Arbeitsgemeinschaft mit aktuellem Thema:
Mathematical Quasicrystals
Mathematisches Forschungsinstitut Oberwolfach
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Talks:

1. **Examples of tilings and cut and project sets**

   Definitions of Delone sets will be given. Standard constructions: cut and project constructions, model sets substitution tilings, return times to a section will be described. For cut and project constructions, the relevance of different assumptions (e.g. dense projections on internal space, injectivity on physical space, Riemann measurability of window) will be discussed, as well as the effect of weakening these hypotheses. All of these concepts will be illustrated using the Penrose example. A basic text for this material is [BG13]. Some additional material can be found in [Mey95], [HKW14, §2], [MS14, §2], [dB81], [Rob96], [Sol97], and [HKW15, Section 6.1].

2. **Dynamics and ergodic theory, I**

   This lecture will introduce (without proof) standard concepts from ergodic theory and dynamics, including the following: Ergodicity and ergodic theorems for actions of $\mathbb{R}$ and $\mathbb{R}^d$. Possible choices and properties of averaging sets, Følner sequences and van Hove sequences, with examples. Minimality and unique ergodicity. Existence of minimal sets in actions on compact spaces, and existence of invariant measures for actions of amenable groups. For amenable groups acting on compact spaces, unique ergodicity (with a measure of full support) implies minimality. Syndetic sets, sets of visit times for minimal actions.

3. **Dynamics and ergodic theory, II**

   Topological conjugacy and measurable conjugacy. Koopman operator and spectral measure. Almost 1-1 extensions and automorphic points. Eigenfunctions, continuous spectrum, discrete spectrum.

   Most of the material for both of these talks can be found in standard textbooks on dynamics, see e.g. [KH95, Pet83, CFS82]. Some of the material on van Hove sequences can be harder to find, see [MR13]. Syndetic sets and perhaps spectral measures are discussed in [Fur81]. See also [BG13, Appendix B].

4. **Pattern spaces**

This talk will begin with a discussion of tiling spaces, along the lines introduced in [Sad08, Chapter 1]. Different methods of tiling Euclidean space (local matching rules, substitutions, and cut and project sets) will be discussed. The tiling metric will be introduced, as well as the definition of the hull of a tiling. Examples should be given. The speaker should emphasize the point of view of realizing the tiling space as a dynamical system, and the notions of finite local complexity and repetitivity should also be introduced. Finally, the speaker should discuss more general pattern spaces (e.g. Delone sets). The notions of the Voronoi and Delone duals of point sets should be presented, as well as a discussion of a metric which generalizes the tiling metric, and can be used to define the 'Chabauty-Fell topology' with which the collection $\mathbf{Cl}(\mathbb{R}^d)$ of closed subsets of $\mathbb{R}^d$ is a compact metric space (see [dlH]). Other references for the material in this talk are [Sol] and [BG13].

5. **Introduction to Čech cohomology**

This talk is intended to provide a beginner's introduction to simplicial and Čech cohomology. The definitions of simplicial complexes, boundary and coboundary maps, and simplicial cohomology should be introduced, with examples. Their geometric significance (e.g. relationship to the fundamental group) should be discussed, and singular cohomology should be briefly mentioned. Direct limits of directed collections of groups should be defined. Finally, the definition of the nerve of a topological space should be given, followed by the definition of Čech cohomology and a discussion of good covers. A sufficient number of examples should be supplied. Good references for this material are [Hat02, Chapters 2 and 3] and [Sad08, Chapter 3].

6. **Gähler and Anderson-Putnam Complexes**

This talk will begin with a discussion of inverse limits of topological spaces, with simple examples (e.g. the p-adic numbers, solenoids). This will be followed by an explanation of collaring, and how to realize a tiling space as an inverse limit of branched manifolds, known as the Gähler complex. Next, the Anderson-Putnam complex for substitution tilings will be defined [AP98], and the advantages of using this construction will be clearly explained. Examples will be given, in one and possibly two dimensions, to show how the inverse limit description of the tiling space associated to a substitution can be used to compute its Čech cohomology, as the direct limit of the cohomologies of the approximants. A good reference for this material is [Sad08, Chapters 2, 3].

7. **Pattern equivariant cohomology**

This talk will be devoted to defining pattern equivariant cohomology following the work of Kellendonk [Kel08, Kel03]. This cohomology theory is a version of de Rham cohomology adapted to pattern and tiling spaces in the sense that it is the cohomology of complexes of (pattern equivariant) smooth forms. For Delone sets with finite local complexity, this cohomology theory is isomorphic to the Čech cohomology, which is treated in a previous talk. This is shown in [KP06] and should be covered in this lecture.

8. **Cohomology for cut and project pattern spaces**

   This talk is devoted to establishing sufficient conditions under which the cohomology of a pattern space associated to a cut and project Delone set is finitely generated. The material should mainly come from [GHK13, §1-4]. Time permiting, an example should be worked out and/or the result for codimension 1 cut and project sets.

9. **Pattern complexity**

   This talk will begin with a discussion of different notions of patterns (see the definition in [Jul10], and the discussion in [HKSW14, Section 1.2]), and pattern complexity, together with a comparison with classical results in the combinatorics of words (e.g. the Morse-Hedlund Theorem [MH38] and Sturmian words [BV00, Section 1.2]). The middle part of the talk will focus on complexity of patterns in cut and project sets, introducing the definitions of canonical and almost canonical acceptance domains and explaining the ideas in Julien's proof of [Jul10, Theorem 3.3], about growth of pattern complexity and the connection with connected components of $\mathrm{reg}(r)$. Finally, connections to the Period Conjecture will be discussed (see [Jul10, Section 5]), as well as a sketch of the proof that minimal complexity is a necessary and sufficient condition for finitely generated cohomology [Jul10, Theorem 5.1].

10. **Perfectly ordered quasicrystals I**

    This talk will discuss the interplay between order (complexity and repetitivity of patterns) and aperiodicity in Delone sets. The speaker will introduce the repetitivity and patch counting functions, $M_X(T)$ and $N_X(T)$, associated to a Delone set X, together with the definitions of what it means for X to be linearly or densely repetitive [LP02]. This will be followed by a discussion of the group of periods of a Delone set, together with the Period Conjecture of Lagarias and Pleasants [LP03, Conjecture 2.2]. It will be shown that if $M_X(T) < T/3$ for any value of T, then X is an ideal crystal [LP02, Theorem 2.2]. This will be followed by an explanation of Lenz's proof that any aperiodic linearly repetitive Delone set is densely repetitive [Len04]. Finally, the speaker will explain the recent characterization of the collection of linearly repetitive cut and project sets [HKW15].

11. **Perfectly ordered quasicrystals II**

    This talk will begin with a discussion of local weight distributions on Delone sets, and an exposition of the proof that linear or dense repetitivity implies the existence of unique averages for these distributions [LP03, Section 5]. Consequences of this fact for the existence of uniform patch frequencies (with error terms) and autocorrelation measures will be presented [LP03, Section 6]. The speaker should also comment about the extent to which repetitivity is a necessary assumption in the above results. Finally, applications to address maps will be discussed [LP03, Section 7].

12. **Quasicrystals and the Poisson summation formula**

For a sufficiently nice function $f : \mathbb{R}^d \to \mathbb{R}$, the Poisson summation formula states that

$$\sum_{x\in\Lambda} f(x) = \frac{1}{|\Lambda|} \sum_{x\in\Lambda^*} \hat{f}(x),$$

where Λ is a lattice, Λ^* is its dual and $|\Lambda|$ is its volume. This equation holds in some nice space of distributions, but for measures supported on discrete sets, as proved by Córdoba, this does not necessarily hold [Cór89]. This talk will go over the problem and the recent result of Lev-Olevskii [LO13].

13. **Diffraction I**

This first talk on diffraction should introduce the basics on diffraction: what it models, the autocorrelation (also known as Patterson function in crystallography), the diffraction measure and how all this is tied to ergodic theory through Dworkin's argument. Hof's work on diffraction for cut and project sets should be described [Hof97] and how this is related to Córdoba's theorem [Cór89] (see also [LO13] and the previous talk). General references are [Lag00, LMS02] and [BG13, Chapters 8,9].

14. **Diffraction II**

The second talk on diffraction will focus on the connection between the dynamical and diffraction spectra. This was introduced in the previous lecture through the work of Dworkin and Hof. The goal of the lecture should be to establish the equivalence of the pure point diffraction spectrum and pure point dynamical spectrum [LMS02] and how the set of eigenvalues is the group generated by the Bragg peaks. A more general approach is given in [Len09] which also deals with the Bombieri-Taylor conjecture. More references include [Lag00, BLM07] and [BG13, Chapters 8,9].

15. **Bi-Lipschitz equivalence I**

This talk is devoted to the question of the existence of Delone sets which are not bi-Lipschitz equivalent to lattices. After explaining the problem and its history, the solution will be sketched via the reduction of Burago-Kleiner [BK98] and McMullen [McM98] to the prescribed Jacobian problem. Then the construction of a concrete Delone set which is not bi-Lipschitz to a lattice will be given, following the treatment of Cortez and Navas [CN14].

16. **Bi-Lipschitz equivalence II**

This talk is about the sufficient condition of Burago and Kleiner [BK02, APCG13] for a Delone set to be bi-Lipschitz equivalent to a lattice, and its applications. A proof of the sufficiency will be given (including a discussion of infinite version of the Hall marriage Lemma). Results on specific Delone sets which are bi-Lipschitz to lattices

[Sol11, HKW14, Hay13] will be surveyed.

17. Bounded displacement equivalence

This talk is devoted to the notion of bounded displacement equivalence. A construction of cut and project sets which are bounded displacement from lattices, arising from canonical acceptance domains, will be given following [DO90]. The implication 'bounded displacement implies bilipschitz' for Delone sets will be explained. The necessary and sufficient condition for bounded displacement to lattices due to Laczkovich [Lac92] will be explained, and results of [HKW14] on existence of cut and project sets which are not BD to lattices, and prevalence of cut and projects sets which are BD to lattices, as well as analogous results of [Sol14] for substitutions, will be reviewed.

18. Deviation of ergodic averages for self-similar tilings

Most tilings and Delone sets of interest define a uniquely ergodic action of $\mathbb{R}^d$ on the associated tiling/pattern space Ω. This means that, for any continuous $f : \Omega \to \mathbb{R}$ and for an increasing sequence $\{B_k\}_k$ of van Hove sets,

$$\frac{1}{\mathrm{Vol}(B_k)} \int_{B_k} \varphi_t(\Lambda)\, dt \longrightarrow \int_{\Omega} f\, d\mu,$$

where μ is the unique $\mathbb{R}^d$-invariant measure on Ω. For $\mathcal{T}$ an aperiodic tiling of $\mathbb{R}^d$ corresponding to a substitution rule, the work of Bufetov-Solomyak [BS13] estimates the rates of convergence of the ergodic averages above. For cut and project sets, such estimates are given in [HKW14]. This talk will go over the result of Bufetov and Solomyak and related results in the literature. Time permiting, the presenter should try to illustrate the result with an example.

19. Gap labeling theorem

In this talk the gap labeling theorem will be surveyed. The physical motivation should be stated and tied to the original conjecture. The different proofs should be mentioned along with comments on the tools used. References are [BHZ00, Bel03, BBG06] and [MS06, Appendix D]. See also [KP00].

20. The Fibonacci Hamiltonian

This talk should survey the results concerning the one-dimensional Schrödinger operator associated with potentials given by aperiodic discrete sets. In particular, it should focus on results obtained in [DGY14] which shows that the spectrum of such operators are "dynamically defined" Cantor sets and other spectral properties. See also [Dam14, §7].

21. Danzer problem

This talk is about the current status of the Danzer problem, a classical problem in elementary geometry [CFG91]. The problem will be stated, and a history of partial results given: negative results of [BW71, SW14] and positive results (also [BW71, SW14]). The dynamical approach ([SW14, Prop. 3.1]) should be explained and the relation to the classification of minimal sets for the action of the affine group in the space of closed subsets of $\mathbb{R}^d$ (see [SW14, §7.3]) should be explained. A sketch of proof of [SW14, Thm. 1.2] should be given.

22. Dense forests

The notion of a Dense forest will be defined following [Bis11]. The construction of a Delone dense forest will be given, following [SW14, §4]. Questions of optimizing the rate in the definition of dense forest will be stated, and results will be surveyed, for both uniformly discrete dense forests, and dense forests whose asymptotic growth is $O(T^d)$.

23. Space of cut and project sets

Marklof and Strömbergsson's [MS14] construction of a space of cut and project sets, with fixed projections, dimensions, and window and variable affine lattice, as a way to topologize and put a measure on spaces of cut and project sets. The map that sends an affine lattice to its cut and project set is continuous at points where the internal space projection of the affine lattice does not intersect the boundary of the window. Scarcity of examples of probability measures on point sets in $\mathbb{R}^n$, which are invariant under the group of volume preserving affine transformations $\mathrm{ASL}_d(\mathbb{R})$. The use of Ratner's theorem to construct such measures.

24. Siegel summation for cut and project sets

The Siegel integration formula is a technique introduced by Siegel [Sie45] in connection with problems in the geometry of numbers, developing ideas of Minkowski. It relates integrals on $\mathbb{R}^d$ with integrals on the space of unimodular d-dimensional lattices, with respect to the natural measure induced by Haar measure on $\mathrm{SL}_d(\mathbb{R})$, and can be used as part of a 'probabilistic method' to prove the existence of lattices with certain properties (notably, as in [Sie45], lattices whose shortest nonzero vector is long). This approach was axiomatized by Veech [Vee98] to general spaces of point-sets in $\mathbb{R}^d$ and recently extended to the space of cut and project sets by Marklof and Strömbergsson [MS14]. In this talk the method will be introduced in the abstract setting (following Veech) and the results of Marklof and Strömbergsson will be presented.

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