Arbeitsgemeinschaft on the Kadison–Singer conjecture

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This is a (tentative) plan for the talks at the Spring 2015 Arbeitsgemeinschaft at Oberwolfach from March 29 – April 4, 2015. The goal will be to cover various themes that collectively fall under the umbrella of topics that arise in the solution of the Kadison–Singer conjecture.

These will include the following major themes:

- (B) Evolution of the problem
- (S) Stability
- (H) Hyperbolicity
- (A) Further Applications
- (E) Extensions

1. Evolution of the problem

   (B1) Introduction to the problem
   (B2) From C^* algebras to vector spaces
   (B3) From C^* algebras to vector spaces
   (B4) From C^* algebras to vector spaces

2. Stablility

   (S1) Univariate stable polynomials
   (S2) Multivariate stable polynomials
   (S3) Kadison-Singer via real stability
   (S4) Kadison-Singer via real stability

3. Hyperbolicity

   (H1) Hyperbolic polynomials
   (H2) Convexity properties
   (H3) Kadison-Singer via hyperbolicity
   (H4) Kadison-Singer via hyperbolicity

4. Further Applications

   (A1) Ramanujan Graphs
   (A2) van der Waerden Conjecture
   (A3) Negative dependence
   (A4) Negative dependence

5. Extensions

   (E1) Lyapunov Theorem
   (E2) Anari–Gharan Extension
   (E3) Anari–Gharan Extension

Evolution of the problem

(B1) Introduction to the problem

Goal: To present the motivation for the original problem
Suggested Plan: Present the material in [10] and then section 2 of [9].

(B2),(B3),(B4) From C^* algebras to vector spaces

Goal: To see the path from the original problem to the combinatorial problem that was eventually solved
Suggested Plan for Talk #1: Introduce ultrafilters and describe pure states, essentially sections 3 and 4 of [9], with results from Appendices as needed.
Suggested Plan for Talk #2: Show the reduction to paving. Sections 5 and 6 of [9] with results from Appendices as needed.
Suggested Plan for Talk #3: Show the reduction from paving to Weaver's conjecture, and other equivalent formulations as time allows. Care should be taken as the original reduction in Weaver's paper [17] is from a result of Akemann-Anderson and not from Anderson paving. Instead, use Theorem 3 of [7]. As time allow, give other equivalent formulations from [8].

Stability

(S1) Univariate stable polynomials

Goal: Introduction of univariate stable polynomial and properties
Suggested Plan: Section 4 of [15].

(S2) Multivariate stable polynomials

Goal: Introduction of multivariate stable polynomial and properties
Suggested Plan: Section 5 of [15]

(S3),(S4) Kadison–Singer via real stability

Goal: To introduce interlacing families, mixed characteristic polynomials, and then prove the bound on the largest root using boundary functions.
Suggested Plan for Talk #1: Sections 3 and 4 of [14]
Suggested Plan for Talk #2: Section 5 and Theorem 1.3/Corollary 1.4 of [14]. Section 6 as time allows.

Hyperbolicity

(H1) Hyperbolic Polynomials

Goal: Introduce hyperbolic polynomials
Suggested Plan: Define hyperbolic polynomials (with examples), hyperbolicity cones, concavity of λ_{min}. Should cover the material in section 2 of [15] (through example 2.7) and section 2 of [16].

(H2) Convex Analysis

Goal: Show convexity properties of hyperbolic polynomials
Suggested Plan: General convexity of compositions: Theorem 3.1, Fact 3.7 through (but not including) Corallary 3.11, and then Corollary 4.6, 4.7 of [3]. As time permits, Alexandrov–Fenchel inequalities [11].

(H3),(H4) Kadison–Singer via hyperbolicity

Goal: Show a proof of the main bound in Kadison–Singer using hyperbolic polynomials.
Suggested Plan for Talk #1: Sections 2-3 of [6]. See [5] for details.
Suggested Plan for Talk #2: Sections 4-5 of [6]. See [5] for details.

Further Applications

(A1) Ramanujan graphs

Goal: Show how the existence of bipartite Ramanujan graphs of any degree.
Suggested Plan: Present [13] assuming that mixed characteristic polynomials are real rooted. Discuss expanders and 2-lifts, introduce interlacing families, show the main theorem for interlacing families, then show how there exists a good 2-lift. Do not prove Heilmann-Lieb, Godsil, or real-rootedness). See `http://cs-www.cs.yale.edu/homes/marcus/talks/bipartite_ramanujan.pdf` as an example.

(A2) van der Waerden

Goal: See how real stable polynomials can be used to solve van der Waerden's conjecture.
Suggested Plan: Present [12].

(A3),(A4) Negative Dependence

Suggested Plan for Talk #1: Section 1, 2 of [4]
Suggested Plan for Talk #2: As much of Section 4 of [4] as time allows.

Extensions

(E1) Lyapunov theorem

Goal: Extend the vector decomposition theorem to arbitrary approximations and to infinite case.
Suggested Plan: Sections 0, 1 and 2 of [1].

(E2),(E3) Anari–Gharan Extension

Goal: To see the extension of the methods used in Kadison–Singer to homogeneous strong Rayleigh measures
Suggested Plan for Talk #1: A *brief* discussion on thin trees to motivate the problem (section 1 of [2] as well as references there), and then define the mixed characteristic polynomials in this regime and show it is real rooted (sections 2 and 3 of [2]).
Suggested Plan for Talk #2: Give the proof of the bound on the largest root (Section 4 of [2]).

References

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[2] Nima Anari and Shayan Oveis Gharan. The Kadison-Singer problem for strongly Rayleigh measures and applications to asymmetric tsp. *arXiv preprint, arXiv:1412.1143*, 2014.

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[8] Peter G Casazza, Matthew Fickus, Janet C Tremain, and Eric Weber. The Kadison-Singer problem in mathematics and engineering: a detailed account. *Contemporary Mathematics*, 414:299, 2006.

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[13] Adam W. Marcus, Daniel A. Spielman, and Nikhil Srivastava. Interlacing families I: bipartite Ramanujan graphs of all degrees. In *Proceedings of the 54th IEEE Symposium on Foundations of Computer Science*, pages 529–537, 2013.

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[15] Robin Pemantle. Hyperbolicity and stable polynomials in combinatorics and probability. *arXiv preprint arXiv:1210.3231*, 2012.

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[17] Nik Weaver. The Kadison–Singer problem in discrepancy theory. *Discrete Mathematics*, 278(1–3):227–239, 2004.

Oberwolfach Arbeitsgemeinschaft:
The Kadison-Singer Conjecture

Date:

29 Mar - 4 Apr 2015 (ID: 1514)

Organizer:

Adam W. Marcus, Yale / Boston

Application for Participation:

The idea of the Arbeitsgemeinschaft is to learn by giving one of the lectures in the program. If you intend to participate, please send your full name and full postal address to

adam.marcus@yale.edu

by **1 February 2015** at the latest.

You should also indicate which talk you are willing to give:

- **First choice: talk no. X**
- **Second choice: talk no. Y**
- **Third choice: talk no. Z**

You will be informed shortly after the deadline if your participation is possible and whether you have been chosen to give one of the lectures. The Arbeitsgemeinschaft will take place at Mathematisches Forschungsinstitut Oberwolfach, Schwarzwaldstrasse 9-11, 77709 Oberwolfach-Walke, Germany.

The Institute covers board and lodging. Travel expenses cannot be covered. Further information will be given to the participants after the deadline.