted representations of A into account in order to determine $\mathrm{Rep}\, A^G$. The proof is somewhat indirect, since one first uses α-induction to prove an equivalence G–Loc$A \simeq \mathrm{Rep}\, A^G \rtimes \mathcal{S}$, of braided crossed G-categories, where $\mathcal{S} \subset \mathrm{Rep}\, A^G$ is the full subcategory of representations of A^G contained in the vacuum representation of A, and $\rtimes$ refers to the Galois extensions of braided tensor categories of [2, 3].

In particular, one finds that A has g-twisted representations for every $g \in G$ and that the sum over the squared dimensions of the simple g-twisted representations for fixed g equals $\dim \mathrm{Rep}\, A$. Since this is a result about the original CFT A, I mentioned that it would be desirable to have direct proofs, avoiding the passage via the orbifold theory A^G.

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Quantum Dimensions in Logarithmic CFT

Thomas Creutzig

Logarithmic vertex operator algebras possess at least one module that is not completely reducible. In these cases not much is known about the relation between torus one-point functions and the representation category. I will review recent progress and discuss a conjecture that makes such a connection.

The rational modular story. In 1988 Verlinde observed that modular properties of characters of rational conformal field theories (CFTs) and the fusion ring of modules are closely connected [17]. For this let $\{M_0, M_1, \ldots, M_n\}$ be the set of inequivalent simple modules of a given CFT with M_0 the vacuum. Then the character of a module is

$$\mathrm{ch}[M](\tau) = \mathrm{tr}_M \left(q^{L_0 - \frac{c}{24}} \right), \qquad q = e(\tau),$$

and physics ensures that it converges on the upper half of the complex plane. Moreover these characters span a vector-valued modular form for the modular group which acts on functions of the upper half plane via Möbius transformations. Especially the transformation $\tau \mapsto -1/\tau$ is called the modular S-transformation and it defines a matrix, the S-matrix, via

$$\mathrm{ch}[M_i](-1/\tau) = \sum_{j=0}^{n} S_{ij} \mathrm{ch}[M_j](\tau) \quad \text{and numbers} \quad {N_{ij}}^{k} := \sum_{\ell=0}^{n} \frac{S_{i\ell} S_{j\ell} \left(S^{-1}\right)_{k\ell}}{S_{0\ell}}.$$

Verlinde's formula is that the ${N_{ij}}^k$ are the structure constants of the Grothendieck ring of the fusion ring. Verlinde gave a short physics argument for this conjecture and he verified it in examples. Not much later, Moore and Seiberg suggested that this conjecture is a consequence of the axioms of rational CFTs [15]. These axioms imply that the representation category of a rational CFT is a modular tensor category. A modular tensor category is especially a braided tensor category and the quantum dimensions, that is the traces over braiding isomorphisms, together with the action of the twist define a projective action of the modular group on the span of simple objects. The Verlinde formula for this second modular group action inside a modular tensor category is true, as for example explained in the book by Turaev [16]. The natural question is whether it is possible to translate the physics to the vertex algebra setting and around ten years ago Huang has finally succeeded to prove the Verlinde formula for rational vertex operator algebras satisfying a few additional conditions [13]. Note that much earlier Faltings came up with a very geometric and much shorter proof in the WZW case [11].

A weak Verlinde formula. Define the map $q_\ell(M_i) := \frac{S_{i\ell}}{S_{0\ell}}$. A corollary of Verlinde's formula is

$$\frac{S_{i\ell}}{S_{0\ell}}\frac{S_{j\ell}}{S_{0\ell}} = \sum_{k=0}^{n} {N_{ij}}^k \frac{S_{k\ell}}{S_{0\ell}}.$$

Meaning that q_ℓ for every $\ell = 0, \ldots, n$ is a one-dimensional representation of the fusion ring. If there is one special simple module M_r with lowest conformal weight, then the quantum character q_r is related to the asymptotics of characters of modules. The reason is that for large imaginary part of τ this character dominates the other ones impliying that

$$\mathrm{qdim}(M_i) := \lim_{\tau\to 0} \frac{\mathrm{ch}[M_i](\tau)}{\mathrm{ch}[M_0](\tau)} = \lim_{\tau\to i\infty} \frac{\mathrm{ch}[M_i](-1/\tau)}{\mathrm{ch}[M_0](-1/\tau)} = \frac{S_{ir}}{S_{0r}} = q_r(M_i).$$

A weak Verlinde formula is that these asymptotic dimensions give a one-dimensional representation of the fusion ring. Our question is whether this weak version is true beyond rationality.

Beyond rationality. After this review of well-known results I want to turn to logarithmic CFTs. The name logarithmic is due to the appearance of logarithmic singularities in the operator product expansion of fields involving reducible but indecomposable modules. There are two cases, logarithmic rational VOAs are those which still only have finitely many simple modules in contrast to logarithmic non-rational ones. In the first case, Miyamoto [14] has proven convergence of characters on the upper half of the complex plane provided a condition called C_2 cofiniteness is satisfied. He has also shown that characters are elements of a vector-valued modular form of mixed weight, but not all elements of this representation are characters. The by far best understood example of a logarithmic rational vertex algebra is the family of (p, q)-triplet algebras. In this case, a Verlinde formula has been conjectured in [12]. Verifying that the quantum dimensions as defined in last section respect fusion in these cases is an exercise. It reveals that quantum dimensions of modules that are elements of the maximal non-trivial ideal of the fusion ring vanish. They are in some sense negligible objects. So that the quantum dimensions capture fusion in the quotient and coincide with theose of some well-known rational vertex algebra.

In the non-rational logarithmic setting the situation of quantum dimensions becomes much richer. Also in this case characters become objects of very modern interest in number theory/modular forms. Characters are of course not modular anymore, but they are sometimes mock modular, sometimes false theta and sometimes expansions of meromorphic Jacobi forms. David Ridout and I, we have developped a conjecture extending the Verlinde formula to this setting [6]. Our first toy example has been the affine vertex superalgebra of gl(1|1) [7], since then it has been applied to many other examples including cases with mock modular forms [2] and most importantly the simple affine vertex algebra of sl(2) at admissible but non-integer level [8, 9].

The triplet vertex algebra has a non-rational subalgebra, the singlet vertex algebra. There characters are built out of partial (or false) theta functions (they are partial sums over lattices). These partial theta functions are usually not modular, but Antun Milas and I, we find modular-like behaviour if we regularize the partial theta function [4]. This means, we have a one-parameter family of objects, parameterized by ϵ, that for $\epsilon = 0$ specialize to the partial theta function and that have modular-like properties if ϵ is not purely imaginery, especially not zero. This regularization allows us to define regularized characters and their asymptotics, the regularized quantum dimensions, follow easily from the modular-like behaviour.

Together with Simon Wood and Antun Milas, we find [5] that depending on the sign of the real part of ϵ we either get continuous functions of ϵ or stripwise constant ones. The continuous part gives the Grothendieck ring, while the stripwise constant one captures fusion on some quotient. The fusion ring on this quotient is again the same as the one of some well-known rational vertex algebra. We take these findings as a strong hint that there is a categorical trace in the representation categories of the singlet algebras that reproduces our results. It is conjectured that the representation categories of the unrolled quantum group of sl(2) at $2p$-th root of unity and the one of the $(p, 1)$-singlet algebra are equivalent [3]. The singlet algebras are cosets of admissible non-integer level affine sl(2) and related algebras [1, 10] and hence should also help understanding these theories.

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Vertex operator algebras and finite groups

Ching Hung Lam

The Moonshine vertex operator algebra $V^\natural$ constructed by Frenkel- Lepowsky-Meurman [2] is one of the most important examples of vertex operator algebras (abbreviated as VOAs). Its full automorphism group is the Monster simple group, the largest member of the 26 sporadic simple groups. The theory of vertex operator algebra also provides a powerful tool for studying the Monster group and some sporadic simple groups. Our aim is to develop several tools for studying the automorphism group of VOA.

Let $V = \bigoplus_{n\in\mathbb{Z}} V_n$ be a VOA and let $Y(a,z) = \sum_{\mathbb{Z}} a_n z^{-n-1}$ be the vertex operator. A VOA V is said to be of CFT type if $V_n = 0$ for $n < 0$ and $\dim V_0 = 1$. In this case, the weight one subspace V_1 has a Lie algebra structure with the Lie bracket given by $[a,b] = a_0 b$, for $a, b \in V_1$. In addition, there is a bilinear form $(\ ,\ )$ on V_1 defined by $(a,b)\cdot 1 = a_1 b$, for $a, b \in V_1$. The form $(\ ,\ )$ is associative in the sense that $(a,[b,c]) = ([a,b],c)$, for any $a,b,c \in V_1$.

Let $0 \neq a \in V_1$. Then the zero mode operator a_0 defines a derivation of V and the map $\exp(a_0) = \sum_{n=0}^{\infty} \frac{a_0^n}{n!}$ is an automorphism of V. Let

$$N = \{\exp(a_0) \mid a \in V_1\}.$$

Then N is a normal subgroup of $Aut(V)$ and N is infinite unless $N = id$. Therefore, we usually assume $V_1 = 0$ when we want to study finite groups. The following is a famous conjecture in VOA theory.

Conjecture 0.1. *Let V be a rational and C_2-cofinite VOA of CFT type such that $V_1 = 0$. Then $Aut(V)$ is a finite group.*

A VOA is *rational* if its module category is semisimple and is *C_2-cofinite* if $\dim(V/C_2(V)) < \infty$, where $C_2(V) = span\{a_{-2}b \mid a, b \in V\}$.

Remark 0.2. If we remove the assumption that V is rational, then there are counterexamples for the conjecture.

Now assume that $V_1 = 0$. Then we lose the Lie algebra V_1 but the weight two space V_2 will then have a commutative (non-associative) algebra structure given by the product $a \times b = a_1 b$, for $a, b \in V_2$. Moreover, there is an associative form on V_2 defined by the relation: $(a, b) \cdot \mathbf{1} = a_3 b$, for $a, b \in V_2$. The most interesting example is again the Moonshine module $V^\natural$ for which the weight two space $V_2^\natural$ is isomorphic to the 196884-dimensional commutative non-associative algebra constructed by Griess [3]. Unfortunately, the algebra V_1 is nonassociative and is very difficult to study in general.

Motivated by Conway's work [1], Miyamoto [11] discovered a simple method for constructing certain automorphisms of a VOA V using some "nice" vertex subalgebras and their fusion rules.

Let $L(c, h)$ be the irreducible highest weight $L(c, 0)$-module of central charge c and highest weight h. It is known that $L(c, 0)$ has a natural simple VOA structure. When $c = \frac{1}{2}$, the simple VOA $L(\frac{1}{2}, 0)$ is rational and it has exactly 3 irreducible modules $L(\frac{1}{2}, 0)$, $L(\frac{1}{2}, \frac{1}{2})$, and $L(\frac{1}{2}, \frac{1}{16})$. Moreover, the fusion rules are known.

Let $(V, Y, \mathbf{1}, \omega)$ be a VOA and let $U \cong L(1/2, 0)$ be a subVOA of V. Since U is rational, we have the decomposition:

$$V = V_U(0) \oplus V_U(\frac{1}{2}) \oplus V_U(\frac{1}{16}),$$

where $V_U(h)$ is the sum of all irreducible U-submodules of V isomorphic to $L(\frac{1}{2}, h)$.

Theorem 0.3 ([11, Theorem 4.7])**.** *Let V be a VOA and $U \cong L(1/2, 0)$ a subVOA of V. Define a linear map $\tau_U : V \to V$ by*

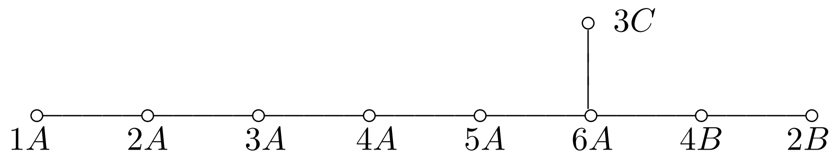
$$\tau_U := \begin{cases} 1 & \text{on } V_U(0) \oplus V_U(\frac{1}{2}), \\ -1 & \text{on } V_U(\frac{1}{16}). \end{cases}$$

Then τ_U is an automorphism of V and $(\tau_U)^2 = 1$. The automorphism τ_U is often called a Miyamoto involution.

When $V = V^\natural$ is the Moonshine VOA, the automorphism τ_U is always an element in the $2A$ conjugacy class of the Monster. Moreover, there is a bijective correspondence between the $2A$-involutions in the Monster and subVOA isomorphic to $L(1/2, 0)$ in $V^\natural$ [4, 12]. This correspondence gives an approach to study certain mysterious phenomena associated with $2A$-involutions of the Monster group by using the theory of VOA. We shall give few examples.

McKay's E_8-observation

It is known [1] that $2A$-involutions of the Monster simple group satisfy a 6-transposition property, i.e., for any two $2A$-involutions x and y, $|xy| \leq 6$. Moreover, the conjugacy class of the product xy belongs to the conjugacy classes $1A, 2A, 3A, 4A, 5A, 6A, 4B, 2B$, or $3C$. John McKay [10] observed that there is an interesting correspondence with the extended E_8 diagram as follows:

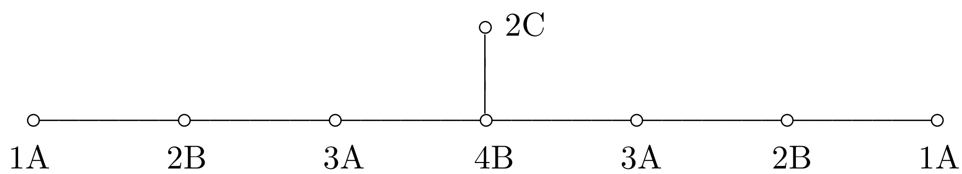


In [9], the above observation has been studied in detail using Miyamoto involutions. It is established that there exists a natural correspondence between the dihedral groups generated by two $2A$-involutions of the Monster and certain sub-VOAs of $V^\natural$ which are constructed naturally by the nodes of the affine E_8 diagram. In addition, a very nice theorem has been proved by Sakuma [13].

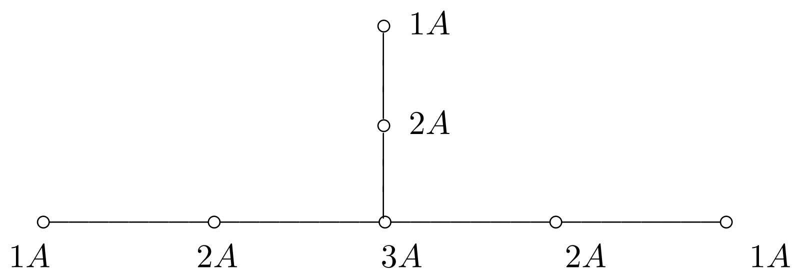
Theorem 0.4 (cf. [13]). *Let V be a unitary VOA of CFT type and $V_1 = 0$. Let $U \cong L(1/2, 0)$ and $W \cong L(1/2, 0)$ be two subVOAs in V. Then $|\tau_U \tau_W| \leq 6$.*

Moreover, there are exactly nine isomorphism classes of Griess subalgebras (subalgebra of V_2) generated by e and f, where $e \in U$ and $f \in W$ are the Virasoro elements of U and W.

The similar method was also used in [5, 6] to study McKay's E_7 and E_6-observation



and



which relate the $\{3, 4\}$-transposition property of the Baby Monster group and affine E_7 diagram and the 3-transposition property of the largest Fischer group and affine E_6 diagram. Moreover, Sakuma-type theorems were proved in [7, 8].

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An Algebraic Conformal Quantum Field Theory Approach to Defects

Marcel Bischoff

(joint work with Yasuyuki Kawahigashi, Roberto Longo, Karl-Henning Rehren)

Inspired by the work of Fuchs, Runkel and Schweigert on CFT on Riemann surfaces with defects and Jones' planar algebra, we study defects in conformal algebraic quantum field theory on Minkowski space.

A completely rational conformal net on the circle gives a modular tensor category and braided subfactors. It was already realized by Ocneanu that this data, which he axiomatized as Ocneanu cells, describes full CFTs, its boundary condition, and defects.

The goal is to show that this structure can be realized using local nets of von Neumann algebras on Minkowski and a quantum double construction.

Work in progress and based on work with Y. Kawahigashi, R. Longo and K.-H. Rehren.

Conformal nets on S^1 and their representations. Let $\mathcal{I}$ be the set of proper intervals $I \subset S^1$ of the circle. A conformal net $\mathcal{A}$ associates with every $I \in \mathcal{I}$ a von Neumann algebra $\mathcal{A}(I)$ on a fixed Hilbert space $\mathcal{H}$, such that $\mathcal{A}(I) \subset \mathcal{A}(J)$ for

$I \subset J$ and $[\mathcal{A}(I), \mathcal{A}(J)] = \{0\}$ for $I \cap J = \emptyset$. We ask that there is a unitary positive-energy representation U of the Möbius group, such that $U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI)$. The vector $\Omega \in \mathcal{H}$ is called the **vacuum** and is asked to be the (up to a phase) unique vector satisfying $U(g)\Omega = \Omega$.

A **representation** π of $\mathcal{A}$ is a family of representations $\pi = \{\pi_I \colon \mathcal{A}(I) \to \mathrm{B}(\mathcal{H}_\pi)\}_{I\in\mathcal{I}}$ on a common Hilbert space $\mathcal{H}_\pi$ which are compatible, i.e. $\pi_J \restriction \mathcal{A}(I) = \pi_I$ for $I \subset J$. Every non-degenerate representation π with $\mathcal{H}_\pi$ separable turns—for every choice of an interval $I_0 \in \mathcal{I}$—out to be equivalent to a representation ρ on $\mathcal{H}$, such that $\rho_J = \mathrm{id}_{\mathcal{A}(J)}$ for $J \cap I_0 = \emptyset$. Then Haag duality implies that ρ_I is an endomorphism of $\mathcal{A}(I)$ for every $I \in \mathcal{I}$ with $I \supset I_0$. Thus we can realize the representation category of $\mathcal{A}$ inside the C* tensor category of endomorphisms $\mathrm{End}(N)$ of the hyperfinite type III factor $N = \mathcal{A}(I)$ and the embedding turns out to be full and replete. We denote this category by $\mathrm{Rep}^I(\mathcal{A})$. In particular, this gives the representations of $\mathcal{A}$ the structure of a tensor category [4]. It has a natural **braiding**, which is completely fixed by asking that if ρ is localized in I_1 and σ in I_2 where I_1 is left of I_2 inside I then $\varepsilon(\rho, \sigma) = 1$ [5].

A special class of conformal nets are so-called **completely rational** ones, which fulfill certain finiteness conditions. Complete rationality of $\mathcal{A}$ implies that the category $\mathrm{Rep}(\mathcal{A})$ is a **unitary modular tensor category (UMTC)** [8].

Example. The $\mathrm{SU}(N)_k$ loop group net $\mathcal{A}_{\mathrm{SU}(N),k}$ is completely rational [10, 11]. If $\rho_\Box \in \mathrm{Rep}\,\mathcal{A}_{\mathrm{SU}(2),k}$ is the fundamental representation, then the associated planar algebra is the Temperley-Lieb-Jones planar algebra of the A_{k+1} subfactor.

Braided subfactors. We consider a type III factor N and a full and replete subcategory $\mathcal{C} \subset \mathrm{End}(N)$, which is a UMTC. Examples come from conformal nets $\mathcal{A}$, where $N = \mathcal{A}(I)$ and $\mathcal{C} = \mathrm{Rep}^I(\mathcal{A})$.

Conjecture 1 (Tannakian duality for UMTCs)**.** *Every UMTC $\mathcal{C} \subset \mathrm{End}(N)$ comes from a conformal net. Equivalently, for every UMTC $\mathcal{C}$ there is a conformal net $\mathcal{A}$, such that $\mathrm{Rep}(\mathcal{A}) \cong \mathcal{C}$.*

Then we can look into subfactors $N \subset M$ with finite index, which are **related to** $\mathcal{C}$. By finite index there is a dual homorphism of the canonical embedding $\iota \colon N \to M$ denoted by $\bar\iota \colon M \to N$. We say $N \subset M$ is related to $\mathcal{C}$ if the composition $\bar\iota \circ \iota \colon N \to N$ is in $\mathcal{C}$. In this case, we call the pair $(N \subset M, \mathcal{C})$ a **braided subfactor**. Using the (unique) rigid structure [9] the pair $\iota, \bar\iota$ gives a planar algebra associated with $N \subset M$ in the sense of [7]. Let $(N \subset M_a, \mathcal{C})$ and $(N \subset M_b, \mathcal{C})$ be two braided subfactors then we say a morphism $\beta \colon M_a \to M_b$ is **related with** $\mathcal{C}$ if $\bar\iota_b \circ \beta \circ \iota_a \in \mathcal{C}$. We say that $(N \subset M_a, \mathcal{C})$ and $(N \subset M_b, \mathcal{C})$ are **Morita equialent** if there is a $\beta \colon M_a \to M_b$ related to $\mathcal{C}$ which is an isomorphism.

Example. If $\mathcal{C} = \mathrm{Rep}^I(\mathcal{A}_{\mathrm{SU}(2),k})$ is the representation category of $\mathrm{SU}(2)_k$, then the Morita classes $[(N \subset M_a, \mathcal{C})]$ are in one-to-one correspondence with A, D, E Dynkin diagrams with Coxeter number $k + 2$.

Remark 2. We get a 2-category whose 0-cells are $N, M_a, M_b, \ldots$ where we might pick a factor $M_\bullet$ for every (Morita) equivalence class $[(N \subset M_\bullet, \mathcal{C})]$, 1-cells are morphisms between this factors related to $\mathcal{C}$, and 2-cells are intertwiners. This data should prescribe different phases, defects between them and intertwiners in CFT (cf. [6]).

Given $\beta\colon M_a \to M_b$ related to $\mathcal{C}$, we get a planar algebra associated with the subfactor $\beta(M_a) \subset M_b$. In the above formulation, this planar algebra describes networks of the defect lines β and the dual defect line $\bar{\beta}$.

Conformal nets on Minkowski space. Let $\mathbb{M} = L_+ \times L_-$ be two-dimensional Minkowski space decomposed as a product of two light rays $L_\pm \cong \mathbb{R}$. Then for $I, J \in \mathcal{I}$ the set $O = I \times J$ is a so-called **double cone** and we can define a net on Minkowski space by $\mathcal{A}_2(O) = \mathcal{A}(I) \otimes \mathcal{A}(J)$, where we see $\mathcal{A}$ (by restriction) as a net on $\mathbb{R} \cong S^1 \setminus \{-1\}$. Locality of $\mathcal{A}$ implies that the algebras $\mathcal{A}_2(O_1)$ and $\mathcal{A}_2(O_2)$ commute if O_1 and O_2 are either mutually space- or timelike. A local extension $\mathcal{B}_2 \supset \mathcal{A}_2$ is roughly speaking an extension $\mathcal{B}_2(O) \supset \mathcal{A}_2(O)$, such that $\mathcal{B}_2$ itself is a net which fulfills spacelike commutativity (Einstein causality). Maximal local extensions are analogy of **full CFTs**, they are characterized as follows (see [2]):

Theorem 3. *If $\mathcal{A}$ is completely rational. Maximal local extensions $\mathcal{B}_2 \supset \mathcal{A}_2$ are in one-to-one correspondence with Morita equivalence classes of $(N \subset M, \mathcal{C})$, where $N = \mathcal{A}(I)$ and $\mathcal{C} = \mathrm{Rep}^I(\mathcal{A})$.*

Defects. Given two local extensions: $\mathcal{B}_{2,\mathrm{L}}, \mathcal{B}_{2,\mathrm{R}} \supset \mathcal{A}_2$ an **$\mathcal{A}$-topological ($\mathcal{A}$-top.) $\mathcal{B}_{2,\mathrm{L}}$–$\mathcal{B}_{2,\mathrm{R}}$ defect** is a non-local extension $\mathcal{D} \supset \mathcal{A}_2$, such that $\mathcal{B}_{2,\mathrm{L/R}}(O_{\mathrm{L/R}})$ commutes with $\mathcal{D}(O)$ if O_L and O_R are space-like left and right from O, respectively.

Theorem 4 ([3]). *Let $\mathcal{A}_2 \subset \mathcal{B}_{2,a}, \mathcal{B}_{2,b}$ maximal extensions corresponding (by Thm. 3) to $(N \subset M_a, \mathcal{C})$ and $(N \subset M_b, \mathcal{C})$, respectively. Then there is a one-to-one correspondence between $\mathcal{A}$-top. $\mathcal{B}_{2,a}$–$\mathcal{B}_{2,b}$-defects and sectors $\beta\colon M_b \to M_a$ related to $\mathcal{C}$.*

Fusion of defects. Inspired by [10, 1] we define the fusion $\mathcal{D} \boxtimes_{\mathcal{B}_{2,b}} \mathcal{E}$ of $\mathcal{D}$ (an $\mathcal{A}$-top. $\mathcal{B}_{2,a}$–$\mathcal{B}_{2,b}$ defect) with $\mathcal{E}$ (an $\mathcal{A}$-top. $\mathcal{B}_{2,b}$–$\mathcal{B}_{2,c}$ defect) using Connes' relative tensor product over $\mathcal{B}_{2,b}(W)$, where W is a left-wedge. $\mathcal{D} \boxtimes_{\mathcal{B}_{2,b}} \mathcal{E}$ turns out to be a $\mathcal{A}$-top. $\mathcal{B}_{2,a}$–$\mathcal{B}_{2,c}$ defect. We have the following result: The decomposition of the fusion $\mathcal{D} \boxtimes_{\mathcal{B}_{2,b}} \mathcal{E}$ into irreducible $\mathcal{A}$-top. $\mathcal{B}_{2,a}$–$\mathcal{B}_{2,c}$ defect corresponds to the decomposition of $\beta_\mathcal{D} \circ \beta_\mathcal{E}$ as irreducible morphisms.

Defects between defects. The final goal ist to understand defects (or intertwiners) between defects. We expect to get the following result: $\mathcal{A}$-top. defects form a (not necessarily strict) 2-category. The category is equivalent to the 2-category obtained by $\mathcal{C}$ and all its braided subfactors (see Remark 2).

The construction is related to a generalization of a quantum double construction. We remember that the planar algebra of an morphism $\beta\colon M_b \to M_a$ related to $\mathcal{C}$ should describe a defect and its dual line. Our expected result gives a concrete "realization" of this planar algebra on Minkowski space, where we only consider diagrams with time like strings.

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Planar algebras in modular tensor categories

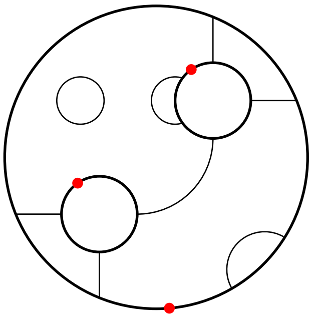
David Penneys
(joint work with André Henriques and James E. Tener)

The motivation for this project is a common generalization of genus zero Segal conformal field theory (CFT) and Jones' planar algebras. In our study of genus zero Segal CFT with topological defect lines, we came across an algebraic structure which generalizes the usual notion of Jones' planar algebras to planar algebras in a modular tensor category.

1. Planar algebras in Vec

Jones' planar algebras [Jon] have proven to be useful in the construction [Pet10, BMPS12] and classification [JMS14] of subfactors. We give a brief definition following [MPS10, BHP12].

Definition 1. A planar algebra is a sequence of vector spaces $\mathcal{P}_\bullet = (\mathcal{P}_n)_{n\geq 0}$ together with an action of the planar operad, i.e., every planar tangle with $k_1, \ldots, k_r$ points on the input disks and k_0 points on the output disk corresponds to a linear map from the unordered tensor product $\mathcal{P}_{k_1} \otimes \cdots \otimes \mathcal{P}_{k_r} \to \mathcal{P}_{k_0}$. For example,



$: \mathcal{P}_3 \otimes \mathcal{P}_5 \to \mathcal{P}_6.$

This data must satisfy the following axioms:

- isotopy invariance: isotopic tangles produce the same multilinear maps,
- identity: the identity tangle (which only has radial strings and no rotation between marked points) acts as the identity transformation, and
- naturality: gluing tangles corresponds to composing maps. When we glue tangles, we match up the points along the boundary disks making sure the distinguished intervals marked by the distinguished dots align.

The following folklore theorem (needing additional adjectives) has made appearances in various forms in [MPS10, Yam12, BHP12] (see also [Kup96, Jon]).

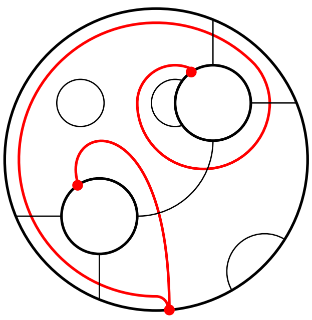
Theorem 2. *Starting with a pair $(\mathcal{C}, X)$ with $\mathcal{C}$ a pivotal category and X a distinguished symmetrically self-dual object, we can construct a canonical planar algebra in* Vec*. Conversely, given a planar algebra in* Vec*, we can construct its pivotal category of projections, where the strand is the distinguished projection.*

These constructions are mutually inverse in the sense that going from pairs to planar algebras back to pairs produces an equivalent pair, and going from planar algebras to pairs back to planar algebras is the identity.

2. Planar algebras in a balanced fusion category $\mathcal{C}$

We now want to relax the condition of working in Vec to working in a given balanced (braided with twists) fusion category $\mathcal{C}$. To define a planar algebra in $\mathcal{C}$, we need additional structure for our planar tangles.

Definition 3. An anchored planar algebra in $\mathcal{C}$ is a sequence of objects $\mathcal{P}_\bullet = (\mathcal{P}_n)_{n\geq 0}$ in $\mathcal{C}$ together with an action of the anchored planar operad, i.e., every anchored planar tangle with $k_1, \ldots, k_r$ points on the input disks and k_0 points on the output disk corresponds to a morphism $\mathcal{P}_{k_1} \otimes \cdots \otimes \mathcal{P}_{k_r} \to \mathcal{P}_{k_0}$. For example,



$: \mathcal{P}_5 \otimes \mathcal{P}_3 \to \mathcal{P}_6.$

There is one anchor line for each input disk, which is a homotopy class of paths from each internal marked point to each external marked point. They are transparent to the ordinary strings of the tangle, but they cannot cross each other.

When $\mathcal{C}$ was Vec, our tensor products were unordered, but in $\mathcal{C}$, order matters. The domain of the corresponding morphism is obtained by ordering the input disks counterclockwise according to anchor line entry. In addition to the previous axioms, we also require:

- twist anchor dependence: the n-string 2π rotation gives the map $\theta_{\mathcal{P}_n}$.
- swap anchor dependence: swapping anchor lines induces a braiding in $\mathcal{C}$.

Similar to the classical classification theorem for planar algebras $\mathcal{P}_\bullet$ and pairs $(\mathcal{C}, X)$ of pivotal categories with distinguished objects, we have the following classification result (again, with additional adjectives).

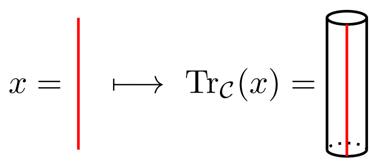
Theorem 4 (Henriques-Penneys-Tener)**.** *Given a balanced fusion category $\mathcal{C}$, a pivotal category $\mathcal{M}$, a braided tensor functor $G : \mathcal{C} \to \mathcal{Z}(\mathcal{M})$, and $m \in \mathcal{M}$ which generates $\mathcal{M}$ as a $\mathcal{C}$-module, there exists a canonical anchored planar algebra $\mathcal{P}_\bullet$ in $\mathcal{C}$. Conversely, we can produce a tuple $(\mathcal{M}, G, m)$ from such an anchored planar algebra in $\mathcal{C}$.*

These constructions are mutually inverse in the sense that going from tuples to anchored planar algebras back to tuples gives an equivalent tuple, and going from anchored planar algebras to tuples back to anchored planar algebras is the identity.

Example 5. One can get examples with $\mathcal{C} \neq \mathsf{Vec}$ as follows. Take $\mathcal{C}$ to be a modular category, and choose a commutative algebra object $a \in \mathcal{C}$. Let $\mathcal{M}$ to be the left a-modules in $\mathcal{C}$, and choose an $m \in \mathcal{M}$ which generates $\mathcal{M}$ as a $\mathcal{C}$-module. When $\mathcal{C} = \mathsf{Rep}(\mathcal{U}_q(\mathfrak{sl}_2))$, the commutative algebras correspond to the A_n, D_{2n}, and E_6 and E_8 Coxeter-Dynkin diagrams [KO02]. Then there are two canonical braided, balanced tensor functors $G_\pm : \mathcal{C} \to \mathcal{Z}(\mathcal{M})$ given by α-induction [BEK01].

3. Ingredients of the proof

We begin by considering Walker's unpublished work studying module 2-categories for a braided tensor category $\mathcal{C}$ as functors $G : \mathcal{C} \to \mathcal{Z}(\mathcal{M})$ as a heuristic. Similar to Ostrik's internal hom for fusion categories [Ost03], we found an internal trace functor $\mathrm{Tr}_\mathcal{C} : \mathcal{M} \to \mathcal{C}$ as the composite functor $\Phi = G^T \circ I$, where G^T is the left adjoint of G (which exists by semi-simplicity and finiteness conditions), and $I : \mathcal{M} \to \mathcal{Z}(\mathcal{M})$ is the induction functor [Müg03]. Diagrammatically, we denote $x \in \mathcal{M}$ as a strand on the plane, and we represent $\mathrm{Tr}_\mathcal{C}(x)$ as a strand on a cylinder.



We call $\mathrm{Tr}_{\mathcal{C}}$ an internal trace because we have two natural isomorphisms called the 'traciators' $\tau_{\pm} : \mathrm{Tr}_{\mathcal{C}}(x \otimes y) \cong \mathrm{Tr}_{\mathcal{C}}(y \otimes x)$ which categorify the notion of a trace (see the left hand side of Figure 1). The idea behind the traciator is that we may lift a one-click rotation from $\mathcal{M}$ into $\mathcal{C}$ to get an isomorphism. However, if we lift the full 2π rotation, we get a twist rather than the identity.

Another important ingredient is a natural multiplication map $\mu : \mathrm{Tr}_{\mathcal{C}}(x) \otimes \mathrm{Tr}_{\mathcal{C}}(y) \to \mathrm{Tr}_{\mathcal{C}}(x \otimes y)$, which has the properties of an associative multiplication.

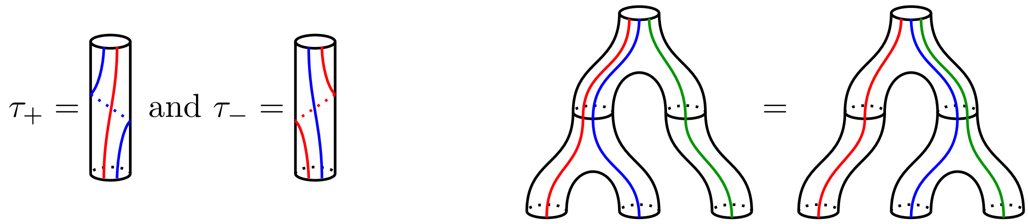
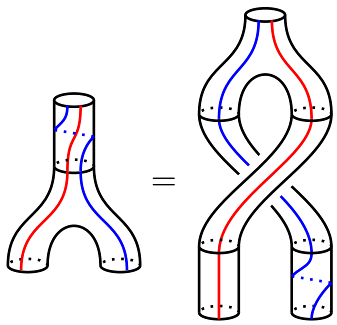


FIGURE 1. The traciators $\tau_{\pm}$ (left) and associativity of the multiplication map μ (right)

The traciator $\tau_{\pm}$ is compatible with the multiplication μ, and the braiding $\beta_{\pm}$ and twists θ in $\mathcal{C}$. We use our diagrams as heuristics to prove many relations. We give one example below, corresponding to the commutative diagram on the right:



$\longleftrightarrow$

$$\begin{array}{ccc} \mathrm{Tr}_{\mathcal{C}}(x) \otimes \mathrm{Tr}_{\mathcal{C}}(y) & \xrightarrow{\mu} & \mathrm{Tr}_{\mathcal{C}}(x \otimes y) \\ \Big\downarrow {\scriptstyle \beta_+ \circ (\mathrm{id}_{\mathrm{Tr}_{\mathcal{C}}(x)} \otimes \theta_{\mathrm{Tr}_{\mathcal{C}}(y)})} & & \Big\downarrow {\scriptstyle \tau_+} \\ \mathrm{Tr}_{\mathcal{C}}(y) \otimes \mathrm{Tr}_{\mathcal{C}}(x) & \xrightarrow{\mu} & \mathrm{Tr}_{\mathcal{C}}(y \otimes x). \end{array}$$

As we developed anchored planar algebras in balanced fusion categories with a view toward a common generalization of genus zero Segal CFT and Jones' planar algebras, we state another result which will play an important role in the theory.

Proposition 6. *The object $A = \mathrm{Tr}_{\mathcal{C}}(1_{\mathcal{M}})$ is a symmetrically self-dual commutative Frobenius algebra object in $\mathcal{C}$ with $\theta_A = 1$.*

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Non-semisimple Tensor Categories and Extended Topological Field Theory

Christopher Schommer-Pries

(joint work with Christopher Douglas, Noah Snyder)

3-Dimensional topological field theory stands in between subfactors and conformal field theory and touches both. On the one hand subfactors give rise to spherical fusion categories, which are well-known to yield 3-dimensional topological field theories (for example via the Barrett-Westbury-Turaev-Viro construction [2]). On the other hand the confomral blocks of a rational conformal field theory give rise to a modular functor, which forms the 2-dimensional part of a 3-dimensional topological field theory.

However we have also seen in Ingo Runkel's and Jürgen Fuchs' talks that rational conformal field theory forms only a small corner of the space of all conformal field theories, and that many of these are associated to non-semisimple tensor cateogries. In this talk I will explain how modern classification techniques allow us to construct fully extended 3-dimesional topological field theories associated to every (possibly non-semisimple) finite tensor category.

We will be concerned with the moduli space of fully extended topological field theories, as in Peter Teichner's talk. Being fully extended, these topological field theories satisfy the strongest possible locality properties. The cost is that we must use the machinery of higher category theory. Thus a d-dimensional topological field theory will mean a symmetric monoidal functor

$$d\mathrm{Bord}^{\mathcal{G}} \to d\mathrm{Vect}$$

from the d-category of cobordism to a preferred target d-category dVect. Here $\mathcal{G}$ represents some geometric or topological structure we impose on our cobordisms. In what follows we will focus on two cases, orientations $\mathcal{G} = or$ and tangential framings $\mathcal{G} = fr$.

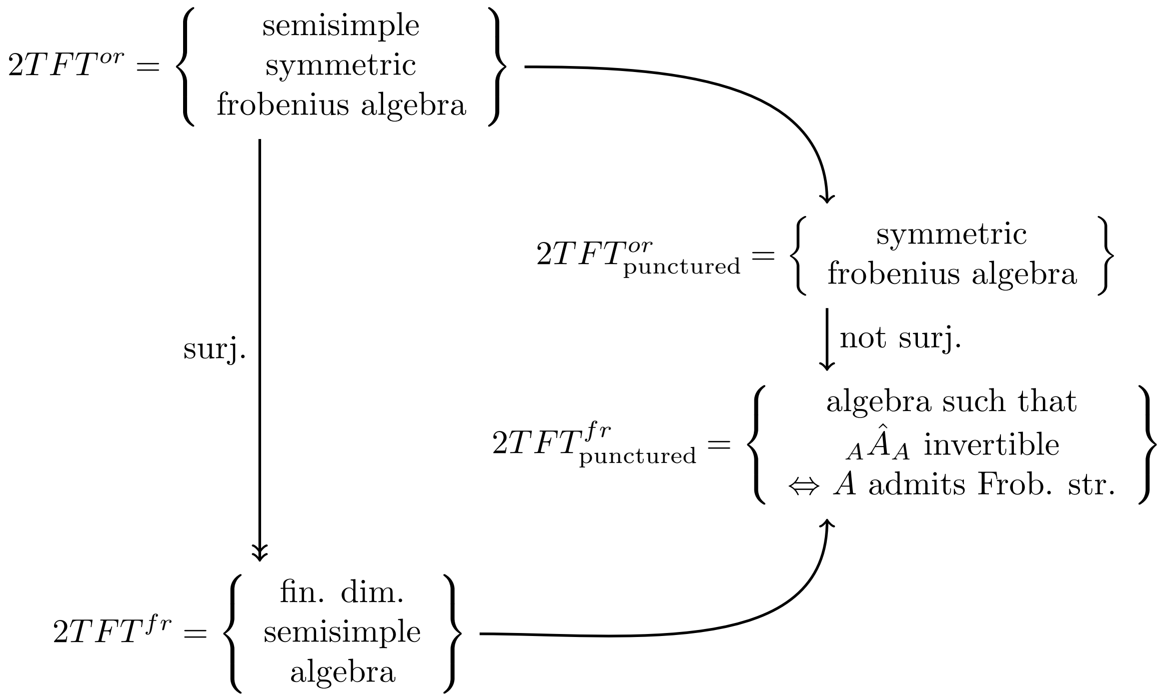
The main classification result that we will use is the cobordism hypothesis [3], which allows us to completely identify the d-groupoid of such field theories. Briefly the tangentially framed topological field theories correspond to the so-called *fully-dualizable* objects, while the oriented theories correspond to equipping these objects with additional structure ($SO(d)$-homotopy fixed points).

In dimension three there is a symmetric monoidal 3-category of finite tensor categories, bimodule categories, functors, and natural transformations. It was shown in [1] that the fully-dualizable objects of this 3-category are precisely the fusion tensor categories. Earlier, the author's Ph. D. dissertation [4] considered the 2-category of algebras, bimodules, and maps and determined that the oriented theories corresponded to symmetric Frobenius algebras, while the fully dualizable objects are finite dimensional semisimple algebras.

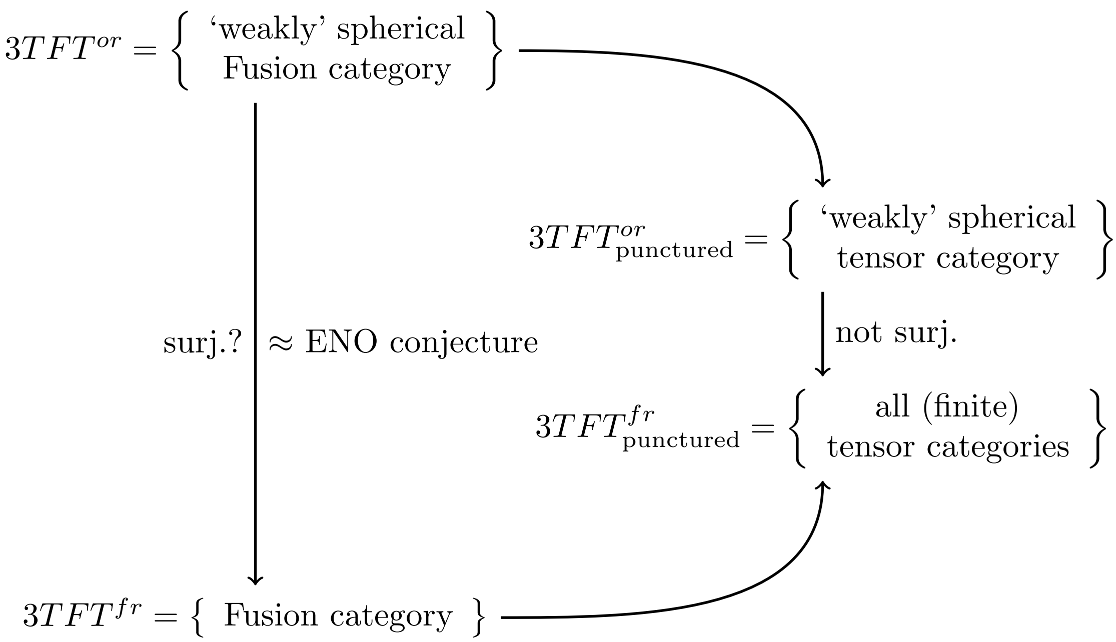
By analyzing the proof of the cobordism hypothesis we obtain more refined information. The proof precedes by considering a filtration which starts with the $(d-1)$-dimensional bordism d-category, and ends with the d-dimensional bordism category. The intermediate stages are given by allowing d-dimensional bordisms which admit Morse functions using only index $\leq k$ critical points, i.e. bordism built using handles up to dimension k.

If we stop just one filtration stage shy of the d-dimensional bordism category, then we get the *punctured bordism d-category.* This is the bordism d-category where the top-dimensional cobrodism must have non-empty out going boundary components.

We can now summarize the classification results for these various kinds of topological field theories. In dimension two we have the following commutative diagram:



While in dimension three we have the following:



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Planar algebras and the Haagerup property

Arnaud Brothier
(joint work with Vaughan Jones)

In this talk, we discuss about analytical properties of standard invariants of subfactors. Those properties are relevant for infinite depth subfactors. This means that the bimodule category $\mathcal{C} \subset \text{Bimod}_{M-M}$ generated by the subfactor $N \subset M$ has infinitely many irreducible objects. We are particularly interested by the Haagerup property which has been defined for subfactors by Popa [5]. Later on an intrinsic definition for standard invariants has been given in [6]. It has recently been proved that the Temperley-Lieb-Jones standard invariants have the Haagerup property [6]. The proof relies on a result for quantum groups [2]. We propose to give a new proof of this result by using exclusively planar algebra technology and the original definition of Popa of the Haagerup property.

Given a subfactor planar algebra $\mathcal{P}$, Curran et al. associated to it a subfactor $N \subset M$ and a symmetric enveloping inclusion $T \subset S$ which coincides with the symmetric enveloping inclusion of the subfactor $N \subset M$ [3, 1]. The planar algebra $\mathcal{P}$ has the Haagerup property if and only if the inclusion $T \subset S$ has the relative Haagerup property. It means that there exists a sequence of normal trace-preserving T-bimodular unital completely positive maps that converge pointwise to the identity for the L^2-norm and are compact in a certain sense. The main idea of this work is to associate to any Hilbert $\mathcal{P}$-module of lowest weight 0 a bimodule ${}_SH_S$ over S and a T-central vector ξ. Using Connes' correspondences such a pair $({}_SH_S, \xi)$ gives us a normal trace-preserving T-bimodular unital completely positive (ucp) map $\phi : S \longrightarrow S$. Let TLJ be a Temperley-Lieb-Jones standard invariant for a fixed loop parameter larger than 2 and $T \subset S$ its associated symmetric enveloping inclusion. Jones constructed a one parameter family of Hilbert TLJ-modules [4]. Those Hilbert modules provide us a net of normal ucp T-bimodular maps $\phi_t : S \longrightarrow S$ that literally fulfill all the assumptions of the relative Haagerup property.

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A classification with subfactors and fusion categories from CFT

Zhengwei Liu

In joint works with Dietmar Bisch and Vaughan Jones [BJ97b, BJ03, BJL, Liu], we achieve the first goal in the classification program initiated by Bisch and Jones in 1997, the classification of singly generated Yang-Baxter relation planar algebras. They are given by Bisch-Jones, BMW and a new one-parameter family of planar algebras. We also have a similar classification for fusion categories from a dimension restriction. We give a skein theoretic construction of the new one-parameter family which overcomes the three fundamental problems: evaluation, consistency, positivity. Infinitely many new subfactors and unitary pivotal spherical fusion categories are obtained.

In the classification, a surprising one-parameter family of planar algebras appeared after Temperley-Lieb-Jones [Jon83], HOMFLYPT [FYH+85, PT88], BMW, the Potts model [Jon93], Bisch-Jones [BJ97a] planar algebras. The first three families arise from quantum groups; the fourth family arise from groups; the fifth family arises from a free product construction. This new one-parameter family can be thought of as the first family from quantum subgroups or conformal inclusions. We construct the q-parameterized planar algebra by skein theory which overcomes the three fundamental problems: Evaluation, Consistency, Positivity.

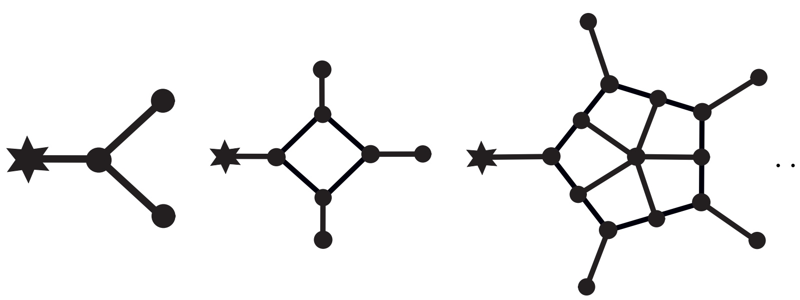
The principal graph of the q-parameterized planar algebra is the Young's lattice. The dimension of a simple object labeled by the Young diagram λ is

$$< \lambda >= \prod_{c\in\lambda} \frac{i(q^{h(c)} + q^{-h(c)})}{q^{h(c)} - q^{-h(c)}},$$

where $h(c)$ is the hook length of the cell c in λ.

This q-parameterized planar algebra contains both the Jones Projection and two universal R matrices for quantum groups of type A. Thus it has one Temperley-Lieb-Jones subalgebra and two Hecke subalgebras of type A.

When $q = e^{\frac{i\pi}{2N+2}}$, (the quotient of) the q-parameterized planar algebra is a subfactor planar algebra, denoted by E_{N+2}. Its principal graph is the sublattice of the Young lattice consisting of Young diagrams whose $(1,1)$ cell has hook length at most N. For $N = 2, 3, 4, ...$, we have

 $\cdots.$

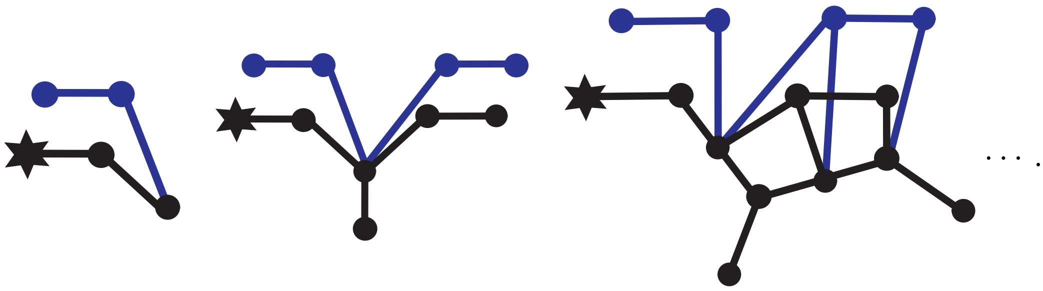
Moreover, we have the following classification result:

Any Yang-Baxter relation planar algebra with 3 dimensional 2-boxes is one of the following: (1) Bisch-Jones; (2) BMW; (3) E_{N+2}, $N \geq 2$, $N \in \mathbb{N}$.

In terms of fusion categories, we have the following classification result:

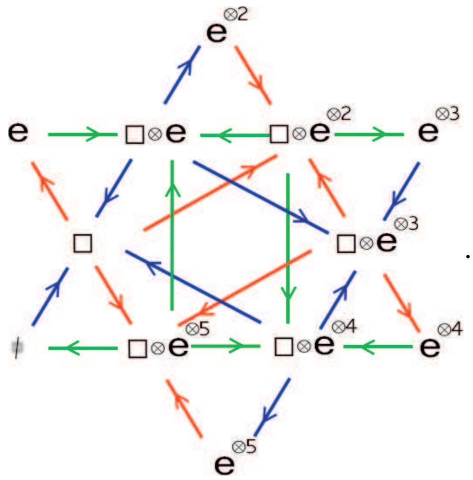
Suppose X is a self-dual object in a unitary pivotal spherical fusion category. If $X^2 = 1 \oplus X_1 \oplus X_2$, for simple objects X_1 and X_2, and $\dim(\hom(X^3, X^3)) \leq 15$, then (the idempotent completion of) the subcategory generated by $\hom(X^2, X^2)$ is (1) unshaded Bisch-Jones, when X_1 is self dual and $E = 1$; (2) BMW, when X_1 is self-dual and $E \neq 1$; (3) E_{N+2} for some N, when X_1 is non-self-dual, where E is computed from the dimensions of the simple objects X, X_1, X_2.

The subfactor planar algebra E_{N+2} has a $D_{2(N+1)}$ symmetry. From the $\mathbb{Z}_2$ symmetry, we obtain another sequence of subfactor planar algebras which is an extension of the near group subfactor planar algebra for $\mathbb{Z}_4$ [Izu93]. The principal graphs for $N = 2, 3, 4, ...$ are given by



From the $\mathbb{Z}_{N+1}$ symmetry, for each subgroup of $\mathbb{Z}_{N+1}$ of odd order, we obtain at least one more subfactor.

We also obtain infinitely many unitary pivotal spherical fusion categories from E_{N+2} for each N. In particular, two of them can be thought of as the representation category of an exceptional subgroups of quantum $SU(N)$ at level $N+2$ and of quantum $SU(N+2)$ at level N which are related to conformal inclusions $SU(N)_{N+2} \subset SU(N(N+1)/2)_1$ and $SU(N+2)_N \subset SU((N+2)(N+1)/2)_1$ respectively. The branching rule is also derived for all N. In particular, the one for $SU(3)_5$ is



which has appeared in other places, e.g. in [Xu98] for conformal inclusions, in [Ocn00] for quantum subgroups. The one for $SU(5)_3$ was known in [Xu98]. The one for $SU(4)_6$ was known in [Ocn00]. We also obtain (non-unitary, pivotal, spherical) fusion categories at other roots of unity.

Questions: Do we have one-parameter families of planar algebras from other conformal inclusions? Do they contribute to polynomial invariants of three manifolds?

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A (co)-homology theory for subfactors and C^*-tensor categories

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(joint work with S. Popa and S. Vaes)

Jones subfactor theory encodes a rich variety of "quantum symmetries". These symmetries are reflected in many objects associated to a subfactor inclusion $M_0 \subset M_1$: systems of higher relative commutants $M_i' \cap M_j$ (standard invariant, λ-lattice, Planar algebra), the structure of the associated bimodules (C^*-tensor category, annular category, Ocneanu tube algebra), as well as a quasi-regular symmetric enveloping algebra inclusion $(M_0 \otimes M_0^o) \subset (M_0 \boxtimes_{e_{-1}} M_0^o)$.

For example, a properly outer action α of a discrete group G with generating set S on a II_1 factor N gives rise to a subfactor inclusion $i : N = M_0 \to M_1 = M \otimes \mathrm{End}(\ell^2(S \cup \{e\})$ given by $i(x) = \sum_{g \in S \cup \{e\}} \alpha_g(x) \otimes P_g$ where P_g denotes the orthogonal projection onto $g \in S \cup \{e\}$. The group G is then precisely the set of irreducible bimodules arising in the Jones tower construction for $M_0 \subset M_1$ and the group algebra product corresponds to fusion of these bimodules. Furthermore, the symmetric algebra inclusion is isomorphic to the crossed product inclusion $(M \otimes M^o) \rtimes_{\alpha\otimes\alpha^o} G$.

It is thus a natural question to understand if here exists a (co)-homology theory that extends the usual group homology theory to these more general symmetries encoded in a subfactor. For definiteness, let us denote by $M_0 \subset M_1$ the subfactor, $T = (M \otimes M^o) \subset (M \boxtimes_{e_{-1}} M^o) = S$ the symmetric enveloping algebra inclusion, $\mathscr{A}$ the affine category and $\mathscr{T}$ the tube algebra. In making this construction, we were motivated by the following ideas:

(1) Representations (modules) over a subfactor inclusion [PV14] correspond to any one of the following objects: (i) S-bimodules generated by their T-central vectors; (ii) modules over $\mathscr{A}$ generated by their vectors of weight 0; (iii) modules over $\mathscr{T}$ generated by their weight 0 vectors.

(2) For a discrete group G, the notion of ℓ^2 cohomology is well defined both for the group and for the (quasi)-regular inclusion $L^\infty(X) \subset L^\infty(X) \rtimes_\sigma G$ for any measure preserving action σ of G on a finite measure space X. Moreover, the values of the associated ℓ^2-Betti numbers are the same. Note that in the case of a subfactor associated to a discrete group these correspond precisely to the Betti numbers of fusion ring and (apart from replacing the commutative ring $L^\infty(X)$ with T) of the symmetric enveloping algebra inclusion.

In each of the cases (1.i), (1.ii), (1.iii), we define the homology in terms of a Hochshild-like complex, and the resulting (co)-homology groups are the same. The complex associated to (1.ii) has a graphical interpretation in terms of the planar algebra. The vector space C_k of k-chains is the linear span of diagrams one obtains if one draws an element x of the planar algebra of the inclusion on the two-sphere S^2 from which one removes n disks $D_1, \ldots, D_n$. One further fixes an "input disk" D inside the sphere, and allows arbitrary non-crossing connections between D and x, insisting that the remaining strings of x are connected to each other (and surround the disks D_j in a general way). The differential ∂_k of this complex associates to such a diagram the alternating sum of diagrams in which one of the disks D_j has been filled in.

The complex (C_k, ∂_k) is an acyclic differential complex: $\partial_{k-1} \circ \partial_k = 0$ and moreover for a suitable homotopy $h : C_k \to C_{k+1}$, $\partial_{k+1} h + h\partial_k = 0$. Each C_k is a left $\mathscr{A}$-module.

Given a right $\mathscr{A}$-module V, we define the homology groups H_k^V with values in V to be the homology of the differential complex $V \otimes_{\mathscr{A}} C_*$.

Theorem 1. The complex C_* is exactly a Hochshild complex for the augmented algebra $\mathscr{A}$ relative to the subalgebra $\mathscr{B}$ consisting of higher relative commutant spaces. Moreover, the sense of (1.i) the complex C_* is related to the relative Hochshild complexes associated to the symmetric enveloping inclusion $T \subset S$ and the inclusion $Z \subset \mathscr{T}$, where Z is the subalgebra of $\mathscr{T}$ generated by central projections associated to the irreducibles.

Using our approach one can define L^2-homology (taking for V the space $L^2(p\mathscr{A})$, with p a minimal projection corresponding to the unit of the fusion algebra $\mathscr{F} = p\mathscr{A}p \subset \mathscr{A}$. One can then define L^2-Betti numbers as $\beta_* = \dim_{W^*(\mathscr{F})} H_*^{L^2(pA)}$.

Theorem 2. Let G be a discrete group generated by a finite set S and W be a G-module. Consider the subfactor inclusion associated to a discrete group G, S, and let V be subfactor representation associated to W. Then $H_*^V = H_*(G;V)$ is the usual group homology. Similarly, the L^2-betti numbers of the subfactor inclusion are the same as the ℓ^2-Betti numbers of G.

We are able to make a number of computations, choosing for them the most convenient of the various definitions. Among them is:

Theorem 3. Let V be the augmentation representation of $\mathscr{A}$ associated to a Temperley-Lieb-Jones subfactor inclusion with index greater than 4. Then $H_k^V = 0$ if $k = 0, 1, 2$.

Corollary. For V the augmentation representation, $H_k^V \neq HH_k(\mathscr{F})$ in general.

Indeed, the left hand side is trivial in the case of Temperley-Lieb-Jones subfactor inclusion with index bigger than 4, while the right hand side is a polynomial algebra whose (augmented) Hochshild homology is nonzero.

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