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Mathematical Aspects of General Relativity

Organised by
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Abstract. The general theory of relativity is a remarkably accurate theory of gravitation, describing phenomena from the level of isolated bodies to the universe as a whole. The mathematical study of this theory leads to fascinating problems connecting the areas of partial differential equations, geometry and topology with physics. The talks of the workshop illustrated the rapid progress in this subject over the last few years.

Mathematics Subject Classification (2010): 83-XX,35-XX,53Cxx.

Introduction by the Organisers

The workshop *Mathematical Aspects of General Relativity* was organized by Mihalis Dafermos (Cambridge/Princeton), Jim Isenberg (Eugene) and Hans Ringström (Stockholm). The almost 50 participants represented a wide selection of different research areas connected to the general theory of relativity, and roughly half of them gave talks at the workshop.

Black holes have long been at the centre of interest in general relativity. Two of the most fundamental open problems in the subject are the question of the *exterior* stability of the classical vacuum black hole solutions (Schwarzschild and Kerr) and the question of the *interior* structure of generic black hole solutions. The latter is related to the question of *strong cosmic censorship*, originally formulated by Penrose.

In the previous meeting of this series, definitive results concerning the property of the wave equation on the general subextremal Kerr background were presented. Interest has moved to the actual problem of linearised gravity. In his talk, Holzegel presented a proof of the stability of Schwarzschild in linearised gravity and gave an overview of the current status quo for the problem of black hole stability.

The extremal case has continued to attract much interest, after the discovery of the Aretakis instability, discussed already at the last meeting of the series. New results were presented by Aretakis concerning stability properties of semilinear equations on extremal Reissner–Nordström, which become quite subtle in view of the non-decay of transversal derivatives. His talk also presented a general characterization of null hypersurfaces admitting conservation laws for the wave equation.

Concerning the interior structure of black holes, Luk presented the theorem that any dynamical vacuum black hole which settles down to Kerr in the exterior will have a piece of Cauchy horizon in its interior across which the metric is continuously extendible. In particular, if the Kerr family is indeed proven stable in the exterior, this will falsify some original formulations of strong cosmic censorship which required that for generic initial data, the maximal Cauchy development be inextendible as a continuous Lorentzian metric. Interestingly, the latter inextendibility problem was not previously known even for the case of Schwarzschild; a proof of this was in fact presented in the talk of Sbierski.

Trapped null geodesics play an important role in the black hole stability problem. It is interesting to know whether Schwarzschild is in some sense uniquely characterized by the structure of its photon sphere of trapped null geodesics. The talk of Cederbaum presented such a result.

A problem intimately related to stability issues is the problem of excluding non-trivial "time periodic" solutions of the Einstein equations. Previously, it has been shown that under the assumption of analyticity, such solutions must necessarily be stationary. Schlue presented some recent results where the assumption of analyticity is dropped.

Again on the theme of stability, Huneau in her talk returned to the classical problem of stability of Minkowski space, but in the context where spacetime is assumed to have a translational symmetry. The data are thus not asymptotically flat in the traditional sense, and the monumental theorem of Christodoulou and Klainerman does not apply. Huneau presented a proof of "almost global existence", using a modified harmonic gauge.

Finally, the study of the problem of black hole stability has inspired renewed interest in understanding the propagation of waves on simpler examples of spacetimes without trapping or event horizons. Tataru presented some upcoming results which in particular characterize the spectral obstructions to decay properties on such spacetimes.

The first step in studying solutions of Einstein's equations via the Cauchy problem is to understand the parametrization and the construction of sets of initial data which satisfy the Einstein constraint equations. The conformal method has long been recognized to be an important tool for carrying out such analyses, and there have been talks regarding the successes and the limitations of its application in all six of the Oberwolfach meetings on mathematical relativity. However, there is still much to learn about what the conformal method can and cannot do, and what might replace it in studying those solutions of the constraints for which this method is ineffective.

Allen's talk considered a class of initial data sets for which the conformal method is effective. He discussed constant mean curvature (CMC) sets which are asymptotically hyperbolic and which are shear-free (an asymptotic condition which is needed for evolving data into spacetime solutions with well-defined null infinity structure). His talk introduced a generalized notion of weakly asymptotically hyperbolic data, and showed that the conformal method readily constructs and parametrizes CMC shear-free data sets of this sort which satisfy the constraints. In Dilts' talk, the focus was on asymptotically Euclidean (AE) data sets. He described his recent results with Maxwell which elucidate the Yamabe classes for AE geometries, and used these results together with the conformal method to study CMC and near-CMC solutions.

While the conformal method has generally been very useful in working with CMC and near-CMC data sets, its success to date in working with far-from-CMC data sets has been much more limited. Holst discussed some of the far-from-CMC cases in which the conformal method has proven to be useful, and noted the techniques used for proving these results. He also noted the observation of Ngo, which shows that in fact these far-from-CMC results can be obtained from near-CMC results by a scaling argument. On the other hand, Nguyen's talk presented new results which do seem to indicate far-from-CMC cases in which the conformal method leads to solutions of the constraints. These results are an extension of the limit equation criterion of Gicquaud and his collaborators. Besides obtaining far-from-CMC cases in which the conformal method produces solutions, Nguyen's also finds other far-from-CMC cases in which the conformal method produces no solutions, and still others in which multiple solutions are obtained. The conformal method is expected to be ineffective for studying general far-from CMC initial data sets, and Maxwell has begun to develop ideas for going beyond the conformal method. Some of these ideas were presented in his talk.

Apart from constructing and parametrizing initial data, one would like to understand its global properties. While there has been considerable work on the total mass and energy-momentum of AE data sets, much less is understood about angular momentum. Based on some of his work on quasi-local quantities, Wang discussed in his talk some new ideas on defining angular momentum, both globally and quasi-locally, and illustrated some of the desirable properties of his new definitions.

Though not directly related to general relativity, geometric heat flows have been very successfully used as analytic tools for studying general relativistic problems such as the Penrose inequality relating the mass of AE initial data sets and the areas of black holes contained in such data sets. In Huisken's talk, he discussed some of his recent new ideas concerning mean curvature flow; in particular, he discussed a comprehensive approach for evolving past singularities in such flows on AE 3 dimensional geometries using surgery.

The solutions to Einstein's equations that are used to model the universe are exactly spatially homogeneous and isotropic. However, it is clear that there are spatial inhomogeneities in the universe. It is therefore of interest to analyze the effect of the inhomogeneities on the solution (the so-called backreaction). Robert Wald presented a result giving quite general conditions ensuring that the backreaction, if it occurs, behaves like matter satisfying the weak energy condition. This is an indication that the effects of dark energy cannot be obtained as a result of backreaction, contrary to what has been suggested by several authors. Piotr Bizon described new results on the instability of anti-de Sitter space. In particular, he presented a simplified system (the so-called resonant system), which models the behaviour of solutions to the spherically symmetric, five dimensional Einstein-massless scalar field system in the case of a negative cosmological constant. Bizon's analysis yields the conclusion that the resonant system develops an oscillatory singularity in finite time. Since the actual system of interest is well modelled by the resonant system, this gives an indication of how to proceed to prove instability in the anti-de Sitter setting. In higher dimensions, the argument justifying the introduction of Einstein's field equations admit a wider range of equations, the so-called Lovelock theories of gravity. Harvey Reall presented results concerning the mathematical properties of these theories. In particular, he discussed the issues of causality, hyperbolicity and the occurrence of shock formation in these theories. Frans Pretorius reviewed recent progress that has been made in the subject of ultra-relativistic collisions, in particular in the context of formation of black holes.

One ingredient in the current models of the universe is dark matter. One important justification for its existence is based on studies of rotation curves for galaxies. It is therefore of interest to obtain good models for galaxies, in particular for their rotation curves. In his talk, Håkan Andréasson described models for disc galaxies. In particular, he presented a large class of flat axially symmetric solutions to the Vlasov-Poisson system with flat rotation curves, something which was previously not expected in the absence of dark matter. In many contexts, it is of interest to model matter using kinetic theory. However, the methods normally used in dealing with the resulting equations in the non-general relativistic setting are sometimes not so appropriate in the general relativistic setting. Jacques Smulevici described how to apply vector field methods to transport equations in order to obtain a perspective more suited to generalization. Describing how a free boundary of a fluid evolves over time is of great interest in astrophysics, where such a model can be used to describe, e.g., a star. However, controlling the behaviour at the boundary is quite delicate. Todd Oliynyk presented a priori estimates for solutions to the equations describing a relativistic liquid body. Deriving such estimates is the first step in obtaining a local existence theory for the relevant system of equations. In the study of cosmological singularities, it has long been conjectured that "matter doesn't matter". This is a conjecture which comes with a few caveats; certain matter models do matter, and they have to be excluded. However, in the case of the Einstein-Euler system, there is a wide range of equations of state for which the matter is expected not to matter. Florian Beyer presented a construction of a family of solutions to the Einstein-Euler system for which the conjecture holds. Peter Hintz presented global existence results for solutions to quasi-linear wave equations on Kerr-de Sitter spacetimes. Some ingredients of the argument were a compactification of the spacetime at future infinity, microlocal analysis and a Nash-Moser iteration.

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Workshop: Mathematical Aspects of General Relativity

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Abstracts

C^0 stability of the Kerr Cauchy horizon

Jonathan Luk
(joint work with Mihalis Dafermos)

The two-parameter family of Kerr spacetimes are explicit solutions to the Einstein vacuum equations

$$Ric(g) = 0$$

which are parametrized by (M, a), the mass and the specific angular momentum respectively. For $|a| < M$, these spacetimes contain black hole regions, which are not connected to future null infinity via causal curves. The exterior regions, i.e., the complement of the black holes, of these spacetimes are widely conjectured to be nonlinearly dynamically stable.

It turns out that the interior regions of the black holes are very different for the Schwarzschild case (when $a = 0$) and for the case with non-vanishing specific angular momentum (i.e., $0 < |a| < M$). In the former case, the maximal globally hyperbolic development of Schwarzschild data terminates with a spacelike singularity, and the spacetime is in fact inextendible as a Lorentzian manifold with a continuous metric [7]. On the other hand, when $0 < |a| < M$, the maximal globally hyperbolic development of Kerr data terminates with a smooth Cauchy horizon. As a consequence, the spacetime is extendible non-uniquely(!) as a smooth solution to the Einstein vacuum equations. Partly due to this unsettling breakdown of determinism, Penrose conjectured that this behavior is non-generic:

Conjecture 1 (Strong cosmic censorship conjecture)**.** *For generic asymptotically flat initial data for the Einstein vacuum equations, the maximal globally hyperbolic development is inextendible as a suitably regular Lorentzian manifold.*

In particular, if the strong cosmic censorship conjecture is true, then the aforementioned breakdown of determinism associated to the smooth Cauchy horizon should be unstable against small perturbations of Kerr spacetime. Moreover, it is often expected that a perturbation of Kerr initial data would moreover give rise to a "Schwarzschild-type singularity", which motivates the following formulation of the conjecture:

Conjecture 2 (C^0 formulation of the strong cosmic censorship conjecture)**.** *For generic asymptotically flat initial data for the Einstein vacuum equations, the maximal globally hyperbolic development is inextendible as a Lorentzian manifold with a continuous metric.*

In a joint work with Mihalis Dafermos, we study the question of the stability of the Kerr Cauchy horizon (without any symmetry assumptions), posed as a characteristic initial value problem where the initial data are perturbations of the Kerr event horizons. We show in particular that in this setting, there are no "Schwarzschild type singularities". More precisely, we have the following theorem:

Theorem 1 (Dafermos-L. [3, 4, 5]). *Consider the characteristic initial value problem on two affine complete null hypersurfaces $\mathcal{H}_A^+$ and $\mathcal{H}_B^+$ transversely intersecting at a 2-sphere S_0. Let (M_A, a_A), (M_B, a_B) be such that $0 < |a_A| < M_A$ and $0 < |a_B| < M_B$. Suppose the data on both $\mathcal{H}_A^+$ and $\mathcal{H}_B^+$ are smooth and approach the event horizons of the Kerr spacetimes with parameters (M_A, a_A) and (M_B, a_B) respectively at a sufficiently fast rate. Assume moreover that the data are everywhere close to that of the Kerr spacetime with parameters[1] (M_A, a_A). Then the maximal globally hyperbolic development to the data on $\mathcal{H}_A^+ \cup \mathcal{H}_B^+$ has a bifurcate Cauchy horizon across which the metric is continuously extendible. Moreover, the metric is everywhere C^0-close to the Kerr metric with parameters (M_A, a_A).*

An analogue of this theorem is previously known for the Einstein-Maxwell-scalar field system in *spherical symmetry* for characteristic initial data approaching the event horizon of the Reissner-Nordström spacetime [1, 2]. In contrast, Theorem 1 holds without any symmetry assumptions.

On one hand, the proof of Theorem 1 uses estimates in studying "weak null singularities" in [6]. This is because (see Conjecture 3 below) the spacetime may potentially be singular near the Cauchy horizon and we need to capture the null structure of the system of equations in order to control the solution in this low regularity setting. On the other hand, the proof also draws on insights obtained in studying the linear wave equation in the interior of the Kerr black hole. In particular, in order to close the argument, it is important to prove an integrated local energy decay estimate.

We remark that while we have not spelt out the precise decay rates in Theorem 1 in this abstract, they are consistent with (and in fact much weaker than) the expected rate of approach for spacetimes arising from small perturbations of the Cauchy data of Kerr spacetime. Thus, if the exterior region of the Kerr spacetime is indeed stable as is widely expected, then our theorem gives an open set of initial data whose maximal globally hyperbolic developments are extendible with continuous Lorentzian metrics. We therefore obtain the following corollary:

Corollary 1. *If the exterior region of the Kerr spacetime is stable (with quantitative decay rates), then the maximal globally hyperbolic developments of small perturbations of the 2-ended Kerr initial data have bifurcate Cauchy horizons across which the metrics are continuously extendible. In particular, the C^0-formulation of the strong cosmic censorship conjecture is false.*

Moreover, the proof of the main theorem can also be "localized" near the event horizon for data which asymptotes to Kerr but are not necessarily close to Kerr everywhere. In this case, one does not get a global Cauchy horizon, but can still guarantee that there is a non-trivial component of the Cauchy horizon "sufficiently near timelike infinity". This applies for instance to spacetimes which are far away from Kerr but approach Kerr in the exterior region, as is expected to occur in astrophysical systems.

[1] This in particular demands that (M_A, a_A) and (M_B, a_B) are close to each other.

Corollary 2. *Consider the characteristic initial value problem with two smooth null hypersurfaces* $\mathcal{H}^+$ *and* $\underline{H}$ *intersecting on a 2-sphere* S_0*. Assume that* $\mathcal{H}^+$ *is affine complete and the data on* $\mathcal{H}^+$ *approach that of the Kerr event horizon sufficiently fast. Suppose that* $\underline{L}'$ *is a geodesic null generator of* $\underline{H}$ *and for* ϵ *sufficient small define* $\underline{H}' \subset \underline{H}$ *to be* $\underline{H}' := \cup_{\tau\in[0,\epsilon]}\varphi_\tau(S_0)$*, where* φ_τ *is the one-parameter family of diffeomorphisms generated by* $\underline{L}'$*. Then there exists* $\epsilon > 0$ *sufficiently small such that the causal future of* $\mathcal{H}^+\cup\underline{H}'$ *has a non-trivial component of the Cauchy horizon across which the metric is continuously extendible.*

While our main theorem guarantees that the metric can be extended continuously beyond the Cauchy horizon, it does not give much further information regarding the regularity/singularity of the Cauchy horizon as the null boundary of the maximal globally hyperbolic development. However, the proof of the theorem, which requires the use of estimates that degenerate at the Cauchy horizon, suggests that for generic perturbations of Kerr spacetime, the Cauchy horizons may in fact be "weak null singularities". This motivates the following conjecture:

Conjecture 3. *For a generic subclass of perturbations in Theorem 1, the maximal globally hyperbolic development is inextendible with a Lorentzian metric in* $C^0 \cap W^{1,2}_{loc}$*.*

If this is indeed the case, then any continuous extensions of the metric (which exist by Theorem 1) cannot be made sense of as a weak solution to the Einstein vacuum equations. In particular, one can hope that the following weaker formulation of the strong cosmic censorship conjecture, due to Christodoulou, may still hold:

Conjecture 4. *For generic asymptotically flat initial data for the Einstein vacuum equations, the maximal globally hyperbolic development is inextendible as a weak solution to the Einstein vacuum equations.*

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Ultra-relativistic Collisions and Black hole formation

Frans Pretorius

I will review what has been learnt in recent years about the ultra-relativistic collision problem in classical general relativity. This area has connections to Super-Planck scale particle collisions and the interaction of gravitational shock waves, and addresses questions pertinent to cosmic censorship, black hole formation and the Hoop conjecture. I will discuss some open questions and problems for future research.

Mean curvature flow with surgery in asymptotically flat 3-manifolds

Gerhard Huisken
(joint work with Simon Brendle)

Spacelike maximal slices of Lorentzian manifolds modelling isolated gravitating systems that satisfy the weak energy condition lead to asymptotically flat Riemannian 3-manifolds $(N^3, \bar{g})$ of non-negative scalar curvature $R \geq 0$. We assume that $(N^3, \bar{g})$ is an *exterior domain*, i.e. the exterior of some compact set in N^3 is diffeomorphic to the complement of a ball in R^3 s via a coordinate chart $\{x^i\}$ where the metric and its derivative satify suitable decay conditions towards the Euclidean metric and the boundary $\partial N^3 = \dot{\bigcup}_{1\leq L} \Sigma_i^2$ consists of finitely many minimal surfaces Σ_i. We assume that there are no other closed minimal surfaces contained in $(N^3, \bar{g})$, such that ∂N^3 may be interpreted as the outermost horizon of the isolated system. It is known that the condition $R \geq 0$ ensures that all components of the boundary are topological spheres.

The lecture outlines recent work by Simon Brendle and the author on mean curvature flow with surgery of embedded, meanconvex 2-surfaces in Riemannian 3-manifolds in [6], [1]. Smooth mean curvature flow is the quasi-linear parabolic evolution system

$$\frac{d}{dt}F = \overrightarrow{H}$$

for the position of an evolving hypersurface, $\overrightarrow{H}$ being the mean curvature vector. "Surgery" is a precise quantitative algorithm removing small cylindrical necks from the surface just before a singularity occurs, thereby reducing the curvature by a large factor and allowing a continuation of the flow after surgery. When applying the results of [6], [1] in an *exterior domain* as desribed above we obtain in particular:

Starting from any large coordinate sphere $M_0^2 = \partial B_r(0)$ in an exterior domain $(N^3, \bar{g})$ (or indeed from any embedded, meanconvex hypersurface that is homologous to the coordinate spheres near infinity), there exists a mean curvature flow $M_t^2, 0 \leq t < T \leq \infty$ with at most finitely many surgeries sweeping out the region interior to the initial surface. If N^3 has no boundary, the flow becomes extinct in finite time $T < \infty$, otherwise the flow will become smooth after some time $T_0 < \infty$ and converges smoothly to ∂N^3 as $t \to \infty$.

The proof is based on a priori estimates for the geometry of the evolving hypersurfaces that allow classification and quantitative control of all singularities. In particular we use a convexity estimate from [7], non-collapsing estimates from [2] and a gradient estimate from [3]. In addition we prove a pseudo-locality estimate and a self-improvement estimate for evolving necks to enable the surgery algorithm from [7].

As the scale at which surgery is performed tends to zero, the number of surgeries may tend to infinity. It was shown however in [5] and [4] that in the limit these flows with surgery converge to the level-set (weak) solution of mean curvature flow. This information is combined with the regularity theory of meanconvex mean curvature flow due to B. White [9] to prove large time regularity and convergence to weakly stable minimal surfaces of the flow with surgery.

If the initial data of the flow tend to the sphere at ∞, an ancient solution to mean curvature flow as in [8] will result.

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Quasi-local angular momentums and their limits at infinity

Mu-Tao Wang

(joint work with Po-Ning Chen, Shing-Tung Yau)

The notion of angular momentum is of most fundamental importance in any branch of physics. However, there have been great difficulties in finding physically acceptable definition of this concept in general relativity except for a few cases. In this talk, I introduced new definitions of angular momentum at both the quasi-local and total level [1, 2]. The construction was based on previous work on quasi-local mass and optimal isometric embeddings [4, 5], which anchors the reference system. The new definition of total angular momentum satisfies desirable properties such as tge invariance in the Kerr spacetime and the conservation along the Einstein equation. At last, a new Bondi type mass loss formula and a new definition of total angular momentum were introduced in the asymptotically hyperbolic setting [3].

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Volumetric Momentum and the Conformal Method

David Maxwell

The conformal method is the workhorse tool for generating solutions of the Einstein constraint equations. Despite its effectiveness in generating constant mean curvature solutions of the Einstein constraint equations, there is a growing body of evidence that even in vacuum and without a cosmological constant, the method is not an effective parameterization in the far-from CMC regime [2][1]. Hence we are lead to examine alternative possibilities, starting with the CMC conformal method as a starting point. The work presented here is is part of a program [3][4][5] aimed at finding an alternative.

As a first step to doing this, it is helpful to frame the conformal method in terms that emphasize the conformal geometry and deemphasize the role played by any particular representative of a conformal class. To describe this, suppose M^n is a compact manifold, and let α be a fixed volume form on M, which we will call the volume gauge. Now suppose (g_{ab}, K_{ab}) are a metric and second fundamental form on M. These determine unambiguously a conformal class $[g_{ab}]$ and a mean curvature $\tau = g^{ab}K_{ab}$. These are two of the parameters of the conformal method; the third is a little more involved to describe and depends on the choice of volume gauge α.

A *conformal momentum* on M is an equivalence class of pairs (g_{ab}, σ_{ab}) where g_{ab} is a metric and σ_{ab} is transverse traceless with respect to g_{ab}, i.e. σ_{ab} is symmetric, trace-free, and divergence-free with respect to g_{ab}. If ϕ is a positive function then

$$(g_{ab}, \sigma_{ab}) \sim (\phi^{q-2} g_{ab}, \phi^{-2} \sigma_{ab}) \tag{1}$$

where $q = 2n/(n-2)$ is the critical Sobolev exponent for dimension n. Conformal momenta can be thought of as elements of the cotangent space of the set of conformal classes, modulo diffeomorphisms.

The conformal momentum of (g_{ab}, K_{ab}) is computed as follows. Let A_{ab} be the trace-free part of K_{ab} and use York splitting to decompose

$$A_{ab} = \sigma_{ab} + \frac{1}{2N}(\mathbb{L}W)_{ab} \tag{2}$$

where σ_{ab} is transverse traceless, W^a is a vector field, $\mathbb{L}$ is the conformal Killing operator, and where N is the ratio of the metric volume form ω_g with the volume gauge α, so $N = \omega_g/\alpha$. The conformal momentum of (g_{ab}, K_{ab}) measured by α is the equivalence class of (g_{ab}, σ_{ab}).

The conformal method can then be framed as follows. Given a conformal class $\mathbf{g}$, a conformal momentum $\boldsymbol{\sigma}$, and a mean curvature τ, find a solution of the vacuum Einstein constraint equations $(\overline{g}_{ab}, \overline{K}_{ab})$ such that

(i) $[\overline{g}] = \mathbf{g}$.
(ii) The conformal momentum of $(\overline{g}_{ab}, \overline{K}_{ab})$ measured by α is $\boldsymbol{\sigma}$.
(iii) $\overline{g}^{ab}\overline{K}_{ab} = \tau$.

When τ is constant, the conformal method is effective; in all but certain non-generic and easily detected cases, a conformal data set $(\mathbf{g}, \boldsymbol{\sigma}, \tau)$ generates exactly one solution of the constraints. In the far-from CMC regime, things are poorly understood, and there is a lack of examples to aid generating good conjectures. In [4] we examined how the the conformal method parameterizes $(S^1)^{n-1}$ symmetric solutions of flat Kasner spactimes, and were able to completely describe the situation in the far from CMC setting. Specifically, we examined data of the form $(\mathbf{g}_0, \mu\boldsymbol{\sigma}_0, \tau)$ where $\mathbf{g}_0$ is the conformal class of the flat metric on the torus, μ is a constant, $\boldsymbol{\sigma}_0$ is a certain conformal momentum, τ is an arbitrary function of one factor S^1, and the volume gauge α similarly depends only on the single factor. In most settings, such conformal data determines exactly one $(S^1)^{n-1}$ symmetric slice of a flat Kasner spacetime. However, if $\mu = 0$ then no solution is generated, except for certain mean curvatures all satisfying

$$\tau_* = \frac{\int \tau \overline{N} \omega_{\overline{g}}}{\int \overline{N} \omega_{\overline{g}}} = 0 \tag{3}$$

where $\overline{g}_{ab}$ is the solution metric and $\overline{N} = \omega_{\overline{g}}/\alpha$. The difficulty with this condition is that it is hard to detect from $(\mathbf{g}_0, \mu\boldsymbol{\sigma}_0, \tau)$, and one must essentially know the solution metric $\overline{g}_{ab}$ before one can verify if $\tau_* = 0$ or not.

It turns out that τ_* can be interpreted (up to multiplication by a dimensional constant) as a kind of momentum for volume forms completely parallel to conformal momentum. The *volumetric momentum* of (g_{ab}, K_{ab}) measured by α is $-2\frac{n-1}{n}\tau_*$ where we decompose

$$\tau = \tau_* + \frac{1}{N}\mathrm{div} V \tag{4}$$

for some vector field V^a. The one-parameter families of solutions for the flat Kasner spacetimes are then signaled by the condition that both the conformal and the volumetric momentum vanish. This suggests seeking alternatives to the conformal method that prescribe conformal and volumetric momentum directly, and efforts in this direction can be found in [5].

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New evidence for the instability of the anti-de Sitter spacetime

Piotr Bizoń
(joint work with Maciej Maliborski and Andrzej Rostworowski)

In [1] we gave numerical evidence that anti-de Sitter (AdS) spacetime is unstable against black hole formation for a large class of arbitrarily small perturbations. More precisely, we showed that for a perturbation with amplitude ε a black hole forms on the timescale $\mathcal{O}(\varepsilon^{-2})$. Using nonlinear perturbation analysis we conjectured that the instability is due to the turbulent cascade of energy from low to high frequencies. Since the computational cost of numerical simulations rapidly increases with decreasing ε, our conjecture was based on extrapolation of the observed scaling behavior of solutions for small (but not excessively so) amplitudes, which left some room for doubts whether the instability will persist to arbitrarily small values of ε. In my talk I described the new work [5] in which we validated and reinforced the above extrapolation with the help of a recently proposed resonant approximation [2, 3, 4]. The key feature of this approximation is that the underlying infinite dynamical system (referred to as the resonant system) is scale invariant: if its solution with amplitude 1 does something at time t, then the corresponding solution with amplitude ε does the same thing at time t/ε^2. Moreover, the latter solution remains close to the true solution (starting with the same initial data) for times $\lesssim \varepsilon^{-2}$ (provided that the errors due to omission of higher order terms do not pile up too rapidly). Thus, by solving the resonant system we were able to probe the regime of arbitrarily small perturbations (whose outcome of evolution is beyond the possibility of numerical verification).

Using the analyticity strip method [6] we showed that for typical initial data the solution of the resonant system develops an oscillatory singularity in finite time. We also gave numerical evidence that this solution acts as a universal attractor for blowup. This result hints at a possible route to establishing instability of AdS under arbitrarily small perturbations. The key open question is how to transfer this blowup result from the resonant system to the full system. Nonetheless, the fact that solutions of the resonant system blow up in finite time (for typical initial data) strongly indicates that the corresponding solutions of the full system collapse on the timescale $\mathcal{O}(\varepsilon^{-2})$.

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Linear Stability of the Schwarzschild Metric under Gravitational Perturbations

Gustav Holzegel

(joint work with Mihalis Dafermos, Igor Rodnianski)

The two-parameter Kerr-family of axisymmetric stationary black hole solutions to the vacuum Einstein equations,

$$Ric\,[g] = 0 \tag{1}$$

for a Lorentzian metric g, plays a central role in general relativity. Establishing its non-linear stability, however, remains a fundamental open problem. Important progress has been made in the past ten years by considering the linear scalar wave equation on fixed Kerr backgrounds [6, 1, 10, 7]. In particular, the subtle interplay of the physical phenomena of trapping, superradiance and the redshift and their role for the dispersion of waves on black hole spacetimes has now been mathematically fully understood [7].

In this talk, we move away from the scalar wave equation and consider the actual linearization of the vacuum Einstein equations (1) with respect to the Schwarzschild metric, which is a one-parameter subfamily of the Kerr family. Clearly, this will form a key ingredient for the non-linear stability problem. More specifically, we linearize the following equations which fully capture the analytical content of (1):

$$div_g W = 0 \quad \text{the Bianchi equations for the Weyl tensor } W, \tag{2}$$

(3) $\nabla\Gamma + \Gamma\Gamma = W$ the structure equations for the connection coefficients Γ.

Linearizing (2) and (3) with respect to Schwarzschild leads to a complicated coupled system for the linearized dynamical fields $\left(\Gamma^{(1)}, W^{(1)}\right)$. A particularly tractable form of the equations is obtained if (2) and (3) are first decomposed with respect to a null-frame as in [3] and then linearized. We call the resulting system the system of gravitational perturbations.[1] It is easily seen to be well-posed for general asymptotically flat characteristic initial data on a double null cone.[2]

We collect three key insights before stating our main theorem, which will assert that the Schwarzschild metric is linearly stable.

(I) The diffeomorphism invariance of the full non-linear theory reflects itself in special "pure gauge" solutions of the linearized system. These can be identified and understood explicitly. Adding them to a solution may be thought of as the same solution expressed in different coordinates.

(II) The fact that the Schwarzschild family embedds as a subfamily into the larger Kerr family is reflected in the fact that there are special solutions of the linearized system describing evolution to "linearized Kerr". Again, these solutions can be identified (in fact at the level of initial data) and understood explicitly.

(III) There exists gauge-invariant quantities (i.e. quantities remaining invariant upon adding a solution from (I)) of the system of gravitational perturbations which satisfy decoupled (wave) equations. This has long been known for the null-curvature components α and $\underline{\alpha}$ (the extremal complex Newman-Penrose scalars) but it was not known whether solutions to the Bardeen-Press equations satisfied by them remain uniformly bounded.

We are now ready to give an informal version of the main theorem:

Theorem 1. *Consider a solution of the equations of linearised gravity around Schwarzschild arising from general asymptotically flat characteristic initial data on a double null cone.*

(i) *Quantitative boundedness and inverse polynomial decay holds for the gauge invariant quantities, in fact for general solutions of the Teukolsky equations.*

(ii) *In a gauge determined by initial data, all quantities of the system of linearised gravity remain bounded by a constant times their initial values.*

(iii) *In a gauge determined by the future, i.e. after addition of a pure gauge solution normalised to the event horizon behaviour, all quantities of the system decay inverse polynomially to a member of the 4-dimensional family*

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[1] This system is closely related to gravitational perturbations studied in the Newman-Penrose formalism in the physics literature [2]. In general, there is a large physics literature studying gravitational perturbations starting with the work of [9], which is however restricted mostly to the study of individual modes. In particular, no statement of uniform boundedness or decay of general perturbations is known.

[2] The choice of characteristic data is merely for convenience, in particular to simplify the constraint equations.

of standard linearised Kerr solutions. The final linearised Kerr solution can be read off from initial data and the pure gauge solution normalised to the future is bounded by initial data.

We conclude with a few words about the proof. Details can be found in [5]. The first important observation is the existence of a second order transformation of the aforementioned curvature components α and $\underline{\alpha}$, which transforms solutions of the Bardeen-Press equation into solutions of the well-known Regge-Wheeler equation [4].[3] On the other hand, it is well etablished (see e.g. [8]) how one can obtain boundedness and decay for solutions to the Regge-Wheeler equations exploiting the geometric insights for the scalar wave equation.[4] Once decay for the transformed quantities has been established one can use transport equations to prove decay for α and $\underline{\alpha}$ themselves. This yields (i) in the main theorem.

To obtain (ii), the basic technique is to integrate the (linearized) null-structure and Bianchi equations as transport equations. Care has to be taken with respect to the weights appearing near the horizon and null-infinity. In particular, the redshift effect needs to be exploited.

Finally, for (iii) one needs to construct, from the dynamical solution, a pure gauge solution which forces all geometric quantities in the new gauge to decay. Remarkably, this can be done by solving an ODE (for the linearized lapse) along the event horizon. The estimates obtained in (ii) are sufficiently strong to guaranteee boundedness of the pure gauge solution that has been added.

Final Remark. While one needs to generalize Theorem 1 to gravitational perturbations of the Kerr family in order to address the full non-linear stability problem, there is an interesting non-linear setting which can already be studied in the context of Theorem 1: The evolution of axisymmetric initial data with zero angular momentum is expected to converge to a member of the Schwarzschild family, hence the linearization in Theorem 1 is in principle sufficient. Work on this problem is currently in progress.

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Backreaction in Cosmology

Robert M. Wald
(joint work with Stephen Green)

My talk reviewed work with Stephen Green [1]-[4] on backreaction by small scale inhomogeneities in cosmology. The context for this phenomenon concerns a situation where the actual spacetime metric g_{ab} has large curvature fluctuations on small scales but, nevertheless, g_{ab} can be well approximated by a metric $g^{(0)}_{ab}$ that does not have large curvature fluctuations. Although our analysis is valid in a much more general context, the main situation we have in mind is where g_{ab} is the actual metric of the universe and $g^{(0)}_{ab}$ is a metric with FLRW symmetry.

The issue at hand in whether the small scale inhomogeneities of g_{ab} can contribute nontrivially to the dynamics of $g^{(0)}_{ab}$. A priori, this is possible even though $\gamma_{ab} \equiv g_{ab} - g^{(0)}_{ab}$ is assumed to be small: Einstein's equation for g_{ab} contains *derivatives* of γ_{ab}, which need not be small even when γ_{ab} is small. Consequently, the Einstein tensor, G_{ab}, of g_{ab} need not be close to the Einstein tensor, $G^{(0)}_{ab}$, of $g^{(0)}_{ab}$. Thus, although g_{ab} is a assumed to be an exact solution of Einstein's equation with some stress-energy source T_{ab}, it is possible that $g^{(0)}_{ab}$ may not be close to a solution to Einstein's equation with a suitably averaged stress-energy source $T^{(0)}_{ab}$. If this occurs, we say that there is a substantial *backreaction* effect of the small scale inhomogeneities on the effective dynamics of $g^{(0)}_{ab}$.

In order to analyze such backreaction effects, we needed an approximation scheme where γ_{ab} may be assumed to be small but derivatives of γ_{ab} need not be small. We developed such a scheme by adopting to the non-vacuum case a framework proposed by Burnett [5], which itself is a mathematically precise version of Isaacson's [6, 7] treatment of backreaction of gravitational radiation. Our framework considers a one-parameter family of metrics $g_{ab}(\lambda)$ that satisfies the following assumptions:

(i) Einstein's equation holds for all $\lambda > 0$, i.e., we have

$$G_{ab}(g(\lambda)) + \Lambda g_{ab}(\lambda) = 8\pi T_{ab}(\lambda), \tag{1}$$

where $T_{ab}(\lambda)$ satisfies the weak energy condition, i.e., for all $\lambda > 0$ we have $T_{ab}(\lambda)t^a(\lambda)t^b(\lambda) \geq 0$ for all vectors $t^a(\lambda)$ that are timelike with respect to $g_{ab}(\lambda)$.

(ii) There exists a smooth positive function $C_1(x)$ on M such that

$$|\gamma_{ab}(\lambda, x)| \leq \lambda C_1(x), \tag{2}$$

where $\gamma_{ab}(\lambda, x) \equiv g_{ab}(\lambda, x) - g_{ab}(0, x)$.

(iii) There exists a smooth positive function $C_2(x)$ on M such that

$$|\nabla_c \gamma_{ab}(\lambda, x)| \leq C_2(x). \tag{3}$$

(iv) There exists a smooth tensor field μ_{abcdef} on M such that

$$\text{wlim}_{\lambda \to 0} \left[\nabla_a \gamma_{cd}(\lambda) \nabla_b \gamma_{ef}(\lambda)\right] = \mu_{abcdef}, \tag{4}$$

where "wlim" denotes the weak limit.

Assumptions (i)–(iv) allow us to rigorously derive an equation by $g^{(0)}_{ab}$ of the form of Einstein's equation with an additional source, $t^{(0)}_{ab}$, which is given by an explicit formula in terms of μ_{abcdef}. Thus, $t^{(0)}_{ab}$ may be interpreted as the effective stress-energy produced by the small scale inhomogeneities. In [1], we then proved two theorems constraining $t^{(0)}_{ab}$:

Theorem 1. *Given a one-parameter family $g_{ab}(\lambda)$ satisfying assumptions (i)–(iv) above, the effective stress-energy tensor $t^{(0)}_{ab}$ is traceless,*

$$t^{(0)a}{}_a = 0. \tag{5}$$

Theorem 2. *Given a one-parameter family $g_{ab}(\lambda)$ satisfying assumptions (i)–(iv) above, the effective stress-energy tensor $t^{(0)}_{ab}$ satisfies the weak energy condition, i.e.,*

$$t^{(0)}_{ab} t^a t^b \geq 0 \tag{6}$$

for all t^a that are timelike with respect to $g^{(0)}_{ab}$.

In essence, these theorems show that only those small scale metric inhomogeneities corresponding to gravitational radiation can have a significant backreaction effect; see [1] for further discussion. In particular, it should be noted that in the case where $g^{(0)}_{ab}$ has FLRW symmetry, even when short wavelength gravitational radiation is present and backreaction effects are large, the effective stress-energy tensor must be of the form of a $P = \frac{1}{3}\rho$ fluid, and therefore cannot mimic dark energy.

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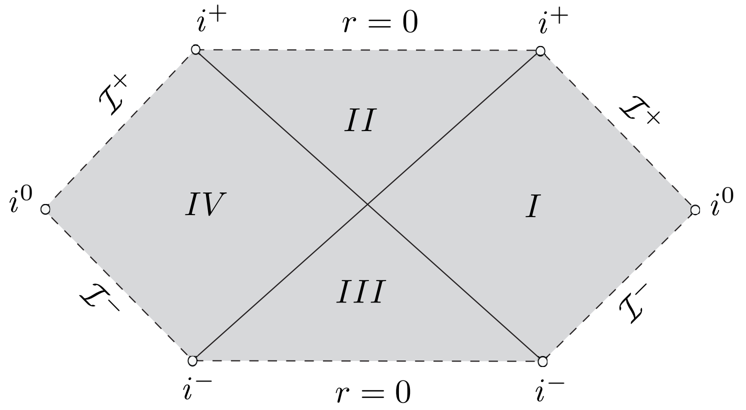
The C^0-inextendibility of the Schwarzschild spacetime

Jan Sbierski

We recall that a connected Lorentzian manifold (M, g) is called C^k-inextendible, if there does not exist a connected Lorentzian manifold $(\tilde{M}, \tilde{g})$ (of the same dimension as M) with a C^k-regular metric $\tilde{g}$ in which M isometrically embeds as a proper subset.

Here, we consider the *maximal analytic Schwarzschild spacetime*, the Penrose diagram of which is given below:



It is well-known that the maximal analytic Schwarzschild spacetime is C^2-inextendible. This follows directly from the observation, that every future inextendible timelike geodesic is either *i)* future complete or *ii)* the Kretschmann scalar blows up along the geodesic. In [2], the following stronger statement is proven:

Theorem 1. *The maximal analytic Schwarzschild spacetime is C^0-inextendible.*

Before we discuss some elements of the proof, we briefly put the above theorem in the wider context: The investigation of low-regularity inextendibility results of Lorentzian manifolds is motivated by the strong cosmic censorship conjecture, which can be stated as follows:

(2) For *generic* asymptotically flat initial data for the vacuum Einstein equations $Ric(g) = 0$, the maximal globally hyperbolic development is inextendible as a suitably regular Lorentzian manifold.

Note that in the above formulation, the regularity class, in which the maximal globally hyperbolic development is conjectured to be generically inextendible, is not fixed. The physical motivation of the conjecture suggests, however, that the maximal globally hyperbolic development should be generically inextendible in all regularity classes that admit a local existence result for the Einstein equations (in a weak form). The recent resolution of the bounded L^2 curvature conjecture, [1], thus suggests that one should prove inextendibility in a regularity class that is in particular rougher than C^2. So far, however, nearly all known inextendibility results for Lorentzian manifolds are at the level of C^2. This motivates the study of low-regularity inextendibility criteria which is initiated in [2].

The proof of Theorem 1 is by contradiction; one assumes that there is a C^0-extension of the maximal analytic Schwarzschild spacetime $(M_{\max}, g_{\max})$, i.e., a connected Lorentzian manifold $(\tilde{M}, \tilde{g})$ of the same dimension as $M_{\max}$ with a continuous metric $\tilde{g}$ together with an isometric embedding $\iota : M_{\max} \hookrightarrow \tilde{M}$, where $\iota(M_{\max})$ is a proper subset of $\tilde{M}$. Under these assumptions, one shows that there is a timelike curve leaving the maximal analytic Schwarzschild spacetime:

Lemma 3. *There exists a timelike curve* $\tilde{\gamma} : [0,1] \to \tilde{M}$ *such that* $\tilde{\gamma}\big([0,1)\big) \subseteq \iota(M_{\max})$ *and* $\tilde{\gamma}(1) \in \tilde{M} \setminus \iota(M_{\max})$.

Let $\gamma := \iota^{-1} \circ \tilde{\gamma}\big|_{[0,1)}$ and let us without loss of generality assume that γ is future directed in $(M_{\max}, g_{\max})$. This timelike curve can leave $M_{\max}$ then only 'through' region I, II, or IV. Since region I and IV are isometric, it suffices to distinguish the following two cases:

(1) **There exists an $s_0 \in (0,1)$ such that $\gamma\big|_{[s_0,1)}$ is contained in region I.**

We recall that the timelike diameter of a time-oriented Lorentzian manifold (M, g) with a continuous metric g is given by

$$\operatorname{diam}_t(M) := \sup_{\substack{p,q \in M \\ q \in I^+(p,M)}} \ \sup_{\substack{\sigma:[0,1]\to M \text{ future directed timelike} \\ \text{curve with } \sigma(0)=p \text{ and } \sigma(1)=q}} \left\{ \int_0^1 \sqrt{-g\big(\dot{\sigma}(s), \dot{\sigma}(s)\big)}\, ds \right\}.$$

Here, $I^+(p, M)$ denotes the future of p in M. One now chooses a time-oriented neighbourhood $\tilde{U}$ of $\tilde{\gamma}(1)$ in $\tilde{M}$ such that in particular

$$\operatorname{diam}_t\Big(I^+\big(\tilde{\gamma}(s), \tilde{U}\big) \cap \bigcup_{s<s'<1} I^-\big(\tilde{\gamma}(s'), \tilde{U}\big)\Big) < \infty \qquad \text{for all } s \text{ close to } 1.$$

On the other hand one shows

$$\operatorname{diam}_t\Big(I^+\big(\gamma(s), M_{\max}\big) \cap \bigcup_{s<s'<1} I^-\big(\gamma(s'), M_{\max}\big)\Big) = \infty \qquad \text{for all } s \text{ close to } 1.$$

The future one-connectedness of region I, that is, that any two future directed timelike curves with the same endpoints are homotopic with fixed endpoints via timelike curves, then ensures that

$$\iota\Big(I^+\big(\gamma(s), M_{\max}\big) \cap \bigcup_{s<s'<1} I^-\big(\gamma(s'), M_{\max}\big)\Big) \subseteq I^+\big(\tilde{\gamma}(s), \tilde{U}\big) \cap \bigcup_{s<s'<1} I^-\big(\tilde{\gamma}(s'), \tilde{U}\big)$$

holds for all s close to 1, which yields a contradiction.

(2) **There exists an $s_0 \in (0,1)$ such that $\gamma\big|_{[s_0,1)}$ is contained in region II.**

We introduce the *spacelike diameter* of a globally hyperbolic Lorentzian manifold (N, g) with a continuous metric g, defined by

$$\operatorname{diam}_s(N) := \sup_{\substack{\Sigma \text{ Cauchy} \\ \text{hypersurface of } N}} \ \sup_{p,q\in\Sigma} \ \inf_{\substack{\gamma:[0,1]\to\Sigma \\ \text{piecewise smooth curve} \\ \text{with } \gamma(0)=p \text{ and } \gamma(1)=q}} \int_0^1 \sqrt{g\big(\dot{\gamma}(s), \dot{\gamma}(s)\big)}\, ds\,.$$

One can prove the following theorem:

Theorem 4. *Let* $(\tilde{U}, \tilde{g})$ *be a* $(d+1)$*-dimensional time-oriented Lorentzian manifold with a* C^0*-regular metric and* $\tilde{N} \subseteq \tilde{U}$ *an open and globally hyperbolic subset. Moreover, assume that* $\tilde{N}$ *is precompact in* $\tilde{U}$ *and that* $\psi : \mathbb{R}^d \supseteq B_2(0) \hookrightarrow \tilde{U}$ *is a smooth embedding of* $B_2(0)$ *such that* $\psi|_{B_1(0)} : B_1(0) \hookrightarrow \tilde{N} \subseteq \tilde{U}$ *is a Cauchy hypersurface in* $(\tilde{U}, \tilde{g})$.

Then one has $\mathrm{diam}_s(\tilde{N}) < \infty$.

The final step now is to choose a time-oriented neighbourhood $\tilde{U} \subseteq \tilde{M}$ of $\tilde{\gamma}(1)$ and to construct a globally hyperbolic $N \subseteq M_{\max}$ with $\mathrm{diam}_s(N) = \infty$ (this subset N 'touches the curvature singularity at $r = 0$') such that $\iota(N) =: \tilde{N}$ is precompact in $\tilde{U}$ and one can find a Cauchy hypersurface which one can slightly extend as an embedded hypersurface (this is the second assumption of Theorem 4). The above theorem then implies the contradiction $\mathrm{diam}_s(\tilde{N}) < \infty$. This concludes the proof of Theorem 1.

Let us conclude by mentioning that an interesting open problem in the realm of C^0-extensions is to show the conjectured C^0-inextendibility of cosmological spacetimes with a big bang singularity (in particular of the FRW spacetimes).

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Stability in exponential time of Minkowski space-time with a space-like translation symmetry

Cécile Huneau

In vacuum Einstein equations can be written

$$R_{\mu\nu} = 0. \tag{1}$$

A trivial solution in $\mathbb{R}^{3+1}$ is given by Minkowski space-time equipped with the Minkowski metric

$$m = -(dt)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2.$$

In [4], Christodoulou and Klainerman proved that this solution is stable in the following sense : for initial data $(\mathbb{R}^3, \bar{g}, K)$ sufficiently smooth, such that $\bar{g}$ is closed to e, K small, and asymptotically flat, the Cauchy development is geodesically complete, and the solution converges at infinity to Minkowski solution. An other proof of the stability has been given later by Lindblad and Rodnianski in harmonic gauge (see [7]). In [6], we are also interested in the stability of Minkowski space-time, but under a particular symmetry assumption.

The translation symmetry, studied by Choquet-Bruhat and Moncrief in [3] allows to reduce the $3 + 1$ dimensional problem to a $2 + 1$ dimensional one. More precisely, we look for solutions of the $3+1$ vacuum Einstein equation, on manifolds of the form $\Sigma \times \mathbb{R}_{x_3} \times \mathbb{R}_t$, where Σ is a 2 dimensional manifold, equipped with a metric of the form

$$\mathbf{g} = e^{-2\phi} g + e^{2\phi}(dx^3)^2,$$

where ϕ a scalar function, and g a Lorentzian metric on $\Sigma \times \mathbb{R}$, all quantities being independent of x^3. For these metrics, Einstein vacuum equations are equivalent to the $2+1$ dimensional system

$$\begin{cases} \Box_g \phi = 0 \\ R_{\mu\nu} = 2\partial_\mu \phi \partial_\nu \phi, \end{cases} \tag{2}$$

where $R_{\mu\nu}$ is the Ricci tensor associated to g. Here we will work in the case $\Sigma = \mathbb{R}^2$. Then a particular solution is given by Minkowski solution itself. It corresponds to $\phi = 0$ and g equals to the Minkowski metric in dimension $2+1$. A natural question one can ask is the stability of this solution.

In [6] we prove the existence of solutions in exponential time : for initial data for ϕ in some weighted Sobolev spaces, of size ε small, there exist solutions to (2) for times $t \leq \exp\left(\frac{C}{\sqrt{\varepsilon}}\right)$. We recall the definition of weighted Sobolev spaces

$$\|u\|_{H^m_\delta} = \sum_{|\beta| \leq m} \|(1+|x|^2)^{\frac{\delta+|\beta|}{2}} D^\beta u\|_{L^2}.$$

Theorem 1. *Let* $0 < \varepsilon < 1$. *Let* $N \geq 40$, $\frac{1}{2} \leq \delta \leq 1$ *and* $0 < \rho < \frac{1}{2}$. *Let* $(\phi_0, \phi_1) \in H^{N+1}_\delta \times H^N_{\delta+1}$ *such that*

$$\|\phi_0\|_{H^{N+1}_\delta} + \|\phi_1\|_{H^N_{\delta+1}} = \varepsilon$$

There exists a constant C *such that if* $T \leq \exp\left(\frac{C}{\sqrt{\varepsilon}}\right)$ *and* ε *is small enough, there exist a coordinate system* (t, x_1, x_2) *and a solution* (ϕ, g) *of* (2) *on* $[0,T] \times \mathbb{R}^2$ *such that*

$$(\phi, \partial_t \phi)|_{t=0} = (\phi_0, \phi_1),$$

and we have the estimates

$$|g_{\alpha\beta} - m_{\alpha\beta}| \lesssim \varepsilon,$$

$$|g_{\alpha\beta} - m_{\alpha\beta}| \lesssim \frac{\varepsilon}{(1+t)^{\frac{1}{2}-\rho}}, \quad \text{for } r \leq \frac{t}{2},$$

where $m_{\alpha\beta}$ *is Minkowski metric on* $\mathbb{R}^{2+1}$.

For a more precise statement of Theorem 1, we refer to [6].

Comments on this theorem.

- The initial data for g must satisfy the constraint equations. The only freedom in this solving is the choice of the initial hypersurface. The construction of solutions to the constraint equations for this problem is done in [5].
- The perturbations we consider are not asymptotically flat in $3+1$ dimension, since asymptotic flatness is not compatible with a translation spacelike symmetry.

- The method used to prove this theorem is by using a wave gauge. In this sense it is similar to Lindblad and Rodnianski proof of the stability of Minkowski. However the weak decay of the solutions to the wave equation in $2+1$ dimension makes the problem a little more intricate.
- The solutions we construct do not tend to Minkowski metric at space-like infinity. Instead they tend to

$$g_b = -dt^2 + dr^2 + (r + b(\theta)(r-t))^2 d\theta^2.$$

This behaviour can be seen as a generalisation of Einstein-Rosen waves (see [2] or [1]). In the Fourier decomposition $b(\theta) = b_0 + b_1 \cos(\theta) + b_2 \sin(\theta) + \ldots$, the component b_0 corresponds to the deficit angle, and the vector (b_1, b_2) can be seen as the ADM linear momentum. Both are imposed by the resolution of the constraint equations.

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Nonexistence and Nonuniqueness Results for solutions to the Vacuum Einstein Conformal Constraint Equations

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In general relativity, a space-time is a $(n+1)-$dimensional Lorentzian manifold $(\mathcal{M}, h)$ (i.e, h has signature $- + + \ \ldots \ +$), with $n \geq 3$ which satisfies the Einstein equations

$$\text{Ric}^h_{\mu\nu} - \frac{R_h}{2} h_{\mu\nu} = \frac{8\pi\mathcal{G}}{c^4} T_{\mu\nu}, \tag{1}$$

where Ric^h and R_h are respectively the Ricci and the scalar curvatures of of h, $\mathcal{G}$ is Newton's constant, c is the speed of light and T is the stress-energy tensor of non-gravitational fields (i.e. matter fields, electromagnetic field...).

Einstein equations are roughly speaking hyperbolic of order 2. Hence all solutions can be obtained from their initial values at some "time t=0", the metric $\hat{g}$ induced on a Cauchy hypersurface $M \subset \mathcal{M}$, and its initial velocity, the second fundamental form $\hat{K}$ of the embedding $M \subset \mathcal{M}$. By the Gauss and Codazzi equations, the choice of $(M, \hat{g}, \hat{K})$ from (1) must satisfy the so-called Einstein constraint equations. In the vacuum case, i.e. when $T \equiv 0$, these equations are

$$\begin{aligned} R_{\hat{g}} - |\hat{K}|^2_{\hat{g}} + \left(\mathrm{tr}_{\hat{g}}\hat{K}\right)^2 &= 0, \\ \hat{K} - d_{\hat{g}}\ \mathrm{tr}_{\hat{g}}\hat{K} &= 0. \end{aligned} \tag{2}$$

Constructing and classifying solutions of this system is an important issue. For a deeper discussion of (2) , we refer the reader to the excellent review article [1]. One of most efficient methods to find initial data satisfying (2) is the conformal method developed by Lichnerowicz [9] and Y. Choquet-Bruhat-Jr. York [2]. The idea of this method is to effectively parameterize the solutions to (2) by some reasonable parts and then solve for the rest of the data. More precisely, we assume given some seed data: a Riemannian manifold (M, g) which we will assume compact, a mean curvature τ (a function), a transverse-traceless tensor σ (i.e. a symmetric, trace-free, divergence-free $(0, 2)$-tensor). Then we look for a positive function φ and a $1-$form W such that

$$\hat{g} = \varphi^{N-2}g, \quad \hat{K} = \frac{\tau}{n}\varphi^{N-2}g + \varphi^{-2}(\sigma + LW)$$

is a solution to the vacuum Einstein constraint equations (2). Here $N = \frac{2n}{n-2}$ and L is the conformal Killing operator defined by

$$LW_{ij} = \nabla_i W_j + \nabla_j W_i - \frac{2}{n}\nabla^k W_k g_{ij},$$

where ∇ is the Levi-Civita connection associated to the metric g.

Equations (2) can be reformulated in terms of φ and W as follows:

$$\frac{4(n-1)}{n-2}\Delta_g\varphi + R_g\varphi = -\frac{n-1}{n}\tau^2\varphi^{N-1} + |\sigma + LW|^2_g\varphi^{-N-1} \quad \text{[Lichnerowicz eq.]}, \tag{3a}$$

$$-\frac{1}{2}L^*LW = \frac{n-1}{n}\varphi^N d\tau \quad \text{[vector eq.]}, \tag{3b}$$

where Δ_g is the nonnegative Laplace operator and L^* is the formal L^2-adjoint of L. These coupled equations are called *the conformal constraint equations.*

During the past decades, many existence and uniqueness results for (3) were proven. When τ is constant, the system (3) becomes uncoupled (since $d\tau \equiv 0$ in the vector equation) and a complete description of the situation was achieved by J. Isenberg [7]. The near CMC case (i.e. when $d\tau$ is small) was addressed soon after. Most results can be found in [1]. For arbitrary τ however, the situation appears much harder and only two methods exist to tackle this case. The first one, obtained by Holst-Nagy-Tsogtgerel [6] and Maxwell [10], shows that the system (3) admits a solution, provided g has positive Yamabe invariant and $\sigma \not\equiv 0$ is small enough. The second one, introduced by Dahl-Gicquaud-Humbert [3], states that if τ has constant sign and if *the limit equation*

$$-\frac{1}{2}L^*LV = \alpha\sqrt{\frac{n-1}{n}}|LV|\frac{d\tau}{\tau} \tag{4}$$

has no non-zero solution V, for all values of the parameter $\alpha \in [0,1]$, then the set of solutions (φ, W) to (3) is not empty and compact. This criterion holds true e.g. when (M,g) has $\mathrm{Ric} \leq -(n-1)g$, with $\left\|\frac{d\tau}{\tau}\right\|_{L^\infty} < \sqrt{n}$ (see also [5] for an extension of this result to asymptotically hyperbolic manifolds). An unifying point of view of theses results is given in [4] and [12].

Conversely, nonexistence and nonuniqueness results for (3) are fairly rare. The only model of nonuniqueness of solutions is constructed on the $n-$torus by D. Maxwell [11] while the only nonexistence result, achieved by J. Isenberg-Murchadha [8] and later strengthened in [3] and [4], states that the system (3) with $\sigma \equiv 0$ has no solution when $\mathcal{Y}_g \geq 0$ and $d\tau/\tau$ is small enough. This assertion together with experimentations on the torus led D. Maxwell to pose a question that whether the non-zero assumption of σ is a necessary condition for existence of solution to the conformal equations (3) with positive Yamabe invariant (see [11]).

In this study, we first give another version of the main theorem in [3], which allows α in the limit equation (4) to be set to 1. More precisely, we show that

Theorem 1 (Control of the parameter)**.** *If τ has constant sign, then at least one of the following assertions is true*

(i) *The conformal constraint equations* (3) *admits a solution (φ, W) with $\varphi > 0$. Furthermore, the set of solutions $(\varphi, W) \in W^{2,p}_+ \times W^{2,p}$ is compact.*

(ii) *There exists a nontrivial solution $W \in W^{2,p}$ to the limit equation*

$$-\frac{1}{2}L^*LW = \sqrt{\frac{n-1}{n}}|LW|\frac{d\tau}{\tau}. \tag{5}$$

(iii) *For all continuous function $f > 0$ the (modified) conformal constraint equations*

$$\frac{4(n-1)}{n-2}\Delta\varphi + f\varphi = -\frac{n-1}{n}\tau^2\varphi^{N-1} + |LW|^2\varphi^{-N-1} \tag{6a}$$

$$-\frac{1}{2}L^*LW = \frac{n-1}{n}\varphi^N d\tau \tag{6b}$$

has a (non-trivial) solution $(\varphi, W) \in W^{2,p}_+ \times W^{2,p}$. Moreover if the corresponding Yamabe invariant $\mathcal{Y}_g > 0$, there exists a sequence $\{t_i\}$ converging to 0 *s.t. the conformal constraint equations* (3) *associated to data $(g, t_i\tau, \sigma)$ has at least two solutions.*

Comparing with the original version of Dahl-Gicquaud-Humbert, the price to pay for control of the parameter ($\alpha = 1$) in (4) is the addition of (iii). However, this assertion is necessary by the following theorem.

Theorem 2 (Nonexistence of solution)**.** *Assume that there exists* $c = c(g) > 0$ *s.t.* $\left|L\left(\frac{d\tau}{\tau}\right)\right| \le c\left|\frac{d\tau}{\tau}\right|^2$*. Let* V *be a given open neighborhood of the critical set of* τ*. If* $\sigma \not\equiv 0$ *and supp*$\{\sigma\} \subsetneq M \setminus V$*, then neither the conformal constraint equations* (3) *nor the limit equation* (5) *associated to initial data* $(g, \tau^a, \frac{\sigma}{\epsilon a})$ *admits (nontrivial) solution, provided* $a^{-1}, \epsilon a > 0$ *are small enough.*

It is worthy noting that [3, Proposition 1.6] provides the existence of such assumptions. In fact, our proof for Theorem 2 is the extension of arguments in [3, Proposition 1.6].

As direct consequences of Theorem 1 and 2, we also obtain the following results.

Corollary 3 (An answer to Maxwell's question)**.** *Let* (M, g, τ) *be given in assumptions of Theorem 2. If* $\mathcal{Y}_g > 0$*, then the conformal constraint equations* (3) *associated to* $(g, \tau^a, 0)$ *has a (nontrivial) solution for all* $a > 0$ *large enough.*

Corollary 4 (Nonuniqueness of solutions)**.** *Assume that* $(M, g, \tau, \sigma, a, \epsilon)$ *is given in Theorem 2. If* $\mathcal{Y}_g > 0$*, then there exists a sequence* $\{t_i\}$ *converging to* 0 *s.t. the conformal constraint equations* (3) *associated to initial data* $(g, t_i\tau^a, \frac{\sigma}{\epsilon a})$ *has at least two solutions.*

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The Conformal Constraint Equations on Asymptotically Euclidean Manifolds

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In general relativity, spacetime is described as a Lorentzian manifold with the metric satisfying the Einstein equations, which relate curvature with matter and energy. On a spacelike (i.e., Riemannian) hypersurface, the Gauss and Codazzi equations reduce these to the constraint equations, which in the vacuum case are

$$R + (\mathrm{tr}K)^2 - |K|^2 = 0$$
$$\mathrm{div}K - \nabla(\mathrm{tr}K) = 0,$$

where R is the scalar curvature of the induced metric, K is the second fundamental form of the hypersurface, and all quantities are taken with respect to the induced metric. Choquet-Bruhat proved that, given a Riemannian manifold and a second fundamental form, there is a spacetime satisfying the Einstein equations with that manifold as a hypersurface, with appropriate induced metric and second fundamental form.

Since a metric and symmetric 2-tensor have 12 components (in 3 dimensions) but there are only 4 constraint equations, the constraint equations are underdetermined. Thus, a valuable goal is to parameterize all solutions of the constraint equations. The main tool towards this goal, and, indeed, finding solutions of the constraint equations in general, is called the conformal method.

I will present the Hamiltonian conformal thin sandwich version of this method, as described by Maxwell in [1]. In this method, one is given a metric g (representing a conformal class), a function $N > 0$ (representing the densitized lapse), a symmetric, trace-free, divergence-free 2-tensor σ (representing a conformal momentum with respect to the given representative g), and a function τ (representing the mean curvature of the embedding). One then seeks to find all solutions (ϕ, W) (a function and vector respectively) of the conformal constraint equations,

$$-8\Delta\phi + R\phi + \frac{2}{3}\tau^2\phi^5 = \left|\sigma + \frac{1}{2N}LW\right|\phi^{-7} \tag{1}$$

$$-\frac{1}{2}L^*\frac{1}{2N}LW = \frac{2}{3}d\tau\phi^6, \tag{2}$$

where L is the conformal Killing operator,

$$LW := \nabla_i W_j + \nabla_j W_i - \frac{2}{3}\nabla_k W^k g_{ij}.$$

Given such (ϕ, W), then

$$\bar{g} = \phi^4 g$$
$$\bar{K} = \phi^{-2}\left(\sigma + \frac{1}{2N}LW\right) + \frac{\tau}{3}\bar{g}$$

solve the constraint equations. The hope for a parameterization is that, given (g, N, σ, τ), such (ϕ, W) exist are are unique. Such is obviously not true, eg. if $R, \tau, \sigma \equiv 0$, but one hopes such cases are simple, as it is in the given example.

Asymptotically Euclidean (AE) manifolds are useful for describing Riemannian hypersurfaces of spacetimes containing compact objects such as stars. An AE manifold contains a compact set, outside of which the manifold is diffeomorphic to some finite number of Euclidean spaces minus a ball; each such component is called an end. On each end, the metric and its derivatives are assumed to decay at some rate (usually polynomial in the radial coordinate) to the Euclidean metric. For the constraint equations, we also want K to decay sufficiently fast to zero. Thus, we need g, σ, τ and W to decay to zero, while ϕ, N need to decay to constants.

Before we can understand the full system (1)-(2), we must first understand each equation individually. The vector equation (2) is well understood on AE manifolds. Since there are no decaying conformal Killing fields on (sufficiently regular) AE manifolds, the operator $L^* \frac{1}{2N} L$ is an isomorphism on appropriately weighted Sobolev or Hölder spaces. Thus, given a ϕ, the vector equation is always uniquely solvable.

The Lichnerowicz equation (1) is more complicated. We have the following results.

- (D., Gicquaud, Isenberg) The Lichnerowicz equation is solvable if and only if there is a $\psi > 0$ such that $\psi^4 g$ has scalar curvature $-\tau^2$.
- (D., Maxwell) Such a ψ exists if and only if the zero set of τ^2 is Yamabe positive.

The Yamabe constant is usually given only for manifolds, but we generalize it to

$$Y(V) = \inf \frac{\int_M |\nabla u|^2 + Ru^2}{\|u\|_6^2},$$

where the infimum is taken over $\{u : u \not\equiv 0, u|_{M\setminus V} = 0\}$. An important note is that sufficiently "small" sets (which include neighborhoods of infinity) are Yamabe positive.

The simplest case for the full system is the constant mean curvature (CMC) case. Since τ must decay, the only such constant is $\tau \equiv 0$. The vector equation in this case is fixed, and so $W \equiv 0$. Thus, only the Lichnerowicz equation must be solved. By our previous result, there is a solution if and only if (M, g) is Yamabe positive. This turns out to be a topological restriction; as for closed manifolds, not all AE manifolds allow Yamabe positive metrics. Indeed, the restrictions are essentially the same. For instance, T^3 with a point removed does not allow a Yamabe positive AE metric.

The next simplest is perturbations off these solutions, i.e., the near-CMC case. Choquet-Bruhat, Isenberg and York showed one can perturb off these solutions using the implicit function theorem. However, this case is still restricted to Yamabe positive manifolds, and requires $|\tau| + |d\tau|$ to be sufficiently small.

A usual strategy for solving the full system is to use a fixed point theorem. Given a ϕ, one solves the vector equation for W, then uses that W to find a solution to the Lichnerowicz equation. Under certain conditions, one can show this map has a fixed point. The rest of the results are proved using this method.

A different kind near-CMC condition leads to solutions for all Yamabe classes. If $\|d\tau\|$ is sufficiently small compared to τ^2, then the conformal constraint equations have a solution, as proven by D., Isenberg, Mazzeo and Meier. Unfortunately, since τ must decay to zero, there are not τ's with $\|d\tau\|$ arbitrarily small compared to τ^2, and so this condition may be vacuous, except trivially ($\tau \equiv 0$).

For arbitrary τ, we also showed that for (M, g) Yamabe positive and σ sufficiently small, there is a solution. However, a later result of Nguyen showed that one can scale τ and σ in opposite directions if one scales the solutions as well. Thus, these solutions are really rescalings of the near-CMC perturbation solutions.

The final general result is the first that guarantees that Yamabe nonpositive AE manifolds allow any solutions of the conformal constraint equations. For large $r > 0$, τ's that are constant on B_r and vanish outside B_{2r} and behave well in between give rise to solutions of the conformal constraint equations. Perturbations of these τ's similar lead to solutions, so the zero set of τ is not essential.

Other techniques, such as the limit equation of Dahl, Gicquaud and Humbert have been attempted, but so far have not lead to any results.

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The vector field method for transport equations with applications to classical and relativistic systems

Jacques Smulevici

(joint work with David Fajman, Jérémie Joudioux)

The vector field method of Klainerman [1] is a very powerful tool to obtain robust decay estimates for solutions to wave equations. The aim of our work is to explain how such a method can be adapted to the study of kinetic transport equations. Consider for instance the relativistic transport equations

$$\left[\left(m^2 + |v|^2\right)^{1/2} \partial_t + v^i \partial_{x^i}\right] f = 0, \tag{1}$$

where the parameter $m \geq 0$ is the mass of the particles and $f = f(t, x, v)$ with $x \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$ if $m > 0$ corresponding to *massive* particles , $v \in \mathbb{R}^n \setminus \{0\}$ if $m = 0$, corresponding to *massless* particles. Since (1) is a transport equation, f is preserved along the characteristics associated to the equation. However, the macroscopic quantities obtained by integrating f in v, such as

$$\rho[f](t, x) \equiv \int_v f(t, x, v) \frac{dv}{\sqrt{m^2 + |v|^2}}, \tag{2}$$

are only conserved as functions of t in L^1_x, and will enjoy decay properties as $t \to +\infty$ in L^∞_x. To prove this, the standard method, which follows earlier work of Bardos-Degond for the classical transport operator [2], consists in writing explicitly the solution in terms of its initial data using the conservation of f along characteristics, and then estimating directly the v-integral in (2). For the massive case $m > 0$, this leads to an estimate of the form, for all $t > 0$ and all $x \in \mathbb{R}^n_x$,

$$\rho[|f|](t,x) \leq \frac{C(V)}{t^n} ||f(t=0)||_{L^1(\mathbb{R}^n_x \times \mathbb{R}^n_v)},$$

where $C(V)$ is a constant depending on an upper bound V of the size of the support in v of the initial data, for instance, assuming the data to be smooth and compactly supported,

$$V = \sup \{\lambda \in \mathbb{R}_+ \,:\, \exists (x,v) \in \mathbb{R}^n_x \times \mathbb{R}^n_v \,:\, \lambda = |v| \,\text{and}\, f(0,x,v) \neq 0\}.$$

Note that $C(V) \to +\infty$ as $V \to +\infty$, so that, unless more refined estimates are used, this method requires compact support of the initial data to work. We prove instead the estimate

Theorem 1 (Decay estimates for velocity averages of massive distributions, see [3]). *For any regular distribution function f solution to (1) with $m > 0$, any $x \in \mathbb{R}^n$ and any $t \geq \sqrt{1+|x|^2}$, we have*

$$\rho[|f|](t,x) \leq \frac{C}{(1+t)^n} \sum_{\substack{|\alpha| \leq n \\ Z^\alpha \in \widehat{\mathbb{P}}^{|\alpha|}}} \left|\left| \hat{Z}^\alpha(f)_{|H^n_1 \times \mathbb{R}^n_v} v_\alpha \nu_1^\alpha \right|\right|_{L^1\left(H^n_1 \times \mathbb{R}^n_v\right)}, \tag{3}$$

where H^n_1 denotes the unit hyperboloid $H^n_1 := \left\{(t,x) \in \mathbb{R}_t \times \mathbb{R}^n_x \,/\, 1 = t^2 - x^2\right\}$, $\hat{Z}^\alpha(f)_{|H^n_1 \times \mathbb{R}^n_v}$ is the restriction to $H^n_1 \times \mathbb{R}^n_v$ of $\hat{Z}^\alpha(f)$, $v_\alpha \nu_1^\alpha$ is the contraction of the 4-velocity $(\sqrt{m^2+|v|^2}, v^i)$ with the unit normal ν_1 to H^n_1 and where the $\hat{Z}^\alpha$ are differential operators obtained as a composition of $|\alpha|$ vector fields of the algebra $\widehat{\mathbb{P}}$.

The algebra of vector fields $\widehat{\mathbb{P}}$ is obtained by taking the *complete lifts* of the usual Killing vector fields of Minkowski space, a classical operation in differential geometry. For instance, the complete lift of a rotation vector field $x^i \partial_{x^j} - x^j \partial_{x^i}$ is given by the vector field $x^i \partial_{x^j} - x^j \partial_{x^i} + v^i \partial_{v^j} - v^j \partial_{v^i}$.

Note that in the above estimates, there is no requirements of compact support in v of the initial data. Moreover, using finite speed of propagation type arguments, one can easily see that for solutions arising from smooth initial data of compact support in x and decaying sufficiently fast in v (but not necessarily of compact support in v) given at $t = 0$, the norm on the right-hand side of (3) is finite, so that the usage of hyperboloids is mostly technical.

In the case of massless particles ($m = 0$), a similar estimate holds with the decay rates being weaker near the light-cone, as in the case of the wave equation.

In the second part of the talk, I then presented several applications of these decay estimates to the Vlasov-Nordström systems, also contained in our article [3]. This system is composed of a relativistic transport equation coupled to a scalar wave equation. When the particles are massless, our decay estimates lead to sharp asymptotics of the solutions for all data in dimension $n \geq 4$ and for small data in dimension $n = 3$ where a version of the null condition is uncovered and exploited to close the estimates. This should be compared with the work of

Dafermos on the study of compactly supported small data solutions spherically symmetric massless Einstein-Vlasov system [4] and its recent, major, extension to the full stability of the Minkowski space for the same system without symmetry assumptions by Taylor [5].

In the massive case, our method allows us to obtain sharp asymptotics of the solutions in dimension $n \geq 4$ under some small data assumptions. Even with the extra decay coming from the high dimensions, a special, new treatment is needed to close the high order estimates. The case $n = 3$ requires a refinement of our methods due to slower decay of the non-linear terms. One possible approach consists in using *modified vector fields*, which are perturbations of the standard vector fields with the perturbations depending themselves on the solutions, in order to improve the commutation relations. This strategy has already been implemented to study small data solutions of Vlasov-Poisson system. For this classical system, one can use a vector field method similar to the one described above. Again, an approach using the standard vector fields (this time associated with the Galilean invariance of the equations) can only handle the dimensions $n \geq 4$. However, we showed in [6] how modified vector fields could be used to treat the $n = 3$ case and it is very likely that these techniques can be extended to treat the $n = 3$ Vlasov-Nordström system.

As a conclusion, we believe that we now possess sufficiently robust techniques to study other non-linear systems beyond the Vlasov-Nordström system, such as the massive Einstein-Vlasov equations (or the massless Einstein-Vlasov equations without any compactness assumptions on the initial data).

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Local Energy Decay for Scalar Fields on Time Dependent Non-trapping Backgrounds

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(joint work with Jason Metcalfe, Jacob Sterbenz)

The aim of this work is to contribute to the understanding of local energy decay bounds for the wave equation on non-trapping, asymptotically flat space-times. From a relativistic perspective, this is a stepping stone toward understanding the similar problem on black hole backgrounds.

This is a well understood question if one considers small perturbations of the Minkowski space-time. Our aim here, instead, is to consider large perturbations. In that, our goals are two-fold.

First we consider the *stationary* case, i.e. where the coefficients are time independent. There we provide a full spectral characterization of the local energy decay estimates in terms of the eigenvalues and resonances of the corresponding elliptic problem, which can be viewed as poles/singular points of an associated resolvent operator. There are three such objects which are of interest to us:

- Complex eigenvalues outside the continuous spectrum $\mathbb{R}$ and in the lower half-space,
- Zero eigenvalues/resonances, and
- Nonzero resonances embedded inside the continuous spectrum.

Our main result here asserts that local energy decay holds iff none of these three obstructions occurs.

One significant simplification occurs in the *symmetric* case, where no nonzero resonances can occur inside the continuous spectrum. There our results are consistent with the standard spectral theory for self-adjoint elliptic operators. In that case, we also consider the problem of continuity of our spectral assumptions along one parameter families of operators; the main idea being that complex eigenvalues can only emerge via the zero mode. If complex eigenvalues do occur, a slightly more complicated picture emerges, and the flow splits into two finite dimensional subspaces where exponential growth, respectively decay occurs, and a bulk part with uniform energy bounds.

Secondly, we study the case of *time dependent* operators and show that the results in the stationary case are stable with respect to perturbations. More precisely, we consider almost symmetric, almost stationary operators which satisfy a quantitative zero spectral assumption uniformly in time, but allowing for eigenvalues off the real axis. Then we establish an exponential trichotomy, splitting the energy space as a direct sum of three subspaces as follows:

- A finite dimensional subspace of spatially localized, exponentially increasing solutions, associated to eigenvalues in the lower half-space
- A finite dimensional subspace of spatially localized, exponentially decreasing solutions, associated to eigenvalues in the upper half-space

- A remaining infinite dimensional subspace of bounded energy solutions with good local energy bounds.

If there are no eigenvalues off the real axis then only the last subspace is nontrivial, and local energy decay holds globally.

One key intermediate step of our approach here is to establish a weaker bound, which we call *two point local energy decay*, and which uses energy bounds at both ends of the time interval. This has the advantage that it allows for nonreal eigenvalues; however, it prohibits nonzero resonances in the symmetric case. The proof of this weaker bound is naturally split into three ranges:

- *Low time frequencies*, where we argue perturbatively starting from the zero resolvent bound.
- *Medium time frequencies*, where we rely heavily on Carleman type estimates.
- *Large time frequencies*, where we use positive commutator estimates based on the nontrapping property of the associated Hamilton flow.

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Causality, hyperbolicity and shock formation in Lovelock theories of gravity

Harvey Reall

(joint work with Norihiro Tanahashi, Benson Way, Giuseppe Papallo)

The Einstein equation relates the curvature of spacetime to the energy-momentum tensor of matter:

$$G_{ab} + \Lambda g_{ab} = 8\pi T_{ab} \tag{1}$$

The form of the LHS of this equation is dictated by Lovelock's theorem [1]. This states that, in four dimensions, the most general symmetric, divergence-free, second rank tensor that is a function only on the metric and its first and second derivatives, is a linear combination of the Einstein tensor and a cosmological constant term.

In $d > 4$ dimensions, this result is not valid and Lovelock showed [1] that additional terms can appear on the LHS:

$$\sum_{p\geq 2} k_p \delta^{ac_1\ldots c_{2p}}_{bd_1\ldots d_{2p}} R_{c_1c_2}{}^{d_1d_2} \ldots R_{c_{2p-1}c_{2p}}{}^{d_{2p-1}d_{2p}} = 8\pi T^a_b \tag{2}$$

where k_p are constants. The antisymmetry ensures that the sum is finite: $2p+1 \leq d$. We normalize so that $k_1 = -1/4$ and define $k_0 = \Lambda$ so that we recover (1) when $d = 4$. Theories with the $p \geq 2$ terms are referred to as Lovelock theories. The Einstein equation is obtained only if one adds the additional criterion that the equation of motion should be linear in second derivatives of the metric, i.e. that the equation of motion is quasilinear.

Causal properties of a PDE are determined by its characteristic hypersurfaces. In GR, a hypersurface is characteristic iff it is null. It has been known for some time that this is not the case in Lovelock theories: characteristic hypersurfaces can be spacelike or null so gravity can propagate faster, or slower, than light [2, 3]. However, rather little is known about the properties of these characteristic hypersurfaces so in Ref. [4] we determined such surfaces for various solutions of Lovelock theories.

First we considered Ricci flat spacetimes with a Weyl tensor of algebraic type N. Any such spacetime is a solution of any Lovelock theory. We showed that there exist $d(d-3)/2$ Lorentzian metrics, which we called "effective metrics", such that a surface is characteristic iff it is null w.r.t. one of the effective metrics. (Note that $d(d-3)/2$ is the number of degrees of freedom of the gravitational field in d dimensions. Equivalently, it is the number of distinct physical polarizations of a graviton.) The null cones of the effective metrics form a nested set so, for this class of spacetimes, causality is determined by the effective metric with the outermost null cone.

Second we considered static, spherically symmetric, black hole solutions of Lovelock theories. We determined the characteristic hypersurfaces by considering the equations governing linearized perturbations of such black holes. Such perturbations can be decomposed into scalar, vector and tensor parts, and each satisfies a decoupled "master equation". From this one can determine the characteristic surfaces. For each type of perturbation (scalar, vector or tensor) one can define an effective metric. A hypersurface is characteristic iff it is null w.r.t. one of these effective metrics.

For a large enough black hole (compared to the length scales defined by the Lovelock coupling constants), the effective metrics are all Lorentzian and their null cones for a nested set. However, for a small black hole it can happen that one of the effective metrics becomes degenerate at a certain radius, outside the event horizon, and changes signature at smaller radius. This implies that the equation of motion is not hyperbolic in such spacetimes. The initial value problem for linear perturbations is not well-posed when this happens.

Such spacetimes are stationary so the violation of hyperbolicity is present for all time. However, it is interesting to ask whether hyperbolicity-violation can occur dynamically, i.e., if one starts from initial data for which the equation of motion is hyperbolic, can the equation become non-hyperbolic under time evolution? The answer is yes: one can consider a large black hole solution, for which the equation of motion is hyperbolic everywhere outside the horizon. Inside the black hole, the equation of motion becomes non-hyperbolic in a region near the singularity. Hence if one starts with the black hole initial data on a surface of constant t (i.e. an Einstein-Rosen bridge) then the equation of motion is initially hyperbolic but becomes non-hyperbolic after a certain time. Now, in analogy with cosmic censorship, we can ask whether this phenomenon is *generic*, i.e., what happens if we perturb the initial data? Ongoing work with G. Papallo indicates that generic linear perturbations blow up immediately inside the region where hyperbolicity is violated. This suggests that nonlinear effects may prevent, generically, violation of hyperbolicity.

In Ref. [5] we discussed shock formation in Lovelock theories. We considered solutions which are smooth apart from a discontinuity in curvature across a hypersurface. Such a hypersurface is necessarily characteristic. We showed that the amplitude of the discontinuity is governed by a transport equation: an ODE along each bicharacteristic curve. For GR, this equation is linear but in a Lovelock theory it is nonlinear. One can show that the solution will blow up in finite time if the initial amplitude is large enough, unlike in GR. We argued that this is analogous to shock formation in a compressible perfect fluid. However, unlike the case of a fluid (in 3+1 dimensions), it seems to be a large data effect. Indeed, a heuristic argument suggests that Minkowski spacetime is nonlinearly stable in Lovelock theories.

The most important outstanding issue concerning these theories is their well-posedness. Can one establish local existence and uniqueness of solutions, and continuous dependence on initial data? Of course one would have to restrict to initial data for which the equation of motion is hyperbolic. Alternatively, could one demonstrate that these theories are not well-posed?

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A priori estimates for the relativistic Euler equations

Todd A. Oliynyk

The relativistic Euler equations are defined by

$$\nabla_\mu T^{\mu\nu} = 0$$

where

$$T^{\mu\nu} = (\rho + p)v^\mu v^\nu + pg^{\mu\nu}$$

is the perfect fluid stress energy tensor,

$$g = g_{\mu\nu}dx^\mu dx^\nu$$

is a Lorentzian metric of signature $(-,+,+,+)$, ∇_μ is the Levi-Civita connection of $g_{\mu\nu}$, v^μ is the fluid four-velocity normalized by

$$g_{\mu\nu}v^\mu v^\nu = -1,$$

ρ is the proper energy density of the fluid, and p is the pressure.

Barotropic liquids are characterised by equations of state

$$\rho = \rho(p)$$

satisfying

$$\rho(0) = \rho_0 > 0$$

for some positive constant ρ_0. Since timelike matter-vacuum boundaries for fluid bodies with compact support are defined by the vanishing of the pressure, it follows that liquid bodies must have a jump discontinuity in the proper energy density at the fluid-vaccuum interface.

The free nature of the matter-vacuum boundary presents severe analytic difficulties that must be overcome in order to establish the local-in-time existence and uniqueness of solutions representing dynamical compact liquid bodies. In the non-relativistic setting, these analytic difficulties have been handled and a number of existence and uniqueness results are available; for example, see [5, 2, 1, 3, 4, 6] and references therein. In contrast, there is only one general, local-in-time existence and uniqueness result that applies to relativistic liquids and is given in [8]. In that article, the local existence and uniqueness of solutions is established using the theory of symmetric hyperbolic systems. The energy estimates derived from the symmetric hyperbolic theory involve a derivative loss that is repaired using a Nash-Moser iteration scheme. The derivative loss is due to sub-optimal estimates near the boundary and results in a high requirement on the regularity of the initial data in order to close the iteration scheme. We also note that derivative loss in the energy estimates has consequence beyond regularity issues such as in implementing summation-by-parts numerical schemes that are based on the energy estimates.

Here, we report on a new method for deriving a priori estimates for sufficiently smooth solutions of the relativistic Euler equations that represent dynamical compact liquid bodies. A precise statement of the a priori estimates can be found in [7]. The approach we take to establishing a priori estimates starts by showing that, in Lagrangian coordinates, sufficiently smooth solutions of the relativistic

Euler equations satisfy a system of non-linear wave equations and acoustic boundary conditions. The advantage of our wave formulation is that it is well suited to deriving energy estimates without derivative loss in the presence of a free matter-vacuum boundary. This is due, in part, to the wave structure of the equations, and in part, to the nature of the acoustic boundary conditions. Indeed in [7], we first establish a local existence and uniqueness theory for linear systems of wave equations with acoustic boundary conditions. This linear theory then provides the key technical result needed to establish our a priori estimates. We anticipate that this linear theory may also be of independent interest as it can be applied more generally to other systems of wave equations with acoustic boundary conditions.

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Weakly asymptotically hyperbolic manifolds and the constraint equations

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(joint work with James Isenberg, John M. Lee, Iva Stavrov Allen)

We discuss a weak version of asymptotic hyperbolicity, recently introduced in [1], and present results of [2], in which solutions to the Einstein constraint equations are constructed in the weakly asymptotically hyperbolic setting. We emphasize that [1, 2] contain a number of results beyond those presented here.

First, recall the "usual" notion of asymptotic hyperbolicity: Assume M is the interior of a compact 3-manifold $\overline{M}$ with boundary ∂M and fix a C^∞ *defining function* $\rho\colon \overline{M} \to [0,\infty)$ such that $\rho^{-1}(0) = \partial M$ and $d\rho \neq 0$ along ∂M. A Riemannian metric g is $C^{k,\alpha}$ *conformally compact* if $\overline{g} = \rho^2 g$ extends to a $C^{k,\alpha}$ metric on $\overline{M}$; such a metric is $C^{k,\alpha}$ *asymptotically hyperbolic* if $|d\rho|_{\overline{g}} \to 1$ as $\rho \to 0$, thus ensuring that the curvature operator of g approaches $-\mathrm{Id}$ at ∂M.

In general relativity, asymptotically hyperbolic metrics arise in the study of asymptotically flat spacetimes. If such a spacetime is conformally compact (of suitable regularity) then the metric g induced on a spacelike slices transversely meeting future null infinity is asymptotically hyperbolic. If K is the second fundamental form induced on such a slice, then (g, K) necessarily satisfy the *Einstein constraint equations*

$$R[g] - |K|_g^2 + (\mathrm{tr}_g K)^2 = 0, \qquad \mathrm{Div}_g K - d(\mathrm{tr}_g K) = 0. \tag{1}$$

(For simplicity we work in vacuum; in [2] Maxwell fields and fluid sources are also considered.) Our goal here is to construct *hyperboloidal initial data sets*—solutions (g, K) to (1) whose asymptotic geometry is compatible with having spacetime development admitting a conformal compactification at future null infinity.

We restrict to the *constant-mean-curvature* (*CMC*) case; under our sign convention this implies that $K = -g + \Sigma$, where $\mathrm{tr}_g \Sigma = 0$. We construct initial data sets by means of the *conformal method*: We fix a *free data set* (λ, μ), consisting of a Riemannian metric λ and a symmetric traceless covariant 2-tensor μ, and seek a solution to (1) of the form

$$g = \phi^4 \lambda \qquad K = -g + \phi^{-2}(\mu + \mathcal{D}_\lambda W); \tag{2}$$

here the function ϕ and vector field W are unknown, and $\mathcal{D}_\lambda W = \frac{1}{2}\mathcal{L}_W \lambda - \frac{1}{3}(\mathrm{Div}_\lambda W)\lambda$. The fields (g, K) given by (2) satisfy (1) if ϕ and W satisfy

$$L_\lambda W = -\mathrm{Div}_\lambda \mu, \qquad \Delta_\lambda \phi = \frac{1}{8} R[\lambda]\phi - \frac{1}{8}|\mu + \mathcal{D}_\lambda W|_\lambda^2 \phi^{-7} + \frac{3}{4}\phi^5, \tag{3}$$

where $L_\lambda W = \mathcal{D}_\lambda^* \mathcal{D}_\lambda W = -\mathrm{Div}_\lambda (\mathcal{D}_\lambda W)^\sharp$ is the self-adjoint *vector Laplace operator* and our convention for the scalar Laplacian is $\Delta_\lambda \phi = \mathrm{tr}_\lambda \mathrm{Hess}_\lambda \phi$.

The existence and regularity of solutions to (3) has been previously studied under the hypothesis that λ is C^2 asymptotically hyperbolic in [6, 7]. These works make clear that if the free data λ and μ satisfy $\overline{\lambda} = \rho^2 \lambda, \overline{\mu} = \rho\mu \in C^\infty(\overline{M})$, then the resulting fields $\overline{g} = \rho^2 g$ and $\overline{\Sigma} = \rho\Sigma$ are smooth on M, but may not extend smoothly to $\overline{M}$. Rather, "typical" CMC solutions to (1) have *polyhomogeneous* asymptotic expansions at $\rho = 0$, given in terms of powers of both ρ and $\log \rho$.

The origins of the log terms in these expansions is examined in [5], where it is shown that if (g, K) is a solution to (1) with asymptotically hyperbolic geometry, then in order for any corresponding spacetime development to admit a conformal infinity it is necessary that the *shear-free condition*

$$\overline{\Sigma}\big|_{\partial M} = \left[\mathrm{Hess}_{\overline{g}}(\rho) - \frac{1}{3}(\Delta_{\overline{g}}\rho)\overline{g}\right]_{\partial M} \tag{4}$$

be satisfied. This suggests that, when working in the conformally compact setting, the condition (4) be required for initial data to be considered physically reasonable.

In [5] it is shown that the CMC solutions to (1) constructed in [6] from C^∞ conformally compact free data does not generically (with respect to the "unphysical" $C^\infty(\overline{M})$ topology) satisfy (4). However, if we consider the "physical" $C^k(M)$ topology given by the metric g we have the following.

Theorem 1 (joint with Iva Stavrov Allen; see [3])**.** *Suppose (g, K) is a polyhomogeneous asymptotically hyperbolic CMC solution to* (1)*. Then for each sufficiently small $\varepsilon > 0$ there exists a solution $(g_\varepsilon, K_\varepsilon)$ to* (1) *that satisfies* (4)*, is smoothly conformally compact (i.e. $\overline{g}_\varepsilon, \overline{\Sigma}_\varepsilon \in C^\infty(\overline{M})$), and is such that $g_\varepsilon \to g$ and $K_\varepsilon \to K$ in the $C^k(M)$ topology for any $k \geq 0$.*

The gap between the $C^k(\overline{M})$ and $C^k(M)$ topologies motivates us to seek a space of metrics g on M (preferably an open subset of a Banach space) such that (i) the shear-free condition (4) is closed, (ii) is compatible with elliptic theory suitable for solving (3), and (iii) is as weak as possible—in particular, we would like to consider tensor fields having considerably less regularity at ∂M than in the interior M. We remark that the weighted spaces $C^{k,\alpha}_\delta(M) = \rho^\delta C^{k,\alpha}(M)$ are insufficient—if the weight δ is sufficiently high for $\mathrm{Hess}_{\overline{g}}(\rho)$ to be defined at ∂M, then the conformal structure at ∂M cannot vary continuously.

To define a suitable class of metrics, we fix a background metric $\overline{h}$ on $\overline{M}$ such that $|d\rho|_{\overline{h}} = 1$ along ∂M, and denote by $\overline{\nabla}$ the associated Levi-Civita connection. We then consider metrics $g = \rho^{-2}\overline{g}$ on M such that

$$\overline{g} \in C^{k,\alpha}_2(M), \qquad \overline{\nabla}\overline{g} \in C^{k-1,\alpha}_3(M). \tag{5}$$

The regularity (5) implies that $\overline{g} \in C^{0,1}(\overline{M})$, and thus that $|d\rho|_{\overline{g}} \in C^{0,1}(\overline{M})$. If (5) holds and if $|d\rho|_{\overline{g}} = 1$ along ∂M, then we say that g is *weakly $C^{k,\alpha}$ asymptotically hyperbolic*; denote the collection of such metrics by $\mathcal{M}^{k,\alpha;1}$.

The mapping properties of elliptic operators arising from asymptotically hyperbolic metrics are studied in [9, 4, 8], with applications to (1) addressed in [6]; all these results assume metrics are at least C^2 conformally compact. In [1] we generalize the Fredholm results of [8] to the weakly asymptotically hyperbolic case.

Theorem 2. *Suppose $\mathcal{P}$ is a second-order geometric* (*in the sense of* [8]) *self-adjoint elliptic operator determined by $g \in \mathcal{M}^{k,\alpha;1}$; let $\breve{\mathcal{P}}$ be the corresponding operator on hyperbolic 3-space $\mathbb{H}$. Suppose also that there exists compact $K \subset M$ such that $\|u\|_{L^2(M)} \leq C\|\mathcal{P}u\|_{L^2(M)}$ for all $u \in C^\infty_c(M \setminus K)$. Finally, suppose $\breve{\mathcal{P}} \colon C^{k,\alpha}_\delta(\mathbb{H}) \to C^{k-2,\alpha}_\delta(\mathbb{H})$ is Fredholm. Then so is $\mathcal{P} \colon C^{k,\alpha}_\delta(M) \to C^{k-2,\alpha}_\delta(M)$.*

One may in fact explicitly compute those values of δ for which $\mathcal{P}$ is Fredholm; see [1]. In [2] we use Theorem 2 to construct weakly initial data.

Theorem 3. *If $(\lambda, \mu) \in \mathcal{M}^{k,\alpha;1} \times C^{k-1,\alpha}_1(M)$ then there exists unique ϕ, W solving* (3) *such that (g, K) given by* (2) *satisfy* (1)*, with $g \in \mathcal{M}^{k,\alpha;1}$ and $\Sigma \in C^{k-1,\alpha}_1(M)$.*

The solutions to (1) given by Theorem 3 are much less regular than those constructed in [6]—if $g \in \mathcal{M}^{k,\alpha;1}$ then $\mathrm{Hess}_{\overline{g}}(\rho) \in L^\infty(\overline{M})$ and the shear-free condition need not even be defined. Thus we introduce $\mathcal{M}^{k,\alpha;2}$, those metrics $g \in \mathcal{M}^{k,\alpha;1}$ such that $\overline{\nabla}^2\overline{g} \in C^{k-2,\alpha}_4(M)$; for such metrics $\mathrm{Hess}_{\overline{g}}(\rho) \in C^{0,1}(\overline{M})$.

We furthermore introduce a conformally covariant tensor $\mathcal{H}_{\overline{\lambda}}(\rho)$ that characterizes the shear-free condition and seek solutions to (3) with $\mu = \rho^{-1}\mathcal{H}_{\overline{\lambda}}(\rho) + \nu$.

Theorem 4. *If $(\lambda, \nu) \in \mathcal{M}^{k,\alpha;2} \times C^{k-1,\alpha}_2(M)$ then there exists unique ϕ, W solving* (3) *such that (g, K) satisfy* (1) *and* (4)*, with $g \in \mathcal{M}^{k,\alpha;2}$ and $\Sigma \in C^{k-1,\alpha}_1(M)$.*

In fact $\overline{\Sigma} = \rho\Sigma \in C^{0,1}(\overline{M})$, thus ensuring (4) is defined. Furthermore, $(\lambda, \nu) \mapsto (g, K)$ is a continuous projection to an appropriate space of hyperboloidal initial data.

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Linear and non-linear wave equations on black hole backgrounds

Stefanos Aretakis

(joint work with Yannis Angelopoulos and Dejan Gajic)

One of the main outstanding open problems in mathematical general relativity is the stability of the *Kerr black hole family* in the context of the initial value problem to the Einstein-vacuum equations. The Kerr family is a two-parameter family $\mathcal{M}_{a,M}, 0 \leq |a| \leq M$, of stationary axisymmetric spacetimes which contain astrophysically relevant black hole regions.

As a first step in resolving the above conjecture one needs to investigate the evolution of solutions to linear and non-linear wave equations on such backgrounds. In this direction, Dafermos and Rodnianski [12], Blue and Andersson [1] and Tataru and Tohaneanu [21] have independently provided quantitative decay rates for solutions to the wave equation in the exterior region of slowly rotating Kerr black holes. Dafermos, Rodnianski and Shlapentokh-Rothman [13] presented analogous rates for the general subextremal Kerr family. These decay rates hold for solutions and all their derivatives up to and including the event horizon. Related non-linear results have been obtained by Luk [16] and Yang [22]. Moreover, recent work of Dafermos, Holzegel and Rodnianski has established the stability of the linearized

Einstein equations around the *Schwarzschild* sub-family which corresponds to the limit $|a| \to 0$.

This report addresses the above stability conjecture in the *extremal limit* $|a| \to M$ by investigating the evolution of linear and non-linear scalar fields on such backgrounds. *Extremal black holes* are characterized by the vanishing of the *surface gravity* on the event horizon. Geometrically, this means that if V is Killing and null normal on the horizon then

$$\nabla_V V = 0.$$

For the problem at hand, the fundamentally new aspect (compared to the subextremal case) is the degeneracy of the so-called *redshift effect* on the horizon.

For simplicity in this report we will only consider the extremal Reissner–Nordström (eRN) spacetimes, which constitute a one-parameter spherically family of spherically symmetric (extremal) black holes $\mathcal{M}_M$, $M > 0$, which satisfy the Einstein–Maxwell equations.

Part I: The linear wave equation on eRN

The story begins with the following Morawetz (integrated local energy decay) estimate [8, 9] for solutions ψ to the wave equation on eRN:

$$\int_0^\infty dt \int_{\Sigma_t} (r - r_{hor}) \cdot (r - r_{trap})^2 \cdot r^{-\sigma} \cdot |\partial\psi|^2 \leq C_\sigma \int_{\Sigma_0} |\partial\psi|^2, \tag{1}$$

where $\sigma > 4$ and $\{r = r_{hor}\}$ is the location of the event horizon and $\{r = r_{trap}\}$ of the trapped geodesics.

The degenerate at infinity factor $r^{-\sigma}$ can be removed by restricting to hyperboloidal slices Σ_t, which terminate at null infinity $\mathcal{I}^+$, and inserting the growing weight r on the right hand side. This result was first demonstrated by Dafermos and Rodnianski [11] and has led to the establishment of dispersive estimates for a wide class of hyperbolic equations.

The degenerate factor $(r - r_{trap})^2$ is a feature of hyperbolic trapping that takes place in the intermediate region $\{r_{hor} + \epsilon \leq r \leq R\}$ for some large $R > 0$ and small ϵ and can be removed by adding the initial energy of higher order derivatives of ψ on the right hand side.

On the other hand, the degenerate at the event horizon factor $(r - r_{hor})$ is an entirely new aspect of the problem in the extremal case. Let us postpone the detailed analysis of this factor and focus first on the corollaries of this estimate.

It can be shown that if $\epsilon > 0$ then all (translation-invariant) derivatives $\partial^k\psi$ decay for $r \geq r_{hor} + \epsilon$ asymptotically towards the future. We further obtain $|\psi| \to 0$ along the event horizon $\mathcal{H}$. On the other hand, higher order stability results do **not** hold along $\mathcal{H}$. Indeed, it can be shown that if Y is a translation-invariant transversal to the horizon vector field then the quantity

$$H[\psi] = \int_{S_v} Y\psi + \frac{1}{M}\psi$$

is **conserved**, i.e. independent of v, where S_v is a foliation of spherical sections of the horizon. Futher extensions of this conservation law have been obtained in [10, 15, 17].

The above immediately imply that for generic initial data we have $Y\psi$ **does not decay** along the horizon. Furthermore, we obtain the following **blow-up** result: $|\partial^k\psi| \to \infty$, $k \geq 2$ along $\mathcal{H}$.

In collaboration with Angelopoulous and Gajic [7] we have recently shown that the degeneracy of the Morawetz estimate (1) at the horizon can be removed if one loses derivatives (reminiscent of the structure of hyperbolic trapping) and if the conserved quantity $H[\psi] = 0$. If, on the other hand, $H[\psi] \neq 0$ then $\int_{r_{hor}\leq r\leq r_{hor}+\epsilon} |\partial\psi|^2 = \infty$. This work crucially uses an appropriate singular vector field construction and exploits a special structure of the wave operator on extremal backgrounds. On other hand, we have shown shown that degenerate horizons exhibit higher order stable trapping since no higher order Morawetz estimate holds: for generic smooth initial data (supported away from $\mathcal{H}$) we have $\int_{r_{hor}\leq r\leq r_{hor}+\epsilon} |\partial^k\psi|^2 = \infty$, $k \geq 2$.

Part II: Non-linear wave equations with null condition on eRN

Clearly all previous instabilities pose serious difficulties in proving global existence for non-linear equations. The first global well-posedness result for non-linear wave equations on such backgrounds was established by Angelopoulos [2] by restricting in spherical symmetry.

The general case is significantly harder. Specifically, in view of the growth of several higher order derivatives of ψ, one needs to derive *improved decay* for the quantities that do decay (e.g. for ψ itself). The required improved rates had previously been predicted in the numerical analysis of Reall et al [14] and Ori [19].

In collaboration with Angelopoulos and Gajic, we have derived a new *physical space method* [6] that allows us to obtain sharp quantitative decay rates. This result is also of use for the study of linear waves in the interior of the black hole region (c.f. upcoming work of Gajic).

In fact, in an upcoming series of papers [3, 5, 4] we have shown that our method allows us to obtain *sharp lower and upper bounds for the radiation field* on spherically symmetric asymptotically flat spacetimes. Tataru [20] had previously obtained sharp estimates for the radiation field in a more general context using, however, a large number of derivatives. Our method avoids the use of the fundamental solution and hence it is geared towards non-linear applications.

Returning to the study of non-linear wave equation on degenerate backgrounds, in collaboration with Angelopoulos we have obtained the following result:

Consider the equation

$$\Box_g\psi = A(\psi)g^{\alpha\beta}\partial_\alpha\psi\partial_\beta\psi,$$

for A a bounded function and with small enough data given on a spacelike hypersurface crossing the event horizon $\mathcal{H}^+$. Then, ***without the assumption of***

***spherical symmetry**, we have global well-posedness in the domain of outer communications up to and including $\mathcal{H}^+$ of an extremal Reissner–Nordström spacetime, and the following asymptotic behaviour:*

(1)
$$\|\psi(t)\|_{L^\infty} \leq \frac{1}{t^{1-\delta}} \text{ as } t \to \infty, \quad \|\partial\psi\|_{L^\infty} \leq C,$$

(2)
$$|\partial^k \psi| \to \infty \text{ for } k \geq 2 \text{ across the event horizon.}$$

Numerical results on the instability of eRN in the fully non-linear context have been obtain by Murata, Reall and Tanahashi [18].

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Photon sphere uniqueness and the static n-body problem

Carla Cederbaum

Generalizing a phenomenon well-know in Schwarzschild and other spherically symmetric spacetimes, we give a geometric definition of *photon spheres* in static asymptotically flat spacetimes [2]. Photon spheres are relevant in the analysis of black hole stability and in gravitational lensing. We then use this definition to prove that static vacuum asymptotically flat spacetimes possessing a single [2] or multiple photon spheres – together with Gregory J. Galloway [3] – must be isometric to the Schwarzschild spacetime in the exterior region of the photon sphere. In particular, multiple photon spheres cannot occur in the same static vacuum asympt. flat spacetime.

The two methods used in these two approaches can be extended to the electrostatic electro-vacuum setting, which has been done by Yazadjiev and Lazov [6] for a single and in joint work with Gregory J. Galloway [4] for multiple photon spheres, respectively. Here, the unique electro-vacuum asymptotically flat spacetime possessing an electrically charged photon sphere is the Reissner-Nordström spacetime, which is again spherically symmetric.

The uniqueness proofs in [2, 6] adapt and generalize the single (electro-)static black hole uniqueness proofs going back to Israel and will not be further discussed here. The proofs in [3, 4] modify and generalize arguments given for static black hole uniqueness by Bunting and Masood-ul-Alam [1] in the vacuum and by Masood-ul-Alam [5] in the electro-vacuum case, see below.

The uniqueness results for multiple photon spheres [3, 4] can easily be extended to include additional non-degenerate Killing black hole horizons. They can be re-interpreted as saying that there are no (electro-)static configurations of $k \in \mathbb{N}$ black holes and $n \in \mathbb{N}$ 'very compact' bodies with $k + n > 1$. Here, a body is considered 'very compact' if it is surrounded by a photon sphere; a property that astrophysicists expect to hold for suitably compact bodies.

In the following, we will restrict our attention to the non-charged case for simplicity of the exposition. We define a *photon sphere* $\mathfrak{P}^3$ in a (standard) static spacetime $(\mathbb{R} \times M^3, -N^2 dt^2 + g)$ to be a timelike umbilic hypersurface on which the static *lapse function* N – the length of the static Killing vector field – is constant. Here, umbilicity captures that any null geodesic initially tangent to $\mathfrak{P}^3$ is tangent to $\mathfrak{P}^3$ throughout. Constancy of N ensures that every null geodesic tangent to $\mathfrak{P}^3$ has constant potential energy $\log N$, or, equivalently, that its energy E – and color/frequency ν – as observed by the static observers is constant. The latter property is essential to characterize photon spheres as will be shown in joint work with Gregory J. Galloway elsewhere.

From this definition and the vacuum Einstein equations, we then derive quasi-local geometric properties of a photon sphere in a static vacuum spacetime:

Proposition 1 (Cederbaum [2]). *Let $(\mathbb{R}\times M^3, -N^2dt^2+g)$ be a static vacuum spacetime and let $(\mathfrak{P}^3,p)\hookrightarrow(\mathbb{R}\times M^3,-N^2dt^2+g)$ be a photon sphere. Write*

$$(1)\qquad \left(\mathfrak{P}^3,p\right)=\left(\mathbb{R}\times\Sigma^2,-N^2dt^2+\sigma\right)=\bigcup_{i=1}^{I}\left(\mathbb{R}\times\Sigma_i^2,-N_i^2dt^2+\sigma_i\right),$$

where each $\mathfrak{P}_i^3=\mathbb{R}\times\Sigma_i^2$ is a connected component of $\mathfrak{P}^3$. Then the embedding $(\Sigma^2,\sigma)\hookrightarrow(M^3,g)$ is totally umbilic with constant mean curvature H_i on the component Σ_i^2. The scalar curvature of the component (Σ_i^2,σ_i), ${}^{\sigma_i}\mathrm{R}$, is a non-negative constant, namely ${}^{\sigma_i}\mathrm{R}=\frac{3}{2}H_i^2$. Moreover, the normal derivative of the lapse function N in direction of the outward unit normal ν to Σ^2, $\nu(N)$, is also constant on every component (Σ_i^2,σ_i), $\nu(N)_i:=\nu(N)|_{\Sigma_i^2}$. For each $i\in\{1,\dots,I\}$, either $H_i=0$ and Σ_i^2 is a totally geodesic flat torus or Σ_i^2 is an intrinsically and extrinsically round CMC sphere for which the above constants are related via

$$(2)\qquad N_iH_i=2\nu(N)_i,\quad (r_iH_i)^2=\frac{4}{3},$$

where $r_i:=\sqrt{\frac{|\Sigma_i^2|_{\sigma_i}}{4\pi}}$ denotes the area radius *of Σ_i^2.*

Using Proposition 1, we obtain the following theorem:

Theorem 1 (Cederbaum–Galloway [3]). *Let $(\mathbb{R}\times M^3,-N^2dt^2+g)$ be a static vacuum asymptotically flat spacetime that possesses a (possibly disconnected) photon sphere $(\mathfrak{P}^3,p)\hookrightarrow(\mathbb{R}\times M^3,-N^2dt^2+g)$, arising as the inner boundary of $\mathbb{R}\times M^3$. Let m denote the ADM-mass of (M^3,g). Then $m>0$ and $(\mathbb{R}\times M^3,-N^2dt^2+g)$ is isometric to the region $\{r\geq 3m\}$ exterior to the photon sphere $\{r=3m\}$ in the Schwarzschild spacetime of mass m. In particular, $(\mathfrak{P}^3,p)$ is connected and a cylinder over a topological sphere.*

Before sketching the proof of Theorem 1, let us very quickly review the proof by Bunting–Masood-ul-Alam [1]. In short, they double the asympt. flat static 3-manifold (M^3,g) across its black hole inner boundary $\cup_{i=1}^{I}\Sigma_i^2$ to obtain a new manifold $(\bar{M}^3,\bar{g})$ which is smooth away from a finite set of gluing 2-surfaces Σ_i^2, $C^{1,1}$ across them, and has two asympt. flat ends. They then conformally modify the manifold $(\bar{M}^3,\bar{g})$ such that the original asymptotic end transforms to have vanishing ADM-mass and the doubled end can be one-point compactified. By construction, the new manifold $(\tilde{M}^3,\tilde{g})$ has vanishing scalar curvature, is geodesically complete, and is asympt. flat with vanishing ADM-mass. By the rigidity statement of the positive mass theorem – more precisely, a weak version due to Bartnik –, the conformally modified manifold $(\tilde{M}^3,\tilde{g})$ must be isometric to Euclidean space. In other words, the original manifold (M^3,g) is conformally flat.

Combining this with the static equations, it follows that $(\mathbb{R}\times M^3, -N^2dt^2+g)$ is necessarily isometric to the Schwarzschild spacetime.

For the proof of Theorem 1, we proceed as follows: Each photon sphere component Σ_i^2 is assigned a *Schwarzschild mass* $\mu_i := r_i/3 > 0$ computed from its area radius r_i. We then show via Proposition 1 that the neck $(2r_i, \mu_i]\times\mathbb{S}^2$ of the Schwarzschild spatial slice $(2r_i,\infty)\times\mathbb{S}^2$ with metric $\varphi_i(r)^{-2}dr^2+r^2\Omega$ can be glued to (M^3,g) across Σ_i^2 in a $C^{1,1}$ fashion. Here, Ω is the canonical metric on the unit sphere and $\varphi_i(r)=\sqrt{1-2\mu_i/r}$ as usual. In order to glue the lapse function N of (M^3,g) to the Schwarzschild lapse function φ_i across Σ_i^2, more care needs to be taken: We exploit the lapse scaling invariance of the static vacuum equations $\triangle N=0$, $\nabla^2 N = N\,\mathrm{Ric}$, and glue N to $3m_i\varphi_i/r_i$, with $m_i := \int_{\Sigma_i^2}\nu(N)_i\,d\sigma_i/4\pi$ the pseudo-Newtonian mass of Σ_i^2. In this way, we obtain a new static vacuum asympt. flat 3-manifold $(\hat{M}^3,\hat{g})$ with black hole inner boundary. This manifold $(\hat{M}^3,\hat{g})$ is smooth away from the gluing 2-surfaces Σ_i^2 and $C^{1,1}$ across them. The Bunting–Masood-ul-Alam method can then be applied to $(\hat{M}^3,\hat{g})$ after ensuring that the conformal factor stays positive. The claim of Theorem 1 follows.

In a forthcoming paper, the author will combine the ideas described above with new geometric and PDE arguments, in particular a new class of metrics generalizing the Schwarzschild class of metrics, to prove the following theorem:

Theorem 2 (Cederbaum, to appear)**.** *Let (M^n,g) be a smooth, asymptotically flat Riemannian manifold of non-negative scalar curvature and ADM-mass m and let $N: M^n\to\mathbb{R}^+$ be harmonic function on (M^n,g) that tends to 1 at infinity. Assume that (M^n,g) has an inner boundary $\cup_{i=1}^I\Sigma_i^{n-1}$ such that each $(\Sigma_i^{n-1},\sigma_i)$ is umbilic, has constant mean curvature H_i and constant scalar curvature ${}^{\sigma_i}\mathrm{R}>0$. Assume moreover that $N|_{\Sigma_i^{n-1}} =: N_i$ and its normal derivative $\nu(N)|_{\Sigma_i^{n-1}} =: \nu(N)_i$ are constant on Σ_i^{n-1} and that there exist constants $0\le c_i<(n-1)/(n-2)$ such that*

$$c_i\nu(N)_i = H_iN_i\left(1-\frac{n-2}{n-1}c_i\right) \quad \textit{and} \quad H_i^2 = c_i\,{}^{\sigma_i}\mathrm{R}, \tag{3}$$

$$\sum_{i=1}^{I} m_i = m, \quad \textit{where} \quad m_i := \frac{1}{4\pi}\int_{\Sigma_i^2}\nu(N)_i\,d\sigma_i. \tag{4}$$

Then (M^n,g) is isometric to n-dim. Schwarzschild-Tangherlini of mass m.

Theorem 2 can be applied to re-prove static vacuum black hole uniqueness in $n+1$ spacetime dimensions (reproducing a result by Gibbons, Ida, and Shiromizu), and to prove static vacuum photon sphere uniqueness in $n+1$ spacetime dimensions, generalizing Theorem 1. It does not appeal to the full static vacuum equations but instead to a generalization of the assertions in Proposition 1.

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Non-existence of time-periodic vacuum spacetimes

Volker Schlue
(joint work with Spyros Alexakis, Arick Shao)

In my talk I presented a recent result that rules out genuinely time-periodic behaviour in general relativity.

The question of existence of time-periodic dynamics arose in the study of isolated self-gravitating systems, in particular the 2-body problem in general relativity. In the earliest treatment due to Einstein, Infeld and Hoffman [10] it was found that in the context of the post-Newtonian approximation circular (and thus time-periodic) orbits are possible. It was only later with a complete understanding of the higher orders in the post-Newtonian expansion, due to Damour, Deruelle and Blanchet, see e.g. [9, 4], that circular orbits could be ruled out, at least in the context of approximations. It is also clear from their work (c.f. derivation of the "radiation reaction force") that the underlying mechanism which prevents periodic motion is the emission of gravitational waves.

In our approach we consider asymptotically flat spacetimes $(\mathcal{M}^{3+1}, g)$ which are solutions to the Einstein vacuum equations in the exterior of a spatially compact set (which can be thought of as containing the sources of the emitted gravitational waves); see Figure 1 (L). We recall that a detailed description of the asymptotics (towards null infinity) of dynamical vacuum spacetimes was obtained by Christodoulou and Klainerman in [5], see also [6].

We shall use a purely geometric notion of "time-periodicity", c.f. [3], which can also be localised to a neighborhood of infinity.

Definition 1. *An asymptotically flat spacetime* $(\mathcal{M}, g)$ *is called* time-periodic *if it admits a discrete isometry* φ *with time-like orbits, i.e. a map* φ *such that* $\varphi(p) \in I^+(p)$*, and* $\varphi^* g = g$.

In [1] we have obtained the following non-existence result:

Theorem 1. *Any asymptotically flat spacetime* $(\mathcal{M}, g)$ *arising as a solution to the vacuum equations* $Ric(g) = 0$ *from regular initial data, which is assumed to be time-periodic (near infinity), is in fact* stationary *near infinity.*

The theorem asserts that there exists a time-like Killing vectorfield near infinity, namely a time-like vectorfield T such that $\mathcal{L}_T g = 0$ on $\mathcal{D}$, see Figure 1 (R). Thus any discrete isometry with time-like orbits is in fact induced by a continuous isometry, and thus genuinely time-periodic solutions do not exist, at least in a neighborhood of infinity. Here the domain $\mathcal{D}$ is an *arbitrarily small* neighborhood of infinity, which as we shall see is related to the positive mass property of a non-trivial spacetime.

In the cosmological (spatially closed) setting, there are conclusive results that rule out time-periodic solutions due to Galloway [11]; (in that setting one can even prove that the spacetime is static). In the asymptotically flat setting the first results are due to Papapetrou [12]. More recently, Bičák, Scholtz, and Tod obtained a precursor of the above theorem [3], however under the very restrictive assumption of analyticity of the spacetime at infinity. Theorem 1 in particular removes the analyticity assumption and applies under physically relevant conditions.

As already indicated, the heuristic reason Theorem 1 is true is energy dissipation: Any self-gravitating system should lose energy due to the emission of gravitational waves, and thus approach a stationary state. We are thus led to the more ambitious question if a dynamical spacetime is stationary under a weaker "no radiation" condition:

Definition 2. *An asymptotically flat spacetime is called* non-radiating *if the Bondi mass is constant along null infinity.*

The Bondi mass $M(u)$ is a non-negative number associated to each "retarded time" u, and known to be dynamically non-increasing which justifies its interpretation as the amount of energy contained in the system at time u; see [6, 7] for the relevant concepts.

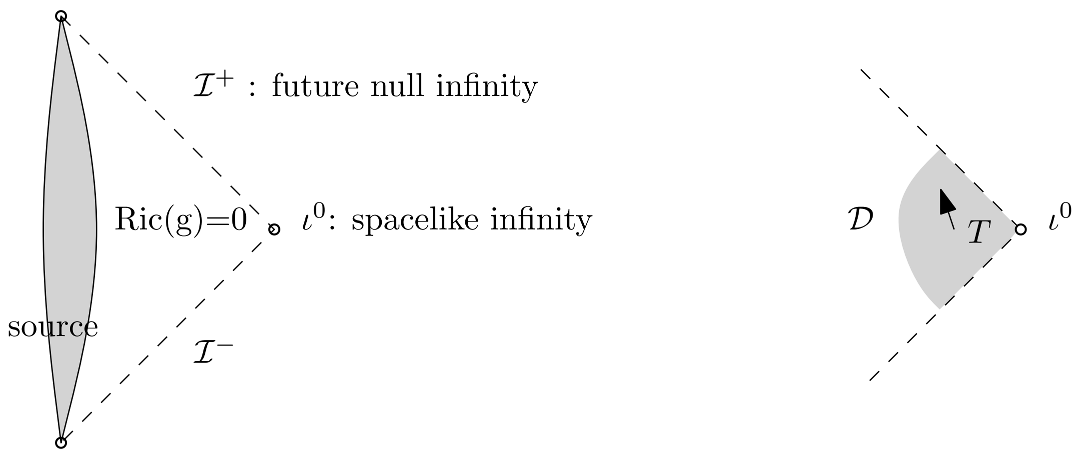

FIGURE 1. Left: Self-gravitating system in general relativity. Right: Stationary domain in a neighborhood of infinity.

In [1] we have proven that non-radiating spacetimes are indeed time-independent far away from the sources, at least if an additional regularity is assumed at infinity.

Theorem 2. *Any asymptotically flat vacuum spacetime* $(\mathcal{M}, g)$ *arising from regular initial data, which is assumed to be non-radiating, is stationary near infinity, provided the spacetime is also smooth at infinity.*

Here "smooth at infinity" means that all geometric quantities admit a complete asymptotic expansion along null infinity in powers of $1/r$, which is well behaved towards spacelike infinity. Note that this assumption is not needed in Theorem 1, where all regularity properties of null infinity are inherited from regularity assumptions on the data by virtue of time-periodicity. It remains an open question if Theorem 2 is true without additional regularity assumptions at infinity.

The proofs of Theorems 1, 2 rely crucially on a uniqueness result for the extension of a "candidate" Killing vectorfield from infinity in Ricci flat manifolds. In [2] we have proven a unique continuation from infinity result for linear waves on asymptotically flat spacetimes, which seems of independent interest:

Theorem 3. *Let* $(\mathcal{M}, g)$ *be an asymptotically flat spacetime with* positive mass, *and* $L = \Box_g + a \cdot \nabla + V$ *a linear operator with decaying coefficients. Suppose* ϕ *is a solution to* $L\phi = 0$ *which moreover is assumed to vanish to all orders at infinity. Then* $\phi \equiv 0$ *in a neighborhood of infinity.*

The theorem in particular allows for a localisation to an arbitrarily small neighborhood of spacelike infinity. This is intimately related to the positivity property of the mass of the spacetime, and it can easily seen to be false in the case of vanishing mass, namely for solutions to linear wave equations on Minkowski space. The proof of Theorem 3 exploits the behaviour of null geodesics in spacetimes with positive mass, which can be seen to "bend more quickly" towards null infinity, than in the case of vanishing, or negative mass. This approach is reminiscent of ideas of Penrose, who sought to characterise the positive mass property purely by the behaviour of null gedesics near null infinity [13]; see also [8].

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"Matter does not matter" for self-gravitating perfect fluids?

Florian Beyer
(joint work with Philippe G. LeFloch)

It is a longstanding conjecture in general relativity that the gravitational dynamics close to a cosmological singularity is governed by an effective equation where rapidly decaying spatial derivatives terms in the field equations are neglected relative to time derivatives; this phenomenon is sometimes referred to as *velocity term dominance* [7]. Similarly, it is conjectured that terms involving matter variables become small (except for certain extreme forms of matter, e.g. scalar fields and stiff fluids [8]): the *matter does not matter* hypothesis. In this short abstract here, we discuss a result [4] where the relationship and relative significance of these two conjectures are studied and compared for the first time.

Theorem 1. *Pick arbitrary data functions* $v^0_* > 0$, v^1_*, $1 > k > 0$, P_*, Q_*, Q_{**} *in* $C^\infty(T^1)$, *and constants* $\gamma \in (1,2)$ *and* $M_{**} \in \mathbb{R}$ *such that*

$$M_*(x) = M_{**} + \int_0^x \Big(2k(\xi)e^{2P_*(\xi)}Q_{**}(\xi)Q'_*(\xi) - k(\xi)P'_*(\xi) - \frac{2\gamma v^1_*(\xi)(v^0_*(\xi))^{\frac{1-2\gamma}{\gamma-1}}}{\gamma-1}\Big)d\xi,$$

is 2π*-periodic. Moreover set*

$$\hat{v}^1_*(x) = v^1_*(x)e^{\frac{2-\gamma}{2(\gamma-1)}M_*(x)}.$$

Then there exists a constant $\delta > 0$ *and a solution of the Einstein-Euler equations with equation of state* $p = (\gamma-1)\rho$ *determined by the following properties:*

(1) *The metric is Gowdy symmetric and, in appropriate coordinates* (t,x,y,z), *asymptotically local Kasner for the data functions* k, M_*, P_*, Q_*, Q_{**}.
(2) *The fluid is Gowdy symmetric, i.e., the fluid vector field* v^α *is of the form*

$$v^\alpha = v^0(t,x)\partial^\alpha_t + v^1(t,x)\partial^\alpha_x,$$

and the functions $v^0, v^1 \in C^\infty((0,\delta] \times T^1)$ have the property that for each sufficiently large integer q there exists a constant $C > 0$ such that

$$\left\| t^{-\mu^1_{[\mathrm{F}]}} \left(v^0(t) t^{-\Gamma} - v^0_* \right) \right\|_{H^q(T^1)} + \left\| t^{-\mu^2_{[\mathrm{F}]}} \left(v^1(t) t^{-2\Gamma} - \hat{v}^1_* \right) \right\|_{H^q(T^1)} \leq C$$

for all sufficiently small $t > 0$. Here $\mu^1_{[\mathrm{F}]}, \mu^2_{[\mathrm{F}]} > 0$ are some exponents and $\Gamma := \frac{1}{4}\left(3\gamma - 2 - (2-\gamma)k^2\right)$.

Let us first clarify some of the terms and assumptions. We restrict to Gowdy symmetric [5] fluids and spacetimes. Since the spatial topology is assumed as T^3, this means that the spatial dependence of all functions is described by one spatial coordinate $x \in T^1$. In fact, previous studies of the analogue vacuum situation suggest that the behavior of more general classes of solutions can be strongly oscillatory and therefore beyond the reach of current mathematical techniques. Regarding the first statement of our theorem, we call a Gowdy symmetric spacetime *asymptotically local Kasner* for data k, M_*, P_*, Q_*, Q_{**} if, with respect to given coordinates (t,x,y,z), (i) the metric has the form

$$\begin{aligned} g =& g_{00}(t,x)dt^2 + 2g_{01}(t,x)dtdx + g_{11}(t,x)dx^2 \\ &+ R(t,x)\big(e^{P(t,x)}(dy + Q(t,x)dz)^2 + e^{-P(t,x)}dz^2\big), \end{aligned}$$

where each function is assumed to be smooth, and (ii) if for each sufficiently large integer q there exists a constant $C > 0$ such that

$$\begin{aligned} &\left\| t^{-\mu^1_{[\mathrm{G}]}} \left(g_{00}(t) t^{-(k^2-1)/2} + e^{M_*} \right) \right\|_{H^q(T^1)} + \left\| t^{-\mu^1_{[\mathrm{G}]}} D \left(g_{00}(t) t^{-(k^2-1)/2} \right) \right\|_{H^q(T^1)} \\ &+ \left\| t^{-\mu^2_{[\mathrm{G}]}} \left(g_{11}(t) t^{-(k^2-1)/2} - e^{M_*} \right) \right\|_{H^q(T^1)} + \left\| t^{-\mu^2_{[\mathrm{G}]}} D \left(g_{11}(t) t^{-(k^2-1)/2} \right) \right\|_{H^q(T^1)} \\ &+ \left\| t^{-\mu^3_{[\mathrm{G}]}} g_{01}(t) t^{-(k^2-1)/2} \right\|_{H^q(T^1)} + \left\| t^{-\mu^3_{[\mathrm{G}]}} D g_{01}(t) t^{-(k^2-1)/2} \right\|_{H^q(T^1)} \\ &+ \left\| t^{-\mu^4_{[\mathrm{G}]}} \left(R(t) t^{-1} - 1 \right) \right\|_{H^q(T^1)} + \left\| t^{-\mu^4_{[\mathrm{G}]}} D \left(R(t) t^{-1} \right) \right\|_{H^q(T^1)} \\ &+ \left\| t^{-\mu^5_{[\mathrm{G}]}} \left(e^{P(t)} t^k - e^{P_*} \right) \right\|_{H^q(T^1)} + \left\| t^{-\mu^5_{[\mathrm{G}]}} D \left(e^{P(t)} t^k \right) \right\|_{H^q(T^1)} \\ &+ \left\| t^{-\mu^6_{[\mathrm{G}]}} \left((Q(t) - Q_*) t^{-2k} - Q_{**} \right) \right\|_{H^q(T^1)} + \left\| t^{-\mu^6_{[\mathrm{G}]}} D \left((Q(t) - Q_*) t^{-2k} \right) \right\|_{H^q(T^1)} \\ &\leq C \end{aligned}$$

for all sufficiently small $t > 0$ and some collection of exponents $\mu^i_{[\mathrm{G}]} > 0$. Here we write $D := t\partial_t$. A particular consequence of our hypothesis and the asymptotically local Kasner property is that all solutions of the theorem have a curvature singularity in the limit $t \searrow 0$ and that the above mentioned *velocity term dominance* holds.

Regarding the second statement of the theorem, note that we describe the fluid by a (in general not normalized) timelike vector field v^α in accordance with the formalism in [6], which renders the Euler equations as explicitly symmetric hyperbolic. The anticipated two degrees of freedom of the fluid are represented by the two fluid data functions v_*^0 and v_*^1. Since the restrictions on k and γ imply that the quantity Γ is strictly positive, the second statement of the theorem can be translated as follows: (i) the energy density associated with the fluid diverges in the limit $t \searrow 0$ (as expected) and (ii) the physical fluid velocity measured by freely falling timelike observers approaches zero for $t \searrow 0$ (possibly less expected). While this behavior of the fluid is universal for the class of models covered by our theorem, we remark that some additional freedom in the choice of the data for *half-polarized* Gowdy models (given by $Q_* = \text{const}$) gives rise to a new critical phenomenon which can be interpreted as the competition between highly anisotropic gravitational forces close to the singularity and, by definition, isotropic fluid counter-forces. These details, which are not covered by the theorem above, are described in [4].

In the vacuum limit $v_*^0 \to \infty$, the integral constraint and the equation for M_* in the theorem reduce to their vacuum analogues, cf. e.g. [3]. The asymptotic local Kasner property therefore implies that the metric variables have the same qualitative leading-order behavior in the vacuum and in non-vacuum case. However, the fact that the fluid data occur explicitly in the above equation for M_* implies that it is impossible to match a non-vacuum solution of the theorem with a vacuum solution of the theorem in a way that the two metrics are asymptotically local Kasner for the *same data*. Nevertheless, it *is* possible to match a non-vacuum solution of the theorem with a vacuum solution of the theorem so that the metric variables agree in leading order *at a single fixed spatial coordinate point* x. Hence, while *"matter does not matter"* is therefore not a uniform property, it holds for any single timelike observer who approaches the singularity. In [4] we provide more details of all these statements. A particular fact is that the fluid variables are *less negligible* than spatial derivatives in a well-defined sense which puts the two conjectures above into context.

We conclude with a few technical remarks. The proof of our theorem makes essential use of the Fuchsian theory introduced in [3, 1]. A key new technique was introduced in [4] that allows us to cover the *full* interval $k \in (0, 1)$ by requiring only some finite differentiability of the data. Our theorem can therefore in fact be written more generally for C^l data where l is a sufficiently large integer. Another important remark is that [4] makes significant use of techniques developed in [2] where the Fuchsian method is applied to the wave gauge formalism of Einstein's equations. In this formalism, while the equations are explicitly hyperbolic, they are also significantly more complex than in earlier treatments, where e.g. areal coordinates were used, and subtle cancellations need to be taken into account. Moreover, the analysis of the constraints and the subsidiary system are more involved. We hope that the results in [2] regarding these issues are useful also for other future studies, in particular, of $U(1)$-symmetric vacuum solutions.

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Quasilinear wave equations on Kerr-de Sitter spacetimes

Peter Hintz
(joint work with András Vasy)

In a series of works [8, 7, 9], we have developed a general framework for the global analysis of nonlinear wave equations on geometric classes of Lorentzian manifolds, ultimately based on Vasy's recent breakthrough [13]. The main examples of manifolds that fit into this framework are cosmological spacetimes such as de Sitter and Kerr-de Sitter spacetimes, and perturbations of these. In particular, we establish the global solvability of semilinear and quasilinear wave equations on cosmological black hole spacetimes and obtain the asymptotic behavior of solutions using a novel approach to the global study of nonlinear hyperbolic equations. The key advance is overcoming the problems caused by the *normally hyperbolic trapping*, in the present context realized by Wunsch and Zworski [14], by combining microlocal analysis and a Nash-Moser iteration.

For concreteness, we focus on the special case of Kerr-de Sitter space here, but it is important to keep in mind that the setting is more general. We work on a neighborhood Ω° of the domain of outer communications of a Kerr-de Sitter black hole, see Figure 1, and use a function t_*, roughly equal to the Boyer-Lindquist coordinate t away from the horizons, but finite up to the horizon, to measure decay.

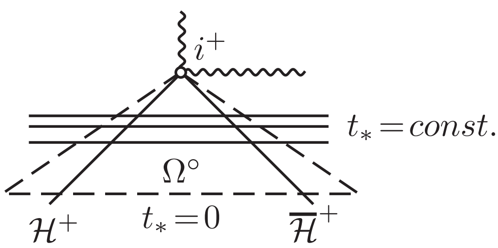

Figure 1. Setup for quasilinear wave equations on Kerr-de Sitter space.

Theorem 1. *Consider the equation*

$$\Box_{g(u,du)} u = q(u, du) \quad \text{in } \Omega^\circ, \tag{1}$$

where $g(0,0)$ is a Kerr-de Sitter metric with cosmological constant $\Lambda > 0$, mass M and very small angular momentum a, i.e. $|a| \ll M$; further $g(u,du)$ does not depend explicitly on t_* (only implicitly via u). Moreover, $q(u,du)$ is a polynomial in u and its derivatives, has no explicit t_*-dependence, and each of its summands contains at least 1 factor of du.

Under these assumptions and for small initial data, the equation (1) has a unique global solution u of the form $u = u_0 + \widetilde{u}$, $u_0 \in \mathbb{R}$, $|\widetilde{u}(t_*)| \lesssim e^{-\alpha t_*}$, with $\alpha = \alpha(\Lambda, M, a) > 0$.

See [9] for details. To our knowledge, this is the first global existence result for quasilinear perturbations of black holes. We note however that Dafermos, Holzegel and Rodnianski [2] have constructed backward solutions for Einstein's equations on the Kerr background; for backward constructions the trapping does not cause difficulties.

Traditionally [10], such non-linear global existence results are established by showing the existence of (almost) conserved energy-type quantities and combining this with the well-known local well-posedness, using a continuous induction argument. Our approach is to instead gain a precise understanding of the linear equation *globally*, i.e. with respect to both global regularity and asymptotics/decay, and then to use an iteration scheme for solving the PDE (1) in which one solves a linear equation globally at each step.

The linear analysis takes place on a compactification of the spacetime at future infinity: We define $\tau = e^{-t_*}$ and add $\tau = 0$ to Ω°, hence obtaining a manifold with boundary Ω; the stationary Kerr-de Sitter metric $g = g(x, dt_*, dx)$ becomes a Lorentzian b-metric $g = g(x, \frac{d\tau}{\tau}, dx)$, and correspondingly the wave operator $\Box_g$ is a b-differential operator, i.e. a linear combination of products of τD_τ and D_x, degenerating in a controlled manner at the boundary $\tau = 0$. Melrose's b-calculus [11] provides powerful microlocal tools to analyze such operators: The natural function spaces are b-Sobolev spaces $H_{\mathrm{b}}^{s,\alpha} = \tau^\alpha H_{\mathrm{b}}^s = e^{-\alpha t_*} H_{\mathrm{b}}^s$, which measure regularity under τD_τ and D_x relative to an exponentially weighted (spacetime!) L^2 space; in particular, *local* $H_{\mathrm{b}}^{s,\alpha}$ regularity near a point in $\tau = 0$ corresponds to *uniform* (in $t_* \to \infty$) regularity in a weighted space in the non-compact picture. The regularity analysis for the linear equation $\Box_g u = f$ then uses the propagation of singularities theorem of Duistermaat and Hörmander [4], propagating regularity from the Cauchy hypersurface along null-geodesics (lifted to the (b-)cotangent bundle, where singularities are measured microlocally), with the caveat that $\tau = 0$ is still 'at infinity', i.e. no null-geodesic in $\tau > 0$ reaches $\tau = 0$ in finite time. However, the null-geodesic flow extends to $\tau = 0$ as well, and in fact exhibits a saddle point structure where the horizons intersect future infinity (Figure 2). A propagation result there, which can be thought of as a microlocal version of the redshift estimates of Dafermos and Rodnianski [3], allows one to conclude $H_{\mathrm{b}}^{s,\alpha}$ regularity there for u for a suitable range of regularities s and weights α. Further, there is normally hyperbolic trapping, and we use the work of Wunsch and Zworski [14] and Dyatlov [6] for the propagation of singularities there; in particular, there is a loss of differentiability in u compared to the usual hyperbolic gain of 1 derivative relative to f.

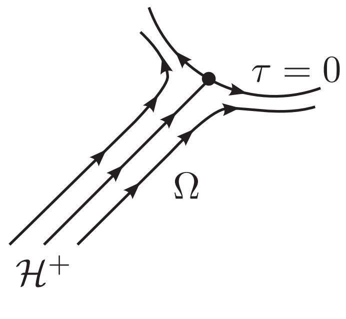

FIGURE 2. Null-geodesic flow near future infinity on the compactified spacetime.

For asymptotics and decay, we use the Mellin transform in τ (equivalently, the Fourier transform in $-t_*$); then, the *resonances*, also known as *quasinormal modes*, which are the poles of the meromorphic continuation of $\widehat{\Box_g}(\sigma)^{-1}$, encode the asymptotic behavior of waves [1, 5, 13]. For scalar waves $\Box_g u = f$ on Kerr-de Sitter backgrounds, we thus obtain $u = u_0 + \widetilde{u}$, where $u_0 \in \mathbb{R}$, due to a simple resonance at 0, and $\widetilde{u} \in H_{\mathrm{b}}^{s,\alpha}$, provided $f \in H_{\mathrm{b}}^{s,\alpha}$, for $\alpha > 0$ small and $s \in \mathbb{R}$ large.

For quasilinear equations of the form (1) then, one needs to analyze regularity and asymptotics for metrics of the form $g = g(u, du)$ with such u; hence, g equals a stationary metric $g(u_0, 0)$ modulo an exponentially decaying perturbation. This was first done in [7], with improvements given in [9] to facilitate tame estimates, which are the crucial ingredients in the Nash-Moser iteration scheme [12] used to deal with the loss of derivatives. —

Our framework directly applies to nonscalar problems as well, and we obtain linear and nonlinear results both for scalar equations and for equations on natural vector bundles. To a large extent, our work is motivated by the black hole stability problem for cosmological spacetimes, and we expect the resolution of this problem to be within reach now: Indeed, in suitable gauges, Einstein's equation becomes a principally scalar quasilinear hyperbolic system which satisfies all the requirements of our framework, possibly except for a natural condition on the resonances, commonly referred to as *mode stability*; the crucial task thus is to find a gauge in which the linearized Einstein equation has well-behaved resonances.

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On the rotation curves for axially symmetric disk solutions of the Vlasov-Poisson system

Håkan Andréasson
(joint work with Gerhard Rein)

The rotation curve of a galaxy depicts the magnitude of the orbital velocities of visible stars or gas particles in the galaxy versus their radial distance from the center. In the pioneering observations by Bosma [2] and Rubin [4] it was found that the rotation curves of spiral galaxies are approximately flat except in the inner region where the rotation curves rise steeply. Independent observations in more recent years agree with these conclusions. The flat shape of the rotation curves is an essential reason for introducing the concept of dark matter. Let us cite from [3]: "Perhaps the most persuasive piece of evidence [for the need of dark matter] was then provided, notably through the seminal works of Bosma and Rubin, by establishing that the rotation curves of spiral galaxies are approximately flat [2, 4]. A system obeying Newton's law of gravity should have a rotation curve that, like the Solar system, declines in a Keplerian manner once the bulk of the mass is enclosed: $V_c \propto r^{-1/2}$."

The last statement is heuristic and it is therefore essential to construct self-consistent mathematical models which describe disk galaxies and study the corresponding rotation curves. For this purpose it is natural to consider the Vlasov-Poisson system which is often used to model galaxies and globular clusters.

The Vlasov-Poisson system is given by

$$\partial_t F + v \cdot \nabla_x F - \nabla_x U \cdot \nabla_v F = 0,$$
$$\Delta U = 4\pi R, \quad \lim_{|x|\to\infty} U(t,x) = 0,$$
$$R(t,x) = \textstyle\int_{\mathbb{R}^3} F(t,x,v)\, dv.$$

Here $F : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}_0^+$ is the density function on phase space of the particle ensemble, i.e., $F = F(t,x,v)$ where $t \in \mathbb{R}$ and $x, v \in \mathbb{R}^3$ denote time, position, and velocity respectively. The mass of each particle in the ensemble is assumed to be equal and is normalized to one. The mass density is denoted by R and the gravitational potential by U. The latter is given by

$$U(t,x) = -\int_{\mathbb{R}^3} \frac{R(t,y)}{|x-y|}\, dy.$$

In this investigation we are interested in extremely flattened axially symmetric galaxies where all the stars are concentrated in the (x_1, x_2)-plane. We therefore assume that

$$F(t,x,x_3,v,v_3) = f(t,x,v)\delta(x_3)\delta(v_3),$$

where from now on $x, v \in \mathbb{R}^2$ and δ is the Dirac distribution. The stars in the plane will only experience a force field parallel to the plane, and the Vlasov-Poisson system for the density function $f = f(t,x,v)$, $x, v \in \mathbb{R}^2$, takes the form

$$\partial_t f + v \cdot \nabla_x f - \nabla_x U \cdot \nabla_v f = 0, \tag{1}$$

$$U(t,x) = -\int_{\mathbb{R}^2} \frac{\rho(t,y)}{|x-y|}\, dy, \tag{2}$$

$$\rho(t,x) = \textstyle\int_{\mathbb{R}^2} f(t,x,v)\, dv. \tag{3}$$

It should be noticed that the system (1)-(3) is not a two dimensional version of the Vlasov-Poisson system but a special case of the three dimensional version where the density function is partially singular.

The above system is solved numerically and a large class of solutions is constructed with the property that the corresponding rotation curves are approximately flat, slightly decreasing or slightly increasing. In addition, the numerically constructed rotation curves are compared with measurements from real galaxies. In [5] data for a number of spiral galaxies belonging to the Ursa Major Cluster are given. The measured rotation curves for the galaxies NGC3877 and NGC3917 are depicted by open circles in Figure 1 and Figure 2 respectively. The uncertainties in the observational data are tabulated in [5] and are shown as error bars. The solid curve corresponds to the numerically constructed solution. It is clear that satisfactory agreement is obtained. This raises the question whether the observed rotation curves for disk galaxies may be explained without introducing dark matter. For details about this investigation we refer to [1].

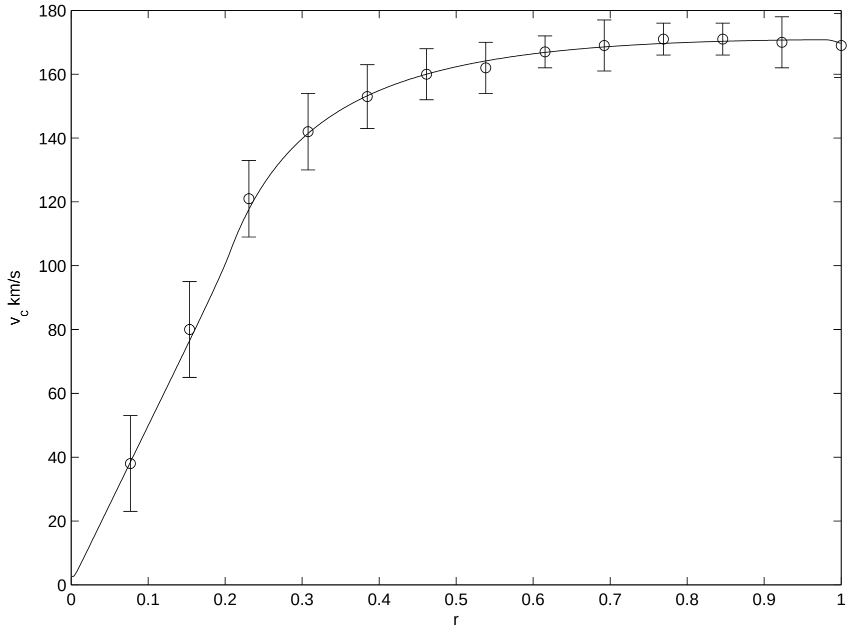

FIGURE 1. Comparison with the galaxy NGC3877

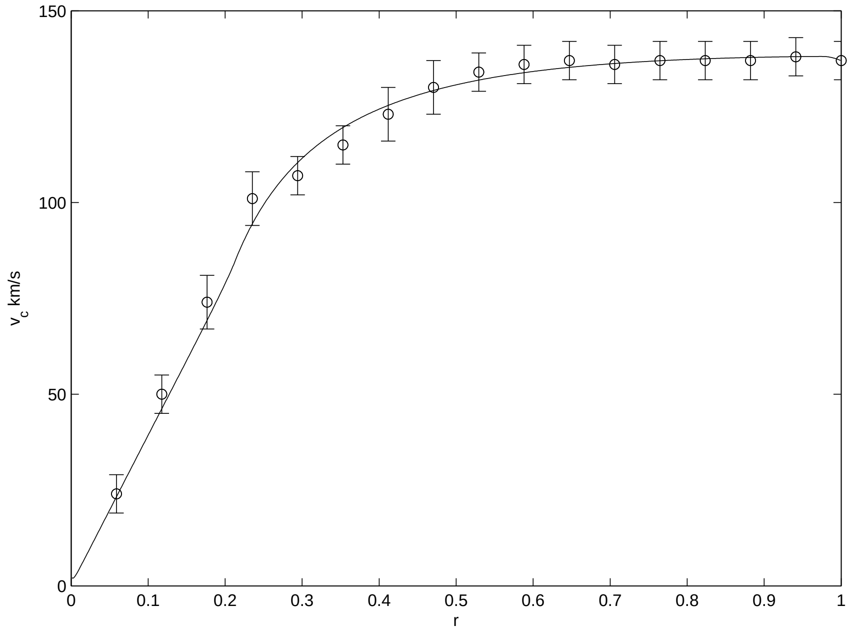

FIGURE 2. Comparison with the galaxy NGC3917

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Conformal properties of the extremal Schwarzschild-de Sitter spacetime

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(joint work with Edgar Gasperín)

The *Schwarzschild-de Sitter spacetime* is the spherically symmetric solution to the Einstein field equations

$$\tilde{R}_{ab} = \lambda \tilde{g}_{ab} \qquad \lambda > 0.$$

given in *static coordinates* (t, r, θ, φ) by

$$\tilde{g} = F(r)dt^2 - F(r)^{-1}dr^2 - r^2\sigma, \qquad F(r) \equiv 1 - \frac{2m}{r} + \frac{1}{3}\lambda r^2$$

where σ is the standard metric on the 2-sphere $\mathbb{S}^2$ and $t \in (-\infty, \infty)$, $r \in (0, \infty)$, $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi)$. This solution reduces to the de Sitter spacetime when $m = 0$ and to the Schwarzschild solution when $\lambda = 0$. Moreover, whenever $0 < -9\lambda m^2 < 1$, the polynomial $r - 2m + \lambda r^3/3$ has two distinct positive roots. These correspond, respectively, to a black hole-like horizon and a Cosmological-like horizon. The extremal Schwarzschild de-Sitter spacetime (eSdS) is then obtained by setting $\lambda = -1/9m^2$. If the extremal condition holds, then the black hole and Cosmological horizons degenerate into a single Killing horizon at $r = 3m$. Moreover, one has that the hypersurfaces of constant coordinate r are spacelike while those of constant t are timelike and there are no static regions. Finally, at $r = 0$ it can be verified that the spacetime has a curvature singularity —in particular, the scalar $\tilde{C}_{abcd}\tilde{C}^{abcd}$, with $\tilde{C}^a{}_{bcd}$ the Weyl tensor of the metric $\tilde{g}$, blows up. The basic conformal structure of the eSdS spacetime and its Penrose diagram has been discussed in [3].

The conformal Einstein field equations are a powerful tool for the analysis of the stability and global properties of asymptotically simple spacetimes —see e.g. [1, 2]. They provide a system of field equations for geometric objects defined on a 4-dimensional Lorentzian manifold $(\mathcal{M}, g)$, the so-called *unphysical spacetime*, which is conformally related to a spacetime $(\tilde{\mathcal{M}}, \tilde{g})$, the so-called *physical spacetime*, satisfying the Einstein field equations. One of the key unknowns in these field equations is the so-called rescaled tensor $d^a{}_{bcd}$ obtained from conformally invariant tensor $\tilde{C}^a{}_{bcd}$ by dividing by the conformal factor. The conformal framework allows to recast global problems in the physical spacetime as local problems in the unphysical one. Despite their use in the analysis of asymptotically simple spacetimes, very little analysis of more complicated solutions to the Einstein field equations (e.g. containing Cosmological or black hole singularities) by means of the conformal Einstein field equations has been carried out.

The eSdS spacetime provides a convenient solution in which to explore the use of the conformal Einstein field equations to analyse global and stability properties of spacetimes containing black holes and singularities. In the project reported in this abstract we have undertaken an analysis of the the eSdS spacetime as a solution to the conformal Einstein field equations.

One of the key features of the extremal Schwarzschild spacetime is that it allows the formulation an asymptotic initial value problem in which suitable Cauchy data is prescribed at the spacelike conformal boundary. The basic pieces of the asymptotic initial data consist of the intrinsic metric of the conformal boundary, a divergence free symmetric and trace free 3-dimensional tensor d_{ij} encoding the electric part of the rescaled Weyl tensor $d^a{}_{bcd}$ and gauge dependent scalar field encoding information about the embedding of the conformal boundary in the unphysical spacetime —in particular, its extrinsic curvature. For the eSdS spacetime there exists a conformal representation of in which each of the sections of the conformal boundary is topologically and metrically $\mathbb{S}^3$. The associated tensor d_{ij} is singular at two points corresponding to the North and South poles of $\mathbb{S}^3$. The singular behaviour of this key piece of the asymptotic data constitutes the essential difficulty in the analysis of the eSdS spacetime as a solution to the conformal field equations. It constitutes an essential obstruction to the reconstruction of the whole spacetime from asymptotic initial data. In view of this singular behaviour a more accurate description of the topology of a given component of the conformal boundary of the eSdS spacetime is that of $\mathbb{S}^3$ with two points removed.

To analyse the evolution of asymptotic initial data for the eSdS spacetime, we have expressed the conformal field equations in terms of a gauge based on the properties of conformal geodesics —a so-called conformal Gaussian system. The essential dynamics of the evolution system is governed by a core system of three equations involving the sole non-vanishing component of the rescaled Weyl tensor, a component of the Schouten tensor and a component of the connection. This core system is of interest on its own as it serves as a model of the mechanism for the formation of curvature singularities —its key equation is a Ricatti equation. We have analysed in detail the properties of this system. We expect that the insights obtained from this analysis can be extended to the discussion of the full conformal field equations.

The conformal representation of the eSdS spacetime obtained from the analysis of the evolution system governed by the core system can be used to analyse nonlinear perturbations of the exact solution in the past domain of dependence of the regular part conformal boundary —this particular analysis is work under progress.

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Overview of Recent non-CMC Results for the Conformal Method

Michael Holst

(joint work with Ali Behzadan, Vyacheslav Kungurtsev, Caleb Meier, Gabriel Nagy, Gantumur Tsogtgerel)

In this lecture, we begin with a brief overview of the 1973-1974 conformal method, and briefly review the CMC (constant mean curvature) and near-CMC results that had been established during the period 1973 through 2007. We then give an overview of the new framework that was developed in 2008 for removing the near-CMC condition, and outline the generalizations made to the framework from 2009 to 2013 (vacuum, rough metrics, manifolds with boundary, AE manifolds, and the limit equation). We review in a some detail some representative results for closed manifolds from 2008-2009, compact manifolds with boundary from 2013-2014, and asymptotically Euclidean manifolds from 2014-2015. We also give a summary of the results for rough metrics in each of these cases through 2015, and describe some results that examine non-uniqueness in the non-CMC case through the use of analytic bifurcation theory. We finish by describing two interesting developments that have substantially changed the direction of the field: the emergence of degeneracies in the so-called far-from-CMC cases (beginning in 2010), and the observation that some non-CMC results can be obtained with implicit function arguments around zero mean curvature, without resorting to near-CMC conditions.

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