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Mini-Workshop: Recent Developments on Approximation Methods for Controlled Evolution Equations

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Abstract. This mini-workshop brought together mathematicians engaged in partial differential equations, functional analysis, numerical analysis and systems theory in order to address a number of current problems in the approximation of controlled evolution equations.

Mathematics Subject Classification (2010): 93C20, 93C25, 65Mxx.

Introduction by the Organisers

The mini-workshop *Recent Developments on Approximation Methods for Controlled Evolution Equations*, organised by *Birgit Jacob* (Wuppertal), *Enrique Zuazua* (Bilbao) and *Hans Zwart* (Twente) was held November 1st – 7th, 2015. This meeting was well attended with 16 participants with broad geographic representation.

Systems modelled by linear ordinary differential equations have long been studied and there exists a wide body of theory and design algorithms dealing with their control. The state describing such a system lies in a finite-dimensional vector space. This setting has its limitations, as many systems of interest, from the point of view of applications to industry and other disciplines, do not fall into this class. A more interesting generalisation is that to systems with an infinite-dimensional state space. This class includes delay systems, and systems modelled by functional differential equations and partial differential equations (PDEs), generally called *evolution equations*. This field finds applications in such diverse areas as aeronautics, mechanical and electrical engineering. Since they appear frequently

as models in these fields of applications, evolution equations with boundary control and boundary observation are of particular interest.
One of the key issues when addressing real applications is the effective control of those systems, which requires of significant effort from the point of view of mathematical analysis.

The talks where grouped into three main themes:

- Modeling and control of *real-live* problems
- Numerical analysis of PDE control
- Theoretical aspects of controller design and approximations for systems described by PDEs

In the first theme the following participants gave talks: *Rob Fey, Aitziber Ibañez, Jarmo Malinen, George Weiss.*

Furthermore, *Athanasios Antoulas, András Bátkai, Umberto Biccari, Nicolae Cîndae, Weiwei Hu, Orest Iftime, Kirsten Morris, Timo Reis* and *Hans Zwart* were the speakers of the second theme.

The last theme was covered by *Björn Augner, Birgit Jacob, Felix Schwenninger* and *Hans Zwart.* Although we have grouped them according to our themes, there was significant overlap between the approaches which stimulated many productive discussions.

The organizers and participants thank the Mathematisches Forschungsinstitut Oberwolfach for providing an inspiring setting for this mini-workshop, which allowed us to concentrate on the mathematics.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, “US Junior Oberwolfach Fellows”.

Mini-Workshop: Recent Developments on Approximation Methods for Controlled Evolution Equations

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Abstracts

Model reduction of large-scale dynamical systems: An overview with applications to evolution equations

Thanos Antoulas

1. MOR and motivating examples

Model order reduction (MOR) seeks to reduce the computational complexity and computational time in the simulation of large-scale dynamical systems by approximations of much lower dimension that produce nearly the same input/output response characteristics.

One version of the problem consists of a given physical system described by known sets of PDEs. These are then appropriately discretized to yield sets of ODEs (Ordinary Differential Equations). The number of the ODEs is reduced, which constitutes *model reduction.* Often the equations (PDEs or ODEs) are replaced by *data*, which can be measured or computed by means of DNS (Direct Numerical Simulation); in this case the reduced system is constructed from the data. Finally, the reduced system is used for simulation, design, or control.

Three motivating examples were mentioned, two based on equations and one on measurements. Example I: Viscous fingering in porous media. In this case the equations of the system are the PDEs (Partial Differential Equations) describing Darcy's law and advection diffusion equations. Example II: Thermal treatment of railroad rails. The equations here are convection diffusion PDEs with inhomogeneous Robin boundary conditions. Example III represents a microstrip device. Such devices consist of thin metallic conductors mounted on a flat dielectric substrate, itself mounted on a ground plate. Common applications are transmission lines, antennas and filters. Microstrip transmission lines are a common way to connect two devices. Such devices are described by means of measured *S-parameters* (Scattering parameter) matrices.

For an overview of MOR methods based on system equations, see [1, 6, 11]. In the sequel we will be concerned with a specific data-driven MOR method, namely that based on the Loewner framework. An overview of the Loewner framework in provided in [5].

2. The Loewner framework

The data-driven MOR framework that we propose is based on the Loewner matrix, which was introduced by Karel Löwner (later Charles Loewner) in the paper: Über monotone Matrixfunctionen, Math. Zeitschrift (1934).

Given a row array $(\mu_j, \mathbf{v}_j)$, $j = 1, \cdots, q$, and a column array $(\lambda_i, \mathbf{w}_i)$, $i = 1, \cdots, k$, the associated *Loewner matrix* whose $(i,j)^{\text{th}}$ entry is: $\mathbb{L}_{i,j} = \frac{\mathbf{v}_i - \mathbf{w}_j}{\mu_i - \lambda_j}$. If there is a known underlying function $\mathbf{g}$, then $\mathbf{w}_i = \mathbf{g}(\lambda_i)$, and $\mathbf{v}_j = \mathbf{g}(\mu_j)$.

Main property. Let $\mathbb{L}$ be a $q \times k$ Loewner matrix built from samples of a rational function $\mathbf{g}(s)$. If $q, k \geq \deg \mathbf{g}$, then $\operatorname{rank} \mathbb{L} = \deg \mathbf{g}$, where the degree of rational function is the maximum between the degrees of the numerator and the denominator polynomials.

The exact or the numerical rank of $\mathbb{L}$ is thus of central importance in this approach, depending on whether we wish to recover the original rational function or an approximant thereof.

2.1. **The Loewner framework and generalizations.** The Loewner framework has been worked out for linear systems with multiple inputs and outputs [10, 9, 5, 2]. It has been extended to parametric systems [4, 7, 8] and recently to nonlinear systems [3].

2.2. **Summary.** In this talk we presented an approach to *data-driven model reduction.* Key tool is the *Loewner pencil* and tangential interpolation. Given input/output data, a singular high order model in descriptor form is constructed *without computation*; this constitutes a natural way to construct models and reduced models. Some features are:

- the SVD of $\mathbb{L}$ provides a trade-off between accuracy and complexity;
- the approach does not force the inversion of $\mathbf{E}$;
- the methodology can deal with many input/output ports as well as parameters and smooth nonlinearities.

The philosophy of this approach is: *collect data and extract the desired information.*

References

[1] A.C. Antoulas, *Model reduction of large-scale dynamical systems* SIAM, Philadelphia, 2005, re-printed 2008.

[2] A.C. Antoulas, *The Loewner framework and transfer functions of singular/rectangular systems*, Applied Math. Letters (2016).

[3] A.C. Antoulas, I.V. Gosea and A.C. Ionita, *Model Reduction of Bilinear Systems In The Loewner Framework*, submitted (Sept. 2015).

[4] A.C. Antoulas, A.C. Ionita, and S. Lefteriu, *On two-variable rational interpolation*, Linear Algebra and Its Applications, **436**: 2889-2915 (2012).

[5] A.C. Antoulas, S. Lefteriu and A.C. Ionita, A tutorial introduction of the Loewner framework in model reduction, in Model Reduction and Approximation for Complex Systems, edited by P. Benner, A. Cohen, M. Ohlberger, and K. Willcox, SIAM, Philadelphia, 2016.

[6] U. Baur, P. Benner, L. Feng, Model order reduction for linear and nonlinear systems: a system-theoretic perspective, Arch. Computat. Methods Eng., Vol 21: 331-358 (2014).

[7] A.C. Ionita and A.C. Antoulas, Parametrized model order reduction from transfer function measurements, in Reduced Order Methods for modeling and computational reduction, Book Series: Modeling, Comutations and Applications, A. Quarteroni, G. Rozza (Editors), Springer Verlag (2013).

[8] A.C. Ionita and A.C. Antoulas, Data-driven parametrized model reduction in the Loewner framework, SIAM J. Scientific Computing, 36(3), A984-A1007, 2014.

[9] S. Lefteriu and A.C. Antoulas: A New Approach to Modeling Multiport Systems from Frequency-Domain Data, IEEE Trans. CAD, 29: 14-27 (2010).

[10] A.J. Mayo and A.C. Antoulas, *A framework for the solution of the generalized realization problem*, Linear Algebra & Applications, 425: 634-662 (2007).
[11] W.H.A. Schilders, H.A. Van Der Vorst, and J. Rommes, *Model order reduction: theory, research aspects and applications*, Springer Verlag 2008.

Boundary Stabilisation of Port-Hamiltonian Systems

Björn Augner
(joint work with Birgit Jacob)

We are interested in stabilisation and control of beam equations. Quite a large class of them may be reformulated in the abstract *port-Hamiltonian* form [5]

$$\frac{\partial x}{\partial t}(t,\zeta) = \sum_{k=0}^{N} P_k \frac{\partial^k (\mathcal{H}x)}{\partial \zeta^k}(t,\zeta) =: (\mathfrak{A}x(t))(\zeta), \quad t \geq 0,\ \zeta \in (0,1)$$

where the states are described by $x(t,\zeta) \in \mathbb{K}^d$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). We assume that $P_k = (-1)^{k+1} P_k^* \in \mathbb{K}^{d\times d}$ are symmetric and skew-symmetric matrices with P_N invertible, and $P_0 \in L_\infty(0,1;\mathbb{K}^{d\times d})$. On the space $X = L_2(0,1;\mathbb{K}^d)$ with inner product $\langle \cdot,\cdot\rangle_X = \langle \cdot, \mathcal{H}\cdot\rangle_{L_2}$ where $\mathcal{H} \in L_\infty(0,1;\mathbb{K}^{d\times d})$ is coercive as operator in $\mathcal{L}(X)$ we define $\mathfrak{A}$ on its maximal domain $D(\mathfrak{A}) = \{x \in X : \mathcal{H}x \in H^N(0,1;\mathbb{K}^d)\}$ and also boundary input and output maps $\mathfrak{B}$ and $\mathfrak{C}$, respectively, which depend on the boundary trace only, i.e. the value of $\mathcal{H}x$ and its derivatives at the boundary, and assume that the *port-Hamiltonian system* $\mathfrak{S} = (\mathfrak{A},\mathfrak{B},\mathfrak{C})$ is *impedance passive*, i.e.

$$\Re\langle \mathfrak{A}x, x\rangle_X \leq \Re\langle \mathfrak{B}x, \mathfrak{C}x\rangle_{\mathbb{K}^{Nd}}.$$

We then consider the evolution equation in boundary control and observation form

$$\begin{aligned}&\frac{d}{dt}x(t) = \mathfrak{A}x(t), \quad x(0) = x_0\\ &\mathfrak{B}x(t) = u(t), \quad \mathfrak{C}x(t) = y(t), \quad t \geq 0\end{aligned}$$

where we couple input and output via a static m-dissipative map $\phi : D(\phi) \subseteq \mathbb{K}^{Nd} \rightrightarrows \mathbb{K}^{Nd}$ and the feedback $\mathfrak{B}x(t) \in \phi(\mathfrak{C}x(t))$ ($t \geq 0$) or with an impedance passive control system, represented by an m-dissipative map $N_c : D(N_c) \subseteq X_c \times \mathbb{K}^{Nd} \rightrightarrows X_c \times \mathbb{K}^{Nd}$ for X_c the state space of the controller, so that

$$\begin{pmatrix} \frac{d}{dt}x_c(t) \\ -y_c(t) \end{pmatrix} \in N_c \begin{pmatrix} x_c(t) \\ u_c(t) \end{pmatrix}, \quad t \geq 0.$$

As it turns out the resulting evolutionary equation is well-posed [4], [1] in the sense that one obtains m-dissipative operators $A_\phi := \mathfrak{A}|_{D(A_\phi)}$ with $D(A_\phi) = \{x \in D(\mathfrak{A}) : \mathfrak{B}x \in \phi(\mathfrak{C}x)\}$ or $\mathcal{A}$ (defined on the product space $X \times X_c$ in a suitable way), respectively. Then, provided that the resulting operators are dissipating enough energy, e.g.

$$\Re\langle \mathcal{A}(x,x_c),(x,x_c)\rangle \leq -\kappa \left|(\mathcal{H}x)(1)\right|^2$$

for systems with $N = 1$, when 0 is a uniformly exponentially stable equilibrium of the control system and the control system is, say, strictly input-passive, then 0 also is a globally uniformly stable equilibrium of the interconnected system [1], [2], [3]. Similar result also hold for higher order systems [1], [2], where for $N = 2$ one needs a dissipation like $|(\mathcal{H}x)(1)|^2 + |(\mathcal{H}x)'(1)|^2 + \left|(\mathcal{H}x)^{(j)}\right|^2$, but less restrictive dissipation condition under more structural conditions, e.g. the Euler-Bernoulli beam structure. Finally we apply the abstract results to the stabilisation of a 1D-wave equation with static nonlinear boundary feedback.

References

[1] B. Augner, *Well-posedness and Stability of Linear Port-Hamiltonian Systems with Nonlinear Boundary Feedback*, submitted (2015).

[2] B. Augner, B. Jacob, *Stability and Stabilization of Infinite-dimensional Linear Port-Hamiltonian Systems*, Evolution Equations and Control Theory (EECT) **3** (2014), 207–229.

[3] H. Ramirez, H. Zwart and Y. Le Gorrec, *Exponential stability of boundary control port Hamiltonian systems with dynamic feedback*, First IFAC Workshop on Control of systems modeled by Partial Differential Equations (CPDE) (2013).

[4] S. Trostorff, *A characterization of boundary conditions yielding maximal monotone operators*, J. Funct. Anal. **267** (2014), 2787–2822.

[5] J.A. Villegas, H. Zwart, Y. Le Gorrec, B. Maschke, *Exponential stability of a class of boundary control systems*, IEEE Trans. Automat. Control **54** (2009), 142–147.

Operator splitting for dissipative delay equations

András Bátkai

(joint work with Petra Csomós, Bálint Farkas)

The Lie–Trotter product formula and its generalizations to nonlinear semigroups play a central role in semigroup theory, as witnessed in the seminal monographs by Goldstein [9] or Engel and Nagel [8]. As it turns out, the Lie–Trotter product formula is widely used in numerical analysis under the name of operator splitting.

Our aim is to investigate the convergence of the Lie–Trotter product formula (or operator splitting) for nonlinear partial differential equations with delay.

Partial differential equations with delay play an important role in modeling physical, chemical, economical, etc. phenomena, since it is quite natural to assume that past occurrences effect the model. For further motivation see for example the monographs by Wu [13] or Bátkai and Piazzera [6].

There has been lots of work describing the asymptotic behavior and regularity of solutions, as well as in the numerical analysis of ordinary differential equations with delay, see for example the monograph by Bellen and Zennaro [7]. The numerical analysis of partial differential equations with delay, however, seems to be in its infancy. We aim to contribute to this topic by analyzing an operator splitting procedure for nonlinear partial differential equations with delay.

The idea of operator splitting is to decompose the differential equation into simpler equations which can be solved in an effective way, and then represent the solution of the original equation using product formulae, like the Lie–Trotter one. For ordinary differential equations, the theory seems to be quite complete, as witnessed in Hairer et al. [10, Section II.4,5]. There has also been enormous progress in the theoretical investigation of splitting procedures for infinite dimensional systems in recent years, see for example the monograph by Holden et al. [11]. See also the recent papers by Bátkai et al. [3, 4, 5], where nonautonomous equations and spatial approximations are also considered. Unfortunately, abstract results analyzing the order of convergence are rather incomplete, and, as it seems, can be applied to delay equations only with considerable difficulty.

The idea to apply splitting procedures to delay equations is the following. Consider, e.g., a delayed reaction–diffusion equation of the form

$$u'(t) = \Delta u(t) + g\left(u(t-1) + \int_{-1}^{0} \eta(\sigma) \cdot u(t+\sigma)\,\mathrm{d}\sigma\right).$$

with some initial and boundary conditions. The delay term appearing here represents the two main classes of possible delays in applications: point delays corresponding to dependence on a single event in the past, and distributed delays (given by an integral term with an integrable kernel η) corresponding to dependence on a whole time period in the past. In our opinion, distributed delays are often more realistic in modeling, but we will not restrict ourselves to them here.

The nonlinearity g can be chosen from a wide range of functions depending on applications, for example a rational function in chemical reactions, or a tanh type function in neural networks, see for example the monograph by Wu [13].

Since the delay term is in a way a "scalar operator" (in the sense that it does not mix the spatial variables), it is natural to decompose the equation into two sub-problems: the heat equation

$$w'(t) = \Delta w(t),$$

and a scalar-valued delay equation

$$v'(t) = g\left(v(t-1) + \int_{-1}^{0} \eta(\sigma) \cdot v(t+\sigma)\,\mathrm{d}\sigma\right).$$

Both equations can be solved numerically in an effective way. Note that the second equation becomes here an ordinary differential equation with delay, hence the methods described in Bellen and Zennaro [7] can be applied.

Our aim is to put this example in an abstract theoretical perspective explaining the convergence and analyze the order of convergence in some special cases. The results obtained will be published in detail in [2].

References

[1] A. Bátkai, P. Csomós, and B. Farkas, *Operator splitting for nonautonomous delay equations*, Computers & Mathematics with Applications **65**, 315–324 (2013).

[2] A. Bátkai, P. Csomós, B. Farkas, *Operator splitting for dissipative delay equations*, Preprint, 2015, `arXiv:1009.1981`

[3] A. Bátkai, P. Csomós, B. Farkas, and G. Nickel, *Operator splitting for non-autonomous evolution equations*, J. Funct. Anal. **260**, 2163–2190 (2010).

[4] A. Bátkai, P. Csomós, B. Farkas, and G. Nickel, *Operator splitting with spatial-temporal discretization*, Operator Theory: Advances and Applications, **221**, 161–171 (2012).

[5] A. Bátkai, P. Csomós, and G. Nickel, *Operator splittings and spatial approximations for evolution equations*, J. Evol. Equ. **9**, 613–636 (2009).
[6] A. Bátkai and S. Piazzera, *Semigroups for delay equations* (Research Notes in Mathematics, vol. 10, A K Peters Ltd., Wellesley, MA, 2005), p. 272.
[7] A. Bellen and M. Zennaro, *Numerical methods for delay differential equations* (Numerical Mathematics and Scientific Computation, The Clarendon Press Oxford University Press, New York, 2003), p.410.
[8] K.-J. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations* (Graduate Texts in Mathematics, vol. 194, Springer-Verlag, New York, 2000)
[9] J. A. Goldstein, *Semigroups of linear operators and applications* (Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1985) p. 245.
[10] E. Hairer, C. Lubich, and G. Wanner,*Geometric numerical integration, Structure-preserving algorithms for ordinary differential equations* (Springer Series in Computational Mathematics, vol. 31, Springer-Verlag, Berlin, 2006)
[11] H. Holden, K. H. Karlsen, K.-A. Lie, and N. H. Risebro, *Splitting methods for partial differential equations with rough solutions* (EMS Series of Lectures in Mathematics, European Mathematical Society (EMS), Zürich, 2010)
[12] P.D. Lax, R.D. Richtmyer, *Survey of the stability of linear finite difference equations.*, Comm. Pure Appl. Math. **9**, 267–293 (1956).
[13] J. Wu, *Theory and applications of partial functional-differential equations* (Applied Mathematical Sciences, vol. 119, Springer-Verlag, New York, 1996)

Boundary controllability for a one-dimensional heat equation with two singular inverse-square potentials

Umberto Biccari

Let $T > 0$ and set $Q := (0,1) \times (0,T)$. We prove boundary controllability for the following one-dimensional heat equation on the domain Q

$$(1) \qquad \begin{cases} u_t - u_{xx} - \dfrac{\mu_1}{x^2}u - \dfrac{\mu_2}{(1-x)^2}u = 0 & \text{in } Q \\ u(0,t) = f(t),\ u(1,t) = 0 & \text{in } (0,T) \\ u(x,0) = u_0(x) & \text{in } (0,1) \end{cases}$$

presenting two singular inverse-square potentials with singularities located on the boundary. In particular, the main result of this work will be the following Theorem, presented in [1].

Theorem 1. *Let μ_1 and μ_2 be two real numbers such that $\mu_1, \mu_2 \leq 1/4$. For any time $T > 0$ and any initial datum $u_0 \in L^2(0,1)$, there exists a control function $f \in L^2(0,T)$ such that the solution of (1) satisfies $u(x,T) = 0$.*

The upper bound for μ_1 and μ_2 is required for the well-posedness of the problem, and is related to a multi-polar Hardy-Poincaré inequality.

As it is by now classical, for proving Theorem 1 we will apply the Hilbert Uniqueness Method; hence the controllability property will be equivalent to the observability of the adjoint system associated to (1), namely

$$(2)\qquad \begin{cases} v_t + v_{xx} + \dfrac{\mu_1}{x^2}v + \dfrac{\mu_2}{(1-x)^2}v = 0 & \text{in } Q \\ v(0,t) = v(1,t) = 0 & \text{in } (0,T) \\ v(x,T) = v_T(x) & \text{in } (0,1). \end{cases}$$

This observability inequality will involve the normal derivative of the solution of (2) in the point from which we aim to control. In particular, we will prove the following inequality

$$(3)\qquad \int_0^1 v(x,0)^2\,dx \le C_T \int_0^T \left[x^{2\lambda_1} v_x^2\right]\Big|_{x=0} dt.$$

where we introduce a weighted normal derivative in order to compensate the singularity at $x = 0$. The inequality above, in turn, will be obtained as a consequence of a Carleman estimate for the solution of (2).

This work extends previous results on the internal controllability for equations of the type of (1), presented in [2], [3] and [4], to the case of boundary controllability.

References

[1] U. Biccari, *Boundary controllability for a one-dimensional heat equation with two singular inverse-square potentials*, Submitted.

[2] C. Cazacu, *Controllability of the heat equation with an inverse-square potential located on the boundary*, J. Control Optim., **52(4)** (2014), 2055–2089.

[3] S. Ervedoza, *Control and stabilization properties for a singular heat equation with an inverse-square potential*, Comm. Part. Diff. Eq., **33(11)** (2008), 1996–2019.

[4] J. Vancostenoble and E. Zuazua, *Null controllability for the heat equation with singular inverse-square potential*, J. Funct. Anal., **254(7)** (2008), 1864–1902.

Numerical controllability of the wave equation using time-space finite elements

Nicolae Cîndea

(joint work with Carlos Castro, Arnaud Münch)

This talk aims to present some recent results concerning the approximation of controls of minimal L^2-norm for the wave equation using a finite elements discretization of the space-time domain [2]. More precisely, the following wave equation with distributed control is considered:

$$(1)\qquad \begin{cases} y_{tt}(x,t) - \Delta y = v(x,t)\mathbf{1}_\omega(x), & (x,t) \in \Omega \times (0,T) \\ y(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T) \\ y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x), & x \in \Omega, \end{cases}$$

where Ω is an open domain of $\mathbb{R}^n$ and $\omega \subset \Omega$ is an open and non-empty subset of Ω. The null controllability in time $T > 0$ consists, for every $(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$, in finding a control $v \in L^2(\omega \times (0,T))$ such that

$$y(x,T) = y_t(x,T) = 0, \qquad \forall x \in \Omega.$$

In order to approach such a control, the first step of the proposed method is to write the optimality condition corresponding to the minimization of the functional $J^\star$, which appears in Hilbert Uniqueness Method (HUM), as a mixed formulation. This is possible by introducing a Lagrange multiplier for tacking into account that the dual variable φ (the functional $J^\star$ is minimized with respect to φ) verifies the wave equation. The obtained mixed formulation is well posed as a consequence of the exact observability of the adjoint system associated to (1).

In a second step, the mixed formulation is numerically approximated using the finite elements method. More exactly, C^1 finite elements are employed for the approximation of the dual variable φ and C^0 Lagrange finite elements for the approximation of the Lagrange multiplier λ.

Finally, since the space-time domain is discretized with no differentiation between the temporal and the spatial variables, this method is very appropriate to tackle the controllability of the wave equation using controls supported in time-dependent domains. Several numerical simulations are discussed and confirm the theoretical analysis of the proposed method. A similar strategy can be employed for the wave equation with boundary control [1].

References

[1] N. Cîndea, A. Münch, *A mixed formulation for the direct approximation of the control of minimal L^2-norm for linear type wave equation*, Calcolo **32** (2015), 100–120.

[2] C. Castro, N. Cîndea, A. Münch, *Controllability of the linear one-dimensional wave equation with inner moving forces*, SIAM J. Control Optim., **52**, No. 6, (2014), 4027–4056.

Controlling nonlinear resonances in thin-walled structures: modeling, simulations, and experiments

Rob H.B. Fey

Thin-walled structures are often used in engineering practice because in their (axial, in-plane) loading direction, they have a high stiffness-mass ratio and a high (static) loading capacity. However, in the transversal direction, they may be quite flexible. Although the structure will be generally designed in such a way that static buckling will not occur, dynamic periodic base excitation of the structure still may lead to dynamic buckling (i.e. resonances) of the structure. This may lead to malfunctioning, fatigue, damage, or even collapse of the structure. In this work, first, nonlinear resonances of an archetype thin-walled structure will be studied in the low frequency range, both numerically and experimentally. Subsequently, they will be attenuated using control. The structure consists of a vertical slender beam, which carries a top mass. The beam is loaded statically by the weight of the top mass, which is below the static buckling load, and dynamically by a vertical periodic base excitation. Piezoelectric (PZT) patches are attached on both sides of the beam, which can locally exert a control moment. The transversal displacement and velocity halfway the length of the beam are measured by means of a laser vibrometer. First, a single mode model of the uncontrolled structure is derived. The transversal displacement field is approximated by an analytical shape function with a corresponding generalized degree of freedom. This shape function is fitted to the lowest eigenmode of the system, which is obtained using finite element analysis. The beam is assumed to be inextensible. This implies that the axial displacement field and the transversal displacement field are kinematically coupled. Furthermore, the beam has a small geometric imperfection. It is assumed that this imperfection can be approximated by the analytical shape function mentioned above as well. The model is geometrically nonlinear due to 3rd order Taylor series approximations of the inextensibility constraint and the beam curvature. After formulating expressions for the kinetic energy, the potential energy, and the non-conservative forces, the equation of motion of the system in terms of the generalized degree of freedom is derived using Lagrange's equation. Viscous damping is accounted for by a linear and a quadratic term. Now, branches of periodic solutions are calculated by solving two-point boundary value problems in combination with path following techniques using the base excitation frequency as the varying parameter. In the uncontrolled case, the model predicts a severe 1/2 subharmonic (parametric) resonance, a harmonic resonance, and several superharmonic resonances. The largest resonances show softening behavior. On the periodic solution branches, cyclic fold bifurcations and period doubling bifurcations are met. The steady-state responses are also measured using stepped frequency sweep-up and stepped frequency sweep-down experiments. The numerical and experimental responses match very well. Now, a (model-based) feedback linearizing controller in combination with a linear controller is implemented to suppress the severe transversal resonances using the PZT patches. The linear controller consists of a gain in combination with a lead filter. Experiments show that near dangerous base disturbance frequencies, the vibration levels are reduced up to a factor 300! It can be concluded that the implemented controller very successfully suppresses all resonances of the uncontrolled system.

Observer-based Feedback Stabilizaiton of a Thermal Fluid

Weiwei Hu

We discuss the problem of designing an observer-based feedback law which locally stabilizes a two dimensional thermal fluid modeled by the Boussinesq equations. The investigation of stability for a fluid flow in the free convection problem is important in the theory of hydrodynamical stability (see [11]). The challenge of stabilization of the Boussinesq equations arises from the stabilization of the Navier-Stokes equations and its coupling with the convection-diffusion equation for temperature. The controllability and stabilizability of the Navier-Stokes equations and Boussinesq equations have been widely discussed (see [4], [5], [6], [7], [10]). In this talk, we consider a boundary feedback stabilization problem. In particular, we consider a finite number of controls acting on a portion of the boundary through Robin boundary condition and construct a linear Luenberger observer based on point observations of the linearized system. The sensor locations for the measurement are determined by the geometric structure of the feedback functional gains (see [1], [2], [3]). It can be shown that the nonlinear system coupled with the observer through the feedback law is locally exponentially stabilizable. Numerical results are provided to illustrate the idea and suggest areas for future research.

References

[1] J. Borggaard, J. A. Burns, and L. Zietsman. *On using LQG performance metrics for sensor placement*, in Proceedings of the American Control Conference, 2381–2386. IEEE, 2011.
[2] J. A. Burns, X. M. He, and W. Hu. *Control of the Boussinesq equations with implications for sensor location in energy efficient buildings*, in Proceedings of the American Control Conference, IEEE, 2012, 46–53.
[3] J. A. Burns and B. B. King. *Optimal sensor location for robust control of distributed parameter systems*, in Proceedings of the 33rd IEEE Conference on Decision and Control, IEEE, 1994, 3967–3972.
[4] O. Yu. Imanuvilov E. Fernandez-Cara, S. Guerrero and J.-P. Puel. *Local exact controllability of the Navier–Stokes system*, Journal de mathematiques pures et appliquees, **83**, 1501–1542, 2004.
[5] A. V. Fursikov and O. Y. Imanuvilov. *Local exact boundary controllability of the Boussinesq equation*, SIAM Journal on Control and Optimization, **36**(2): 391–421, 1998.
[6] A. V. Fursikov and O. Y. Imanuvilov. *Exact controllability of the Navier-Stokes and Boussinesq equations*, Russian Mathematical Surveys, **54**(565), 1999.
[7] A. V. Fursikov. *Stabilizability of two-dimensional Navier-Stokes equations with help of a boundary feedback control*, Journal of Mathematical Fluid Mechanics, **3**(3):259–301, 2001.
[8] L. S. Hou and S. S. Ravindran. *A penalized Neumann control approach for solving an optimal Dirichlet control problem for the Navier-Stokes equations*, SIAM Journal on Control and Optimization, **36**(5):1795–1814, 1998.
[9] L. S. Hou and S. S. Ravindran. *Numerical approximation of optimal flow control problems by a penalty method: Error estimates and numerical results*, SIAM Journal on Scientific Computing, **20**(5):1753–1777, 1999.
[10] O. Yu. Imanuvilov. *Controllability of evolution equations of fluid dynamics*, in Proceedings on the International Congress of Mathematicians: Madrid, August 22-30, 2006: Invited lectures, 1321–1338, 2006.
[11] D. D. Joseph. *Stability of Fluid Motions, Vol. II*, Springer-Verlag, New York, 1976.

Guidance by repulsion model: analysis, some simulations and optimal control

Aitziber Ibañez
(joint work with Ramón Escobedo, Enrique Zuazua)

The most usual strategy to find an effective way to solve a guidance problem is the "Follow me" strategy, which is based on the effect of the attraction that one agents (or particle) exerts on the others. The major novelty of our work [1] is that we use a repulsion force (i. e., "move away from me" strategy) to solve a guidance problem.

We can find several examples of how different guidance problems can be solved using the previously mentioned "Follow me" strategy [2], [3], [4], [5]. Repulsion forces, like the one used in our work, are mostly used for collision avoidance or interception, like in [6]. In a very recent work [7] they show how this kind repulsion force can be used to describe a defender-intruder interaction, where the defender exerts a repulsion force in the intrude to expel it away from a protected target.

In the recent work [1] we describe a "guidance by repulsion" phenomenom describing the behaviour of two agents, a driver and an evader. The first one follows the evader but cannot be arbitrarily close to it, and the evader tries to move away from the driver beyond a short distance. The driver can display a circumvention motion around the evader, in such a way that the trajectory of the evader is modified due to the repulsion that the driver exerts on the evader.

We show how the evader can be guided to any target on the plane or to follow a sufficiently smooth path by controlling just one parameter which modifies its behaviour, activating or deactivating the circumvention mode and selecting the clockwise/counterclockwise direction of the circumvention motion.

We propose different open loop strategies to drive the evader from any given point to another assuming that both switching the control and keeping it on (i. e. when $\kappa \neq 0$) has a cost, and we end presenting a feedback law which prevents the excessive use of the circumvention mode with a significant reduction of the cost.

References

[1] Escobedo, R., Ibañez, A., Zuazua, E. *Optimal strategies for driving a mobile agent in a guidance by repulsion model.* arXiv preprint arXiv:1509.05008. (2015).

[2] Albi, G., Bongini, Cristiani E., Kalise D.. *Invisible control of self-organizing agents leaving unknown environments.* (2015)

[3] Borzi A., Wongkaew S. *Modelling and control through leadership of a refined flocking system.* Math. Mod. Meth. Apl. Sci. 25 (2015).

[4] Caponigro M., Fornasier M., Piccoli B., Trélat E. *Sparse stabilization and optimal control of the Cucker-Smale model.* Math. Cont. Related Fields 3 (2013).

[5] Caponigro, M., Fornasier, M., Piccoli B., Trélat E. . *Sparse stabilization and control of alignment models.* Mathematical Models and Methods in Applied Sciences, (2015).

[6] Muro C., Escobedo R., Spector L., Coppinger R.P. *Wolf-pack (Canis lupus) hunting strategies emerge from simple rules in computational simulations.* Behav. Proc. 88 (2011).

[7] Wang J., Li W. *Motion patterns and phase-transition of a defender-intruder problem and optimal interception strategy of the defender.* Comm Nonlinear Sci. Numer. Simulat. 27 (2015).

Model reduction by moment matching for infinite dimensional systems

Orest V. Iftime

(joint work with Tudor C. Ionescu)

Infinite-dimensional systems are a wide class of systems that stems from modelling of phenomena and processes. Approximation of infinite-dimensional systems involves finding a finite-dimensional model of the original system with an irrational transfer function, satisfying appropriate constraints. During the past decades, considerable advances have been made in this direction. One popular approach is based on finite-element methods, resulting in large scale models which are difficult to analyse and control. Alternatively, model reduction procedures can be performed directly on the infinite-dimensional systems. One way to approximate an infinite-dimensional system is to perform modal truncation, i.e., keep a finite number of terms from the modal decomposition of the transfer function of the given system. The drawbacks of this method consist of the difficulty of computing the transfer function and of the fact that the zeros of the given system are lost, see e.g., [1]. Another method is based on the of the controllability and observability Gramians, where one can obtain reduced order models by balanced truncation, optimal or sub-optimal Hankel normapproximations [2, 4]. Balanced proper orthogonal decomposition and balanced truncation coincide when using a particular balanced realization for infinite-dimensional systems [3].

We study the problem of model reduction by moment matching for a class of infinite-dimensional systems, based on the unique solution of an operator Sylvester equation [5]. The solution yields a class of parametrized, finite-dimensional, reduced order models that match a set of prescribed moments of the given system. We show that, by properly choosing the free parameters, additional constraints are met, e.g., pole placement, preservation of zeros. One of the main advantages of this method is the preservation of a finite number of zeros of the given infinite-dimensional system, yielding better approximations (including the case of minimum phase systems).

References

[1] R.F. Curtain and K. Morris, *Transfer functions of distributed parameter systems*, Automatica **45**, (2009), 1101–1116.

[2] K. Glover, R.F. Curtain, and R. Partington, *Realisation and approximation of linear infinite dimensional systems with error bounds*, SIAM J. Control and Optim, **26(4)**, no. 4, (1988), 863–898.

[3] R.F. Curtain and H.J. Zwart, *An introduction to infinite-dimensional linear systems theory*, New York: Springer-Verlag, 1995.

[4] O.V. Iftime and A.J. Sasane, *Sub-optimal Hankel norm approximation problem: A frequency-domain approach*, Systems and Control Letters **52(1)**, (2004), 75–83.

[5] O.V. Iftime and T.C. Ionescu, *Time-domain moment matching for Riesz-spectral, infinite-dimensional systems*, in preparation.

Optimal actuator and observation location for time-varying systems on a finite-time horizon

Birgit Jacob

(joint work with Xueran Wu, Hendrik Elbern)

The choice of the location of controllers and observations is of great importance for designing control systems and improving the estimations in various practical problems. For time-varying systems in Hilbert spaces, the existence and convergence of the optimal location based on linear-quadratic control on a finite-time horizon is studied. More precisely, for an initial-time $t_0 \in [a, b]$ we consider the time-varying system described by

$$x_r(t) = T(t, t_0)x(t_0) + \int_{t_0}^{t} T(t, s)B_r(s)u(s)ds, \quad t_0 \leq t \leq b,$$

where $T(t, s)$ is a mild evolution operator on a separable Hilbert space H and B_r is a strongly measurable essentially bounded function with $B(s) \in L(U, H)$ a.e., where U is a finite-dimensional Hilbert space. Further we assume that B_r depends on a parameter $r \in \Omega$. The linear-quadratic optimal control problem is the question of finding for $x_0 \in X$ a control $u_0 \in L^2(t_0, b; U)$ which minimizes the cost functional

$$J_r(t_0, x_0, u) = \langle x(b), Gx(b)\rangle + \int_{t_0}^{b} \|C(s)x(s)\|^2 + \langle u(s), R(s)u(s)\rangle ds.$$

As the initial state x_0 is not fixed we define the optimal location by

$$\hat{\ell}(t_0) = \inf_{r\in\Omega} \max_{\|x_0\|=1} \min_{u\in L^2(t_0,b;U)} J_r(x_0, u),$$

For time-invariant systems on an infinite time horizon the optimal location problem has been studied in [1, 2]. In this talk we aim to extend these results to time-varying system on a finite-time horizon and we tread the problem for stochastical systems as well. The optimal location of observations for improving the estimation of the state at the final time, based on Kalman filter, is considered as the dual problem to the LQ optimal problem of the control locations. Further, the existence and convergence of optimal locations of observations for improving the estimation at the initial time, based on Kalman smoother is discussed.
The obtained results are applied to a linear advection-diffusion model with a special extension of emission rates.

References

[1] N. Darivandi, K. Morris and A. Khajepour, *An algorithm for LQ optimal actuator location*, Smart Mater. Struct. **22**(3), (2013), 035001.

[2] K. Morris, *Linear-quadratic optimal actuator location*, IEEE Trans. Autom. Control, **56** (2011) 113–124.

Non-linear dynamic stabilising control of port-Hamiltonian systems

Yann Le Gorrec

(joint work with Hector Raminez, Hans Zwart)

Port-Hamiltonian systems can be used to describe ordinary and partial differential equations (pde's) with control and observation. Since in standard port-Hamiltonian systems there is no internal damping, the stabilisation has to be done via the control. For a port-Hamiltonian system described by a pde, linear damping at the boundary of the spatial domain is well-studied, and there are easy testable conditions for exponential stability, see e.g. [2, Chapter 9] or [1, 3, 4].

However, these conditions limit the class of controllers. For instance, these controllers need to have direct feed through. If these controllers don't satisfy these conditions, exponential stability cannot be achieved. Furthermore, it is interesting to know if similar results can be obtained when applying non-linear control. Since any physical controller will have some non-linear behaviour, this question is also practically relevant. In what follows we consider non linear controllers inducing asymptotic stability. The system we consider is the port-Hamiltonian system described by

$$\frac{\partial x}{\partial t} = P_1 \frac{\partial}{\partial \zeta} (\mathcal{H}x), \quad t \geq 0, \zeta \in (0,1)$$

with boundary condition (homogeneous and control)

$$W_{B,1} \begin{pmatrix} \mathcal{H}x(1) \\ \mathcal{H}x(0) \end{pmatrix} = 0, \qquad W_{B,2} \begin{pmatrix} \mathcal{H}x(1) \\ \mathcal{H}x(0) \end{pmatrix} = u(t),$$

and boundary observation

$$W_C \begin{pmatrix} \mathcal{H}x(1) \\ \mathcal{H}x(0) \end{pmatrix} = y(t).$$

The energy is given by $E(t) = \frac{1}{2}\int_0^1 x^T (\mathcal{H}x)\, d\zeta$. It is assumed that for $u \equiv 0$ the homogeneous systems is governed by a contraction semigroup, and for classical solutions the following balance equation holds

$$\dot{E}(t) = u^T(t) y(t).$$

The dynamic, non-linear controller is of the form

$$\dot{x}_c(t) = \begin{pmatrix} 0 & I \\ -I & -R \end{pmatrix} \begin{pmatrix} \dot{P}(x_{c,1}) \\ K_2 x_{c,2} \end{pmatrix} + \begin{pmatrix} 0 \\ I \end{pmatrix} u_c(t), \quad y_c(t) = \; K_2 x_{c,2}$$

where $x_c = \binom{x_{c,1}}{x_{c,2}} \in \mathbb{R}^{n_c}$, and R, K_2 are symmetric and (strictly) positive. The energy of this system equals $E_c(t) = P(x_{c,1}) + \frac{1}{2} x_{c,2}^T K_2 x_{c,2}$. Hence for mechanical systems, $x_{c,1}$ would be positions and $x_{c,2}$ the velocities. It can be shown that

$$\dot{E}_c(t) = -x_{c,2}^T(t) R x_{c,2}(t) + u_c^T(t) y_c(t).$$

The control system is connected to the port-Hamiltonian system via $u = -y_c$ and $u_c = y$. By doing this it is easy to see that the total energy is bounded, and among others this shows that the solution of the closed loop system exists globally. Furthermore, for smooth initial conditions, the solution is classical. Abstractly, the connected system can be written as $\dot{z}(t) = Az(t) + N(z(t))$, in which N represents the non-linear part. For smooth initial conditions, we can show that $\dot{z}(t)$ is uniformly bounded. By the above equation, and the fact that A has compact resolvent, it follows that $\{z(t), t \geq 0\}$ is relatively compact. Hence by LaSalle's theorem the solution will converges to the largest invariant set contained in $\dot{E}(t) \equiv 0$. A simple observability condition now gives that the closed loop system is asymptotically stable.

References

[1] B. Augner and B. Jacob, *Stability and stabilization of infinite-dimensional linear port-Hamiltonian systems*, Evolution Equations and Control Theory (EECT), **3** (2014) 207–229.
[2] B. Jacob and H. Zwart, *Linear Port-Hamiltonian Systems on Infinite-dimensional Spaces*, Birkhäuser (2012)
[3] H. Ramirez, Y. Le Gorrec, A. Macchelli, and H. Zwart, *Exponential stabilization of boundary controlled port-Hamiltonian systems with dynamic feedback*, IEEE Transactions on Automatic Control, **59** (2014) 2849–2855.
[4] J. Villegas, H. Zwart, Y. Le Gorrec, and B. Maschke, *Exponential stability of a class of boundary control systems*, IEEE Transactions on Automatic Control, **54** (2009), 142–147.

Webster's equation in acoustic waveguides

Jarmo Malinen

Long wavelength acoustics in lossy, curved tubular waveguides can be approximated by the generalised Webster's model, consisting of the PDE

$$\frac{1}{c^2\Sigma(s)^2}\frac{\partial^2\psi}{\partial t^2} = \frac{1}{A(s)}\frac{\partial}{\partial s}\left(A(s)\frac{\partial\psi}{\partial s}\right) - \frac{2\pi\alpha W(s)}{A(s)}\frac{\partial\psi}{\partial t} \quad \text{for } s \in [0,\ell], t \geq 0,$$

together with the appropriate boundary conditions at $s = 0, \ell$. Here A represents the spatially varying intersectional area of the waveguide, the function Σ is a correction due to curvature, and the function W in the dissipation term is due to different inner and outer curvature radii of the waveguide. The solution $\psi = \psi(s,t)$ is the velocity potential, and the acoustic pressure as well as the (perturbation) velocity can be obtained from it by partial derivatives.

Two problems related to Webster's model are considered, and their solutions are presented in the special case $\alpha = 0$ and $\Sigma(s) = 1$. Firstly, the control $A(0)\phi_s(0,t) = i(t)$ and observation $p(t) = \rho\phi_t(\ell,t)$ are used, and it is shown that the impedance transfer function from i to p does not have zeroes if A is real analytic in a neighbourhood of ℓ.

Secondly, first order perturbations are derived for the eigenvalues λ^2 of the resonance problem

$$\left(\frac{\lambda}{c}\right)^2\Psi = \frac{1}{A(s)}\frac{\partial}{\partial s}\left(A(s)\frac{\partial\Psi}{\partial s}\right) \text{ for } s \in [0,\ell]$$

for small perturbations $A = A_0 + \epsilon B$ of a nominal area function A_0. As an application, local spectral inversion problem is considered. Given (measurement) data about the some of the eigenvalues λ_j^2 relating to a true area function A (of which we only have a guess, i.e., $A_0 \approx A$), how can be find a first order correction B to A_0?

As a practical application of both the problems, modelling of speech acoustics of vowels is discussed.

Using approximations in controller design for infinite-dimensional systems

Kirsten Morris

Many control systems are modelled by partial differential equations (PDE's). The state of such systems evolves on an infinite-dimensional space. There are essentially two approaches to controller design for systems modeled PDE's: direct and indirect. In direct controller design, the original model is used design the controller. The resulting controller will often be infinite-dimensional and in a subsequent step, order reduction is used to obtain a finite-dimensional controller of acceptable order. In indirect controller design, a finite-dimensional approximation of the system is obtained at the start and controller design is based on this finite-dimensional approximation.

The chief drawback of direct controller design is that a representation of the solution suitable for calculation is required. For many practical examples, this is not possible. For this reason, indirect controller design is generally used in practice. The hope is that the controller designed using the finite-dimensional approximation has the desired effect on the original system. That this method is not always successful was first documented in Balas [1], where the term *spillover* was introduced. Spillover refers to the phenomenon that a controller which stabilizes a reduced-order model may not stabilize the original model. Systems with infinitely many poles either on or asymptoting to the imaginary axis are notorious candidates for spillover effects. Conditions under which the indirect approach to controller design works have been obtained and are presented. The difference between approximation of parabolic and of hyperbolic equations is illustrated by several simple equations.

The performance of controlled partial differential equations depends on the location of controller hardware, the sensors and actuators. A mechatronic approach, where controller design is integrated with actuator location, can lead to better performance without increased controller cost. The best locations may be different from those chosen based on physical intuition. Furthermore, since it is often difficult to move hardware, and trial-and-error is laborious when there are multiple sensors and actuators, analysis is crucial. Approximations to the governing equations, often of very high order, are required and this complicates not only controller design but also optimization of the hardware locations. Care needs to be taken in formulating the joint optimization/ controller design problem in order to obtain correct results. Numerical issues will be briefly discussed.

References

[1] M. J. Balas. Active Control of Flexible Systems. *Jour. of Optimization Theory and Applications*, Vol. 25, No. 3, 415–436, 1978.

[2] A. Bensoussan. Optimization of sensors' location in a distributed filtering problem. ed. R.F. Curtain, *Stability of Stochastic Dynamical Systems*, Springer, 62–84,1972.

[3] K.K. Chen and C.W. Rowley. H_2-optimal actuator and sensor placement in the linearised complex Ginzburg-Landau system. *J. Fluid Mech.* , 681:241–260, 2011.

[4] R. L. Clark and C. R. Fuller. Optimal placement of piezoelectric actuators and polyvinylidene fluoride error sensors in active structural acoustic control approaches. *J. Acous. Soc. Amer.* , 92(3):1521 – 1533, 1992.
[5] N. Darivandi, K. A. Morris and A. Khajepour. An algorithm for LQ optimal actuator location. *Smart Materials & Struct.,* 22, 2013.
[6] F. Fahroo and M. A. Demetriou. Optimal actuator/sensor location for active noise regulator and tracking control problems. *J. Comp. Appl. Math.*, 114(1):137–158, 2000.
[7] D. Kasinathan and K. A. Morris. H_∞-optimal actuator location. *IEEE Trans. Auto. Cont.*, 58(10), 2013.
[8] K.A. Morris, M. Demitriou and S.D Yang. Using H_2-control performance metrics for infinite-dimensional systems. *IEEE Trans. Auto. Control*, Vol. 60, No. 2, pg. 450–462, 2015.
[9] K. A. Morris. Linear-quadratic optimal actuator location. *IEEE Trans. Auto. Control*, 56:113–124, 2011.
[10] K. A. Morris. Control of systems governed by partial differential equations. ed. W. S. Levine, *IEEE Control Theory Handbook*, CRC Press, 2010.
[11] S. D Yang and K. A. Morris. Comparison of Actuator Placement Criteria for Control of Structural Vibrations. *Jour. of Sound and Vibration*, Vol. 353, pg. 1–18, 2015.

Balanced truncation model reduction for infinite-dimensional linear systems

Timo Reis

(joint work with Mark. R. Opmeer, Tilman Selig, Winnifried Wollner)

Balanced truncation is one of the most popular model reduction methods for finite-dimensional input-output-systems governed by ordinary differential equations, i.e.,

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t),\end{aligned}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$. This technique relies on the solution of the observability and controllability Lyapunov equations

$$\begin{aligned}AP + PA^\top + BB^\top &= 0,\\ A^\top P + PA + C^\top C &= 0\end{aligned}$$

for positive semi-definite matrices $P, Q \in \mathbb{R}^{n\times n}$.

We consider this method for infinite-dimensional systems, i.e., A, B, C, P and Q are operators acting on Hilbert spaces. Motivated by the ADI method for the determination of Gramian matrices [1], we develop a numerical scheme for the finite-rank approximation of the Gramian operators associated to infinite-dimensional systems. This enables us to determine finite-dimensional approximations of infinite-dimensional systems. Error bounds in the $\mathcal{H}_\infty$ norm are provided. Our results will be illustrated by means of a boundary-controlled heat equation and delay-differential equations.

References

[1] J. R. Li and J. White, *Low-rank solution of Lyapunov equations*, SIAM Rev., **46**(4) (2004), 693-713.

[2] T. Reis, M.R. Opmeer and W. Wollner, *Finite-rank ADI iteration for operator Lyapunov equations*, SIAM J. Control Optim., **51**(5) (2013), 4084-4117 .

[3] T. Reis and T. Selig, *Balancing transformations for infinite-dimensional systems with nuclear Hankel operator*, Integr. Equ. Oper. Theory, **79**(1) (2013), 67-105.

Results around the Cayley Transform Problem

Felix L. Schwenninger

(joint work with Hans Zwart)

This work deals with the infinite-dimensional analogue of the following question. *Are the powers of the matrix*

$$\mathrm{Cay}(A) := (A + \mathrm{I})(A - \mathrm{I})^{-1} \tag{1}$$

(where I *denotes the identity) uniformly bounded if it is assumed that* $A \in \mathbb{C}^{N\times N}$*,* $N \in \mathbb{N}$*, has a uniformly bounded matrix exponential, i.e.,* $\sup_{t\geq 0} \|e^{tA}\| < \infty$ *?*

This can be rephrased as the question of stability of the *Crank–Nicolson* scheme. In the infinite-dimensional situation, A is assumed to be the generator a bounded C_0-semigroup $(e^{tA})_{t\geq 0}$ on a Banach space X, which implies that $\mathrm{Cay}(A)$ defines a bounded operator and thus, one can again study whether $\sup_{n\in\mathbb{N}} \|\mathrm{Cay}(A)^n\| < \infty$.

In contrast to the elementary matrix case, the answer is 'no' for general Banach spaces. On the other hand, it is 'yes' if $(e^{tA})_{t\geq 0}$ is bounded analytic. For Hilbert spaces X, the question still remains open, but in this case, there are 'more positive answers'; e.g., contraction semigroups and uniformly continuous semigroups. See, e.g., [3] for a detailed overview and references of the mentioned results.

We show that, in order to answer the question for Hilbert spaces, it suffices to consider exponentially stable C_0-semigroups $(e^{tA})_{t\geq 0}$ with $\sup_{t\geq 0} \|e^t e^{tA}\| < \infty$.

It is well known that this problem is strongly related to the *Inverse Generator Problem*, i.e., the question if A^{-1} generates a bounded C_0-semigroup, whenever A generates a bounded C_0-semigroup and A^{-1} exists as a densely defined operator [1, 2, 4]. We show that both problems are indeed equivalent and can be reduced to the special case of exponentially stable semigroups. The presented results were derived in [3].

References

[1] A.M. Gomilko, *The Cayley transform of the generator of a uniformly bounded* C_0*-semigroup of operators. Ukraïn. Mat. Zh.*, 56(8) (2004), 1018–1029.

[2] A.M. Gomilko, H. Zwart, and Y. Tomilov. *On the inverse operator of the generator of a* C_0*-semigroup.* Mat. Sb., 198(8) (2007), 35–50.

[3] F.L. Schwenninger. *On Functional Calculus Estimates.* PhD thesis, University of Twente, 2015, available at `http://dx.doi.org/10.3990/1.9789036539623`.

[4] H. Zwart. *Is* A^{-1} *an infinitesimal generator?* In Perspectives in operator theory, Banach Center Publ., Polish Acad. Sci., Warsaw, **75** (2007), 303–313.

Coupled passive systems and strong stabilization of the SCOLE system using tuned mass dampers

George Weiss
(joint work with Xiaowei Zhao)

Coupled infinite-dimensional systems have attracted much interest in recent years. From the vast literature, we mention here only the thesis of Villegas [5], the books of Lasiecka [4] and Jacob and Zwart [3]. In [6] we have developed a theory for the regularity, controllability and observability of coupled linear systems consisting of an infinite-dimensional strictly proper subsystem Σ_d and a finite-dimensional subsystem Σ_f connected in feedback. In [6] we assume that the feedthrough matrix of Σ_f is invertible and certain (not too restrictive) algebraic conditions hold - this is relevant for the current work. In this work we study the stability of a feedback system as in [6] where the subsystems are impedance passive. We are interested in two main issues: (1) the strong stability of the operator semigroup associated with Σ_c, (2) the input-output stability (meaning transfer function in H^∞) of Σ^c. We show that these properties hold under relatively mild and verifiable assumptions, that include the approximate observability of both systems.

As an application we tackle a problem that is relevant for the vibration suppression of wind turbine towers. We demonstrate that a Tuned Mass Damper (TMD) can be used to stabilize the SCOLE (NASA Spacecraft Control Laboratory Experiment) model. The SCOLE system is a well known model for a flexible beam with one end clamped and the other end connected to a rigid body. Originally it has been developed to model a mast carrying an antenna on a satellite, see Littman and Markus [1, 2]. This framework is suitable also to model wind turbine towers, since these are typically clamped at the bottom while the upper end of the tower is linked to the heavy nacelle.

To approach the practical engineering problem of stabilizing a wind-turbine tower, we should use the natural energy state space for the SCOLE model, where only strong stability can be achieved (because the control and observation operators are bounded).

We design and analyze a TMD control system to reduce the vibration of the SCOLE model. A TMD consists of a large mass, connected to the top of the structure to be stabilized via springs and dampers. The idea is that the TMD is tuned to a particular structural frequency and thus will resonate and dissipates input energy via the dampers when the structure is excited at that particular frequency. In this paper we analyze this technique in one plane only, so that there is only spring and one damper, connected in parallel between the rigid body of the SCOLE system and the mass component of the TMD. This mass component can be either put on a trolley moving along a rail, or it can be hanged via cables.

The mathematical model of the SCOLE system with a trolley Σ_c is the following set of equations:

$$\rho(x)w_{tt}(x,t) + (EI(x)w_{xx}(x,t))_{xx} = 0, \quad (x,t) \in (0,l) \times [0,\infty),$$
$$w(0,t) = 0,\ w_x(0,t) = 0,$$

$$mw_{tt}(l,t) - (EIw_{xx})_x(l,t) = F_e(t) + k_1(p(t) - w(l,t)) + d_1(p_t(t) - w_t(l,t)),$$
$$Jw_{xtt}(l,t) + EI(l)w_{xx}(l,t) = T_e(t),$$
$$m_1 p_{tt}(t) = k_1(w(l,t) - p(t)) + d_1(w_t(l,t) - p_t(t)),$$

where the subscripts t and x denote derivatives with respect to the time t and the position x. The first four equations are the non-uniform SCOLE model Σ_d while the last equation describes the trolley system Σ_f. Σ_d and Σ_f are interconnected through the transverse speed of the nacelle $y = w_t(l,t)$ and the force

$$u = k_1(p(t) - w(l,t)) + d_1(p_t(t) - w_t(l,t))$$

generated by the trolley system. l, ρ and EI denote the beam's height, mass density function and flexural rigidity while w denotes its transverse displacement. $\rho, EI \in C^4[0,l]$ are assumed to be strictly positive functions. The parameters $m > 0$ and $J > 0$ are the mass and the moment of inertia of the rigid body. F_e and T_e are the force and torque inputs acting on the rigid body. The state of Σ_d at the time t is

$$z(t) = [z_1(t)\ z_2(t)\ z_3(t)\ z_4(t)] = [w(\cdot,t)\ w_t(\cdot,t)\ w_t(l,t)\ w_{xt}(l,t)].$$

Its natural energy state space is

$$X^d = \mathcal{H}_l^2(0,l) \times L^2[0,l] \times \mathbb{C}^2,$$

where $\mathcal{H}_l^2(0,l) = \{h \in \mathcal{H}^2(0,l) \mid h(0) = 0,\ h_x(0) = 0\}$. Here $\mathcal{H}^n$ $(n \in \mathbb{N})$ denote the usual Sobolev spaces. The natural norm on X^d is (twice the physical energy)

$$\|z(t)\|^2 = \int_0^l EI(x)|z_{1xx}(x,t)|^2 dx + \int_0^l \rho(x)|z_2(x,t)|^2 dx + m|z_3(t)|^2 + J|z_4(t)|^2.$$

In the trolley system Σ_f, $m_1 > 0$, $k_1 > 0$ and $d_1 > 0$ are the mass, spring constant and damping coefficient. p and p_t are the position and transverse velocity of the mass component of the trolley. The state of Σ_f is defined as

$$q(t) = [q_1(t)\ q_2(t)] = [p(t) - w(l,t)\ p_t(t)]$$

with state space $X^f = \mathbb{C}^2$, on which the norm is defined as $\|q(t)\|^2 = k_1|q_1(t)|^2 + m_1|q_2(t)|^2$. The state space of the SCOLE-trolley coupled system Σ_c is X_c, the product of the two state spaces of the subsystems.

Our results are that the SCOLE system with trolley Σ_c with input $[F_e\ T_e]$, state $[z\ q]$ and output $[u\ w_{xt}(l,t)\ y]$ is well-posed and regular on the energy state space X^c and that Σ_c is strongly stable on X^c.

References

[1] W. Littman and L. Markus, Stabilization of a hybrid system of elasticity by feedback boundary damping, *Ann. Mat. Pura Appl.* **152** (1988), 281–330.

[2] W. Littman and L. Markus, Exact boundary controllability of a hybrid system of elasticity, *Archive for Rational Mechanics and Analysis* **103** (1988), 193–235.

[3] B. Jacob and H.J. Zwart, *Linear Port-Hamiltonian Systems on Infinite-Dimensional Spaces*, Birkhäuser, Basel, 2012.

[4] I. Lasiecka, *Mathematical Control Theory of Coupled PDEs*, SIAM, Philadelphia, CBMS-NSF Regional Conf. Series 75, 2002.

[5] J.A. Villegas, A Port-Hamiltonian Approach to Distributed Parameter Systems, PhD Dissertation, University of Twente, 2007.
[6] X. Zhao and G. Weiss, Controllability and observability of a well-posed system coupled with a finite-dimensional system, *IEEE Trans. Aut. Control* **56** (2011), 88–99.

Numerical approximations for port-Hamiltonian systems

Hans Zwart

An important subclass of port-Hamiltonian systems are those described by (linear) partial differential equations with control and observation at the boundary. A canonical example is the wave equation

$$\rho w_{tt} = \operatorname{div}\left(T \operatorname{grad}(w)\right)$$

on an e.g. two dimensional spatial (bounded) domain Ω. There are two ways in which this system can be presented as a port-Hamiltonian system, namely using abstract differential equations on Hilbert spaces or via differential forms. We begin with the abstract differential equation. For that we take the state

$$x = \begin{pmatrix} \rho w_t \\ T\operatorname{grad}(w) \end{pmatrix}$$

and write the partial differential equation as

$$\frac{\partial x}{\partial t} = \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\rho} & 0 \\ 0 & T \end{pmatrix} x = \mathcal{J}\mathcal{H}x.$$

The $\mathcal{J}$ clearly gives the skew-symmetric structure associated to the conservation of energy for the wave. This means that the operator $\mathcal{J} : e \mapsto f$ is skew symmetric with respect to the inner product in $L^2(\Omega)$.
The second representation of the wave equation uses differential forms. Thus $x, \dot{x}$, and $\mathcal{H}x$ are seen as k-forms. The above mentioned $\mathcal{J}$ becomes now the mapping

$$\mathcal{J}_{df} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} := \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

mapping the zero-form e_1 and the 1-form e_2 to the 2-form f_1 and the 1-form f_2. The mapping $\mathcal{H}$ is also replaced. The skew-symmetry of $\mathcal{J}_{df}$ is now with respect to the wedge product.

There are now two research questions. The first is to understand the precise relation between these two ways of reformulation the wave equation. A start in solving this problem is given by Arnold et al in [1]. In this paper we can also see how the differential forms can be used in numerics. This is also the approach taken by van der Schaft et al, see e.g. [2]. What is missing is an error analysis.

References

[1] D. Arnold, R. Falk and P. Winther *Finite element exterior calculus: from Hodge theory to numerical stability*, Bulletin of the AMS, **47** (2010), 281–354.
[2] M. Seslija, J. Scherpen and A. van der Schaft, *Explicit simplicial discretization of distributed-parameter port-Hamiltonian systems*, Automatica, **50** (2014), 369–377.

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