Mathematisches Forschungsinstitut Oberwolfach

Report No. 5/2016

DOI: 10.4171/OWR/2016/5

Algebraic Cobordism and Projective Homogeneous Varieties

Organised by
Stefan Gille, Edmonton
Marc Levine, Essen
Ivan Panin, St. Petersburg
Alexander Vishik, Nottingham

31 January – 6 February 2016

Abstract. The aim of this workshop was to bring together researchers in the theory of projective homogeneous varieties with researchers working on cohomology theories of algebraic varieties, so that the latter can learn about the needs in an area of successful applications of these abstract theories and the former can see the latest tools.

Mathematics Subject Classification (2010): 20Gxx, 14F42.

Introduction by the Organisers

Recent years have seen a new wave of results on quadratic forms and projective homogeneous varieties over semisimple algebraic groups. This development can be traced back at least in part to the introduction of methods from stable homotopy theory into the field, culminating in Voevodsky's proof of the Milnor conjectures and the proof of Rost and Voevodsky of the Bloch-Kato conjecture. As central as these two results are, the progress in the field is not limited to this. The introduction of new cohomological methods spawned by the push to prove the Milnor- and the Bloch-Kato conjectures has transformed the study of projective homogeneous varieties.

Much of this new impetus is due to the systematic extension of well-known cohomological techniques from algebraic topology, such as cohomology operations, to the setting of algebraic geometry. These tools are in turn descendants of the recent research in cohomology theories of algebraic varieties, including the universal oriented theory, algebraic cobordism, as well as a number of newly studied unoriented theories, such as hermitian K-theory, Witt theory and symplectic cobordism.

The aim of the workshop *Algebraic cobordism and projective homogeneous varieties* organized by Stefan Gille (Edmonton), Marc Levine (Essen), Ivan Panin (St. Petersburg), and Alexander Vishik (Nottingham) has been to bring together researchers in the theory of projective homogeneous varieties with researchers working on cohomology theories of algebraic varieties, so that the latter can learn about the needs in an area of successful applications of these abstract theories and the former can see the latest tools.

The workshop has been attended by about 50 researchers from Europe, North- and South-America and Asia, about 1/3 of them working on motives and/or $\mathbb{A}^1$-homotopy theory, 1/3 on quadratic forms and related topics as algebraic groups and projective homogeneous varieties, and 1/3 in both of these areas. There have been 20 one hour talks. As was the intention of the workshop the organizers have taken some effort to keep the balance between talks which presented latest developments in motivic cohomology or $\mathbb{A}^1$-homotopy theory, and talks which discussed recent applications to projective homogeneous varieties. New and strong results have been reported, which caused active discussion and interaction among the participants. New scientific connections and collaboration groups were formed.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, "US Junior Oberwolfach Fellows". Moreover, the MFO and the workshop organizers would like to thank the Simons Foundation for supporting David B. Leep in the "Simons Visiting Professors" program at the MFO.

Workshop: Algebraic Cobordism and Projective Homogeneous Varieties

Table of Contents

Abstracts

Serre's conjecture II for semisimple groups of type E_7

Philippe Gille

Serre's original conjecture II (1962) states that the Galois cohomology set $H^1(k,G)$ vanishes for a semisimple simply connected algebraic group G defined over a perfect field k of cohomological dimension $\mathrm{cd}(k) \leq 2$ [10, §4.1] . In other words, it predicts that all G-torsors (or principal homogeneous spaces) over $\mathrm{Spec}(k)$ are trivial. Serre extended the conjecture to imperfect fields in 1994 [10, II.3.1] and we discuss implicitly here that conjecture.

For example, if A is a central simple algebra defined over a field k and $c \in k^\times$, the subvariety

$$X_c := \{\mathrm{nrd}(y) = c\} \subset \mathrm{GL}_1(A)$$

of elements of reduced norm c is a torsor under the special linear group $G = \mathrm{SL}_1(A)$ which is semisimple and simply connected. If $\mathrm{cd}(k) \leq 2$, we expect then that this G-torsor is trivial, i.e. $X_c(k) \neq \emptyset$. By considering all scalars c, we expect then that the reduced norm map $A^\times \to k^\times$ is surjective.

For function fields of complex surfaces, this follows from the Tsen-Lang theorem given that the reduced norm is a homogeneous form of degree $\deg(A)$ in $\deg(A)^2$-indeterminates [10, II.4.5]. The general case of the surjectivity of reduced norm maps was established in 1981 by Merkurjev and Suslin [12, th. 24.8]. This fact characterizes essentially the fields of cohomological dimension ≤ 2 (and more generally fields of separable cohomological dimension ≤ 2).

Throughout its history, the evidence for and progress towards establishing conjecture II has been gathered by either considering special classes of fields, or by looking at the implications that the conjecture would have on the classification of algebraic groups.

From the groups point of view, the strongest evidence for the validity of the conjecture is given by the case of the classical groups (and type G_2 and F_4) established in 1995 by Bayer and Parimala [1] (and also Berhuy/Frings/Tignol for the generalization to imperfect fields [2]).

From the point of view of fields, we know that the conjecture holds in the case of imaginary number fields (Kneser, Harder, Chernousov see [9, §6]), and more recently for function fields of complex surfaces. For exceptional groups with no factors of type E_8, the relevant reasonings and references are given in [4]. A general proof for all types using deformation methods was given recrntly by He/de Jong/Starr [8]. This result has a clear geometric meaning: If $G/\mathbb{C}$ is a semisimple simply connected group and X a smooth complex surface, then a G-torsor over X (or a G-bundle) is locally trivial with respect to the Zariski topology.

For exceptional groups (trialitarian, type E_6, E_7 and E_8), the general conjecture is still open in spite of some considerable progress [3, 4, 5]. In the talk we discussed our approach of the conjecture by means of the study of cohomology classes arising from finite diagonalizable subgroups. The precise statement is the following.

Theorem *Let G/k be a semisimple simply connected group satisfying the hypothesis of the extended conjecture II. Let $f : \mu_n \to G$ be a k–group homomorphism. Then the induced map*

$$f_* : k^\times/k^{\times n} \to H^1(k, G)$$

is trivial.

If μ_n is central in G, this follows from the norm principle [5, th. 6] (or [6, th. 5.3]). The new thing is then the generalization to the non-central case. This result shall appear in the monography in preparation [7]. It is a key ingredient for proving the Serre's conjecture II in the E_7 case.

References

[1] E. Bayer-Fluckiger, R. Parimala, *Galois cohomology of the classical groups over fields of cohomological dimension* ≤ 2, Invent. math. **122** (1995), 195–229.

[2] G. Berhuy, C. Frings et J.-P. Tignol, *Galois cohomology of the classical groups over imperfect fields*, J. Pure Appl. Algebra **211** (2007), 307–341.

[3] V. Chernousov, *The kernel of the Rost invariant, Serre's Conjecture II and the Hasse principle for quasi-split groups*, Math. Ann. **326** (2003), 297–330.

[4] J.-L. Colliot-Thélène, P. Gille and R. Parimala, *Arithmetic of linear algebraic groups over 2-dimensional geometric fields*, Duke Math. J. **121** (2004), 285–341.

[5] P. Gille, *Cohomologie galoisienne des groupes quasi-déployés sur des corps de dimension cohomologique* ≤ 2, Compositio Math. **125** (2001), 283–325.

[6] P. Gille, *Serre's conjecture II: a survey*, Quadratic forms, linear algebraic groups, and cohomology, 41–56, Dev. Math. **18** (2010), Springer, New York.

[7] P. Gille, *Groupes algébriques sur des corps de dimension cohomologique séparable* ≤ 2, monography in preparation.

[8] A. J. de Jong, X. He, J.M. Starr, *Families of rationally simply connected varieties over surfaces and torsors for semisimple groups*, Pub. Math. I.H.É.S. **114** (2011), 1–85.

[9] V. P. Platonov and A. Rapinchuk, *Algebraic groups and number theory*, Pure and Applied Mathematics **139** (1994), Academic Press.

[10] J-P. Serre, *Cohomologie galoisienne*, cinquième édition révisée et complétée, Lecture Notes in Math. **5**, Springer-Verlag.

[11] J-P. Serre, *Cohomologie galoisienne: Progrès et problèmes*, Séminaire Bourbaki, exposé 783 (1993-94), Astérisque **227** (1995).

[12] A. A. Suslin, *Algebraic K-theory and the norm–residue homomorphism*, J. Soviet. **30** (1985), 2556–2611.

Framed motives of algebraic varieties

Grigory Garkusha

(joint work with Ivan Panin)

In [7] Voevodsky developed the machinery of framed correspondences and framed (pre)sheaves. Based on the theory, we introduce and study framed motives of algebraic varieties in [3] as well as linear framed motives in [4]. Framed correspondences form a category, denoted by $Fr_*(k)$, whose objects are those of Sm/k (smooth separated schemes of finite type over a field k). Morphisms between $X, Y \in Sm/k$ are given by $\bigsqcup_{n\geq 0} Fr_n(X, Y)$, where each $Fr_n(X, Y)$ consists of framed correspondences of level n in the sense of Voevodsky [7]. In order to get zero object in $Fr_*(k)$, we slightly modify the category by gluing empty correspondences $\{0_n \in Fr_n(X,Y)\}_{n\geq 0}$ and setting $Fr_+(X,Y) := \bigvee_{n\geq 0} Fr_n(X,Y)$. In this way we get a pointed category $Fr_+(k)$.

Let Γ^{op} be the category of finite pointed sets and pointed maps in the sense of Segal [6]. Then the functor $A \in \Gamma^{\mathrm{op}} \to Fr_+(X, Y \otimes A)$ with $Y \otimes A := \sqcup_{A\setminus *} Y$ is a Γ-space and its Segal's symmetric S^1-spectrum is denoted by $Fr_+^{S^1}(X,Y)$. We then enrich Sm/k over symmetric spectra by taking $Fr_+^{S^1}(X,Y)$ as a spectrum of morphisms between $X, Y \in Sm/k$. The resulting spectral category is denoted by $\mathcal{F}r_+^{S^1}(k)$. The right $\mathcal{F}r_+^{S^1}(k)$-modules are also called *spectral presheaves with spectral framed transfers.*

Theorem 1. *$\mathcal{F}r_+^{S^1}(k)$ is Nisnevich excisive in the sense of* [2]. *Its associated ringoid $\pi_0(\mathcal{F}r_+^{S^1}(k))$ is an additive category, denoted by $\mathbb{Z}F_*(k)$, whose morphisms are described as free Abelian groups freely generated by framed correspondences with connected support.*

Corollary 2. $\mathrm{Mod}\,\mathcal{F}r_+^{S^1}(k)$ *enjoys a stable projective model structure whose weak equivalences are those inducing isomorphisms of Nisnevich sheaves of stable homotopy groups. Its homotopy category, denoted by $SH_{loc}^{fr}(k)$, is compactly generated triangulated with $\{Fr_+^{S^1}(-,Y)\}_{Y\in Sm/k}$ compact generators.*

For every $Y \in Sm/k$ there is a distinguished framed correspondence $\sigma_Y \in Fr_1(Y,Y)$ defined as the quadruple $(Y\times 0, \mathbb{A}^1_Y, pr_{\mathbb{A}^1}, pr_Y)$. It induces an endomorphism $\sigma_Y : Fr_+^{S^1}(-,Y) \to Fr_+^{S^1}(-,Y)$ and we set

$$Fr^{S^1}(-,Y) = colim(Fr_+^{S^1}(-,Y) \xrightarrow{\sigma_Y} Fr_+^{S^1}(-,Y) \xrightarrow{\sigma_Y} \cdots).$$

We can similarly define $\mathbb{Z}F^{S^1}(-,Y)$ as

$$colim(\mathbb{Z}F_*^{S^1}(-,Y) \xrightarrow{\sigma_Y} \mathbb{Z}F_*^{S^1}(-,Y) \xrightarrow{\sigma_Y} \cdots),$$

where $\mathbb{Z}F_*^{S^1}(X,Y)$ stands for the Eilenberg–Mac Lane spectrum of $\mathbb{Z}F_*(X,Y)$.

The *framed motive $M_{fr}(Y)$ of a smooth algebraic variety $Y \in Sm/k$* is a spectral presheaf with spectral framed transfers defined as $Fr^{S^1}(\Delta^\bullet \times -, Y)$. The *linear framed motive* $LM_{fr}(Y)$ of $Y \in Sm/k$ is the spectrum $\mathbb{Z}F^{S^1}(\Delta^\bullet \times -, Y)$.

Theorem 3. *For every $Y \in Sm/k$, $M_{fr}(Y)$ is a schemewise Ω-spectrum in positive degrees. Moreover, if k is infinite and perfect, then it is $\mathbb{A}^1$-local in $SH_{S^1}(k)$.*

We define the *motivic model structure* on $\mathrm{Mod}\,\mathcal{F}r_+^{S^1}(k)$ by localizing the local model structure with respect to collections of arrows $\{pr_Y : Fr_+^{S^1}(-, Y\times\mathbb{A}^1) \to Fr_+^{S^1}(-,Y)\}$ and $\{\sigma_Y : Fr_+^{S^1}(-,Y) \to Fr_+^{S^1}(-,Y)\}$, $Y \in Sm/k$. Its homotopy category is denoted by $SH_{S^1}^{fr}(k)$ and is called the *category of framed motives.*

Theorem 4. *$SH_{S^1}^{fr}(k)$ is compactly generated triangulated with $\{M_{fr}(Y)\}_{Y\in Sm/k}$ compact generators. Furthermore, it is equivalent to the full subcategory of spectral presheaves with spectral framed transfers whose Nisnevich sheaves of stable homotopy groups are $\mathbb{A}^1$- and σ-invariant whenever k is infinite and perfect.*

The next result computes homology of framed motives.

Theorem 5. *If k is infinite and perfect and $Y \in Sm/k$, then the spectrum $H\mathbb{Z} \wedge M_{fr}(Y)$ has locally stable homotopy type of $LM_{fr}(Y)$. In partucular, homology of $M_{fr}(Y)$ is computed locally as homology of the complex $\mathbb{Z}F(\Delta^\bullet \times -, Y)$.*

Let $\mathbb{G} = Cyl(\iota)/(-, pt)_+$ with $Cyl(\iota)$ the mapping cylinder for the map $\iota : (-, pt)_+ \to (-, \mathbb{G}_m)_+$ sending pt to $1 \in \mathbb{G}_m$ and let $M^{\mathbb{G}}_{fr}(X)$ be the $(S^1, \mathbb{G})$-bispectrum $(M_{fr}(X), M_{fr}(X)(1), \ldots)$, each term of which is a twisted framed motive of X. If we take a Nisnevich local replacement $M_{fr}(X)(n)_f$ of each $M_{fr}(X)(n)$, we arrive at a bispectrum $M^{\mathbb{G}}_{fr}(X)_f = (M_{fr}(X)_f, M_{fr}(X)(1)_f, \ldots)$.

The following theorem is proved in [1].

Theorem 6. *If k is infinite and perfect and $X \in Sm/k$, then $M^{\mathbb{G}}_{fr}(X)_f$ is a motivically fibrant bispectrum.*

The preceding result together with a recent theorem of Neshitov [5] implies

Corollary 7. *If $char(k) = 0$ then $\pi^{\mathbb{A}^1}_{-*,-*}(M^{\mathbb{G}}_{fr}(pt))(pt) = K^{MW}_*(k)$.*

References

[1] A. Ananyevskiy, G. Garkusha, I. Panin, *Cancellation theorem for framed motives of algebraic varieties*, preprint arXiv:1601.06642.

[2] G. Garkusha, I. Panin, *K-motives of algebraic varieties*, Homology, Homotopy Appl. **14**(2) (2012), 211–264.

[3] G. Garkusha, I. Panin, *Framed motives of algebraic varieties (after V. Voevodsky)*, preprint arXiv:1409.4372.

[4] G. Garkusha, I. Panin, *The triangulated category of linear framed motives $DM^{eff}_{fr}(k)$*, in preparation.

[5] A. Neshitov, *Framed correspondences and the Milnor–Witt K-theory*, J. Inst. Math. Jussieu, to appear.

[6] G. Segal, *Categories and cohomology theories*, Topology **13** (1974), 293–312.

[7] V. Voevodsky, *Notes on framed correspondences*, unpublished, 2001. Also available at math.ias.edu/vladimir/files/framed.pdf

The Shareshian-Wachs Conjecture

Patrick Brosnan

(joint work with Timothy Chow)

The following extended abstract describes joint work with Timothy Chow (Center for Communications Research, Princeton) proving the Shareshian-Wachs conjecture [4]. I remark that there is another, completely independent and very interesting, proof of the conjecture given by M. Guay-Paquet [8].

The Stanley-Stembridge Conjecture. Suppose $G = (V, E)$ is a finite graph (with vertex set V and edge set E). A *coloring* of G is a map $\kappa : V \to \mathbb{Z}_+$ such that $\kappa(v) \neq \kappa(w)$ if v and w are adjacent. Write $C(G)$ for the set of all colorings. Let Λ denote the $\mathbb{C}$-algebra of all symmetric functions in infinitely many variables $x_1, x_2, \ldots$. For a coloring $\kappa \in C(G)$, we set $x_\kappa := \prod_{v \in V} x_{\kappa(v)}$. R. Stanley defined the chromatic symmetric function

$$X_G(x) := \sum_{\kappa \in C(G)} x_\kappa. \tag{1}$$

Suppose $n \in \mathbb{Z}_+$. A *Hessenberg function* for n is a non-decreasing sequence $m_1, \ldots, m_n$ of positive integers such that, for all i, $i \leq m_i \leq n$. Given a Hessenberg sequence $\mathbf{m}$, let $G(\mathbf{m}) = (V, E)$ denote the graph with vertex set $V = \{1, \ldots, n\}$ and with i and j adjacent for $i < j$ if and only if $j \leq m_i$. In this language, we can formulate the following long-standing conjecture of Stanley and J. Stembridge [10, 11] .

Conjecture 1 (Stanley-Stembridge). *Suppose $G = G(\mathbf{m})$ for a Hessenberg function $\mathbf{m}$. Then $X_G(x)$ is a non-negative sum of elementary symmetric functions.*

Remark 2. In fact, Stanley and Stembridge conjecture something which seems more general. But, in [7], Guay-Paquet proved that the general conjecture reduces to Conjecture 1.

The Shareshian-Wachs polynomial. Now suppose $G = (V, E)$ is a graph with $V \subset \mathbb{Z}_+$. For a coloring κ of G define

$$\operatorname{asc}(\kappa) := \#\{(v, w) \in V^2 : v < w, \kappa(v) < \kappa(w)\}.$$

In [9], J. Shareshian and M. Wachs prove the following remarkable theorem.

Theorem 3 (Shareshian-Wachs). *Suppose $G = G(\mathbf{m})$. Then the*

$$X_G(x, t) := \sum_{\kappa \in C(G)} t^{\operatorname{asc} \kappa} x_\kappa$$

is a polynomial in $\Lambda[t]$.

Examples 4. Write e_k for the k-th elementary symmetric function.

(1) If $\mathbf{m} = (1, 2, \ldots, n)$, then $X_{G(\mathbf{m})}(x, t) = n! e_n$.
(2) If $\mathbf{m} = (2, 3, 3)$, then $X_{G(\mathbf{m})}(x, t) = e_3 + t(e_3 + e_2 e_1) + t^2 e_3$.
(3) If $\mathbf{m} = (n, n, \ldots, n)$, then $X_{G(\mathbf{m})}(x, t) = \sum_{w \in S_n} t^{\ell(w)} e_n$.

Hessenberg varieties. Suppose $\mathbf{m} = (m_1, \ldots, m_n)$ is a Hessenberg function. Let s be an $n \times n$-matrix in $\mathfrak{g} := \mathfrak{gl}_n$. Let X denote the variety of complete flags F in n-dimensional space. Set

$$\mathscr{H}(\mathbf{m}, s) := \{F \in X : \forall i, sF_i \subset F_{m_i}\}.$$

This is called the *Hessenberg variety* of type $\mathbf{m}$. These varieties were introduced by DeMari, Procesi and Shayman in [5]. In fact, [5] studies a generalization of the varieties defined above inside the variety $\mathscr{B}$ of Borel subgroups of an arbitrary reductive group. In my work with Chow, only the type A case appears.

Write $\mathfrak{g}^{\mathrm{rs}}$ for the Zariski open subset of $\mathfrak{g}$ consisting of regular semi-simple matrices. Then [5] shows that, for $y \in \mathfrak{g}^{\mathrm{rs}}$, $\mathscr{H}(\mathbf{m}, y)$ is smooth. Moreover, the centralizer $Z(y)$ of y, which is a maximal torus in $G = \mathbf{GL}_n$ acts on $\mathscr{H}(\mathbf{m}, y)$. And the fixed point set $\mathscr{H}(\mathbf{m}, y)^{Z(y)}$ coincides with $X^{Z(y)}$. Note that the set $X^{Z(y)}$ is a torsor for the Weyl group $W \cong S_n$ of $Z(y)$. By Białynicki-Birula, it follows that the cohomology $H^*(\mathscr{H}(\mathbf{m}, y))$ is freely generated by one element for each fixed point [3]. Thus, $H^*(\mathscr{H}(\mathbf{m}, y))$ is (non-canonically) freely generated by one element for every element of W.

Tymoczko's Dot Action. Pick $y \in \mathfrak{g}^{\mathrm{rs}}$, and set $T = Z(y)$. In [13], J. Tymoczko defines an action (called the *dot* action) of the Weyl group W of T on the equivariant cohomology $H_T^*(\mathscr{H}(\mathbf{m}, y))$. Moreover, Tymoczko shows that this action descends to a W action (also called the *dot* action) on $H^*(\mathscr{H}(\mathbf{m}, y))$.

Frobenius Character. For each positive integer n write $p_n := \sum x_i^n$ for the power-sum symmetric function, and for each partition $\lambda = (\lambda_1, \dots, \lambda_k)$ write $p_\lambda := \prod p_{\lambda_i}$. For $w \in S_n$, write $\lambda(w)$ for the partition corresponding to the cycle decomposition of w. Then for a representation V of S_n, set

$$\mathrm{ch} V := \frac{1}{n!} \sum_{w \in S_n} p_{\lambda(w)}.$$

It is well-known (see [6]), that ch is then an isomorphism from the space of $\mathbb{C}$-valued class functions of S_n to the space Λ_n of symmetric functions of degree n. We write ω for the involution on Λ that takes $\mathrm{ch}V$ to $\mathrm{ch}V \otimes \mathrm{sgn}$.

Shareshian-Wachs

The main result of my joint paper [4] is the following theorem which was conjectured by Shareshian and Wachs in [9].

Theorem 5. *Suppose* $\mathbf{m} = (m_1, \dots, m_n)$ *is a Hessenberg function, and* $y \in \mathfrak{g}^{\mathrm{rs}}$*. Then we have*

$$\omega X_{G(\mathbf{m})}(t) = \sum_{k \in \mathbb{Z}} t^k \mathrm{ch} H^{2k}(\mathscr{H}(\mathbf{m}, y)).$$

As mentioned above, Guay-Paquet has given an independent proof in [8]. While the proof in [4] is geometric, the main ideas behind [8] are combinatorial. The crucial tool in Guay-Paquet's proof is a theorem of Aguiar, Bergeron and Sottile on the universality of a certain Hopf algebra of quasi-symmetric functions [1]. It is, in fact, very tempting to try to combine the ideas from both proofs to search for a proof of Conjecture 1.

Sketch of Brosnan-Chow proof. I now sketch the proof, but I warn the reader that the sketch is not in the same order as it is in our paper.

The first step in the proof of Theorem 5 from [4] is to realize that Tymoczko's dot action is nothing other than the monodromy action coming from the family of Hessenberg varieties over $\mathfrak{g}^{rs}$. To make this someone more precise, set $\mathscr{H}(\mathbf{m}) := \{(F, s) \in X \times \mathfrak{g} : \forall i, sF_i \subset F_{m_i}\}$. Then the second projection gives a map π :

$\mathscr{H}(\mathbf{m}) \to \mathfrak{g}$. We prove that the dot action is induced by the action of $\pi_1(\mathfrak{g}^{rs})$ (a group isomorphic to the braid group) on the cohomology of a fiber.

If $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a partition of n, then we write $S_\lambda := S_{\lambda_1} \times \cdots \times S_{\lambda_k}$. A subgroup of S_n of this form is called a *Young subgroup*. It is easy to see that, if V is a (virtual) representation of S_n, then the numbers $\dim V^{S_\lambda}$ determine V up to isomorphism.

Write $\mathfrak{g}^{\mathrm{r}}$ for the subset of regular matrices. These are matrices whose Jordan blocks have distinct eigenvalues. A matrix $s \in \mathfrak{g}^{\mathrm{r}}$ is said to be of type λ if λ is the partition corresponding to the sizes of the Jordan blocks.

The next step in the proof is to show that the local monodromy near a matrix s of type λ acts on the cohomology of $\mathscr{H}(\mathbf{m}, y)$ for $y \in \mathfrak{g}^{\mathrm{rs}}$ by the restriction of the dot action to S_λ.

We then prove a theorem related to the local invariant cycle theorem of [2], which shows that the cohomology of the special fiber of a degeneration surjects onto the local monodromy invariants of the cohomology of the nearby smooth fibers. Roughly speaking, our theorem says that the local invariant cycle map is an isomorphism if and only if the cohomology of the special fiber is palindromic.

We then prove that, for $s \in \mathfrak{g}^r$ of type λ, the cohomology groups $H^*(\mathscr{H}(\mathbf{m}, s))$ are palindromic. This is done by appealing to several results: (1) an explicitly computation of these groups obtained by Tymoczko in [12], (2) a reciprocity law which we prove which allows us to compare Tymoczko's computation to the Shareshian-Wachs polynomial $\omega X_{G(\mathbf{m})}(t)$, (3) a theorem of Shareshian-Wachs stating that the polynomial $\omega X_{G(\mathbf{m})}(t)$ is palindromic.

It then follows that, for s as above, $H^*(\mathscr{H}(\mathbf{m}, s)) \cong H^*(\mathscr{H}(\mathbf{m}, y))^{S_\lambda}$. Then the comparison proved before between the cohomology of $H^*(\mathscr{H}(\mathbf{m}, s))$ and $\omega X_{G(\mathbf{m})}(t)$ is used to prove the theorem.

References

[1] Marcelo Aguiar, Nantel Bergeron, and Frank Sottile. Combinatorial Hopf algebras and generalized Dehn-Sommerville relations. *Compos. Math.*, 142(1):1–30, 2006.

[2] A. A. Beĭlinson, J. Bernstein, and P. Deligne. Faisceaux pervers. In *Analysis and topology on singular spaces, I (Luminy, 1981)*, volume 100 of *Astérisque*, pages 5–171. Soc. Math. France, Paris, 1982.

[3] A. Białynicki-Birula. Some theorems on actions of algebraic groups. *Ann. of Math. (2)*, 98:480–497, 1973.

[4] P. Brosnan and T. Y. Chow. Unit Interval Orders and the Dot Action on the Cohomology of Regular Semisimple Hessenberg Varieties. *ArXiv e-prints*, November 2015.

[5] F. De Mari, C. Procesi, and M. A. Shayman. Hessenberg varieties. *Trans. Amer. Math. Soc.*, 332(2):529–534, 1992.

[6] William Fulton. *Young tableaux*, volume 35 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1997. With applications to representation theory and geometry.

[7] M. Guay-Paquet. A modular relation for the chromatic symmetric functions of (3+1)-free posets. *ArXiv e-prints*, June 2013.

[8] M. Guay-Paquet. A second proof of the Shareshian–Wachs conjecture, by way of a new Hopf algebra. *ArXiv e-prints*, January 2016.

[9] J. Shareshian and M. L. Wachs. Chromatic quasisymmetric functions. *ArXiv e-prints*, May 2014.

[10] Richard P. Stanley. A symmetric function generalization of the chromatic polynomial of a graph. *Adv. Math.*, 111(1):166–194, 1995.

[11] Richard P. Stanley and John R. Stembridge. On immanants of Jacobi-Trudi matrices and permutations with restricted position. *J. Combin. Theory Ser. A*, 62(2):261–279, 1993.

[12] Julianna S. Tymoczko. Linear conditions imposed on flag varieties. *Amer. J. Math.*, 128(6):1587–1604, 2006.

[13] Julianna S. Tymoczko. Permutation actions on equivariant cohomology of flag varieties. In *Toric topology*, volume 460 of *Contemp. Math.*, pages 365–384. Amer. Math. Soc., Providence, RI, 2008.

Grothendieck ring of maximal orthogonal Grassmannian

Nikita Karpenko

We refer to [4] for a more detailed exposition.

For an integer $n \geq 1$, let $\varphi : V \to F$ be a non-degenerate quadratic form of dimension $2n+1$ over a field F (of arbitrary characteristic). The vector space V of definition of φ is a vector space over F of dimension $2n+1$. Let X be the maximal orthogonal Grassmannian of φ, i.e., X is the F-variety of n-dimensional totally isotropic subspaces in V. We write $K(X)$ for the Grothendieck ring of X.

Proposition 1. *Assume that the index of the even Clifford algebra $C_0(\varphi)$ is maximal: $\operatorname{ind} C_0(\varphi) = 2^n$. Then the topological filtration on $K(X)$ coincides with the gamma filtration.*

We use Proposition 1 to prove the following

Theorem 2. *For φ as in Proposition 1, let S be the Severi-Brauer variety of the division algebra $C_0(\varphi)$. Then the topological filtration on $K(X_{F(S)})$ coincides with the gamma filtration.*

It would be very interesting to completely understand the gamma filtration on $K(X_{F(S)})$. Note that for any field extension $L/F(S)$, the change of field homomorphism $K(X_{F(S)}) \to K(X_L)$ is an isomorphism preserving the filtration. So, if this helps, it is enough to perform the computation, say, over an algebraically closed field. Actually, it is even enough to do it for a maximal orthogonal Grassmannian $\bar{X}$ over $\mathbb{C}$. One may consider the additive basis of the group $K(\bar{X})$ given by the Schubert classes; the ring structure is determined by the K-theoretical Littlewood-Richardson formulas obtained in [2]. Alternatively, one may describe the ring $K(\bar{X})$ by generators and relations in the spirit of the description of $\operatorname{CH} \bar{X}$ provided in [7], taking for generators the special Schubert classes $e_i \in K(\bar{X})$. (Unfortunately, the relations on e_i in $K(\bar{X})$ look more complicated than in $\operatorname{CH} \bar{X}$.)

Of particular interest is to understand the position of the special Schubert classes in the filtration. More specifically, let us consider the class $e_n \in K(X_{F(S)})$ of the special Schubert variety of the lowest dimension (corresponding to the class of a rational point on the quadric).

Conjecture 3. *For $n \geq 8$, the special Schubert class $e_n \in K(X_{F(S)})$ does not belong to the term number $n+1-[(n+1)/2]$ of the gamma filtration.*

Our interest to Conjecture 3 is explained by the fact that it implies

Conjecture 4. *For any even $m \geq 18$, the index of the $[m/4]$-th orthogonal Grassmannian of any generic m-dimensional quadratic form φ/F in I_q^3 is equal to $2^{[m/4]}$.*

(We refer to [3, §9.B] for the definition of I_q^3.)

Conjecture 4 actually implies that any generic m-dimensional quadratic form φ/F in I_q^3, for any even $m \geq 18$, has the property of [1, Theorem 4.2]: if this is not the case, i.e., if φ_E does contain a proper even-dimensional subform ψ of trivial discriminant for some finite field extension E/F of odd degree, then, possibly replacing ψ by its complement, we have $d := (\dim \psi)/2 \leq [m/4]$ and ψ becomes hyperbolic over an extension of degree dividing 2^{d-1}. Therefore φ_E acquires Witt index $\geq [m/4]$ over an extension of L of degree dividing $2^{[m/4]-1}$, i.e., the index of the $[m/4]$-th orthogonal Grassmannian of φ divides $2^{[m/4]-1}$.

Remark 5. The number $n+1-[(n+1)/2]$ in Conjecture 3 is optimal for $n=8$. Indeed, if φ is an 18-dimensional quadratic form of trivial discriminant and trivial Clifford invariant and Y is its projective quadric, since the Chow groups $\mathrm{CH}^i Y$ are torsion-free for $i \leq 3$, [5], the class $l_8 \in K(Y)$ is in the 4-th term of the topological filtration, cf. [6, Theorem 3.10]. Since $l_0 = l_8 \cdot h^8$, it follows that l_0 is in the term number $4+8=12$. Therefore e_8 is in the term number $4 = 12-n$. Conjecture 3 claims that e_8 is not in the term number 5.

References

[1] Chernousov, V., and Merkurjev, A. Essential dimension of spinor and Clifford groups. *Algebra Number Theory 8*, 2 (2014), 457–472.

[2] Clifford, E., Thomas, H., and Yong, A. K-theoretic Schubert calculus for $\mathrm{OG}(n, 2n+1)$ and jeu de taquin for shifted increasing tableaux. *J. Reine Angew. Math. 690* (2014), 51–63.

[3] Elman, R., Karpenko, N., and Merkurjev, A. *The algebraic and geometric theory of quadratic forms*, vol. 56 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2008.

[4] Karpenko, N. Around 16-dimensional quadratic forms in I_q^3. Preprint, 29 November 2015, 12 pages. Available on the web page of the author.

[5] Karpenko, N. Chow groups of quadrics and index reduction formula. *Nova J. Algebra Geom. 3*, 4 (1995), 357–379.

[6] Karpenko, N. A. Algebro-geometric invariants of quadratic forms. *Algebra i Analiz 2*, 1 (1990), 141–162.

[7] Vishik, A. On the Chow groups of quadratic Grassmannians. *Doc. Math. 10* (2005), 111–130 (electronic).

On p-group actions on smooth projective varieties

Olivier Haution

Let us first mention a classical result in algebraic topology.

Theorem (Conner-Floyd and Atiyah-Bott). *An orientation preserving diffeomorphism of odd prime power order on a closed oriented manifold of positive dimension cannot have exactly one fixed point.*

This statement was first conjectured by Conner-Floyd [CF64, §45], then proved by Atiyah-Bott [AB68, Theorem 7.1], and later reproved by Conner-Floyd [CF66, (8.3)]. Our purpose is to discuss results of this type in algebraic geometry. The letter k will denote an algebraically closed field. An action of a group on a k-variety will mean an action by k-automorphisms. The first result is:

Theorem 1. *Let p a prime number different from the characteristic of k. Let G be a finite abelian p-group acting on a smooth projective k-variety X without zero-dimensional component. Then the set $X(k)^G$ cannot consist of a single element.*

The next example shows that the assumption on the characteristic of k is necessary.

Example 2. Assume that the characteristic of k is two, and let the group $G = \mathbb{Z}/2$ acts on $X = \mathbb{P}^1_k$ by $[x : y] \mapsto [y : x]$. Then the set $X(k)^G$ is the single point $[1 : 1]$.

The proof of Theorem 1 uses an equivariant version of a construction introduced by Rost in the context of the degree formula [Ros08]. The proof also relies on the following proposition, where the notation CH_G stands for the G-equivariant Chow group defined by Totaro and Edidin-Graham [EG98], and the degree of an element of $\mathrm{CH}_G(X)$ is defined as the degree of its image in the Chow group $\mathrm{CH}(X)$.

Proposition 3. *Let G be a finite p-group acting on a projective k-variety X. Assume that G is abelian or that the characteristic of k is p. If $X(k)^G = \varnothing$, then the degree of any element of $\mathrm{CH}_G(X)$ is divisible by p.*

A *characteristic number* of a smooth projective k-variety is the degree of a monomial in the Chern classes of its tangent bundle. The tangent bundle of a smooth k-variety equipped with a G-action, being a G-equivariant vector bundle, has Chern classes in the G-equivariant Chow group. Thus Proposition 3 provides an effective way of proving the existence of fixed points of G-actions (for G a finite p-group, abelian if the characteristic of k is not p): if some characteristic number of a smooth projective k-variety with a G-action is prime to p, then G fixes a k-point.

The equivariant Chow group may be replaced by equivariant K-theory in Proposition 3, at the expense of a more restrictive assumption on the group G:

Proposition 4. *Let G be a finite p-group acting on a projective k-variety X. Assume that G is cyclic or that the characteristic of k is p. If $X(k)^G = \varnothing$, then the Euler characteristic $\chi(X, \mathcal{F})$ of any G-equivariant coherent $\mathcal{O}_X$-module $\mathcal{F}$ is divisible by p.*

The next example shows that it is necessary to assume that G is cyclic, as opposed to merely abelian as in Proposition 3.

Example 5. Assume that the characteristic of k is not two. We make the group $G = \mathbb{Z}/2 \times \mathbb{Z}/2$ act on $X = \mathbb{P}^1_k$ via the commuting involutions $[x : y] \mapsto [y : x]$ and $[x : y] \mapsto [-x : y]$. Then $X(k)^G = \varnothing$. As predicted by Proposition 3, the only characteristic number of X is even. On the other hand, the conclusion of Proposition 4 does not hold, since $\chi(X, \mathcal{O}_X) = 1$.

Proposition 4 may sometimes be used to prove the existence of fixed points when Proposition 3 does not suffice. For instance, let X be a smooth projective stably rational k-variety. Let G be a finite p-group, cyclic if the characteristic of k is not p, acting on X. Since $\chi(X, \mathcal{O}_X) = 1$, it follows from Proposition 4 that G fixes a k-point of X. This was already observed by Serre [Ser09, §7.4, Remark] in the special case of a rational smooth projective surface in characteristic p, using Smith's theory and the Artin-Schreier sequence.

We declare two smooth projective k-varieties equivalent if their collections of characteristic numbers (indexed by the monomials in the Chern classes) coincide. The set of equivalence classes forms a ring (the product is induced by the cartesian product of varieties, and the sum by the disjoint union, see [ELW15, p.708]). This ring does not depend on the field k, in fact Merkurjev proved that it coincides with the Lazard ring $\mathbb{L}$, the coefficient ring of the universal commutative one-dimensional formal group law [Mer02, Theorem 8.2]. Thus to each smooth projective variety X corresponds a class $[X] \in \mathbb{L}$. When the characteristic of k is zero, the ring $\mathbb{L}$ may be identified with the coefficient ring $\Omega(\operatorname{Spec} k)$ of the algebraic cobordism of Levine-Morel, and $[X]$ is the cobordism class of the morphism $X \to \operatorname{Spec} k$ [LM07, Remark 4.3.4, Theorem 4.3.7].

Using an observation of Esnault-Levine-Wittenberg [ELW15, Lemma 5.4], we deduce from Proposition 4 the following theorem, which is the analog of a theorem of Conner-Floyd in algebraic topology [CF66, Theorem 8.1].

Theorem 6. *Let G be a finite p-group acting on a smooth projective k-variety X. Assume that G is cyclic or that the characteristic of k is p. If $X(k)^G = \varnothing$, then the class $[X]$ is divisible by p in the Lazard ring $\mathbb{L}$.*

This work was supported by the DFG Grant HA 7702/1-1.

References

[AB68] Michael F. Atiyah and Raoul Bott. A Lefschetz fixed point formula for elliptic complexes. II. Applications. *Ann. of Math. (2)*, 88:451–491, 1968.

[CF64] Pierre E. Conner and Edwin E. Floyd. *Differentiable periodic maps.* Ergebnisse der Mathematik und ihrer Grenzgebiete, N. F., Band 33. Academic Press Inc., Publishers, New York; Springer-Verlag, Berlin-Göttingen-Heidelberg, 1964.

[CF66] Pierre E. Conner and Edwin E. Floyd. Maps of odd period. *Ann. of Math. (2)*, 84:132–156, 1966.

[EG98] Dan Edidin and William Graham. Equivariant intersection theory. *Invent. Math.*, 131(3):595–634, 1998.

[ELW15] Hélène Esnault, Marc Levine, and Olivier Wittenberg. Index of varieties over Henselian fields and Euler characteristic of coherent sheaves. *J. Algebraic Geom.*, 24(4):693–718, 2015.

[LM07] Marc Levine and Fabien Morel. *Algebraic cobordism*. Springer Monographs in Mathematics. Springer, Berlin, 2007.

[Mer02] Alexander S. Merkurjev. Algebraic oriented cohomology theories. In *Algebraic number theory and algebraic geometry*, volume 300 of *Contemp. Math.*, pages 171–193. Amer. Math. Soc., Providence, RI, 2002.

[Ros08] Markus Rost. Notes on the degree formula, 2001, *unpublished, available at* `https://www.math.uni-bielefeld.de/~rost/degree-formula.html`.

[Ser09] Jean-Pierre Serre. How to use finite fields for problems concerning infinite fields. In *Arithmetic, geometry, cryptography and coding theory*, volume 487 of *Contemp. Math.*, pages 183–193. Amer. Math. Soc., Providence, RI, 2009.

Isotropy Problem for Quadratic Forms over Function Fields of Quadrics

Stephen Scully

Let p and q be anisotropic quadratic forms of dimension ≥ 2 over a general field F, and let P and Q denote the projective F-quadrics defined by p and q, respectively. A central problem in the algebraic theory of quadratic forms is that of determining necessary and sufficient conditions in order for q to be become isotropic over the function field $F(p)$ of the quadric P. In more geometric language, this is equivalent to asking for necessary and sufficient conditions in order for there to exist a rational map $P \dashrightarrow Q$ over F. As the known results in low dimensions confirm, this is a highly complex problem, and it is rather unreasonable to hope for a complete solution in the general case. Nevertheless, one can still look for something weaker in the same broad direction. In this talk, we discussed the problem of determining restrictions of a very general nature on the *isotropy index* (i.e., the largest dimension of a totally isotropic subspace) of q after scalar extension to $F(p)$. Building on earlier results of Cassels, Pfister, Knebuch and Fitzgerald, the key breakthrough was made here by Hoffmann, who in 1995 discovered an upper bound for this index which is expressed solely in terms of the dimensions of p and q. More specifically, Hoffmann showed that if the characteristic of F is *not* 2, then

$$\mathfrak{i}_0(q_{F(p)}) \leq \max(\dim(q) - 2^{l(p)}, 0),$$

where $2^{l(p)}$ is the largest power of 2 strictly less than $\dim(p)$ ([2]). The characteristic restriction was later removed by Hoffmann and Laghribi in [3]. Hoffmann's bound cannot be improved without imposing further conditions on p and q or permitting the use of additional invariants. Despite its far-reaching consequences, it was necessary for the further development of the theory to search for a more refined statement which would retain a sufficient degree of generality. The needed refinement of Hoffmann's result was proposed by Izhboldin, who conjectured that

$$\mathfrak{i}_0(q_{F(p)}) \leq \max(\dim(q) - \dim(p) + \mathfrak{i}_1(p), 0),$$

where $\mathfrak{i}_1(p)$ is the *first higher isotropy index* of p, defined as the isotropy index of p over its own function field, i.e., $\mathfrak{i}_1(p) = \mathfrak{i}_0(p_{F(p)})$. This latter invariant is surprisingly informative, and permits to distinguish between varying levels of complexity among forms of prescribed dimension (for example, in the non-degenerate case, the pair $\big(\dim(p), \mathfrak{i}_1(p)\big)$ determines whether or not p is similar to a Pfister form). Izhboldin's conjecture built upon work of Vishik ([8]), who showed that the integer $\dim(p) - \mathfrak{i}_1(p)$ appearing on the right-hand side of the inequality is a stable birational invariant of P. In 2003, Karpenko and Merkurjev ([5]) proved that Izhboldin's bound is valid under an additional non-degeneracy hypothesis on q. This assumption was later removed by Totaro in [7], and so the inequality is now known to hold in absolute generality. A priori, of course, it only gives an interesting refinement of Hoffmann's result if something can be said concerning the possible values of the integer $\mathfrak{i}_1(p)$. For some time, little was known beyond the upper bound $\mathfrak{i}_1(p) \leq \dim(p) - 2^{l(p)}$ given by Hoffmann's theorem. In [4], however, Karpenko showed that if F has characteristic *not* 2, then

$$\mathfrak{i}_1(p) \leq 2^{v_2(\dim(p) - \mathfrak{i}_1(p))},$$

that is, $\mathfrak{i}_1(p)$ is less than or equal to the largest power of 2 dividing $\dim(p) - \mathfrak{i}_1(p)$. Moreover, it turns out that there are no further restrictions on $\mathfrak{i}_1(p)$ in general (though the existing restrictions are quite non-trivial), and so we are left with a fairly complete picture. Unfortunately, unlike the Karpenko-Merkurjev-Totaro bound on $\mathfrak{i}_0(q_{F(p)})$, it is not known at present whether Karpenko's theorem extends to characteristic 2. Indeed, the known proofs ([4],[9]) make essential use of the known existence of homological Steenrod operations the mod-2 Chow groups of certain algebraic varieties; in characteristic 2, the construction of such operations is still an open problem.

The purpose of this talk was to formulate a new upper bound for $\mathfrak{i}_0(q_{F(p)})$ complementary to that of Karpenko-Merkurjev-Totaro, and to explain its validity in at least two important cases:

Theorem 1. *If, in the above situation, q is diagonalizable, then*

$$(\star) \qquad \mathfrak{i}_0(q_{F(p)}) \leq \max\big(\dim(q) - \dim(p), 2^{v_2(\dim(p) - \mathfrak{i}_1(p))}\big).$$

In other words, inequality $(\star)$ holds if

(1) $\mathrm{char}(F) \neq 2$*, or*
(2) $\mathrm{char}(F) = 2$ *and q is quasilinear (i.e., q is additive).*

The two cases are treated separately using very different approaches, neither of which admits an obvious adaptation to the generically non-singular case in characteristic 2. Nevertheless, we believe the inequality should hold free of any characteristic or degeneracy hypotheses.

While it is possible to be in the situation where the right-hand side of $(\star)$ gives a weaker bound for $\mathfrak{i}_0(q_{F(p)})$ than the Karpenko-Merkurjev-Totaro theorem, there are other situations in which we obtain something more informative. For example, in characteristic $\neq 2$, the bound yields a new, very short proof of a deep result of Fitzgerald concerning the Witt kernel of the function field of a quadric ([1]).

From another point of view, the inequality may be seen as a generalization of the statement of Karpenko's theorem, which is just the case where $p = q$. In particular, Karpenko's theorem extends to quasilinear quadratic forms in characteristic 2 (a result which was obtained earlier by similar means in [6]). This is the only class of anisotropic forms in characteristic 2 for which this statement is currently known.

The characteristic $\neq 2$ case of Theorem 1 is achieved using algebro-geometric methods, namely, by exploiting a certain interaction between splitting patterns and (integral) motivic decompositions of quadrics in this setting. In addition to the aforementioned results of Karpenko-Merkurjev and Karpenko, we make particular use here of the fundamental work of Vishik in this direction ([9]). The quasilinear case, by constast, is done using a more direct algebraic approach. In fact, in this case, the statement is deduced as an easy dimension-theoretic consequence of the existence of a certain direct sum decomposition of the the anisotropic kernel of $q_{F(p)}$ (incidentally, the quasilinear case of Karpenko-Merkurjev-Totaro theorem, originally proved using algebro-geometric methods in [7], also follows in the same way). This decomposition statement is a new kind of result of which no analogue is known in the characteristic $\neq 2$ theory. An interesting feature of this algebraic approach to the isotropy problem is that the integer $\dim(p) - \mathfrak{i}_1(p)$ arises as the dimension of a quadratic form as opposed to a motive, and the largest power of 2 which divides it enters the picture by observing that this quadratic form is divisible by a "quasilinear Pfister form" of dimension $\geq \mathfrak{i}_0(q_{F(p)})$. By contrast, the same 2-power enters the picture in the characteristic $\neq 2$ setting by considering the effect of applying a certain Steenrod operation to a certain cycle on $P \times P$. This gives at least some optimism that a more algebraic approach to the subject, encompassing the characteristic 2 case, may be viable.

References

[1] R.W. Fitzgerald. *Function fields of quadratic forms.* Math. Z., **178** (1981), no. 1, 63–76.

[2] D.W. Hoffmann. *Isotropy of quadratic forms over the function field of a quadric.* Math. Z., **220** (1995), no. 3, 461–476.

[3] D.W. Hoffmann and A. Laghribi. *Isotropy of quadratic forms over the function field of a quadric in characteristic 2.* J. Algebra, **295** (2006), no. 2, 362–386.

[4] N.A. Karpenko. *On the first Witt index of quadratic forms.* Invent. Math., **153** (2003), no. 2, 455–462.

[5] N. Karpenko and A. Merkurjev. *Essential dimension of quadrics.* Invent. Math., **153** (2003), no. 2, 361–372.

[6] S. Scully. *Hoffmann's conjecture for totally singular forms of prime degree.* Preprint (2014), 44 pages.

[7] B. Totaro. *Birational geometry of quadrics in characteristic 2.* J. Algebraic Geom., **17** (2008), no. 3, 577–597.

[8] A. Vishik. *Direct summands in the motives of quadrics.* Preprint (1999), 13 pages.

[9] A. Vishik. *Excellent connections in the motives of quadrics.* Ann. Sci. Éc. Norm. Supér. (4), **44** (2011), no. 1, 183–195.

Stable operations in derived Witt theory and motivic Serre finiteness

Alexey Ananyevskiy
(joint work with Marc Levine, Ivan Panin)

The talk is a report on joint work with Marc Levine and Ivan Panin [1, 2].

Let k be an perfect field of characteristic different from 2. The aim of the project is to describe the motivic stable homotopy groups of spheres with rational coefficients. In topology the corresponding result reads as follows:

$$\pi_n(\mathbb{S}_{\mathbb{Q}}) \cong \left[\begin{array}{ll} \mathbb{Q}, & n = 0, \\ 0, & n \neq 0. \end{array} \right.$$

The claim immediately follows from a celebrated J.-P. Serre's theorem [3] claiming that $\pi_n(\mathrm{S}^m)$ is a finite abelian group with the exception of $\pi_n(\mathrm{S}^n)$ and $\pi_{4n-1}(\mathrm{S}^{2n})$.

In the motivic setting the situation is more complicated. First of all, for every motivic spectrum X one has bigraded homotopy sheaves $\pi_*^{\mathbb{A}^1}(X)_*(-)$ in place of the classic homotopy groups. For every i the sequence of sheaves $\pi_i^{\mathbb{A}^1}(X)_*(-)$ carries a structure of a homotopy module, which is a certain generalization of the notion of Rost cycle module. F. Morel's computations of the first nontrivial homotopy modules for some motivic spheres [4] yield

$$\pi_i^{\mathbb{A}^1}(\mathbb{S})_* \cong \left[\begin{array}{ll} \underline{\mathrm{K}}_*^{\mathrm{MW}}, & i = 0, \\ 0, & i < 0. \end{array} \right.$$

Here $\underline{\mathrm{K}}_*^{\mathrm{MW}}$ is the unramified sheaf of Milnor-Witt K-theory. In particular, for every finitely generated field extension F/k one has an isomorphism

$$\pi_i^{\mathbb{A}^1}(\mathbb{S}_{\mathbb{Q}})_*(F) \cong \left[\begin{array}{ll} \mathrm{K}_*^{\mathrm{M}}(F) \otimes_{\mathbb{Z}} \mathbb{Q} \oplus \mathrm{W}_{\mathbb{Q}}(F), & i = 0, \\ 0, & i < 0, \end{array} \right.$$

where $\mathrm{K}_*^{\mathrm{M}}(F)$ is Milnor K-theory and $\mathrm{W}_{\mathbb{Q}}(F) = \mathrm{W}(F) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the rationalized Witt ring of quadratic forms.

The splitting

$$\mathrm{Hom}_{\mathcal{SH}(k)_{\mathbb{Q}}}(\mathbb{S}_{\mathbb{Q}}, \mathbb{S}_{\mathbb{Q}}) = \mathbb{Q} \oplus \mathrm{W}_{\mathbb{Q}}(k)$$

induces a splitting of the motivic stable homotopy category

$$\mathcal{SH}(k)_{\mathbb{Q}} = \mathcal{SH}(k)_{\mathbb{Q}}^+ \oplus \mathcal{SH}(k)_{\mathbb{Q}}^-,$$

where the plus-part corresponds to $\mathrm{K}_0^{\mathrm{M}}(k) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$ and the minus-part corresponds to $\mathrm{W}_{\mathbb{Q}}(k)$. D.C. Cisinski and F. Déglise [5] described the plus-part of the motivic stable homotopy groups obtaining

$$\pi_i^{\mathbb{A}^1}(\mathbb{S}_{\mathbb{Q}}^+)_j(F) \cong \mathrm{H}_M^{j-i}(F, \mathbb{Q}(j)).$$

Here on the right-hand side one has motivic cohomology groups. Note that for $i = 0$ one has precisely the same answer as above, since $\mathrm{H}_M^j(F, \mathbb{Q}(j)) = \mathrm{K}_{\mathrm{j}}^{\mathrm{M}}(F) \otimes_{\mathbb{Z}} \mathbb{Q}$.

We obtained the following description for the minus-part of the motivic stable homotopy groups.

Theorem 1. *Let F/k be a finitely generated field extension. Then*

$$\pi_i^{\mathbb{A}^1}(\mathbb{S}_{\mathbb{Q}}^-)_j(F) \cong \left[\begin{array}{ll} \mathrm{W}_{\mathbb{Q}}(F), & i=0, \\ 0, & i \neq 0. \end{array}\right.$$

The general strategy of the proof loosely follows the approach of D.C. Cisinski and F. Déglise, which is based on the computation of stable operations and co-operation in algebraic K-theory with rational coefficients. In order to deal with the minus-part we compute the stable operations and cooperation in derived Witt theory. Denote $\mathrm{KW} = \mathrm{BO}[\eta^{-1}]$ the spectrum representing derived Witt groups obtained via localization from the geometric model for the spectrum representing hermitian K-theory described by I. Panin and C. Walter [6]. Then we have the following theorems.

Theorem 2. *Denote $\beta \in \mathrm{KW}^{-8,-4}(\operatorname{Spec} k)$ the Bott element realizing $(8,4)$-periodicity isomorphisms. Then the homomorphism*

$$\mathrm{KW}_{\mathbb{Q}}^{*,*}(\mathrm{KW}_{\mathbb{Q}}) \to \left(\prod_{n\in\mathbb{Z}} \mathrm{KW}_{\mathbb{Q}}^{*,*}(\operatorname{Spec} k)\right)_h$$

given by

$$f \mapsto (\ldots, \beta^2 f(\beta^{-2}), \beta f(\beta^{-1}), f(1), \beta^{-1} f(\beta), \beta^{-2} f(\beta^2), \ldots)$$

is an isomorphism of algebras. Here the subscript denotes the subalgebra generated by homogeneous elements.

Theorem 3. *There is a canonical isomorphism*

$$\mathrm{KW}_{\mathbb{Q}} \wedge \mathrm{KW}_{\mathbb{Q}} \cong \bigoplus_{n\in\mathbb{Z}} \Sigma_T^{4n} \mathrm{KW}_{\mathbb{Q}}.$$

As a straightforward application of the above computations we obtain a rational degeneration of the Brown–Gersten type spectral sequence for the derived Witt groups and the following version of the Chern character isomorphism.

Corollary 4. *Let X be a smooth variety. Then the spectral sequence arising from the homotopy t-structure*

$$E_2^{p,q} = \mathrm{H}^p_{Zar}(X, \underline{\mathrm{W}}^q) \Rightarrow \mathrm{KW}^{p+q,0}(X) \cong \mathrm{W}^{p+q}(X)$$

rationally degenerates yielding isomorphisms

$$\mathrm{W}^n(X) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \bigoplus_{m\in\mathbb{Z}} \mathrm{H}^{4m+n}_{Zar}(X, \underline{\mathrm{W}}_{\mathbb{Q}}).$$

Here $\underline{\mathrm{W}}$ stands for the unramified sheaf of Witt rings.

References

[1] A. Ananyevskiy, *Stable operations and cooperations in derived Witt theory with rational coefficients*, arXiv:1504.04848v2

[2] A. Ananyevskiy, M. Levine, I. Panin, *Witt sheaves and the η-inverted sphere spectrum*, arXiv:1504.04860

[3] J.-P. Serre, *Homologie singulière des espaces fibrés. III. Applications homotopiques*, C. R. Acad. Sci. Paris 232, (1951). 142–144.
[4] F. Morel, $\mathbb{A}^1$*-algebraic topology over a field*, Lecture Notes in Mathematics, 2052. Springer, Heidelberg, 2012.
[5] D.C. Cisinski, F. Déglise, *Triangulated categories of mixed motives,* arXiv:0912.2110v3
[6] I. Panin, C. Walter, *On the motivic commutative spectrum BO*, arXiv:1011.0650

$\mathbb{A}^1$-homotopical classification of principal G-bundles

Marc Hoyois
(joint work with Aravind Asok and Matthias Wendt)

Let k be a commutative ring, G a reductive algebraic group over k, and X a smooth affine k-scheme. We are interested in understanding the set of isomorphism classes of generically trivial G-torsors over X. By a theorem of Nisnevich [7], if k is regular, this is equivalently the set $H^1_{\mathrm{Nis}}(X, G)$ of G-torsors that are trivial locally in the Nisnevich topology.

Theorem 1 ([2, 3]). Let k be an infinite field, G an isotropic reductive k-group, and X a smooth affine k-scheme. Then there is a bijection

$$H^1_{\mathrm{Nis}}(X, G) \simeq [X, BG]_{\mathbb{A}^1},$$

where the right-hand side denotes the set of maps in the $\mathbb{A}^1$-homotopy category over k [6].

If G is GL_n, SL_n, or Sp_{2n}, the above result holds for k any commutative ring which admits a regular ring homomorphism from a Dedekind domain with perfect residue fields.

The usefulness of Theorem 1 stems from the fact that the right-hand side is more amenable to computation, using tools from ($\mathbb{A}^1$-)homotopy theory.

The prototypical case of Theorem 1, when $G = GL_n$ and k is a perfect field, was established by Morel [5]. This was extended to $G = SL_n$ by Asok and Fasel [1], and a simplified proof applying also to $G = Sp_{2n}$ was later found by Schlichting [9], still under the assumption that k is a perfect field. Our approach is completely independent of Morel's and allows us to remove all assumptions on k, except the (obviously necessary) assumption that $H^1_{\mathrm{Nis}}(-, G)$ is $\mathbb{A}^1$-homotopy invariant on smooth affine k-schemes. In other words, our proof of Theorem 1 proceeds in two independent steps:

Theorem 2. Theorem 1 holds for any commutative ring k and k-group scheme G such that $H^1_{\mathrm{Nis}}(-, G)$ is $\mathbb{A}^1$-homotopy invariant on smooth affine k-schemes.

Theorem 3. If k is an infinite field and G is an isotropic reductive k-group, then $H^1_{\mathrm{Nis}}(-, G)$ is $\mathbb{A}^1$-homotopy invariant on smooth affine k-schemes.

The second part of Theorem 1 follows from Theorem 2 and the partial solution of the Bass–Quillen conjecture by Lindel and Popescu [8].

The proof of Theorem 3 is a variant of arguments of Colliot-Thélène and Ojanguren [4], combined with an analog of Quillen's patching theorem for G-torsors.

The proof of Theorem 2 relies on a new characterization of the Nisnevich topology. Recall that a Nisnevich cover is an étale cover that is surjective on k-points for every field k.

Theorem 4. The Nisnevich topology on the category of schemes is generated by the following types of covers:

(1) open covers;
(2) $\{\operatorname{Spec} B \to \operatorname{Spec} A, \operatorname{Spec} A[1/f] \hookrightarrow \operatorname{Spec} A\}$, where $A \to B$ is an étale ring homomorphism inducing an isomorphism $A/fA \cong B/fB$.

On the category of affine schemes, covers of type (2) suffice.

If in (2) we replace $\operatorname{Spec} A[1/f] \hookrightarrow \operatorname{Spec} A$ by an arbitrary open immersion $U \hookrightarrow \operatorname{Spec} A$, requiring $\operatorname{Spec} B \to \operatorname{Spec} A$ to be an isomorphism over the closed complement of U, then the result is well known and goes back to Morel and Voevodsky [6]. Thus, the main innovation of Theorem 4 is that it suffices to consider complements of hypersurfaces, which leads to a simple set of generators for the Nisnevich topology on affine schemes.

References

[1] A. Asok, J. Fasel, *Algebraic vector bundles on spheres*, J. Topology **7** (2014), no. 3, pp. 894–926
[2] A. Asok, M. Hoyois, and M. Wendt, *Affine representability results in $\mathbb{A}^1$-homotopy theory I: vector bundles*, 2015, `http://arxiv.org/abs/1506.07093`
[3] A. Asok, M. Hoyois, and M. Wendt, *Affine representability results in $\mathbb{A}^1$-homotopy theory II: principal bundles and homogeneous spaces*, 2015, `http://arxiv.org/abs/1507.08020`
[4] J.-L. Colliot-Thélène, M. Ojanguren, *Espaces principaux homogènes localement triviaux*, Publ. Math. I.H.É.S. **75** (1992), pp. 97–122
[5] F. Morel, *$\mathbb{A}^1$-algebraic topology over a field*, Lecture Notes in Mathematics, volume 2052, Springer, Heidelberg, 2012
[6] F. Morel and V. Voevodsky, *$\mathbb{A}^1$-homotopy theory of schemes*, Publ. Math. I.H.É.S. **90** (1999), pp. 45–143
[7] Y. A. Nisnevich, *Éspaces homogènes principaux rationnellement triviaux et arithmétique des schémas en groupes réductifs sur les anneaux de Dedekind*, C. R. Acad. Sci. Paris Sér. I Math., **299** (1984), pp. 5–8
[8] D. Popescu, *Polynomial rings and their projective modules*, Nagoya Math. J. **113** (1989), pp. 121–128
[9] M. Schlichting, *Euler class groups, and the homology of elementary and special linear groups*, 2016, `http://arxiv.org/abs/1502.05424`

Motives of twisted flag varieties and modular representations

Kirill Zaynullin
(joint work with B. Calmès and A. Neshitov)

Given a split semisimple linear algebraic group G over a field k, a maximal torus T and a G-equivariant algebraic oriented cohomology theory h in the sense of Levine-Morel, we introduce a relative equivariant category $Mot_{G\to T}$ of Chow motives by setting Homs to be the images of forgetful maps $im(h_G(X \times Y) \to h_T(X \times Y))$, where X, Y are smooth projective G-varieties over k, and composition being the usual correspondence product.

Given a smooth projective G-variety Z, We consider the Yoneda functor F_Z from $Mot_{G\to T}$ to the category of $D = End_{Mot_{G\to T}}(Z)$-modules, mapping $[X]$ to $Hom_{Mot_{G\to T}}([pt],[X])$.

We investigate the cases when F_Z is (fully) faithful if restricted to the category of projective homogeneous G-varieties. In particular, we show that F_Z is always faithful, hence, proving an embedding $End_{Mot_{G\to T}}([G/P]) \hookrightarrow End_D(D^\star_{F,P})$, where D is the formal affine Demazure (Hecke-type) algebra corresponding To a formal group law of the theory h and $D^\star_{F,P}$ is the dual of its parabolic version.

Using the Bott-Samelson basis for $D^*_{F,P}$ and explicit formulas for the action of D on $D^\star_{F,P}$ by Lenart-Z.-Zhong We obtain restrictions on possible motivic decomposition types of relative motives $[G/P]$ and usual Chow motives of versal flags $[E/P]$ (here E is a versal G-torsor over some field extension of k).

Segre Classes and Kempf–Laksov formula for algebraic cobordism

Thomas Hudson
(joint work with Tomoo Matsumura)

Let E be a vector bundle of rank n over a smooth scheme X and $F^\bullet$ a full flag of subbundles $(0 = F^n \subset \cdots F^1 \subset F^0 = E)$. If we consider the Grassmann bundle $Gr_d(E) \overset{\varphi}{\to} X$ parametrizing the rank d subbundles of E, then one defines its Schubert varieties as follows. For every partition $\lambda = (\lambda_1, ..., \lambda_r)$ with $\lambda_1 \leq n-d$ and $r \leq d$, we set

$$X_\lambda := \left\{ x \in Gr_d(E) \mid \dim_k \left(\varphi^* F_x^{\lambda_i - i + d} \cap U_x\right) \geq i,\ i = 1, ...,\ r \right\},$$

where U_x is the restriction over x of U, the universal subbundle of rank d.

A classical result due to Kempf and Laksov ([10]) describes the fundamental classes of these varieties, as elements of $CH^*(Gr_d(E))$, using the following determinant in Chern classes:

$$[X_\lambda]_{CH} = \det\left(c_{\lambda_i+j-i}(E/F^{\lambda_i-i+d} - U)\right)_{1\leq i,j\leq r}. \tag{1}$$

Remark 1. The previous equality can be rephrased as follows:

$$[X_\lambda]_{CH} = \phi\left(t_1^{\lambda_1\cdots} t_r^{\lambda_r} \prod_{1\leq i,j\leq r}\left(1 - \frac{t_i}{t_j}\right)\right),$$

where, after expanding the rational function in t_i's as a power series in $t_1^{\alpha_1}\cdots t_r^{\alpha_d}$, the map ϕ sends each monomial to $c_{\alpha_1}(E/F^{\lambda_1-1+d} - U)\cdots c_{\alpha_r}(E/F^{\lambda_r-r+d} - U)$.

In recent years, as a consequence of the work of Levine and Morel ([12]) on oriented cohomology theories and algebraic cobordism Ω^*, several attempts have been made to extend the results of Schubert calculus to more general theories. In the process, however, all the attention was devoted to flag varieties ([1, 8]) and flag bundles $\mathcal{F}\ell(E)$ ([11, 3, 2, 4]), while Grassmannians and Grassmann bundles have been substantially ignored. Although this makes a lot of sense from the point of view of CH^* and of the Grothendieck ring of vector bundles K^0 (to some extent knowing what happens for $\mathcal{F}\ell(E)$ allows one to recover the story for $Gr_d(E)$), there are a few good reasons that would suggest to look at $\Omega^*(Gr_d(E))$ as well.

- In Ω^* not all Schubert varieties have a well defined fundamental class and all formulas discovered so far depend on the choice of a desingularization. A method based on a different resolution of singularities will produce different expressions.
- Historically Grassmannians have been easier to deal with and their study should give hints on how to address the issues that are open in the flag case. For instance, at present the expressions describing the Schubert classes do not satisfy some expected stability properties: replacing E by $E \oplus \mathcal{O}_X$ alters them.
- From a combinatorial point of view, much more is known about $CH^*(Gr_d(E))$. If, on the one hand, the study of $\Omega^*(Gr_d(E))$ should provide interesting (and hopefully solvable) problems in combinatorics, finding solutions to these problems could help choosing between different desingularizations.
- The formulas associated to $\mathcal{F}\ell(E)$ are, by design, recursive while those coming from $Gr_d(E)$ are closed.

Our attempt in [7] to achieve an analogue of (1) for Ω^* can be seen as the ideal completion of [5, 6], where the same problem was respectively solved for K^0 and even infinitesimal theories. The underlying strategy is the same and essentially relies on three components:

- the geometric input coming from the resolution $\widetilde{X}^{KL}$ used by Kempf and Laksov;
- the explicit identification of the correct analogue of Segre classes $\mathcal{S}_i(E)$;
- an algorithmic procedure, modelled after the one used by Kazarian in [9], which is used to deal with the push-forwards.

Of these aspects the second one proved to be the most challenging, since it requires the direct use of the formal group law of the theory. Let us recall that every oriented cohomology theory A^* is endowed with a formal group law F_A defined over the coefficient ring, which encodes the behaviour of the first Chern class with respect to tensor product. To be more precise, for line bundles L and M over a smooth scheme X one has

$$c_1(L \otimes M) = F_A\Big(c_1(L), c_1(M)\Big),$$

where F_A is a power series in two variables satisfying some properties. Since F_Ω is the universal such object, it happens to be far too complex to be used in explicit computations. However, we observed that for every theory one has the following decomposition:

$$F_A\Big(c_1(L), c_1(M^\vee)\Big) = \big(c_1(L) - c_1(M)\big) P_A\Big(c_1(L), c_1(M)\Big)$$

for some unique power series P_A. This proved sufficient to obtain a description of the power series $\mathcal{S}(E;u) := \sum_{i\in\mathbb{Z}} \mathcal{S}_i(E)u^i$. In fact one has

$$\mathcal{S}(E;u) = \frac{1}{c(E;u)} \cdot \frac{1}{w(E;u^{-1})} \cdot \mathcal{P}(u^{-1}),$$

where:

- $c(E,u) = \sum_{j=0}^n c_j(E)u^j := \prod_{i=1}^n (1+x_i u)$ is the Chern polynomial and the x_i's are the Chern roots of E;
- $w(E,u) = \sum_{j=0}^\infty w_j(\mathbf{x})u^j := \prod_{i=1}^n P_\Omega(u, x_i)$;
- $\mathcal{P}(u) := \sum_{i=0}^\infty [\mathbb{P}^i]u^i$ is assembled by putting together the classes of the projective spaces $[\mathbb{P}^i] \in \Omega^*(\operatorname{Spec} k)$.

Furthermore, we have been able to extend our definition of Segre classes to the classes of virtual bundles and obtain this geometric interpretation.

Theorem 2. *Let V and W be two vector bundles over X, respectively of rank n and m. Consider the projective bundle $\mathbb{P}^*(V) \xrightarrow{\pi} X$ with tautological bundle $O(1)$. Then one has*

$$\pi_*\Big(c_f\big(O(1) \otimes W^\vee\big)\Big) = \mathcal{S}_{m-n+1}(V - W)$$

as elements of $\Omega^*(X)$.

With this result at our disposal we are able to apply the adaptation of Kazarian's machinery to the resolution used in [10]: this gives us the pushforward class $[\widetilde{X}_\lambda^{KL} \to Gr_d(E)]_\Omega$, our candidate for the Schubert class of X_λ.

Theorem 3. *In $\Omega^*(Gr_d(E))$ we have*

$$[\widetilde{X}_\lambda^{KL} \to Gr_d(E)]_\Omega = \phi\left(t_1^{\lambda_1}\cdots t_r^{\lambda_r} \prod_{1\le i<j\le r}(1 - t_i/t_j) \prod_{1\le i<j\le r} P_\Omega(t_j, t_i)\right),$$

where, after expanding the rational function in t_i's as a power series in $t_1^{\alpha_1}\cdots t_r^{\alpha_d}$, the map ϕ sends each monomial to

$$\mathcal{S}_{\alpha_1}(U^\vee - (E/F^{\lambda_1-1+d})^\vee)\cdots\mathcal{S}_{\alpha_r}(U^\vee - (E/F^{\lambda_r-r+d})^\vee)\,.$$

References

[1] B. Calmès, V. Petrov, and K. Zainoulline, *Invariants, torsion indices and oriented cohomology of complete flags*, Ann. Sci.Éc. Norm. Supér. (4) 46, (2013), 405-448.

[2] B. Calmès, K. Zainoulline, C. Zhong *Equivariant oriented cohomology of flag varieties* , Doc. Math., Extra vol.: Alexander S. Merkurjevs sixtieth birthday, (2015), 113-144.

[3] T. Hudson, *A Thom-Porteous formula for connective K-theory using algebraic cobordism*, Journal of K-theory 14, (2014), 343-369.

[4] T. Hudson, *Generalised symplectic Schubert classes*, arXiv:1504.07924, (2015).

[5] T. Hudson, T. Ikeda, T. Matsumura, and H. Naruse, *Determinantal and Pfaffian formulas of K-theoretic schubert calculus*, arXiv:1504.02828v2, (2015).
[6] T. Hudson, and T. Matsumura, *Segre classes and Kempf-Laksov formula in algebraic cobordism*, arXiv:1602.05704, (2016).
[7] T. Hudson, and T. Matsumura, *Kempf-Laksov Schubert classes for even infinitesimal cohomology theories*, arXiv:1602.07953, (2016).
[8] J. Hornbostel, and V. Kiritchenko, *Schubert calculus for algebraic cobordism*, J. Reine Angew. Math. 656, (2011), 59-85.
[9] M. Kazarian, *On Lagrange and symmetric degeneracy loci*, Isaac Newton Institute for Mathematical Sciences Preprint Series, (2000).
[10] G. Kempf, and D. Laksov, *The determinantal formula of Schubert calculus*, Acta Math. 132, (1974), 153-162.
[11] V. Kiritchenko, and A. Krishna, *Equivariant cobordism of flag varieties and of symmetric varieties*, Transform. Groups 18, (2013), 391-413.
[12] M. Levine, and F. Morel, *Algebraic cobordism*, Springer Monographs in Mathematics. Springer, Berlin, (2007).

Homology stability for special linear groups and Euler classes of projective modules

Marco Schlichting

The talk was based on [Sch]. Let A be a commutative ring, let $\mathbb{Z}[A^*]$ be the group ring of the group of units A^* in A with standard $\mathbb{Z}$-basis $\langle a\rangle$, $a \in A^*$, multiplication $\langle a\rangle \cdot \langle b\rangle = \langle ab\rangle$, and $\langle 1\rangle = 1$. Let $I_{A^*} = \ker(\mathbb{Z}[A^*] \to \mathbb{Z} : \langle a\rangle \mapsto 1)$ be the augmentation ideal, and let $[a] = \langle a\rangle - 1 \in I_{A^*}$.

Definition 1. Let A be a commutative ring (with infinite residue fields). We define the graded $\mathbb{Z}[A^*]$-algebra

$$\hat{K}^{MW}_*(A) = \mathrm{Tens}_{\mathbb{Z}[A^*]}\, I_{A^*} / \,\mathrm{Steinberg}$$

as the quotient of the tensor algebra of I_{A^*} over the group ring $\mathbb{Z}[A^*]$ modulo the ideal generated by the Steinberg relations $[a] \otimes [1-a]$ for $a, 1-a \in A^*$.

For instance,

$$\hat{K}^{MW}_0(A) = \mathbb{Z}[A^*], \quad \hat{K}^{MW}_1(A) = I_{A^*}, \quad \hat{K}^{MW}_2(A) = I_{A^*} \otimes_{A^*} I_{A^*} / \,\mathrm{Steinberg}\,.$$

Denote by $K^{MW}_n(A)$ the Milnor-Witt K-theory of A as defined by Hopkins-Morel [Mor12].

Theorem 2. *Let A be a commutative local ring. If A is not a field assume that the cardinality of its residue field is at least 4. Then there is a natural map of graded rings $\hat{K}^{MW}(A) \to K^{MW}(A)$ which is an isomorphism*

$$\hat{K}^{MW}_n(A) \cong K^{MW}_n(A) \quad \textit{for } n \geq 2.$$

Let A be a commutative ring, and let $SL_n(A) = \ker(\det : GL_n(A) \to A^*)$ be the nth special linear group of A, for $n \geq 1$. When $n = 0$, we let $SL_0(A)$ be the *discrete set* A^*, and we set $SL_n(A) = \emptyset$ for $n < 0$. We prove the following.

Theorem 3. *Let A be a commutative local ring with infinite residue field. Then there are isomorphisms of A^*-modules for all $n \geq 0$*

$$H_i(SL_nA, SL_{n-1}A; \mathbb{Z}) \cong \begin{cases} 0 & i < n \\ \hat{K}_n^{MW}(A) & i = n. \end{cases}$$

This theorem is the SL_n-analogue of a theorem of Nesterenko-Suslin [NS89]. A version of the theorem was proved for fields of characteristic zero by Hutchinson-Tao [HT10].

Let R be a noetherian ring with infinite residue fields, and let P be an oriented projective R-module of rank n. We define the "Euler class" $e(P)$ of P as a certain Zariski cohomology class

$$e(P) \in H^n_{Zar}(R, \mathcal{K}_n^{MW})$$

where $\mathcal{K}_n^{MW}$ denotes the Zariski sheaf associated with the presheaf $A \mapsto \hat{K}_n^{MW}(A)$. Using the previous theorem, we can show the following.

Theorem 4. *Let R be a commutative noetherian ring of dimension $n \geq 2$. Assume that all residue fields of R are infinite. Let P be an oriented rank n projective R-module. Then*

$$P \cong Q \oplus R \Leftrightarrow e(P) = 0 \in H^n_{Zar}(R, \mathcal{K}_n^{MW}).$$

Using $\mathbb{A}^1$-homotopy theory, a version of this theorem was proved by Morel [Mor12] for smooth algebras over infinite perfect fields.

References

[HT10] Kevin Hutchinson and Liqun Tao. Homology stability for the special linear group of a field and Milnor-Witt K-theory. *Doc. Math.*, (Extra volume: Andrei A. Suslin sixtieth birthday):267–315, 2010 .

[Mor12] Fabien Morel. *$\mathbb{A}^1$-algebraic topology over a field*, volume 2052 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2012.

[NS89] Yu. P. Nesterenko and A. A. Suslin. Homology of the general linear group over a local ring, and Milnor's K-theory. *Izv. Akad. Nauk SSSR Ser. Mat.*, 53(1):121–146, 1989.

[Sch] Marco Schlichting. Euler class groups, and the homology ofelementary and special linear groups. arXiv:1502.05424v3.

Linkage properties for fields

Karim Johannes Becher

Quaternion algebras are said to be *linked* when they can be written with a slot in common. A field (of characteristic different from 2) is said to be linked if any pair of quaternion algebras is linked. In my talk I consider the property of *n-linkage* for a field, that is, that any set of n quaternion algebras over the field are linked. For $n = 2$ this means usual linkage, for $n = 3$ I refer to this property as *triple linkage*.

Elman-Lam showed in 1973 that any linked field has u-invariant 1,2,4 or 8 and that the value 8 occurs if and only if there exists an anisotropic 3-fold Pfister form. As a partial converse, any field with u-invariant 1, 2 or 4 is linked. A typical example of a linked field with u-invariant 8 is the iterated power series field $C(\!(x)\!)(\!(y)\!)(\!(z)\!)$. I show that a field with triple linkage does have u-invariant at most 4, which raises the question whether the converse holds. The result shows that triple linkage is strictly stronger than linkage. Furthermore, I give examples for fields having triple linkage, or even n-linkage for any n.

Endomorphisms of the equivariant motivic sphere

Jeremiah Heller
(joint work with David Gepner)

Let k be a field and G a finite group whose order is coprime to $\mathrm{char}(k)$. Write $G\mathrm{Sm}/k$ for the category of smooth G-schemes over k. In [1], Voevodsky introduces equivariant $\mathbb{A}^1$-homotopy theory, see also [3]. If V is a representation, the associated motivic representation sphere is $T^V := \mathbb{P}(V\oplus 1)/\mathbb{P}(V)$. The stable equivariant $\mathbb{A}^1$-homotopy category $\mathrm{SH}_G(k)$ is obtained by inverting the motivic representation sphere T^{ρ_G}, where ρ_G is the regular representation. This agrees with the category constructed in [5], other variants of stable equivariant $\mathbb{A}^1$-homotopy are constructed in [6, 4]. This is a tensor triangulated category, with unit the sphere spectrum $\mathbb{S}_k$.

If E is an object of $\mathrm{SH}_G(k)$, write $\underline{\pi}_n^G(E)$ for the presheaf on Sm/k,

$$\underline{\pi}_n^G(E)(X) = [S^n \wedge \Sigma^\infty X_+, E]_{\mathrm{SH}_G(k)}.$$

Below, if $H \subseteq G$ is a subgroup, $WH = NH/H$ and (H) denotes the conjugacy class of H.

Theorem 1 ([2])**.** *Let Y be a based motivic G-space. There is an isomorphism*

$$\underline{\pi}_n^G(\Sigma^\infty Y) \cong \bigoplus_{(H)} \underline{\pi}_n(\Sigma^\infty Y \wedge_{WH} \mathbb{B}WH_+),$$

where (H) ranges over the set of conjugacy classes of subgroups. In particular

$$\mathrm{End}_{\mathrm{SH}_G(k)}(\mathbb{S}_k) \cong \bigoplus_{(H)} \pi_0(\Sigma^\infty \mathbb{B}WH_+)$$

Here $\mathbb{B}G$ denotes the geometric classifying space of G, constructed by Morel-Voevodsky [8] and Totaro [11]. Theorem 1 is a motivic version of tom Dieck's splitting theorem for classical stable equivariant homotopy groups [10]. In the classical setting, tom Dieck's splitting recovers Segal's computation [9] that $\pi_0^G(\mathbb{S}) = A(G)$, the Burnside ring of finite G-sets.

When k is perfect, $\mathrm{char}(k) \neq 2$, Morel [7] computes that $\pi_0(\mathbb{S}_k) = GW(k)$, the Grothendieck-Witt ring of k. The formula in Theorem 1 implies that the endomorphism ring of the equivariant motivic sphere contains a piece which is the most obvious way of combining these two endomorphism rings,

$$\mathrm{End}_{\mathrm{SH}_G(k)}(\mathbb{S}_k) \cong \bigoplus_{(H)} \pi_0(\Sigma^\infty \mathbb{B}WH) \oplus (A(G) \otimes GW(k))\,.$$

In general $\pi_0(\Sigma^\infty \mathbb{B}WH)$ is nonzero and mysterious.

To establish our splitting theorem, we can reduce to the case where Y is concentrated over a single conjugacy class, i.e., there is a subgroup H such that $Y^K \simeq *$ unless $K \in (H)$. In this case, the theorem follows by showing that the composition below is an isomorphism

$$
(1) \qquad \begin{tikzcd}
\pi_*^{WH}(\mathbb{E}WH_+ \wedge Y^H) \arrow[r, "\pi^*"] \arrow[d, "\Theta_{Y,H}"] & \pi_*^{NH}(\mathbb{E}WH_+ \wedge Y^H) \arrow[r, "i_*"] & \pi_*^{NH}(\mathbb{E}WH_+ \wedge Y) \arrow[d, "\omega"', "\cong"] \\
\pi_*^G(Y) & & \pi_*^G(G_+ \wedge_{NH} (\mathbb{E}WH_+ \wedge Y)). \arrow[ll]
\end{tikzcd}
$$

Here ω is the motivic Wirthmüller isomorphism, π^* is induced by $\pi : NH \to WH$ and i_* by the inclusion $Y^H \to Y$.

The key ingredient in the proof of the splitting theorem is provided by a motivic version of the geometric fixed points functor. If $\mathcal{F}$ is a family of subgroups, there is a geometric universal $\mathcal{F}$-space $\mathbb{E}\mathcal{F}$, which generalizes the Morel-Voevodsky, Totaro construction $\mathbb{E}G$. is an unbased motivic G-space characterized by the property that for any X in $G\mathrm{Sm}/k$

$$\mathrm{map}(X, \mathbb{E}\mathcal{F}) = \begin{cases} \emptyset & \text{if } X^H \neq \emptyset \text{ for some } H \notin \mathcal{F}, \\ * & \text{else.} \end{cases}$$

The reduction to motivic G-spaces concentrated at a single conjugacy class mentioned above is achieved by considering spaces of the form $X \wedge \mathbb{E}\mathcal{F}'/\mathbb{E}\mathcal{F}$, where $\mathcal{F} \subseteq \mathcal{F}'$ is a subfamily such that $\mathcal{F}' - \mathcal{F} = \{(H)\}$.

Define

$$\tilde{\mathbb{E}}\mathcal{F} := \mathrm{cofiber}(\mathbb{E}\mathcal{F}_+ \to S^0).$$

For a normal subgroup $N \trianglelefteq G$. Define the geometric fixed points functor by

$$\Phi^N(E) = (\tilde{\mathbb{E}}\mathcal{F}[N] \wedge E)^N,$$

where we write $\mathcal{F}[N]$ be the family of subgroups $\{H \leq G \mid N \not\subseteq H\}$. This is a G/N-spectrum. Smashing with $\tilde{\mathbb{E}}\mathcal{F}[N]$ is a localization, and roughly speaking has the effect of killing off maps from smooth G-schemes X whose stabilizers do not all contain N. This characterization of Φ^N leads to the bottom horizontal arrow in (1) being an isomorphism.

The next result is a characterization of Φ^N which leads to the upper horizontal composition in (1) being an isomorphism. (Note that for ordinary fixed points, $(\Sigma^\infty Y)^N \not\simeq \Sigma^\infty(Y^N)$).

Theorem 2 ([2]). *Let $N \trianglelefteq G$ be a normal subgroup, there is a natural isomorphism in $\mathrm{SH}_{G/N}(k)$*

$$\Sigma^\infty(Y^N) \simeq \Phi^N(\Sigma^\infty Y).$$

References

[1] P. Deligne. Voevodsky's lectures on motivic cohomology 2000/2001. In *Algebraic topology*, volume 4 of *Abel Symp.*, pages 355–409. Springer, Berlin, 2009.

[2] D. Gepner and J. Heller. The endomorphism ring of the equivariant motivic sphere. *in preparation.*

[3] J. Heller, A. Krishna, and P. A. Ostvaer. Motivic homotopy theory of group scheme actions. *ArXiv e-prints*, August 2014.

[4] P. Herrmann. Equivariant Motivic Homotopy Theory. *ArXiv e-prints*, December 2013.

[5] M. Hoyois. The six operations in equivariant motivic homotopy theory. *ArXiv e-prints*, September 2015.

[6] P. Hu, I. Kriz, and K. Ormsby. The homotopy limit problem for Hermitian K-theory, equivariant motivic homotopy theory and motivic Real cobordism. *Adv. Math.*, 228(1):434–480, 2011.

[7] F. Morel. On the motivic π_0 of the sphere spectrum. In *Axiomatic, enriched and motivic homotopy theory*, volume 131 of *NATO Sci. Ser. II Math. Phys. Chem.*, pages 219–260. Kluwer Acad. Publ., Dordrecht, 2004.

[8] F. Morel and V. Voevodsky. $\mathbf{A}^1$-homotopy theory of schemes. *Inst. Hautes Études Sci. Publ. Math.*, (90):45–143 (2001), 1999.

[9] G. B. Segal. Equivariant stable homotopy theory. In *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2*, pages 59–63. Gauthier-Villars, Paris, 1971.

[10] T. tom Dieck. Orbittypen und äquivariante Homologie. II. *Arch. Math. (Basel)*, 26(6):650–662, 1975.

[11] B. Totaro. The Chow ring of a classifying space. In *Algebraic K-theory (Seattle, WA, 1997)*, volume 67 of *Proc. Sympos. Pure Math.*, pages 249–281. Amer. Math. Soc., Providence, RI, 1999.

Hecke-type algebras and equivariant oriented cohomology of flag varieties

Changlong Zhong

1. Introduction

The idea of using combinatorial and algebraic information of root system to study geometric property of flag varieties came from the works of Demazure [10, 11] and Bernstein-Gelfand-Gelfand [3] in 1970s, and then was generalized to equivariant setting by Arabia [1, 2], Kostant-Kumar [13, 14]. The singular cohomology/Chow group and Grothendieck ring were used in this period. Over more generalized cohomology theories, the Bott-Samelson classes depend on the choice of reduced sequences. Surprisingly, the algebraic method still works. This was started in [4, 5], and then continued in [6, 12, 7, 8, 9, 16, 15]. This is the main topic of this talk.

2. Equivariant oriented cohomology of flag varieties

Let k be a field of characteristic 0, and let $\mathfrak{h}$ be an oriented cohomology theory in the sense of Levine-Morel. Let F be the formal group law over $R := \mathfrak{h}(k)$ determined by $\mathfrak{h}$.

Let G be any split, semi-simple linear algebraic group over k, with B any Borel subgroup containing the maximal torus T. Let $\Lambda = T^*$ be the group of characters. Let Σ be the fixed set of roots with $\Pi = \{\alpha_1, ..., \alpha_n\}$ the set of simple roots. Let W be the Weyl group. For any $J \subset \Pi$, let P_J be a parabolic subgroup of G determined by J, $W_J < W$ be the corresponding subgroup. For each $w \in W$, let $\ell(w)$ be the length of w. Let $w_0^J \in W_J$ be the longest element in W_J.

Let Γ be a free abelian group of rank 1, generated by γ. Let $S = R[[\Lambda]]_F$ and $S' = R[[\Lambda \oplus \Gamma]]_F$ be the formal group algebras defined in [6], and denote $Q = S[\frac{1}{x_\alpha}|\alpha \in \Sigma]$, $\quad Q' = S'[\frac{1}{x_\alpha}|\alpha \in \Sigma$. Let $Q_W = Q\sharp R[W], Q'_W = Q\sharp R[W]$ be the twisted product. Q_W and Q'_W act on Q and Q' naturally.

The *formal affine Demazure algebra* $\mathbf{D}_F$ is defined as the R-subalgebra of Q_W generated by S and $X_\alpha := \frac{1}{x_\alpha}(1 - \delta_\alpha)$ for all $\alpha \in \Sigma$. Similarly, in [16], the *formal affine Hecke algebra* $\mathbf{H}_F$ is defined to be the R-subalgebra of Q'_W generated by S' and $T_\alpha := x_\gamma X_\alpha + \delta_\alpha$ for all $\alpha \in \Sigma$. Denote $X_i = X_{\alpha_i}, T_i = T_{\alpha_i}$ for simplicity. For each reduced sequence $I_w = (i_1, ..., i_k)$ of $w \in W$, define X_{I_w} and T_{I_w} as the corresponding product, which will depend on the choice of I_w unless F is the additive or multiplicative formal group law. Then $\{X_{I_w}\}_{w\in W}$ and $\{T_{I_w}\}_{w\in W}$ are S-bases of $\mathbf{D}_F$ and $\mathbf{H}_F$, respectively.

Let $\mathbf{D}_F^*$ be the S-dual of $\mathbf{D}_F$, on which there is an action of $\mathbf{D}_F$, denoted by $\bullet$. There is an explicit defined element $Y_J \in \mathbf{D}_F$.

Theorem 1. [8, 9]

1. *$\mathbf{D}_F^* \cong \mathfrak{h}_T(G/B)$, and $(\mathbf{D}_F^*)^{W_J} \cong \mathfrak{h}_T(G/P_J)$, and the pushforward corresponds to $Y_J \bullet \_ : \mathbf{D}_F^* \to (\mathbf{D}_F^*)^{W_J}$. Moreover, via the isomorphism, there is an explicitly defined element $[pt] \in \mathbf{D}_F^*$ which corresponds to the class $[B/B] \in \mathfrak{h}_T(G/B)$. Indeed, $[pt]$ is a basis of $\mathbf{D}_F^*$ as a $\mathbf{D}_F$-module.*
2. *There are explicitly defined cohomology classes in $\mathbf{D}_F^*$ corresponding to the Bott-Samelson classes.*

3. The $\odot$-action

There is another action of $\mathbf{D}_F$ on $\mathbf{D}_F^*$, denoted by $\odot$, which is not S-linear but commutes with the $\bullet$-action. More precisely, it is restricted from the following $\odot$-action of Q_W on $Q_W^* := Hom(W, Q)$, defined as follows

$$q\delta_w \odot (pf_v) = qw(p)f_{wv}, \quad p, q \in Q, w, v \in W.$$

Therefore, it induces an action of $\mathbf{D}_F$ on $(\mathbf{D}_F^*)^{W_J}$. Such action was studied Knutson, Peterson and Tymoczko for (equivariant) singular cohomology/Chow groups.

Using the $\odot$-action and by reinterpreting Deodhar's work, we prove the following result:

Theorem 2. [15] *We have a chain complex of $\mathbf{D}_F$-module*

$$0 \longrightarrow \mathfrak{h}_T(G/B) \xrightarrow{\partial_0} \bigoplus_{|J|=1} \mathfrak{h}_T(G/P_J) \xrightarrow{\partial_1} \bigoplus_{|J|=2} \mathfrak{h}_T(G/P_J) \xrightarrow{\partial_2} \cdots \tag{3}$$
$$\cdots \xrightarrow{\partial_{n-1}} \mathfrak{h}_T(Spec(k)) \xrightarrow{\partial_n}$$

where the maps are defined as certain alternating sums of push-pull morphisms between cohomology of flag varieties. This complex is exact everywhere, except for $\mathbf{D}_F^*$. *Moreover, at* $\mathbf{D}_F^*$, *the cohomology is of rank* 1 *over* S, *provided* F *is additive, multiplicative, or the hyperbolic formal group law (defined below).*

4. Hyperbolic formal group law and the Kazhdan-Lusztig basis

In this subsection, let $F = F_h = \frac{x+y-xy}{1-\frac{1}{\mu^2}xy}$ over $R_0 := \mathbb{Z}[t, t^{-1}, \mu^{-1}]$ where $\mu = t + t^{-1}$. Let H be the classical Hecke algebra associated to W, which is generated by $\tau_i, 1 \le i \le n$ satisfying the braid relation and the quadratic relation $\tau_i^2 = (t^{-1} - t)\tau_i + 1$. Let C_w be the Kazhdan-Lusztig basis of H.

Let $D_h \subset \mathbf{D}_F$ be the subalgebra generated by $X_\alpha, \alpha \in \Sigma$. It is checked that D_h is indeed a free R_0-module with basis $\{X_{I_w}\}_{w\in W}$. There is an isomorphism $\phi : H \to D_h$ of algebras defined by $\phi(\tau_i) = \mu X_i + t^{-1}$. We then have the following results:

Theorem 4. [15]

(1) *There is a canonical isomorphism* $\phi(H) \odot (Y_J \bullet [pt]) \cong H \otimes_{H_J} R_0$ *where* $[pt]$ *is the cohomology class in* $\mathfrak{h}_T(G/B)$ *determined by* B/B, H_J *is the Hecke algebra associated to* W_J, *and* R_0 *is a* H_J*-module via the map* $H_J \to R_0, \tau_j \mapsto t^{-1}, j \in J$.

(2) $\phi(C_{w_0^J}) = \mu^{\ell(w_0^J)} Y_J$, *where* $C_{w_0^J} \in H$ *is the Kazhdan-Lusztig basis corresponding to* w_0^J.

5. Formal affine Hecke algebra and convolution algebra

Let Z be the Steinberg variety, which admits an action of $G \times k^\times$. Let $\mathfrak{h}$ be any oriented cohomology theory obtained from base change of the algebraic cobordism Ω, that is, $\mathfrak{h}(X) = \Omega(X) \otimes_{\mathbb{L}} R$ where $\mathbb{L}$ is the Lazard ring and $\mathbb{L} \to R$ is the map determining the formal group law F of $\mathfrak{h}$. Then $\mathfrak{h}(Z)$ is well defined, even though Z is a singular variety.

Generalizing results of Lusztig on affine Hecke algebra, we have the following result

Theorem 5. [16] *There is a canonical isomorphism of non-commutative algebras* $\mathbf{H}_F \cong \mathfrak{h}_{G\times k^\times}(Z)$ *where the right hand side is the convolution algebra.*

For each elliptic curve E, there is an elliptic affine Hecke algebra $\mathcal{H}$ defined by Ginzburg-Kapranov-Vasserot, which is a sheaf of algebras over the variety E^{n+1}/W (where W acts on the first n coordinates). We have

Theorem 6. [16] *Let* $\mathcal{H}_{(0)}$ *be the completion of the stalk of* $\mathcal{H}$ *at* $0 \in E^{n+1}$, *then* $\mathcal{H}_{(0)}$ *is isomorphic to the formal affine Hecke algebra* $\mathbf{H}_F$ *where* F *is the elliptic formal group law determined by* E.

References

[1] A. Arabia, *Cycles de Schubert et cohomologie équivariante de K/T*, Invent. Math. 85 (1986), no. 1, 39–52.

[2] ———, *Cohomologie T-équivariante de la variété de drapeaux d'un groupe de Kac-Moody*, Bull. Soc. Math. France 117, no. 2, 129–165, 1989.

[3] I. N. Bernstein, I. M. Gelfand and S. I. Gelfand, Schubert cells and cohomology of the spaces G/P, Russian Math. Surveys 28, 1–26, 1973.

[4] P. Bressler, S. Evens, The Schubert calculus, braid relations and generalized cohomology. Trans. Amer. Math. Soc. 317 (1990), no.2, 799–811.

[5] P. Bressler, S. Evens, Schubert calculus in complex cobordisms. Trans. Amer. Math. Soc. 331 (1992), no.2, 799–813.

[6] B. Calmès, V. Petrov, K. Zainoulline, Invariants, torsion indices and cohomology of complete flags, Ann. Sci. École Norm. Sup. (4) 46 (2013), no.3, 405–448.

[7] B. Calmès, K. Zainoulline, C. Zhong, A coproduct structure on the formal affine Demazure algebra, to appear in Math. Z. (arXiv:1209.1676).

[8] B. Calmès, K. Zainoulline, C. Zhong, Push-pull operators on the formal affine Demazure algebra and its dual, arXiv:1312.0019.

[9] B. Calmès, K. Zainoulline, C. Zhong, Equivariant oriented cohomology of flag varieties, to appear in Documenta Math. (arXiv:1409.7111).

[10] M. Demazure, Invariants symétriques entiers des groupes de Weyl et torsion. Invent. Math. 21 (1973), 287–301.

[11] M. Demazure, Désingularisation des variétés de Schubert généralisées. Ann. Sci. École Norm. Sup. (4) 7 (1974), 53–88.

[12] A. Hoffnung, J. Malagón-López, A. Savage, K. Zainoulline, Formal Hecke algebras and algebraic oriented cohomology theories. Selecta Math. 20 (2014), no.4, 1213–1245.

[13] B. Kostant, S. Kumar, The nil Hecke ring and cohomology of G/P for a Kac-Moody group G. Advances in Math. 62 (1986), 187–237.

[14] B. Kostant, S. Kumar, T-Equivariant K-theory of generalized flag varieties. J. Differential Geometry 32 (1990), 549–603.

[15] C. Lenart, K. Zainoulline, C. Zhong, Parabolic Kazhdan-Lusztig basis, Schubert classes and equivariant elliptic cohomology, in preparation.

[16] G. Zhao, C. Zhong, Geometric representations of the formal affine Hecke algebra, Preprint 2014, arxiv:1406.1283.

The Hurewicz and Conservativity Theorems for $\mathbf{SH}(k) \to \mathbf{DM}(k)$

Tom Bachmann

In classical algebraic topology, the study of the stable homotopy category $\mathbf{SH}$ is greatly facilitated by considering the functor $C_* : \mathbf{SH} \to D(Ab)$ which associated to a spectrum its singular chain complex. This functor is conservative on connective spectra (i.e. those with only finitely many non-vanishing negative stable homotopy groups) and induces an isomorphism on Picard groups. Both of these result follows easily from the so-called Hurewicz theorem, i.e. the fact that if $E \in \mathbf{SH}(k)$ has $\pi_i(E) = 0$ for all $i < 0$ then $H_i C_*(E) = 0$ for all $i < 0$ and also $H_0 C_* E = \pi_0 E$.

The aim of this talk is to explain how one may extend this result to the motivic world. More specifically we ask: what extra assumptions are necessary in order for the natural functor $M : \mathbf{SH}(k) \to \mathbf{DM}(k)$, associating to a spectrum its motive, to be conservative? Answering this question consists in two steps. The first is to establish an analog of the Hurewicz theorem. For this, recall that the categories $\mathbf{SH}(k)$ and $\mathbf{DM}(k)$ afford t-structures and M affords a t-conservative right adjoint U. We write $\pi_i(E)_* \in \mathbf{SH}(k)^\heartsuit$ for the homotopy objects associated to an object $E \in \mathbf{SH}(k)$, and $h_i(M)_* \in \mathbf{DM}(k)^\heartsuit$ for the homotopy objects of a motive $M \in \mathbf{DM}(k)$. There is an induced functor $U : \mathbf{DM}(k)^\heartsuit \to \mathbf{SH}(k)^\heartsuit$ which detects zero objects. With this notation set up, we prove the following theorem.

Theorem 1. *If $E \in \mathbf{SH}(k)$ and $\pi_i(E)_* = 0$ for all $i < 0$, then $h_i(E)_* = 0$ for all $i < 0$, and moreover $Uh_0(ME)_* = \pi_0(E)_*/\eta$.*

Here $\eta : \pi_0(E)_{*+1} \to \pi_0(E)_*$ is the Hopf element. The proof of this result uses an important structure theorem of Déglise [2].

It then remains to find criteria for when $\pi_0(E)_*/\eta = 0$ implies that $\pi_0(E)_* = 0$. In this talk we show how to use results of Levine on Voevodsky's slice filtration [3] to prove the following result.

Theorem 2. *Let k be a field of characteristic zero and finite 2-étale cohomological dimension and $E \in \mathbf{SH}(k)$ connective and slice-connective. Then $ME = 0$ implies that $E = 0$.*

Here slice-connectivity is a technical condition which is implied by compactness, for example. The restriction on the characteristic of the base field is not really relevant. More details can be found in [1].

References

[1] Tom Bachmann. On the Conservativity of Some Functors in Motivic Homotopy Theory, Preprint 2016, arxiv:1506.07375.
[2] Frédéric Déglise. Orientable homotopy modules. *American Journal of Mathematics*, 135(2):519–560, 2013.
[3] Marc Levine. The slice filtration and grothendieck-witt groups, 2010.

Differential and Quadratic Forms in Characteristic Two

Ahmed Laghribi
(joint work with Roberto Aravire and Manuel O'Ryan)

Let F be a field of characteristic 2. For a quadratic form Q over F, we denote by $F(Q)$ the function field of the projective quadric given by Q. If B is a bilinear form over F of underlying vector space V, we denote by $\widetilde{B}$ the (quasilinear) quadratic form defined on V by: $\widetilde{B}(v) = B(v, v)$. The function field of B, denoted $F(B)$, is by definition the function field of the projective quadric given by $\widetilde{B}$. For $a_1, \ldots, a_n \in F^* := F \setminus \{0\}$, let $\langle a_1, \cdots, a_n \rangle$ denote the diagonal bilinear form $\sum_{i=1}^n a_i x_i y_i$.

Let Ω_F^1 be the space of absolute differential forms over F, i.e. the F-vector space generated by symbols dx, $x \in F$, subject to the relations: $d(x+y) = dx+dy$ and $d(xy) = xdy + ydx$ for any $x, y \in F$. For any integer $n \geq 1$, let $\Omega_F^n = \wedge^n \Omega_F^1$ denote the n-th exterior power of Ω_F^1 (we take $\Omega_F^0 = F$). We have an F^2-linear map $F \longrightarrow \Omega^1_F$, given by: $x \mapsto dx$ for $x \in F$. This map extends to the differential operator $d : \Omega^n_F \longrightarrow \Omega^{n+1}_F$ defined by: $d(xdx_1 \wedge \cdots \wedge dx_n) = dx \wedge dx_1 \wedge \cdots \wedge dx_n$.

The well-known Artin-Schreier map $\wp : F \longrightarrow F$, $\wp(a) = a^2 - a$, extends to a well-defined map $\wp : \Omega^n_F \longrightarrow \Omega^n_F / d\Omega^{n-1}_F$ given by:

$$\wp(x\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}) = (x^2 - x)\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n} + d\Omega^{n-1}_F.$$

Let $H^{n+1}_2(F)$ (*resp.* $\nu_F(n)$) be the cokernel of $\wp$ (*resp.* the kernel of $\wp$). For any field extension K/F, we get two group homomorphisms $H^{n+1}_2(F) \longrightarrow H^{n+1}_2(K)$ and $\nu_F(n) \longrightarrow \nu_K(n)$ induced by scalar extension. A natural question that arises consists in computing the kernels of these homomorphisms.

Here we are interested to the first homomorphism $H^{n+1}_2(F) \longrightarrow H^{n+1}_2(K)$, whose kernel is denoted $H^{n+1}_2(K/F)$. Recall that the kernel $H^{n+1}_2(K/F)$ was computed in the following cases:

- (A1) K/F is purely transcendental [2, Lem. 2.17].
- (A2) K/F is quadratic [1, Prop. 6.4], [2, Lem. 2.18].
- (A3) K/F is biquadratic separable [4, Th. 19].
- (A4) $K = F(Q)$, where Q is a bilinear Pfister form of arbitrary dimension [2, Th. 4.1], or a quadratic Pfister form of dimension 2^{m+1} such that: $n \leq m$ [2, Th. 5.5] and [6, Page 655], or $m = 0, 1$ [1, Prop. 6.4] and [3, Th. 1.6].
- (A5) K/F is multiquadratic of separability degree ≤ 2 [5, Prop. 2 and 3].

Our main result is the following theorem which gives a complete computation of the kernel $H^{n+1}_2(K/F)$, where $K = L \cdot F(B)$ is the compositum of a multiquadratic extension L/F of separability degree ≤ 2 with the function field $F(B)$ of a bilinear Pfister form B over F. More precisely, we have:

Theorem 1. *([6]) Let $B = \langle 1, a_1\rangle \otimes \cdots \otimes \langle 1, a_m\rangle$ be an anisotropic bilinear Pfister form over F, $L = F(\sqrt{c_1}, \ldots, \sqrt{c_k})(\alpha)$ for $c_1, \cdots, c_k \in F^*$ such that $[F(\sqrt{c_1}, \ldots, \sqrt{c_k}) : F] = 2^k$, $\wp(\alpha) = a \in F$ and B_L is anisotropic. Let $\eta = \frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_m}{a_m} \in \Omega^m_F$. Then:*

$$H^{n+1}_2(L \cdot F(B)/F) = \begin{cases} \overline{a\nu_F(n)} + \overline{\eta \wedge \Omega^{n-m}_F} + \sum_{i=1}^{k} \overline{\frac{dc_i}{c_i} \wedge \Omega^{n-1}_F} & \text{if } n \geq m, \\ \overline{a\nu_F(n)} + \sum_{i=1}^{k} \overline{\frac{dc_i}{c_i} \wedge \Omega^{n-1}_F} & \text{if } 0 \leq n < m. \end{cases}$$

To interpret this theorem in the language of quadratic forms we recall some definitions. Let $W_q(F)$ (*resp.* $W(F)$) be the Witt group of nonsingular quadratic forms over F (*resp.* the Witt ring of regular symmetric bilinear forms over F). For any integer $n \geq 1$, let $I^n_q(F)$ denote the subgroup $I^{n-1}(F) \otimes W_q(F)$ of $W_q(F)$, where $I^k(F)$ is the k-th power of the fundamental ideal $I(F)$ of $W(F)$, and $\otimes$ is the module action of $W(F)$ on $W_q(F)$ (we take $I^0(F) = W(F)$). Let $\overline{I^n_q}(F)$ (for $n \geq 1$) and $\overline{I^n}(F)$ be the quotients $I^n_q(F)/I^{n+1}_q(F)$ and $I^n(F)/I^{n+1}(F)$, respectively. For any field extension K/F, let $\overline{I^n_q}(K/F)$ (*resp.* $I^n_q(K/F)$) denote the kernel of the homomorphism $\overline{I_q^n}(F) \longrightarrow \overline{I_q^n}(K)$ (*resp.* $I_q^n(F) \longrightarrow I_q^n(K)$) induced by scalar extension. It is well-known by a result of Kato [7] that we have two isomorphisms $e^n : \overline{I_q^{n+1}}(F) \longrightarrow H_2^{n+1}(F)$ and $f^n : \overline{I^n}(F) \longrightarrow \nu_F(n)$ given as follows:

$$e^n(\langle 1, b_1\rangle \otimes \cdots \otimes \langle 1, b_n\rangle \otimes [1,c] + I_q^{n+2}(F)) = c\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n} + d\Omega_F^{n-1} + \wp(\Omega_F^n),$$

$$f^n(\langle 1, b_1\rangle \otimes \cdots \otimes \langle 1, b_n\rangle + I^{n+1}(F)) = \frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n},$$

where $[1,c]$ denotes the binary nonsingular quadratic form $x^2 + xy + cy^2$. Using these isomorphisms, we deduce from Theorem 1 the following corollary:

Corollary 2. *([6]) We keep the same notations and hypotheses as in Theorem 1. Then:*

$$\overline{I_q^{n+1}}(L \cdot F(B)/F) = \begin{cases} \overline{I^n(F) \otimes [1,a]} + \overline{B \otimes I_q^{n-m+1}(F)} + \sum_{i=1}^k \overline{\langle 1, c_i\rangle \otimes I_q^n(F)} & \text{if } n \geq m, \\ \overline{I^n(F) \otimes [1,a]} + \sum_{i=1}^k \overline{\langle 1, c_i\rangle \otimes I_q^n(F)} & \text{if } 0 \leq n < m. \end{cases}$$

As a consequence, we prove the following:

Corollary 3. *([6]) We keep the same notations and hypotheses as in Theorem 1. Then:*

$$I_q^{n+1}(L \cdot F(B)/F) = \begin{cases} I^n(F) \otimes [1,a] + B \otimes I_q^{n-m+1}(F) + \sum_{i=1}^k \langle 1, c_i\rangle \otimes I_q^n(F) & \text{if } n \geq m, \\ I^n(F) \otimes [1,a] + B \otimes W_q(F) + \sum_{i=1}^k \langle 1, c_i\rangle \otimes I_q^n(F) & \text{if } 0 \leq n < m. \end{cases}$$

We also discussed the analogue of Theorem 1 when the bilinear Pfister form B is replaced by a quadratic Pfister form. This is related to the following question:

Question 1. *Let $Q = \langle 1, a_1\rangle \otimes \cdots \otimes \langle 1, a_m\rangle \otimes [1,b]$ be an anisotropic quadratic Pfister form over F. Is it true that*

$$H_2^{n+1}(F(Q)/F) = \begin{cases} \overline{b\frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_m}{a_m} \wedge \nu_F(n-m)} & \text{if } n \geq m \\ 0 & \text{if } n < m? \end{cases}$$

This question has a positive answer when $m = 0, 1$ or $n \leq m$ (see the references mentioned before in the case (A4)). In general, depending on the positive answers to this question, we prove the following:

Theorem 4. *([6]) Let m, n be two integers for which Question 1 has a positive answer. Let $L = F(\sqrt{c_1}, \ldots, \sqrt{c_k})$ be such that $[L : F] = 2^k$, and let $Q = \langle 1, a_1\rangle \otimes \cdots \otimes \langle 1, a_m\rangle \otimes [1,b]$ be an anisotropic quadratic Pfister form over F. Then:*

$$H_2^{n+1}(L.F(Q)/F) = \begin{cases} \sum_{i=1}^k \overline{\frac{dc_i}{c_i} \wedge \Omega_F^{n-1}} & \text{if } Q_L \text{ is isotropic, or} \\ & Q_L \text{ is anisotropic and } n < m, \\ \sum_{i=1}^k \overline{\frac{dc_i}{c_i} \wedge \Omega_F^{n-1}} + \overline{b\frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_m}{a_m} \wedge \nu_F(n-m)} & \\ & \text{otherwise.} \end{cases}$$

Using the Kato's isomorphisms described before, Theorem 4 can be interpreted in the language of quadratic forms to get the analogues of Corollaries 2 and 3.

According to the results that we described, we ask the following question:

Question 2. *Let L/F be a purely inseparable extension of finite degree, B an anisotropic bilinear Pfister form over F, and K/F a separable quadratic extension. Suppose that B is anisotropic over L. Is it true that*

$$H_2^{n+1}(L \cdot K(B)/F) = H_2^{n+1}(L/F) + H_2^{n+1}(K/F) + H_2^{n+1}(F(B)/F)?$$

References

[1] R. Aravire, R. Baeza, *Milnor's K-Theory and quadratic forms over fields of characteristic two,* Comm. Algebra **20** (1992) 1087-1107.

[2] R. Aravire, R. Baeza, *The behavior of quadratic and differential forms under function field extensions in characteristic two,* J. Algebra **259** (2003) 361–414.

[3] R. Aravire, B. Jacob, *$H^1(X,\nu)$ of conics and Witt kernels in characteristic 2,* Contemp. Math., **493**, Amer. Math. Soc., Providence, RI, 2009, 1–19.

[4] R. Aravire, B. Jacob, *The graded Witt group kernel of biquadratic extensions in characteristic two,* J. Algebra **370** (2012), 297-319.

[5] R. Aravire, A. Laghribi, *Results on Witt kernels of quadratic forms for multi-quadratic extensions,* Proc. Amer. Math. Soc. **141** (2013) 4191-4197.

[6] R. Aravire, A. Laghribi, M. O'Ryan, *Graded Witt kernels of the compositum of multi-quadratic extensions with the function fields of Pfister forms,* J. Algebra **449** (2016), 635–659.

[7] K. Kato, *Symmetric bilinear forms, quadratic forms and Milnor K-theory in characteristic two,* Invent. Math. **66** (1982), 493–510.

Equivariant motives of homogeneous varieties

Nikita Semenov

(joint work with Victor Petrov)

Let G denote a split semisimple algebraic group of rank l over a field F and let B be a Borel subgroup of G. Denote by BB the classifying space of B, fix a prime number p, and denote by Ch^* the Chow ring modulo p.

Consider the kernel of the characteristic map (see [1]):

$$I_p = \mathrm{Ker}(\mathrm{Ch}^*(BB) \xrightarrow{c} \mathrm{Ch}^*(G/B)).$$

Victor Kac showed in [2] that the ideal I_p is generated by l homogeneous elements of explicit degrees $d_{1,p}, \dots, d_{l,p}$ depending on the combinatorics of G.

We work in the category of G-equivariant Chow motives with coefficients in $\mathbb{Z}/p$. This category is defined in the same way as the category of ordinary Chow motives (see [3]) but with Chow rings replaced by the equivariant Chow rings of Edidin–Graham, Totaro.

We show (see [4] for details):

Theorem. *The G-equivariant Chow motive $\mathcal{M}_G(G/B)$ of the variety G/B with $\mathbb{Z}/p$-coefficients is isomorphic to*

$$\bigoplus_{i\geq 0} R_{p,G}(G)(i)^{\oplus c_i}$$

for some explicit integers c_i, the G-equivariant motive $R_{p,G}(G)$ is indecomposable and the Poincaré series of $\mathrm{Ch}^*_G(R_{p,G}(G))$ equals $\prod_{i=1}^{l} \frac{1}{1-t^{d_{i,p}}} \in \mathbb{Z}[[t]]$.

The proof uses a reduction to the case of ordinary Chow motives of twisted forms of G/B whose structure was established in [5].

References

[1] A. Grothendieck, *La torsion homologique et les sections rationnelles*, Exposé 5 in Anneaux de Chow et applications, Séminaire Claude Chevalley, 1958.

[2] V. Kac, *Torsion in cohomology of compact Lie groups and Chow rings of reductive algebraic groups*, Invent. Math. **80** (1985), 69-79.

[3] Y. Manin, *Correspondences, motives and monoidal transformations*, Matematicheskij Sbornik **77** (**119**) (1968), no. 4, 475–507.

[4] V. Petrov, N. Semenov, *Rost motives, affine varieties, and classifying spaces*, Preprint 2015, available from `http://arxiv.org/abs/1506.07788`

[5] V. Petrov, N. Semenov, K. Zainoulline, *J-invariant of linear algebraic groups*, Ann. Sci. Éc. Norm. Sup. **41** (2008), 1023–1053.

Transfers for non-stable K_1-functors of classical type

Anastasia Stavrova

Let Sm_k be the category of Noetherian smooth schemes of finite type over a field k. For any presheaf F on Sm_k, we denote its Nisnevich sheafification by F_{Nis}. Let G be a reductive k-group that contains a proper parabolic k-subgroup P, or, equivalently, a non-central subgroup $\mathbb{G}_{m,k}$. For any k-scheme X set

$$E_P(X) = \langle U_P(X), U_{P^-}(X)\rangle \leq G(X),$$

where U_P and U_{P^-} are the unipotent radicals of P and any opposite parabolic subgroup P^-. The quotient $K_1^{G,P}(X) = G(X)/E_P(X)$ is called *the non-stable K_1-functor* associated to G. The functor $K_1^{G,P}$ on affine k-schemes is independent of the choice of a strictly proper parabolic k-subgroup P. If every semisimple normal subgroup of G contains $(\mathbb{G}_m)^2$, then $K_1^{G,P}$ takes values in the category of groups. Also, it satisfies certain injectivity and $\mathbb{A}^1$-invariance theorems [St]. The above-mentioned properties of $K_1^{G,P}$ imply that the functor $K_{1,Nis}^{G,P} = K_{1,Nis}^{G}$ on Sm_k is group-valued, independent of the choice of P, and $\mathbb{A}^1$-invariant.

Let k be a field, and let $H \xrightarrow{\mu} T$ be a homomorphism of algebraic k-groups where T is a torus. Let F be a field extension of k. We say that μ *satisfies Merkurjev's norm principle over F*, if for any étale F-algebra E the standard norm homomorphism $T(E) \xrightarrow{N_{E/F}} T(F)$ satisfies

$$N_{E/F} \circ \mu\big(H(E)\big) \subseteq \mu\big(H(F)\big).$$

Our construction of transfers for $K^G_{1,Nis}$ is based on the construction of norm maps for R-equivalence class groups due to Chernousov and Merkurjev [ChMe]. Note that if G is a simply connected semisimple k-group, then for any field extension F of k one has $K_1^G(F) = G(F)/RG(F)$ by [Gi, Théorème 7.2].

Theorem 1. *Let k be an infinite field. Let G be a simply connected semisimple k-group such that every semisimple normal subgroup of G contains $(\mathbb{G}_{m,k})^2$. Assume that G fits into a short exact sequence of k-group homorphisms*

$$1 \to G \to H \xrightarrow{\mu} T \to 1, \tag{1}$$

where H is a k-rational reductive k-group, T is a k-torus, and μ satisfies Merkurjev's norm principle over every field extension F of k. Then $K^G_{1,Nis}$ takes values in abelian groups, and has transfer homomorphisms

$$f_* : K^G_{1,Nis}(B) \to K^G_{1,Nis}(A),$$

for any pair A, B of essentially smooth k-algebras and any finite flat generically étale k-algebra homomorphism $f : A \to B$, satisfying the following properties.

(a) *Assume that $B = B_1 \times B_2$ is a product of two regular k-algebras, and let $f_1 : A \to B_1$ and $f_2 : A \to B_2$ be the natural maps. Then $f_* = (f_{1*}, f_{2*})$. If, moreover, $f_1 : A \xrightarrow{\cong} B_1$ is a k-algebra isomorphism, then $f_{1*} = ((f_1)^{-1})^*$.*

(b) *For any k-algebra homomorphism $g : A \to A'$ such that A' and $B' = B \otimes_A A'$ are essentially smooth over k, and $f' = f_{A'} : A' \to B'$ is generically étale, the following diagram commutes:*

$$\begin{tikzcd} K^G_{1,Nis}(A) \arrow[r, "g^*"] & K^G_{1,Nis}(A') \\ K^G_{1,Nis}(B) \arrow[u, "f_*"] \arrow[r, "g'^*"] & K^G_{1,Nis}(B') \arrow[u, "(f')_*"'] \end{tikzcd}$$

(c) *If $f : E \to F$ is a finite separable extension of fields essentially smooth over k, then*

$$f_* = N_{E/F} : G(E)/RG(E) \to G(F)/RG(F)$$

is the Chernousov–Merkurjev norm homomorphism [ChMe, p. 187].

The theorem applies if G is a simply connected group of classical type $A_l - D_l$, see [ChMe] (for D_l, one has to assume $\operatorname{char} k \neq 2$ because of the norm principle). However, $K^G_{1,Nis}$ is trivial on Sm_k for groups of type B_l and C_l.

Lemma 2. *Let R be a local domain with an infinite residue field and the fraction field K. Let G, H, T and G', H', T' be reductive group schemes over R, such that H and H' are R-rational, and there are two short exact sequences of R-group scheme homomorphisms* (1) *and*

$$1 \to G' \to H' \xrightarrow{\mu'} T' \to 1. \tag{2}$$

Assume also that $\beta : T' \to T$ is a R-group scheme homomorphism such that

$$\beta(\mu'(H'(F))) \subseteq \mu(H(F)) \tag{3}$$

for any field extension F of K.

(i) There exist an open dense R-subscheme $U \subseteq H'$ and an R-morphism $\eta : U \to H$ such that $\mu \circ \eta = \beta \circ \mu'|_U$, $1_{H'} \in U(R)$, and $\eta(1_{H'}) = 1_H$.

(ii) The induced map $\bar{\eta} : U(R) \to H(R)/\big(RG(K) \cap H(R)\big)$ extends uniquely to a homomorphism

$$\tilde{\beta} : H'(R) \to H(R)/\big(RG(K) \cap H(R)\big)$$

such that $\tilde{\beta}\big(RG'(K) \cap H'(R)\big) = 1$. This homomorphism is independent of the choice of a pair (U, η) as in (i). If $R = K$ is a field, then $\tilde{\beta}$ coincides with the respective map in [ChMe, Lemma 3.2].

(iii) Assume that G is simply connected, G and G' have strictly proper parabolic R-subgroups P and P', and $K_1^{G,P}(R) \to K_1^{G,P}(K)$ is injective. Then $\tilde{\beta}$ induces a homomorphism

$$\hat{\beta} : K_1^{G',P'}(R) \to K_1^{G,P}(R).$$

For any ring homomorphism $f : R \to S$, where S is a local domain with the fraction field E such that $K_1^{G,P}(S) \to K_1^{G,P}(E)$ is injective, the map $\hat{\beta}$ is functorial with respect to f.

Proof. The proof of (i) is exactly the same the one for fields [ChMe, Lemma 3.1]. For (ii), use that by [ChMe, Lemma 3.2] the map $\bar{\eta} : U(K) \to H(K)/RG(K)$ extends uniquely to a homomorphism $\tilde{\beta} : H'(K) \to H(K)/RG(K)$ so that for any $g \in H'(K)$ and $g_1, g_2 \in U(K)$ with $g = g_1 g_2$ one has

$$\tilde{\beta}(g) = \bar{\eta}(g_1)\bar{\eta}(g_2). \tag{4}$$

This $\tilde{\beta}$ is independent of (U, η), and $\tilde{\beta}(RG'(K)) = 1$. We restrict this $\tilde{\beta}$ to $H'(R)$. For (iii), note that since $K_1^{G,P}(R)$ injects into $K_1^{G,P}(K)$ and $E_P(K) = RG(K)$, we have $RG(K) \cap G(R) = E_P(R)$. Since $E_{P'}(R) \leq RG'(K)$, the map $\tilde{\beta}$ induces $\hat{\beta}$. Its functoriality follows from (4). □

The following lemma is proved by induction on $\dim A$.

Lemma 3. *Let A be an essentially smooth k-domain. Let $F_1 : \mathrm{Sm}_A \to$ **Groups** be a presheaf, and let $F_2 : \mathrm{Sm}_A \to$ **Groups** be a Nisnevich sheaf. Assume that for any essentially smooth A-domain R with the fraction field E the map $F_2(R) \to F_2(E)$ is injective, and if R is also Henselian local, then there is a homomorphism*

$$\lambda_R : F_1(R) \to F_2(R),$$

functorial with respect to any homomorphism of essentially smooth local Henselian A-domains R. Then there is a unique homomorphism $\lambda_A : F_1(A) \to F_2(A)$ such *that for any prime henselization $i_p^h : A \to A_p^h$ the following diagram commutes:*

$$
\begin{tikzcd}
F_1(A) \arrow[r, dashed, "\lambda_A"] \arrow[d, "F_1(i_p^h)"] & F_2(A) \arrow[d, "F_2(i_p^h)"] \\
F_1(A_p^h) \arrow[r, "\lambda_{A_p^h}"] & F_2(A_p^h).
\end{tikzcd}
\tag{5}
$$

Proof of Theorem 1. By [ChMe, Lemma 1.2] the values of K_1^G on fields are abelian, hence $K_{1,Nis}^G$ on Sm_k takes values in abelian groups by [St, Theorem 1.4]. For any $f : A \to B$ as in the statement of the theorem, set $T' = R_{B/A}(T_B)$, $G' = R_{B/A}(G_B)$, and $H' = R_{B/A}(H_B)$. These groups form the sequence (2), and for any essentially smooth A-algebra R one has $K_{1,Nis}^{G'}(R) = K_{1,Nis}^G(R \otimes_A B)$. For any A-algebra R, one can define a natural "norm" homomorphism

$$N_{B/A} : T'(R) = T(R \otimes_A B) \to T(A),$$

extending the usual norm in the field case, see [Pa, §2]. This homomorphism defines an A-group scheme homomorphism $\beta = N_{B/A} : T' \to T_A$.

Assume that A is a domain with the fraction field K. Lemma 2 and [St, Theorem 1.4] imply that $F_1 = K_{1,Nis}^{G'}$ and $F_2 = K_{1,Nis}^G$ satisfy Lemma 3. Set

$$f_* = \lambda_A : K_{1,Nis}^{G'}(A) = K_{1,Nis}^G(B) \to K_{1,Nis}^G(A).$$

We extend the definition of f_* to products of domains by additivity. If $f : A \to B$ is a finite separable extension of fields, then $\lambda_A = \lambda_K$ is the map $\hat{\beta}$ of Lemma 2 (iii), and hence coincides with the Chernousov–Merkurjev norm map.

The maps f_* are compatible with any admissible base change $A \to A'$. Indeed, by the commutativity of (5) we can assume that A and A' are henselian local, and transfers are of the form $\hat{\beta}$. Since the toral norm homomorphisms are compatible with base change [Pa], we just need to use the last claim of Lemma 2. It remains to check that f_* satisfies (a) in the statement of the theorem. This follows from Lemma 2 and the multiplicativity and normalization properties of norm maps [Pa]. □

The author's visit to MFO was partially supported by "Native towns", a social investment program of PJSC "Gazprom Neft".

References

[ChMe] V. Chernousov, A. S. Merkurjev, *R-equivalence and special unitary groups*, J. Algebra **209** (1998), 175–198.

[Gi] Ph. Gille, *Le problème de Kneser-Tits*, Sém. Bourbaki **983** (2007), 983-01–983-39.

[Pa] I. Panin, *On Grothendieck – Serre's conjecture concerning principal G-bundles over reductive group schemes: II*, to appear in Izv. Ross. Akad. Nauk Ser. Mat.

[St] A. Stavrova, *Homotopy invariance of non-stable K_1-functors*, J. K-Theory **13** (2014), 199–248.

Chern classes for Morava K-theories

Pavel Sechin

1. What is K-orientability?

Let A^* be any generalized oriented cohomology theory (recall Def.1 below). One can show using Th.2 that the ring of all (not neccesarily additive) operations from the Grothendieck group of vector bundles K_0 to A^* is freely generated by Chern classes. Motivated by this fact, we pose the following question.

Let K be a generalized cohomology theory, perhaps, similar to K_0 in some sense. Is there a notion of K-orientable theories? These theories should be presheaves of rings on the category of smooth varieties satisfying some axioms and supplied with some additional structure. In particular, any K-orientable theory should be equipped with some operations $K \to A^*$ which imitate usual Chern classes. These Chern classes should freely generate the ring of all operations to A^*. More generally we might try to reformulate any property of usual oriented theories (which are now K_0-oriented) for this notion.

Recall that I.Panin and A.Smirnov [1] have shown that an orientation of a theory can be specified by different types of data: the structure of pushforwards for proper morphisms, Thom classes of line bundles or Chern classes of line bundles. If the notion K-orientability exists for some theory K, it is reasonable to ask what is a geometric structure analogous to proper pushforwards which controls this notion?

Unfortunately we are nowhere near to answer these quite general questions. The goal of this talk was to provide a little evidence that the notion $K(n)$-orientability could make sense. Here $K(n)$ is the n-th Morava K-theory (Def.4).

The main result is the existence of a series of operations $c_i : K(n) \to CH^i \otimes \mathbb{Z}_{(p)}$ which satisfy the Cartan formula and generate freely all operations to $CH^* \otimes \mathbb{Z}_{(p)}$. We may interprete this as $CH^* \otimes \mathbb{Z}_{(p)}$ are '$K(n)$-orientable'. At the same time we are able to show that there are much more operations to CH^*/p, except from those generated by operations c_i.

2. Set-up: theories of rational type

The main tool in our construction of Chern classes from Morava K-theories to Chow groups is the classification of operations and poly-operations from a theory of rational type to any orientable theory due to A. Vishik [2], [3].

Fix a field k with char $k = 0$.

Def. 1 (Panin-Smirnov, [1]; Levine-Morel, [4]; Vishik, [2]). *A* **generalized oriented cohomology theory** *(g.o.c.t.) is a presheaf of rings on a category of smooth varieties over k $A^* : Sm/k^{op} \to RING$ supplied with the data of pushforward maps for proper morphisms.*

These structures have to satisfy the following axioms: the projection formula, the projective bundle theorem, $\mathbb{A}^1$-homotopy invariance and the excision (sometimes called localization) axiom (EXCI).

The axiom (EXCI) says that for X smooth, $U \subset X$ an open subset and Z the closed complement $X \setminus U$ we have a right exact sequence:

$$A^*(Z) \to A^*(X) \to A^*(U) \to 0,$$

where the left map is the pushforward (carefully defined for singular Z) and the right map is the restriction map.

Orientation of a theory provides it with Chern classes, which we consider as (non-additive) operations $c_i^A : K_0 \to A^*$. One may associate with each g.o.c.t. a formal group law (FGL) over its ring of coefficients by the following formula: $c_1^A(L \otimes L') = F_A(c_1^A(L), c_1^A(L'))$.

Exl. 2. *Algebraic cobordisms of Levive-Morel Ω^* are the universal g.o.c.t. ([4]). Other examples are K_0 and Chow groups CH^*.*

The corresponding FGL's are the universal one over the Lazard ring $\mathbb{L}$, the multiplicative one $F_m(x, y) = x + y + xy$ and the additive one $F_a(x, y) = x + y$ respectively.

Def. 3 (Levine-Morel, [4])**.** *Let R be a ring, let $\mathbb{L} \to R$ be a ring morphism, which classifies an FGL over R.*

Then $\Omega^* \otimes_{\mathbb{L}} R$ is a g.o.c.t. which is called a **free theory**.

Theories of rational type were introduced by A. Vishik in [2] and are those g.o.c.t. which satisfy an additional axiom (CONST) and have a really strong property (but rather technical to state it here precisely): its values can be described by induction on the dimension of a variety.

The axiom (CONST) for a g.o.c.t. A^* says that for any smooth irreducible variety X the canonical map from $A^*(k(X)) := \lim_{U \subset X} A^*(U)$ to $A := A^*(\mathrm{Spec}\ k)$ is an isomorphism. This allows one to split A^* as presheaf of abelian groups in two summands: $A^* = \tilde{A}^* \oplus A$, where A is a constant presheaf and $\tilde{A}^*$ is a presheaf of elements trivial in generic points.

Th. 1 (Vishik, [2])**.** *Theories of rational type are precisely free theories.*

Though we do not have new examples of cohomological theories as theories of rational type, their intrinsic 'inductive' description allows one to study operations between them in a very efficient way.

Th. 2 (Vishik, [2],[3])**.** *Let A^* be a theory of rational type and B^* be a g.o.c.t.*

Then the set of operations $[A^*, B^*]$ is in 1-to-1 correspondence with the following data:

maps of sets $A^*((\mathbb{P}^\infty)^{\times r}) \to B^*((\mathbb{P}^\infty)^{\times r})$ (the restriction of an operation), which are compatible with a list of morphisms between products of $\mathbb{P}^\infty$. This list consists of partial point inclusions, partial diagonals, partial projections, partial Segre maps and the action of the symmetric group.

3. Operations to Chow groups from Morava K-theories

Fix a prime p.

Def. 4. *Let $n \geq 1$. Denote by $K(n)$ a theory of rational type with the ring of coefficients $\mathbb{Z}_{(p)}$ and the logarithm of the FGL $\log_{K(n)}(x) = \sum_{i=0}^{\infty} \frac{x^{p^{in}}}{p^i}$.*

It's called the **n-th Morava K-theory**.

Rem. 3. *The first Morava K-theory is isomoprhic to $K_0 \otimes \mathbb{Z}_{(p)}$ as a presheaf of rings (and the Artin-Hasse exponential provides a change of orientation in terms of the first Chern class).*

The main result is as follows.

Th. 4. *There exist a series of non-additive operations $c_i : K(n) \to CH^i \otimes \mathbb{Z}_{(p)}$ for $i \geq 1$, s.t.*

1. *the following Cartan's formula holds:*
$$c_{tot}(x+y) = F^{K(n)}(c_{tot}(x), c_{tot}(y)),$$
where $c_{tot} = \sum_{i\geq 1} c_i$;
2. *any operation $\tilde{K}(n) \to CH^* \otimes \mathbb{Z}_{(p)}$ can be uniquely written as a series in c_i with $\mathbb{Z}_{(p)}$-coefficients.*

Rem. 5. *In a paper by V.Petrov and N.Semenov [5] operations c_1, c_{p^n} appeared as additive operations to Chow groups where summation was changed.*

We construct operations c_i inductively. From Cartan's formula it follows that $\log_{K(n)}(c_{tot})$ is an additive operation to $CH^* \otimes \mathbb{Q}$. It's not hard to prove using Th.2 that the space of additive operations from $K(n)$ to $CH^i \otimes \mathbb{Z}_{(p)}$ is a free module of rank one (and the same is true for operations to $CH^i \otimes \mathbb{Q}$). Comparing the i-th graded component of $\log_{K(n)}(c_{tot})$ with generators of additive operations allows to describe c_i in terms of c_j with $j < i$ as operations to $CH^i \otimes \mathbb{Q}$. The technical difficulty of the proof is left to show that these operations are in fact integral, which is done by the use of Th.2.

References

[1] I. Panin, A. Smirnov *Push-forwards in oriented cohomology theories of algebraic varieties*, K-Theory Preprint Archives (2000), No. 459.

[2] A. Vishik *Stable and unstable operations in algebraic cobordism*, arXiv preprint arXiv:1209.579 (2012).

[3] A. Vishik *Operations and poly-operations in Algebraic Cobordism*, arXiv preprint arXiv:1409.0741 (2014).

[4] M. Levine, F. Morel *Algebraic cobordism.* Springer Science & Business Media (2007).

[5] V. Petrov, N. Semenov *Morava K-theory of twisted flag varieties*, arXiv preprint arXiv:1406.3141 (2014).

Galois cohomology of simply connected real groups, and generalized Reeder's puzzle

Mikhail Borovoi
(joint work with Zachi Evenor)

Let G be a connected, simply connected, absolutely simple algebraic group over the field of real numbers $\mathbb{R}$. Let $H \subset G$ be a connected, simply connected, semisimple $\mathbb{R}$-subgroup. We consider the homogeneous space $X = G/H$, which is an algebraic variety over $\mathbb{R}$. We ask

Question 1. *How many connected components has $X(\mathbb{R})$?*

This is equivalent to

Question 2. *What is the cardinality of the finite set*

$$\ker[H^1(\mathbb{R}, H) \to H^1(\mathbb{R}, G)],$$

where $H^1(\mathbb{R}, H)$ and $H^1(\mathbb{R}, G)$ denote the corresponding Galois cohomology sets?

We propose a method of answering these questions using our solutions of generalized Reeder's puzzle on Dynkin diagrams.

Let D be a simply-laced graph with vertices numbered by $1, 2, \dots, n$, where "simply-laced" means that the graph has no multiple edges. By a *labeling* of D we mean a vector $\boldsymbol{a} = (a_1, \dots, a_n)$, where $a_i = 0$ or $a_i = 1$. In other words, at each vertex i of D we write a numerical label $a_i = 0, 1$.

For a vertex i and a labeling $\boldsymbol{a}$, the *move* M_i of Reeder's puzzle changes a_i (from 0 to 1 or from 1 to 0), if vertex i has *odd* number of neighbors j in D with $a_j = 1$, and does nothing if i has *even* number of such neighbors. We say that two labelings $\boldsymbol{a}, \boldsymbol{a}'$ are *equivalent* if one can obtain $\boldsymbol{a}'$ from $\boldsymbol{a}$ by a finite sequence of moves. The corresponding equivalence classes of labelings are the orbits of a certain Coxeter group. To solve the puzzle means to describe these orbits.

This puzzle was introduced by Mark Reeder [4] in 2005. In 2011 Chih-wen Weng [5] announced a solution in the case when the graph D is a tree.

When G is as above, and also compact and simply-laced, the set $H^1(\mathbb{R}, G)$ is in a bijection with the set of orbits of Reeder's puzzle for the Dynkin diagram D of G. In order to treat the case when G is not simply-laced or not compact, we generalize Reeder's puzzle. We solve generalized Reeder's puzzle for all simply connected absolutely simple $\mathbb{R}$-groups G, which gives an explicit description of $H^1(\mathbb{R}, G)$, permitting one to answer Questions 1 and 2. Our results have appeared in [2].

Note that the cardinalities $\#H^1(\mathbb{R}, G)$ were recently computed by Jeffrey Adams [1]. Later Borovoi and Timashev [3] proposed a combinatorial method based on the notion of Kac diagram, permitting one to compute easily $\#H^1(\mathbb{R}, H)$ for any semisimple $\mathbb{R}$-group H, not necessarily simply connected. However, it seems that neither of these alternative approaches permits one to answer Questions 1 and 2 about $(G/H)(\mathbb{R})$, except for the case when $H^1(\mathbb{R}, G) = 1$ (which happens only when $G = \mathrm{SL}(n)$ or $G = \mathrm{Sp}(2n)$).

References

[1] J. Adams, *Galois and θ cohomology of real groups,* `arXiv:1310.7917[math.GR]`.

[2] M. Borovoi and Z. Evenor, *Real homogenous spaces, Galois cohomology, and Reeder puzzles,* `arXiv:1406.4362[math.GR]`

[3] M. Borovoi and D. A. Timashev, *Galois cohomology of real semisimple groups,* `arXiv:1506.06252[math.GR]`.

[4] M. Reeder, *Level-two structure of simply-laced Coxeter groups,* J. Algebra **285** (2005), 29–57.

[5] C.-w. Weng, *Reeder's puzzle on a tree,* `jupiter.math.nctu.edu.tw/~weng/powerpoint/DMEC_2011.pdf`.

Reporter: Tom Bachmann

Participants

Prof. Dr. Alexey Ananyevskiy
School of Mathematics
Institute for Advanced Study
1 Einstein Drive
Princeton, NJ 08540
UNITED STATES

Tom Bachmann
Mathematisches Institut
LMU Muenchen
Theresienstrasse 39
80333 München
GERMANY

Prof. Dr. Sanghoon Baek
Department of Mathematical Sciences
KAIST
291 Daehak-ro, Yuseong-gu
Daejeon 305-701
KOREA, REPUBLIC OF

Prof. Dr. Karim Johannes Becher
Departement Wiskunde & Informatica
Universiteit Antwerpen
M.G. 224
Middelheimlaan 1
2020 Antwerpen
BELGIUM

Prof. Dr. Mikhail V. Borovoi
Department of Mathematics
School of Mathematical Sciences
Tel Aviv University
Ramat Aviv, Tel Aviv 69978
ISRAEL

Prof. Dr. Patrick Brosnan
Department of Mathematics
University of Maryland
1301 Mathematics Building
College Park, MD 20743-4015
UNITED STATES

Dr. Baptiste Calmes
Faculté Jean Perrin
Université d'Artois
Rue Jean Souvraz - SP 18
62307 Lens Cedex
FRANCE

Prof. Dr. Vladimir Chernousov
Department of Mathematics & Statistics
University of Alberta
632 Central Academic Bldg.
Edmonton, AB T6G 2G1
CANADA

Dr. Charles De Clercq
LAGA UMR 7539
Sorbonne Paris Cité
Université Paris 13
Case C-3
93430 Villetaneuse Cedex
FRANCE

Dr. Andrei Druzhinin
Chebyshev Laboratory
St. Petersburg State University
14th Line 29B
Vasilyevsky Island
199 178 St. Petersburg
RUSSIAN FEDERATION

Dr. Raphael Fino
Institut de Mathématiques de Jussieu
Paris Rive Gauche (UMR 7586)
Equipe Topologie et Géométrie
Algébrique
Université Pierre et Marie Curie, Paris
VI
4, Place Jussieu
75005 Paris Cedex
FRANCE

Dr. Mathieu Florence
Institut de Mathématiques de Jussieu
Université Paris VII
175, rue du Chevaleret
75013 Paris Cedex
FRANCE

Dr. Grigory Garkusha
Department of Mathematics
University of Wales/Swansea
Singleton Park
Swansea SA2 8PP
UNITED KINGDOM

Prof. Dr. Philippe Gille
Département de Mathématiques
Université Claude Bernard Lyon I
43, Bd. du 11 Novembre 1918
69622 Villeurbanne Cedex
FRANCE

Prof. Dr. Stefan Gille
Department of Mathematics & Statistics
University of Alberta
632 Central Academic Bldg.
Edmonton, AB T6G 2G1
CANADA

Dr. Olivier Haution
Mathematisches Institut
Ludwig-Maximilians-Universität
München
Theresienstrasse 39
80333 München
GERMANY

Dr. Jeremiah Heller
Department of Mathematics
University of Illinois at Urbana
Champaign
1409 West Green Street
Urbana, IL 61801
UNITED STATES

Prof. Dr. Detlev Hoffmann
Fakultät für Mathematik
Technische Universität Dortmund
Vogelpothsweg 87
44227 Dortmund
GERMANY

Dr. Marc Hoyois
Department of Mathematics
Massachusetts Institute of Technology
77 Massachusetts Avenue
Cambridge, MA 02139-4307
UNITED STATES

Dr. Thomas Hudson
Department of Mathematics
KAIST
291 Daehak-ro, Yuseong-gu
Daejeon 305-701
KOREA, REPUBLIC OF

Prof. Dr. Nikita Karpenko
Department of Mathematical and
Statistical Sciences
University of Alberta
Edmonton, Alberta T6G 2G1
CANADA

Dr. Valentina Kiritchenko
Faculty of Mathematics
Higher School of Economics
Vavilova 7
117 312 Moscow
RUSSIAN FEDERATION

Prof. Dr. Amalendu Krishna
Tata Institute of Fundamental Research
School of Mathematics
Homi Bhabha Road, Colaba
400 005 Mumbai
INDIA

Prof. Dr. Boris Kunyavskii
Department of Mathematics
Bar-Ilan University
Ramat Gan 52 900
ISRAEL

Dr. Ahmed Laghribi
Laboratoire de Mathématiques de Lens
Faculté Jean Perrin
Rue Jean Souvraz - SP 18
62307 Lens Cedex
FRANCE

Dr. Ting-Yu Lee
SB IMB
É.P.F.L.
Station 8
1015 Lausanne
SWITZERLAND

Prof. Dr. David B. Leep
Department of Mathematics
University of Kentucky
715 Patterson Office Tower
Lexington, KY 40506-0027
UNITED STATES

Prof. Dr. Marc Levine
Fakultät für Mathematik
Universität Duisburg-Essen
Universitätsstrasse 2
45117 Essen
GERMANY

Prof. Dr. Alexander Luzgarev
Institute of Mathematics & Mechanics
St. Petersburg State University
Petrodvorets
198 904 St. Petersburg
RUSSIAN FEDERATION

Dr. Mark MacDonald
Department of Mathematics & Statistics
University of Lancaster
Fylde College
Bailrigg
Lancaster LA1 4YF
UNITED KINGDOM

Toan Nguyen Manh
Fakultät für Mathematik
Universität Duisburg-Essen
(Mathematikcarrée)
Thea-Leymann-Straße 9
45127 Essen
GERMANY

Prof. Dr. Ivan A. Panin
Laboratory of Algebra at St. Petersburg
Department of V.A. Steklov Institute of Mathematics (P.O.M.I.)
Russian Academy of Sciences
27, Fontanka
191 023 St. Petersburg
RUSSIAN FEDERATION

Prof. Dr. Raman Parimala
Department of Mathematics & Computer Science
Emory University
400 Dowman Drive
Atlanta, GA 30307
UNITED STATES

Prof. Dr. Victor A. Petrov
St. Petersburg Branch of Steklov Mathematical Institute
Fontanka 27
191 011 St. Petersburg
RUSSIAN FEDERATION

Prof. Dr. Anne Queguiner-Mathieu
Département de Mathématiques
LAGA - CNRS - UMR 7539
Université Paris Nord (Paris XIII)
99, Avenue J.-B. Clement
93430 Villetaneuse Cedex
FRANCE

Dr. Marco Schlichting
Mathematics Institute
University of Warwick
Zeeman Building
Coventry CV4 7AL
UNITED KINGDOM

Dr. Stephen Scully
Department of Mathematical and
Statistical Sciences
University of Alberta
Edmonton, Alberta T6G 2G1
CANADA

Pavel Sechin
Faculty of Mathematics
National Research University
Higher School of Economics
7, Vavilova str.
117 312 Moscow
RUSSIAN FEDERATION

Prof. Dr. Nikita Semenov
Mathematisches Institut
Universität München
Theresienstrasse 39
80333 München
GERMANY

Prof. Dr. Evgeny Shinder
Department of Pure Mathematics
University of Sheffield
Hicks Building
Sheffield S3 7RH
UNITED KINGDOM

Dr. Anastasia Stavrova
Department of Mathematics &
Mechanics
St. Petersburg State University
Starys Petergof
Universitetsky Pt., 28
198 504 St. Petersburg
RUSSIAN FEDERATION

Prof. Dr. Jean-Pierre Tignol
ICTEAM Institute
Université Catholique de Louvain
Ave. Georges Lemaitre 4-6 Bte, L 4.05.1
1348 Louvain-la-Neuve
BELGIUM

Dr. Thomas Unger
School of Mathematics and Statistics
University College Dublin
Belfield
Dublin 4
IRELAND

Dr. Alexander Vishik
Department of Mathematics
The University of Nottingham
University Park
Nottingham NG7 2RD
UNITED KINGDOM

Prof. Dr. Nobuaki Yagita
Department of Mathematics
Ibaraki University
1-1 Bunkyo 2-chome, Mito-Shi
Ibaraki 310-8512
JAPAN

Prof. Dr. Serge A. Yagunov
St. Petersburg Branch of Steklov
Mathematical
Institute of Russian Academy of Science
Fontanka 27
191 023 St. Petersburg
RUSSIAN FEDERATION

Dr. Maria Yakerson
Fakultät für Mathematik
Universität Duisburg-Essen
Universitätsstrasse 2
45117 Essen
GERMANY

Dr. Changlong Zhong
Department of Mathematics
State University of New York at Albany
1400 Washington Avenue
Albany, NY 12222
UNITED STATES

Prof. Dr. Kirill Zainoulline
Department of Mathematics & Statistics
University of Ottawa
Ottawa, Ont. K1N 6N5
CANADA

Dr. Maksim Zhykhovich
Mathematisches Institut
Ludwig-Maximilians-Universität
München
Theresienstrasse 39
80333 München
GERMANY