Oberwolfach Seminar

Topology and Combinatorics of Configuration Spaces

Organizers:

Pavle Blagojević, Carl-Friedrich Bödigheimer, Michael Farber

In many classical situations in geometry or topology the “space of all possible solutions” is naturally a configuration space of finitely many points in some manifold.

- Questions about Riemann surfaces led Hurwitz (1892) to configuration spaces in the plane and thus to the discovery of the braid groups, with their connection to knots and links in 3-manifolds and to mapping class groups, Teichmüller theory and moduli spaces of surfaces found later.
- The Poincaré–Birkhoff problem asks for the existence of orbits in smooth strictly convex billiads, and the Borsuk problem (1957) is about the existence of k-regular embeddings that map any k distinct points in $\mathbb{R}^d$ to k linearly independent vectors in $\mathbb{R}^N$, or $\mathbb{C}^N$.
- Even in homotopy theory, where geometric properties of maps are irrelevant, one uses (1970's) configuration spaces as models for loop spaces of spheres or, more generally, for mapping spaces, to prove stable splitting results like the Snaith splitting; this had an important influence on the development of operads in many areas.
- In robotics, there are interesting conjectures (Kevin Walker 1985) about the geometry of linkages and the topology of the configuration space.
- And the partition problem of Nandakumar and Ramana Rao (2005) asks for a partition of a convex planar body into n convex pieces of equal area and equal perimeter.

All these problems are most naturally treated by methods of algebraic topology.

Lecturers

- Pavle V. M. Blagojević (Belgrade/Berlin/Bonn)
- Carl-Friedrich Bödigheimer (Bonn)
- Michael Farber (London)
- Jelena Grbić (Southampton)
- Günter M. Ziegler (Berlin)

Programme

In this Oberwolfach seminar we intend to develop some of the key concepts and results about configurations spaces. We plan to discuss:

(1) The equivariant topology of the ordered configuration space $F(\mathbb{R}^d, n)$ and its relation to regular embedddings, using its cohomology.
(2) The topology of the unordered configuration space $F(\mathbb{R}^d, n)/\mathfrak{S}_n$ and its relation to homotopy theory.
(3) The classifying spaces of linkages of various types and recent results centered around the solution of the Walker Conjecture.
(4) Methods to construct G-CW models of configuration spaces like $F(\mathbb{R}^d, n)$ with $G = \mathfrak{S}_n$, and $F_{\mathbb{Z}/2}(S^d, n)$ with $G = (\mathbb{Z}/2)^n \rtimes \mathfrak{S}_n$.

We will develop the methods in the lectures far enough in order to discuss solutions of the problems listed above.

Preparatory reading

As a preparation for the seminar it will be useful to consult the following material:

[1] A. Björner, *Subspace arrangements.* in Proc. of the First European Congress of Mathematics (Paris 1992), vol. I, Birkhäuser, 1994, 321–370. (Sec. 1 to 7)

[2] M. Farber, *Invitation to Topological Robotics.* Zürich Lectures in Advanced Mathematics, EMS 2008. (Ch. 1)

[3] E. R. Fadell and S. Y. Husseini, *Geometry and Topology of Configuration Spaces.* Springer Monographs in Mathematics (2001). (Parts I and II).

[4] R. Fox and L. Neuwirth, *The braid groups.* Math. Scandinavica, 10 (1962), 119–126. (Sec. 3 and 4)

Additional references

[1] A. Björner, G. M. Ziegler, *Combinatorial stratification of complex arrangements*, J. Amer. Math. Society 5 (1992), 105–149.

[2] P. Blagojević, G. M. Ziegler, *Convex equipartitions via equivariant obstruction theory.* Israel J. Math. 200 (2014), 49–77.

[3] P. Blagojević, W. Lück, G. M. Ziegler, *Equivariant Topology of Configuration Spaces.* J. Topology, to appear; arXiv:1207.2852.

[4] P. Blagojević, W. Lück, G. M. Ziegler, *On highly regular embeddings.* Trans. Amer. Math. Soc., to appear; arXiv:1305.7483.

[5] P. Blagojević, F. Cohen, W. Lück, G. M. Ziegler, *On highly regular embeddings, II.* arXiv:1410.6052.

[6] C.-F. Bödigheimer, F. Cohen, L. Taylor, *On the homology of configuration spaces.* Topology 28 (1989), 111–123.

[7] C.-F. Bödigheimer, *Stable splittings of mapping spaces.* Algebraic Topology, Proc. Seattle (1985), Springer Lecture Notes in Math. 1286 (1987), 174–187.

[8] F. R. Cohen, *The homology of C_{n+1}-spaces, $n \geq 0$.* Part of "The Homology of Iterated Loop Spaces", Springer Lecture Notes in Math. 533, (1976), 207–351.

[9] M. Farber, *Topology of billiard problems. I, II.* Duke Math. J. 115 (2002), 559–585, 587–621.

[10] M. Farber, S. Tabachnikov, *Periodic trajectories in 3-dimensional convex billiards.* Manuscripta Math. 108 (2002), 431–437.

[11] M. Farber, S. Tabachnikov, *Topology of cyclic configuration spaces and periodic trajectories of multi-dimensional billiards.* Topology 41 (2002), 553–589.

[12] M. Farber, J.-C. Hausmann, D. Schütz, *On the conjecture of Kevin Walker.* J. Topol. Anal. 1 (2009), 65–86.

[13] C. Giusti, D. Sinha, *Fox-Neuwirth cell structures and the cohomology of symmetric groups.* arXiv:1110.4137.

[14] P. May, *The geometry of iterated loop spaces.* Springer Lecture Notes in Math. 271, (1972).

[15] P. May, *The homology of E_∞ spaces.* Springer Lecture Notes in Math. 533, (1976), 1–68.

[16] D. Schütz, *The isomorphism problem for planar polygon spaces.* J. Topol. 3 (2010), 713–742.