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Mini-Workshop: Applied Koopmanism

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Abstract. Koopman and Perron–Frobenius operators are linear operators that encapsulate dynamics of nonlinear dynamical systems *without loss of information*. This is accomplished by embedding the dynamics into a larger infinite-dimensional space where the focus of study is shifted from trajectory curves to measurement functions evaluated along trajectories and densities of trajectories evolving in time. Operator-theoretic approach to dynamics shares many features with an optimization technique: the Lasserre moment–sums-of-squares (SOS) hierarchies, which was developed for numerically solving non-convex optimization problems with semialgebraic data. This technique embeds the optimization problem into a larger primal semidefinite programming (SDP) problem consisting of measure optimization over the set of globally optimal solutions, where measures are manipulated through their truncated moment sequences. The dual SDP problem uses SOS representations to certify bounds on the global optimum. This workshop highlighted the common threads between the operator-theoretic dynamical systems and moment–SOS hierarchies in optimization and explored the future directions where the synergy of the two techniques could yield results in fluid dynamics, control theory, optimization, and spectral theory.

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Introduction by the Organisers

In a remarkable outline of mathematics and its future at the beginning of XX-th century, Poincaré suggested that complicated dynamics governed by *non-linear partial differential equations* can be reduced to and analyzed with the novel (at that time) *linear infinite dimensional* spectral methods advocated by Hilbert and Fredholm, [23]. Prompted by the advance of spectral theory for unitary and unbounded

self-adjoint operators, Poincaré's vision became reality a good two decades later by landmark contributions of Carleman [7], Koopman [11] and von Neumann [27]. Although originally aimed at ergodic theory, these linear operator reductions of dynamical systems have far reaching implications and a much wider, unexpected area of applicability. The modern high potential of computer simulations and accumulation of big data have imposed a reconsideration and full appreciation of Poincaré's bold prophecy. It suffices to note the recent surge of interest for *Carleman linearization*, *spectral analysis of Koopman's operator* or *Koopman-von Neumann mechanics*. From our narrow perspective, we witness today a proliferation of results utilizing this very operator-theoretic approach in the study of dynamical and control systems [5, 9, 12, 22]. The need of integration of apparently disparate efforts into a comprehensive theory was the principal motivation of the mini-workshop. The main focus was the interplay between ergodic theory, operator theory, geometric dynamical systems and convex optimization methods.

Two classes of linear operators were of particular interest for the workshop: Koopman-type operators [5] and Perron–Frobenius-type operators [8, 12]. Koopman (or composition) operator is a linear infinite-dimensional operator that can be defined for any nonlinear dynamical system. The linear operator retains the full information of the nonlinear state-space dynamics. The formalism based on Koopman operator representation holds promise for extension of dynamical systems methods to systems in high-dimensional spaces as well as hybrid systems, with a mix of smooth and discontinuous dynamics. Recently, Koopman operator properties have been intensely studied, and applications pursued in fields as diverse as fluid mechanics and power grid dynamics. Perron–Frobenius operator is also a linear operator, and, when defined in an appropriate function space, the adjoint to the Koopman operator. Physically, the Perron–Frobenius operator is useful in studying propagation of dynamical systems' densities. It has shown major promise for applications such as Lagrangian properties of fluid flows and control and optimization of dynamical systems. Next we describe the main themes of the workshop.

One of the topics that indicates how merging of techniques from optimization and ergodic theory can be useful is the development of dedicated *convex optimization techniques* for the numerical study of dynamical systems. More specifically, we are interested in tailoring the moment-SOS hierarchies of semidefinite programming (SDP) – originally developed for polynomial optimization – to obtain relevant information on the support of invariant measures for dynamical systems with semialgebraic dynamics and constraints. Invariant measures have been studied extensively in dynamical systems theory [12] and Markov decision processes [10] and it is now recognized that key properties of a dynamical system can be assessed by considering only a few moments of a measure transported along the system flow [1]. The constructive proof of the ergodic partition theorem [4, 19] provides characterization of ergodic sets, which are the smallest invariant sets that ensure measurability of partition. Ergodic sets are the supports of ergodic measures, that can in turn be studied via their moments, or their Fourier coefficients in the

periodic case. Here too, the key idea consists in observing the action of invariant measures on a countable number of observables, or test functions, see e.g. [5] or [18]. Even more recently, invariant measures and weak Kolmogorov–Arnold–Moser (KAM) theory have been used to study geometrical properties of the joint spectral radius (JSR) of a set of linear operators [6, 21].

Hierarchies of finite-dimensional convex optimization problems have been introduced in the early 2000s to solve numerically non-convex optimization problems with semialgebraic data, with convergence guarantees [13]. The overall strategy consists of building a family of semidefinite programming (SDP) problems [2] of increasing size, with primal SDP problems relaxing the original polynomial optimization problem in the space of truncated moments of a measure supported on the globally optimal solutions, and dual SDP problems certifying bounds on the global optimum with specific polynomial sum-of-squares (SOS) representations. In the context of polynomial optimization, this is called the moment-SOS hierarchy [15] or sometimes Lasserre's hierarchy [22], and this relies on fundamental results of convex algebraic geometry, see [24] or [3]. The approach has been extended in [14] to optimal control problems on ordinary differential equations (ODEs), and more recently, to construct families of semialgebraic approximations of the support of measures transported along the flow of controlled ODEs [9].

Another topic of interest was the relationship between *geometric properties of dynamical systems and spectral properties of the associated operators*. In fact, the hallmark of the work on the operator-theoretic approach in the last two decades is the linkage between geometrical properties of dynamical systems - whose study has been advocated and strongly developed by Poincaré and followers - with the geometrical properties of the level sets of Koopman eigenfunctions [16,17,19]. The operator-theoretic approach has been shown capable of detecting objects of key importance in geometric study, such as invariant sets, but doing so globally, as opposed to locally as in the geometric approach. It also provides an opportunity for study of high-dimensional evolution equations in terms of dynamical systems concepts [20,25] via a spectral decomposition, and links with associated numerical methods for such evolution equations [26].

Judging by the contents of the lively discussions during lectures or daily ad-hoc seminars, sometimes extended to the late hours of evening, we believe that the workshop was a success. It has offered a timely and unique opportunity of collaboration and exchange of views among experts in operator theory, convex optimization, dynamical systems, and systems control. The seeds of a new research group, strongly bonded by convergent mathematical interests, were laid on this occasion.

The abstracts below offer an accurate picture of the scientific themes touched during the mini-workshop.

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Mini-Workshop: Applied Koopmanism

Table of Contents

Abstracts

Polynomial optimization for analysis of dynamical systems

Amir Ali Ahmadi

The miniworkshop on applied Koopmanism (MFO 1606b) brought together two mathematical communities who both work on computational analysis of dynamical systems but with very different sets of tools. The first camp studies the Koopman operator as a way of linearizing nonlinear dynamical systems but at the price of going to infinite dimensions. The second camp, roughly speaking, uses algebraic techniques in optimization to analyze dynamical systems through automatic construction of Lyapunov functions. As I belong to the second camp and was speaking in an early tutorial session by D. Henrion and myself, I spent my time giving an overview of sum of squares optimization, its connection to semidefinite programming, and its utility in proving Lyapunov inequalities for control problems. In this report, I give a brief summary of my talk.

At the core of most algebraic methods in optimization and control is the simple idea of optimizing over polynomials that take only *nonnegative* values, either globally or on certain regions of the Euclidean space. A multivariate polynomial $p(x) := p(x_1, \ldots, x_n)$ is said to be (globally) *nonnegative* if $p(x) \geq 0$ for all $x \in \mathbb{R}^n$. As an example, consider the task of deciding whether the following polynomial in 3 variables and degree 4 is nonnegative:

$$\begin{aligned} p(x) \quad = \quad & x_1^4 - 6x_1^3x_2 + 2x_1^3x_3 + 6x_1^2x_3^2 + 9x_1^2x_2^2 \\ & -6x_1^2x_2x_3 - 14x_1x_2x_3^2 + 4x_1x_3^3 \\ & +5x_3^4 - 7x_2^2x_3^2 + 16x_2^4. \end{aligned} \tag{1}$$

This may seem like a daunting task (and indeed it is as testing for nonnegativity is NP-hard), but suppose we could "somehow" come up with a decomposition of the polynomial as a sum of squares:

$$\begin{aligned} p(x) \quad = \quad & (x_1^2 - 3x_1x_2 + x_1x_3 + 2x_3^2)^2 + (x_1x_3 - x_2x_3)^2 \\ & +(4x_2^2 - x_3^2)^2. \end{aligned} \tag{2}$$

Then, we have at our hands an *explicit algebraic certificate* of nonnegativity of $p(x)$, which can be easily checked (simply by multiplying the terms out). A polynomial p is said to be a *sum of squares* (SOS), if it can be written as $p(x) = \sum q_i^2(x)$ for some polynomials q_i. What is appealing here is that the question of existence of an SOS decomposition (i.e., the task of going from (1) to (2)) can be cast as a *semidefinite program* (SDP) and be solved efficiently, e.g., by interior point methods. This is because of following equivalence [1] which is in fact very straightforward to prove: A polynomial $p(x)$ of degree $2d$ is a sum of squares if and only if it can be written as

$$p(x) = z^T(x)Qz(x),$$

where $z(x)$ is the vector of all monomials of degree d in x and Q is a positive semidefinite matrix.

The question of *when* nonnegative polynomials admit a decomposition as a sum of squares is one of the central questions of real algebraic geometry, dating back to the seminal work of Hilbert [2], [3], and an active area of research today. This question is commonly faced when one attempts to prove guarantees for performance of algebraic algorithms in optimization and control.

In short, sum of squares decomposition is a sufficient condition for polynomial nonnegativity. It has become quite popular because of three reasons: (i) the decomposition can be obtained by semidefinite programming, (ii) the proof of nonnegativity is in form of an *explicit certificate* and is easily verifiable, and (iii) there is strong empirical (and in some cases theoretical) evidence showing that in relatively low dimensions and degrees, "most" nonnegative polynomials are sums of squares. Contrary to the low-dimensional situation, it is known that in very high dimensions (exceeding thousands of variables with the current bounds and with the degree fixed to four) most nonnegative polynomials are not sums of squares [4]. Fortunately, many problems of interest in control theory have fairly low dimensions.

Let us now explain why establishing nonnegativity of polynomials is an important problem. We briefly present two fundamental application areas: the polynomial optimization problem, and Lyapunov analysis of control systems.

1.1. **The polynomial optimization problem.** The polynomial optimization problem (POP) is currently a very active area of research in the optimization community. It is the following problem:

$$\begin{array}{ll} \text{minimize} & p(x) \\ \text{subject to} & x \in K := \{x \in \mathbb{R}^n \mid g_i(x) \geq 0, h_i(x) = 0\}, \end{array} \tag{3}$$

where p, g_i, and h_i are multivariate polynomials. The special case of problem (3) where the polynomials p, g_i, h_i all have degree one is of course *linear programming*, which can be solved very efficiently. When the degree is larger than one, POP contains as special case many important problems in operations research; e.g., all problems in the complexity class NP, such as MAXCUT, travelling salesman, computation of Nash equilibria, scheduling problems, etc.

A set defined by a finite number of polynomial inequalities (such as the set K in (3)) is called *basic semialgebraic*. By a straightforward reformulation of problem (3), we observe that if we could optimize over the set of polynomials, *nonnegative on a basic semialgebraic set*, then we could solve the POP problem to global optimality. To see this, note that the optimal value of problem (3) is equal to the optimal value of the following problem:

$$\begin{array}{ll} \text{maximize} & \gamma \\ \text{subject to} & p(x) - \gamma \geq 0, \ \forall x \in K. \end{array} \tag{4}$$

Here, we are trying to find the largest constant γ such that the polynomial $p(x)-\gamma$ is nonnegative on the set K; i.e., the largest lower bound on problem (3). For ease of exposition, we only explained how a sum of squares decomposition provides a

sufficient condition for polynomial nonnegativity globally. But there are straightforward generalizations for giving SOS certificates that ensure nonnegativity of a polynomial on a basic semialgebraic set; see, e.g., [1]. All these generalizations are amenable to semidefinite programming and commonly used to tackle the polynomial optimization problem.

1.2. **Lyapunov analysis of dynamical systems.** Numerous fundamental problems in nonlinear dynamics and control, such as stability, invariance, robustness, collision avoidance, controller synthesis, etc., can be turned by means of "Lyapunov theorems" into problems about finding special functions (the *Lyapunov functions*) that satisfy certain sign conditions. The task of constructing Lyapunov functions has traditionally been one of the most fundamental and challenging tasks in control. In recent years, however, advances in convex programming and in particular in the theory of semidefinite optimization have allowed for the search for Lyapunov functions to become *fully automated*. As a simple example, if the task at hand is to establish global asymptotic stability of the origin for a polynomial differential equation $\dot{x} = f(x)$, with $f : \mathbb{R}^n \to \mathbb{R}^n, f(0) = 0$, then the Lyapunov inequalities that a radially unbounded Lyapunov function V would need to satisfy are [5]:

$$\begin{array}{ll} V(x) > 0 & \forall x \neq 0 \\ \dot{V}(x) = \langle \nabla V(x), f(x) \rangle < 0 & \forall x \neq 0. \end{array} \tag{5}$$

Here, $\dot{V}$ denotes the time derivative of V along the trajectories of $\dot{x} = f(x)$, $\nabla V(x)$ is the gradient vector of V, and $\langle ., . \rangle$ is the standard inner product in $\mathbb{R}^n$. If we parametrize V as an unknown polynomial function, then the Lyapunov inequalities in (5) become polynomial positivity conditions. The standard SOS relaxation for these inequalities would then be:

$$V \quad \text{SOS} \qquad \text{and} \qquad -\dot{V} = -\langle \nabla V, f \rangle \quad \text{SOS}. \tag{6}$$

The search for a polynomial function V satisfying these two SOS constraints is a semidefinite program, which, if feasible, would imply[1] a solution to (5) and hence a proof of global asymptotic stability through Lyapunov's theorem.

In general, by restricting attention to polynomial Lyapunov functions one may introduce some conservatism. For example, the origin of the following simple differential equation

$$\begin{aligned} \dot{x} &= -x + xy \\ \dot{y} &= -y. \end{aligned}$$

is globally asymptotically stable and yet there is no polynomial Lyapunov function (of any degree!) that can prove this fact [7].

[1]Here, we are assuming a strictly feasible solution to the SDP (which unless the SDP has an empty interior will be automatically returned by the interior point solver). See the discussion in [6, p. 41].

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Regularizing singular measures for maximum entropy reconstruction

Marko Budišić

(joint work with Mihai Putinar)

Moment problems are a class of inverse problems whose input data are moments: integrals of a basis set against an unknown measure or a distribution. The goal is to characterize the measure in some way, ranging from determining whether there is a (unique) measure that could generate the input data, to reconstruction or approximation of the measure itself.

In applied dynamical systems moment problems most commonly appear as attempts to reconstruct distributions of points in orbits from average values of functions evaluated along trajectories. Additionally, one can phrase the reconstruction of the spectral measure of the Koopman operator as a moment problem. Distributions that appear in both of these settings are often singularly continuous. As a result, they are poorly approximated by classical reconstruction techniques such as Padé approximations, adapted to point mass measures, or maximum entropy approximations, adapted to absolutely-continuous measures, i.e., those that have densities.

This presentation reported on the joint work with Mihai Putinar, described in two papers [1, 2] and in Putinar's contribution to this volume, all of which contain further references. The work extends the reach of the maximum entropy algorithms from absolutely-continuous measures to a general class of measures. Figure 1 represents a schematic description of the algorithm, which starts at the finite set of input moments τ_μ and follows the full arrows to the approximation of the measure μ. The algorithm hinges on the use of the Riesz–Herglotz formula to represent the generating function for the moment sequence as an integral against an auxiliary measure ϕ, which is always absolutely continuous. The procedure calculates a finite set of moments τ_ϕ from a finite set of τ_μ and therefore regularize a moment sequence of a singular measure into a moment sequence that is suitable

as an input to a maximum entropy reconstruction. Once maximum entropy is used to approximate the density $d\phi$, the regularization of measures supported on one-dimensional spaces can be inverted on the density level using Plemelj–Sokhotski formulas, resulting in a density $d\mu$ serving as the approximation to the original unknown measure μ.

In future work we plan to apply this technique to reconstruction of spectral measures of the Koopman operator that arises in chaotic dynamical systems. Additionally, we plan to explore the possibilities for extending the approach to measures with supports in higher-dimensional spaces.

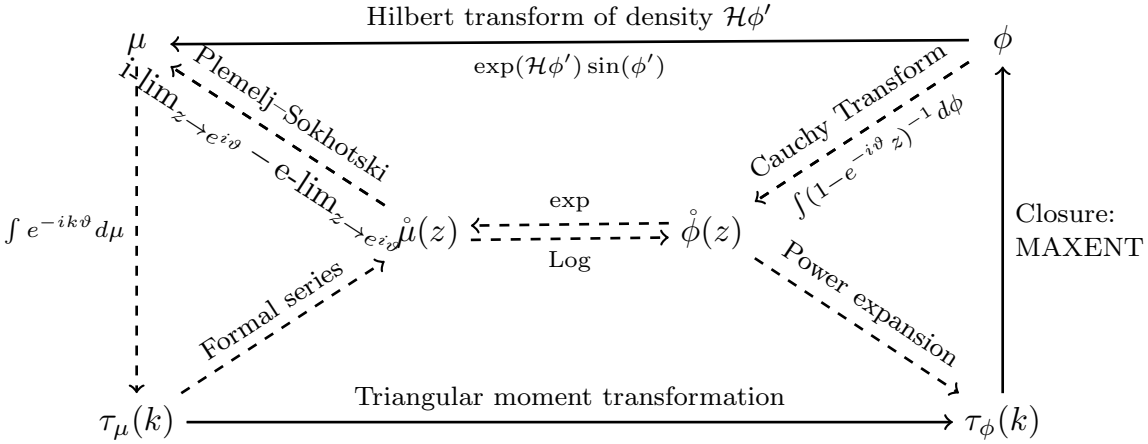

FIGURE 1. Relations between measures μ and ϕ, their moments, and their analytic representations. Full arrows represent steps actually performed in our implementation, while dashed arrows represent analytical justifications that are not numerically evaluated.

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Nonarchimedean linear programming and mean payoff games

STÉPHANE GAUBERT

(joint work with X. Allamigeon, P. Benchimol and M. Joswig)

Tropical linear programs are defined as the images by the valuation of classical linear programs over nonarchimedean ordered fields, like the field of Puiseux series with real coefficients. We survey two recent applications of tropical linear programming:

- bounding the complexity of mean payoff games by the complexity of pivoting methods in linear programming [3, 4];
- bounding the total curvature of the central path [2].

Mean payoff games are a class of deterministic zero-sum two-player perfect information games, in which each player wishes to maximize his average payoff per time unit. The question of the existence of a polynomial-time algorithm to solve these games was raised by Gurvich, Karzanov and Khachyan [8]. These games are among the problems known to be in NP ∩ coNP but not known to be in P.

We next relate this problem to an older question, raised by Dantzig at the time of the invention of the simplex algorithm, in 1947: the existence of a polynomial time pivoting rule.

We show that a positive answer to Dantzig's problem, in a form satisfying a mild restriction, would yield a positive answer to the problem of Gurvich, Karzanov and Khachyan. Given a linear program

$$\min c \cdot x,\ Ax \leq b, x \geq 0$$

a rule is said to be combinatorial if it depends only on the signs of minors of the matrix $\left(\begin{smallmatrix} A & b \\ c & 0 \end{smallmatrix}\right)$.

Theorem 1. *[4] A combinatorial pivoting rule for the simplex algorithm which would run in strongly polynomial time on all non-degenerate linear programs over* $\mathbb{R}$ *would allow one to solve mean payoff games in strongly polynomial time.*

Recall that an algorithm is strongly polynomial if the number of arithmetic operations is bounded polynomially by the number of integers appearing in the instance, and if these arithmetic operations are applied to integers of bitsize polynomially bounded in the size of the input.

This theorem has been extended in [1] and [6] to cover more general semialgebraic pivoting rules.

The proof relies on the equivalence of mean payoff games problems and feasibility problems in tropical linear programming, established in an earlier work with Akian and Guterman [5].

A general unsolved question in linear programming is the existence of a strongly polynomial algorithm (Smale Problem 9). Interior point methods lead only to polynomial time estimates. These methods consists in following an algebraic curve: the central path arising from the logarithmic barrier. In order to analyse the complexity of central path following methods in a way independent of the technicalities of the method, Deza, Terlaky and Zinchenko proposed in [7] a continuous analogue of Hirsch's conjecture, stating that the total curvature of the central path can be bounded linearly in the number of constraints.

We show in [2] that the image by the valuation of the central path over a nonarchimedean field may coincide with the image of a simplex path. This leads to the following counter example to the latter conjecture.

Theorem 2 ([2])**.** *There is a linear program with* $3r+2$ *inequalities in dimension* $2r+2$ *such that the total curvature of the primal-dual central path is* $\Omega(2^r)$.

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The Lasserre hierarchy for polynomial optimization and optimal control

Didier Henrion

The objective of this tutorial talk was to introduce the Lasserre hierarchy for optimizing polynomials and solutions of differential equations with polynomial vector fields and semialgebraic constraints, based on the sketchy lecture notes [1].

The Lasserre hierarchy consists first in reformulating a non-convex problem as a linear optimization problem in a space of probability measures. Then, this infinite-dimensional linear optimization problem is solved approximately with a hierarchy of finite-dimensional semidefinite programming problems (i.e. linear optimization in the cone of non-negative quadratic forms) of increasing size, with convergence guarantees, obtained by exploiting the duality between the cone of positive polynomials and the cone of moments, see [4] for a nice introduction.

A parallel can be drawn with Koopman operator techniques which are also global linearizations. A non-linear differential equation is reformulated as an infinite-dimensional system of linear equations obtained by evaluating observables along trajectories.

The end of tutorial talk focused on applications of the Lasserre hierarchy to approximate the volume of a semialgebraic set [2] and to approximate the region of attraction of a controlled polynomial differential equation [3].

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Semidefinite approximations of the polynomial abscissa

Roxana Hess

(joint work with Didier Henrion, Jean B. Lasserre, Tien Son Pham)

This talk reports on some results of our joint work [3]. Let $\mathcal{Q} \subset \mathbb{R}^n$ be compact and semi-algebraic. The abscissa of a univariate parameterized polynomial

$$p: \ s \mapsto p(q,s) := \sum_{k=0}^{m} p_k(q) s^k \in \mathbb{R}[s], \quad s \in \mathbb{C}, \ q \in \mathcal{Q}, \ p_k \in \mathbb{R}[q]$$

with $p_m \equiv 1$ is the maximal real part of its roots, i.e. the abscissa is the map $a : \mathcal{Q} \to \mathbb{R}, \ q \mapsto a(q) := \max_{k=1,\dots,m} \Re(s_k(q))$, where $s_k(q)$ denote the roots of $p(q,\cdot)$. It occurs, for example, when studying linear differential equations as a measure of the decay or growth rate of the solution. Furthermore, in the space of controller parameters one is interested in the zero sublevel set of the abscissa, called the stability region [2].

The abscissa function is continuous, but in general not locally Lipschitz. This low regularity causes numerical difficulties when designing and optimizing control laws and therefore motivates upper and lower approximations of the abscissa by less complex functions, in our case polynomials of fixed degree.

For the approximation from above we rewrite the set of zeros as the compact semi-algebraic set $\mathcal{Z} := \{(q,x,y) \in \mathbb{R}^n \times \mathbb{R}^2 : q \in \mathcal{Q}, \ p_{\Re}(q,x,y) = p_{\Im}(q,x,y) = 0\}$ with $x = \Re(s), \ y = \Im(s)$ and $p_{\Re}, p_{\Im} \in \mathbb{R}[q,x,y]$ such that $p = p_{\Re} + i p_{\Im}$. Then the abscissa is the optimal solution v of the infinite dimensional linear programming problem

$$\inf_{v \in \mathcal{C}(\mathcal{Q})} \int_{\mathcal{Q}} v(q)\, dq \quad \text{s.t. } v(q) - x \geq 0 \text{ for all } (q,x,y) \in \mathcal{Z},$$

where $\mathcal{C}(\mathcal{Q})$ denotes the space of continuous functions from $\mathcal{Q}$ to $\mathbb{R}$. In order to gain an implementable optimization problem, we build a hierarchy of finite dimensional semidefinite programming problems by replacing the nonnegativity constraint by a specific certificate of positivity as described in [4]. The resulting degree d polynomials v_d are upper approximations of the abscissa function. Furthermore, our

main result states that the sequence $(v_d)_d$ of increasing degree d converges to the abscissa a in L^1 norm on Q.

Constructing a lower bound on the abscissa proved to be much more challenging. We propose two methods: first, a direct one using elementary symmetric functions which is very neat and general, but involves many variables, and second, an approach using the Gauß-Lucas theorem which is computationally faster, but more complicated and subject to assumptions.

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Set oriented numerics for dynamical systems

Oliver Junge

Since the mid-nineties, *set oriented* numerical methods have been developed for the study of the global behavior of dynamical systems [2–4]. These methods can be used to, e.g., approximate different types of invariant sets or invariant manifolds but they also allow to extract statistical information on the dynamical behavior via the computation of invariant measures or almost invariant sets be means of the discretization of the Frobenius-Perron or the Koopman operator.

The methods are based on a multilevel approach which allows to cover the set of interest – e.g. an invariant manifold or the support of an invariant measure – by collections of subsets of state space. Since these methods yield outer approximations and since numerically ill-conditioned long term simulations are avoided, these methods are quite robust. They are similar in spirit to the so-called *cell mapping approach*, see e.g. [9, 11]. However, a significant difference is that in the cell mapping case the numerical effort depends crucially on the dimension of state space whereas for the multilevel schemes the efficiency essentially depends on the complexity of the underlying invariant set.

More recent developments in this area comprise

- the rigorous verification of complicated dynamics in infinite-dimensional systems, [1]: The method combines set-oriented numerical tools for the computation of invariant sets and isolating neighborhoods, the Conley index theory, and analytic considerations. For example, the existence of period points, connecting orbits, and chaotic dynamics in the Kot-Schaffer growth-dispersal model for plants has been shown.

- the reliable quantification of transport rates among different subsets of phase space [5]: Here, the concept of almost invariant sets is combined with invariant manifold and lobe dynamics techniques. The result is a new computational technique for computing key dynamical features, including almost invariant sets, resonance regions as well as transport rates and bottlenecks between regions in dynamical systems. This methodology can be applied to a variety of multibody problems, including those in molecular modeling, chemical reaction rates and dynamical astronomy.
- the construction of value functions for optimal control problems [10]: The method is based on a set oriented approach for the discretization of the problem in combination with graph-theoretic techniques. The central idea is that a discretization of phase space of the given problem leads to an (all source, single destination) shortest path problem on a finite graph.
- the construction of global optimal controllers for perturbed and quantized systems [6–8]: The method is based on a set-oriented discretization of the state space in combination with a new algorithm for the computation of shortest paths in weighted directed hypergraphs. Based on this, techniques for the design of optimal controllers are developed which are based on a piecewise constant approximation of the value function of the underlying optimal control problem. The approach is particularly well suited for problems with highly irregular value function, complicated state constraints and naturally handles hybrid and quantized systems.

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Joint spectral characteristics of matrix semigroups

Raphaël Jungers

Introduction. We introduce the *joint spectral characteristics,* which are numerical quantities that describe the asymptotic behaviour of matrix semigroups. They have found many applications, in particular in Systems and Control. The references mentioned are incomplete, and we apologize for omitting many important ones.

Consider a finite set of m matrices $\mathcal{M} \subset \mathbb{R}^{n\times n}$. (Generalizations exist to complex matrices, infinite sets, etc. our goal here is to keep the discussion straightforward and focus on the ideas.) In many applications (ranging through Engineering, pure Mathematics, Computer Science), one may be interested in properties of products whose factors are from this finite set of matrices. That is, we are interested in the semigroup

$$\mathcal{M}^* = \{A_{i_1} \dots A_{i_t} : \ A_i \in \mathcal{M}\}.$$

Joint spectral characteristics are numerical quantities that give quantitative characterizations of (different aspects of) the asymptotic behaviour of the semigroup $\mathcal{M}^*$. They have emerged quite independently during the second half of the 20th century. These quantities have attracted a lot of attention, not only because of their applications, but probably also because, despite the apparent simplicity of their definition, they turn out to be extremely hard to compute. See for instance [1, 13] for typical complexity results on the topic.

We here give an interpretation of these quantities in terms of *linear discrete time switching systems,* which are systems evolving according to the following law:

$$x(k+1) = A_{\sigma_k} x(k) \quad \sigma_k \in \{1, \dots, m\}. \tag{1}$$

Thus, these systems are not uniquely defined, but any 'switching signal' σ implies a well defined law of evolution for the system. The joint spectral characteristics can in fact be viewed as the rate of growth of the system corresponding to a particular definition of the switching signal.

The joint spectral characteristics. The first one can easily be understood as a robust control quantity: it represents the worst case rate of growth of a switching system:

$$\rho_\infty(\mathcal{M}) = \lim_{t\to\infty} \max_{A\in\mathcal{M}^t} \{||A||^{1/t}\},$$

where $\mathcal{M}^t$ denotes the set of all products of length t. It is commonly referred to as the *Joint Spectral Radius* (JSR in short) of the set $\mathcal{M}$. It has been introduced by Rota and Strang [12]. See [7] for a monograph on the topic.

The second quantity, often called the *p-radius,* finds motivations in functional analysis. See [6, 14] for early work on the topic, and [9] for a more computational

approach. It considers the average norm of *all* the products of length t :

$$\rho_p(\mathcal{M}) = \lim_{t\to\infty} \left[\frac{1}{m^t} \sum_{A\in\mathcal{M}^t} ||A||^p \right]^{1/(pt)}.$$

The next quantity also considers the asymptotic evolution of some average norm among all the products of length t, but, here, the geometric average is taken:

$$\bar{\rho}(\mathcal{M}) = \lim_{t\to\infty} \left[\prod_{A\in\mathcal{M}^t} ||A|| \right]^{1/(tm^t)}.$$

In control, we often call it the *Lyapunov exponent of the system* (1) referring implicitely to a system where (equal) probabilities are appended to each matrix in the set, so that at each time step, one matrix is sampled from the set according to the probabilities. In this context, the Lyapunov exponent provides the rate of growth of the switching system with probability one. See [10,11] for a more formal statement of this result and recent computational approaches.

The joint spectral subradius is the direct counterpart of the JSR, where the *minimal* rate of growth is tracked, instead of the maximal one:

$$\check{\rho}(\mathcal{M}) = \lim_{t\to\infty} \min_{A\in\mathcal{M}^t} \{||A||^{1/t}\}.$$

It represents the minimal open-loop rate of growth that one can hope for System (1). The earliest reference we know of this quantity is [5]. See [4] for a recent computational approach of it.

Finally, the last quantity is also concerned with the smallest possible rate of growth, but now it is assumed that at every step k, one can choose the matrix depending on the present value of $x(k)$. This last joint spectral quantity is thus smaller than the previous one. It has only been introduced formally recently [8], but the reader can find earlier implicit studies of it in [2, 3]. The *stabilizability radius* is defined as follows:

$$\tilde{\rho} = \sup_{x_0\in\mathbb{R}^d} \inf\{\lambda \in \mathbb{R} : \exists\sigma(\cdot),\ M > 0 \text{ s. t. } \forall t \geq 0,\ |x(t)| \leq M\lambda^t |x_0|, x(0) = x_0\}.$$

Further work. Can these quantities be generalized to infinite dimensional linear operators, like the Koopman operator? Can some kind of unique Koopman operator be defined for a given switching system? Normally, the Koopman operator is defined for a single, well defined, function from the state-space to itself, not for a switching system. Yet, we want to try the following definition (for the sake of clarity we restrict here our attention to sets with only two matrices): For a given set of matrices $\mathcal{M} = \{A_0, A_1\}$, consider the *normalized* set $\hat{\mathcal{M}} = \mathcal{M}/\rho(\mathcal{M})$, and suppose that $\hat{\mathcal{M}}^*$ is bounded (this happens generically, and the conditions for this *nondefectiveness* are well understood, see [7]). Then, one can define the following nonlinear operator:

$$f(x) = \begin{cases} A_0 x, & \text{when } \sup_{A\in\hat{\mathcal{M}}^*} ||Ax||_2 = \sup_{A\in\hat{\mathcal{M}}^*} ||AA_0x||_2 \\ A_1 x, & \text{otherwise.} \end{cases} \tag{2}$$

We can then define a Koopman operator corresponding to the function f. Is this the 'right' definition for the Koopman operator of a switching system? Does it bring interesting information on the system?

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Moment-sum-of-squares hierarchies for set approximation and optimal control

Milan Korda

(joint work with D. Henrion, Colin N. Jones)

This talk revolved around the idea of lifting (or embedding) a difficult *nonlinear* problem into a higher dimensional (typically infinite dimensional) space where this problem becomes *linear*. This infinite-dimensional problem is then approximated by a tractable finite-dimensional problem whose solutions provide information about the solutions to the original nonlinear problem.

A particular instance that fits into this framework is the problem of analyzing the nonlinear discrete-time recurrence

$$x_{t+1} = f(x_t). \tag{1}$$

This recurrence is equivalently described by the infinite-dimensional linear Koopman operator

$$v \mapsto \mathcal{K}v := v \circ f$$

for all $v \in \mathcal{F}$, where $\mathcal{F}$ is a suitable space of observables. This infinite dimensional operator is then approximated by a finite-dimensional one (i.e., by a matrix) whose spectral properties approximate the spectral properties of $\mathcal{K}$ which in turn give information on various properties of (1), e.g., on invariant sets, isochrons, ergodic partition, etc.

Lifting for controlled ODEs. In this talk we focused in more detail on a different lift-plus-approximate procedure particularly suited for studying problems involving the continuous-time controlled ODE

$$\dot{x}(t) = \int_U f(x(t), u)\, d\nu_t(u), \tag{2}$$

where $\nu_t \in P(U)$ is the so-called relaxed control, where $P(U)$ is the space of probability measures on a compact control set $U \subset \mathbb{R}^m$. Importantly from an application perspective, by Filippov-Ważewski theorem, the set of trajectories of the non-relaxed ODE $\dot{x}(t) = f(x(t), u(t))$, $u \in L^\infty([0,T]; U)$, is dense in the supremum norm in the set of trajectories of (2).

The nonlinear ODE (2) is lifted into an infinite-dimensional space of nonnegative measures, where it is equivalently described by the so-called Liouville equation

$$\int_{\mathbb{R}^n} v(T,x)\, d\mu_T(x) = \int_{\mathbb{R}^n} v(0,x)\, d\mu_0(x) + \int_{[0,T]\times\mathbb{R}^n\times U} \frac{\partial v}{\partial t} + \nabla v(t,x)\cdot f(x,u)\, d\mu(t,x,u), \tag{3}$$

which is required to hold for all $v \in C^1([0,T]\times\mathbb{R}^b)$. The variables in this equation are the nonnegative measures $(\mu_0, \mu, \mu_T) \in \mathcal{M}(\mathbb{R}^n)\times\mathcal{M}([0,T]\times\mathbb{R}^n\times U)\times\mathcal{M}(\mathbb{R}^n)$.

This equation is a valid lifting only for the finite-time interval $[0,T]$. For the infinite-time interval $[0,\infty)$, a valid lifting reads

$$\beta \int_{\mathbb{R}^n\times U} v(x)\, d\mu(x,u) = \int_{\mathbb{R}^n} v(x)\, d\mu_0(x) + \int_{\mathbb{R}^n\times U} \nabla v \cdot f(x,u)\, d\mu(x,u) \quad \forall v \in C_b^1(\mathbb{R}^n) \tag{4}$$

with variables $(\mu_0, \mu) \in \mathcal{M}(\mathbb{R}^n)\times\mathcal{M}(\mathbb{R}^n\times U)$ and a given discount factor $\beta > 0$.

Equations (3) and (4) are linear equations in the cone of nonnegative measures and the solutions to these equations are in one-to-one correspondence with trajectories of (2).

Approximation. The set of solutions to (3) and (4) over the cone of nonnegative measures is then approximated by a finite-dimensional set of linear equations over either a super-cone or sub-cone of the cone of nonnegative measures. If the set of solutions to the finite-dimensional approximation is larger than that of the original equation, then we obtain a *relaxation* of the original problem. If the set of solutions is smaller, then we obtain a *tightening* of the original problem.

Crucially for practical applications, these finite-dimensional approximations need to tractable in the sense that it is easy to optimize over the set of solutions of this finite-dimensional approximation. One such tractable approximation can be obtained provided that the supports of the measures are constrained to a basic semialgebraic set of the form

$$X = \{x \in \mathbb{R}^n \mid g_i(x) \geq 0,\, i = 1, \ldots, n_g\}$$

with $g_i \in \mathbb{R}[x]$ and provided that the vector field $f(x, u)$ has polynomial entries, i.e., $f \in \mathbb{R}[x, u]^n$. One can then utilize the well-known Lasserre's moment-sum-of-squares hierarchy [5], which provides a sequence of super-cones to $\mathcal{M}(X)$ defined by

$$\mathcal{M}_d^{\sup}(X) = \{\boldsymbol{y} \in \mathbb{R}^{\binom{n+d}{d}} \mid M_d(\boldsymbol{y}) \succeq 0,\; M_d(\boldsymbol{y}, g_i) \succeq 0,$$

where $M_d(\cdot)$ and $M_d(\cdot, \cdot)$ are the so-called moment and localizing matrices and $\succeq$ denotes positive semidefiniteness. The cones $\mathcal{M}_d^{\sup}(X)$ are semidefinite-programming representable for each $d \geq 0$ and form a nested sequence of super-cones of $\mathcal{M}(X)$ indexed by the relaxation order d, i.e.,

$$\mathcal{M}_d^{\sup}(X) \supset \mathcal{M}_{d+1}^{\sup}(X) \supset \mathcal{M}(X).$$

Restricting the set of test functions in either (3) or (4) to the basis spanning the set of all multivariate polynomials up to a degree d, we obtain a set of linear equations and an inclusion to an SDP representable cone $\mathcal{M}_d^{\sup}(X)$ and hence a set over which it is possible to optimize using semidefinite programming.

This lift-plus-approximate scheme can be used to derive a convex characterization of the region of attraction (i.e., the backward reachable set), the forward reachable set (these are the results of [1]) and the maximum controlled invariant set [2] whose finite-dimensional relaxation provide a converging sequence of outer approximations to these sets. These are the results of [1] and [2]. A converging sequence of inner approximations to the region of attraction in the uncontrolled setting was proposed in [4]

If, on the other we try to tighten rather than relax the set of solutions to the Liouville's equation over the cone of nonnegative measures, the situation is not as straightforward. A natural way to do this is to restrict the nonnegative measures to measures having a polynomial density belonging to an SDP representable quadratic module generated by g_i's. This tightening, however, is too restrictive as no measures with polynomial densities solving (3) or (4) typically exist. One than has to resort to a modified lifting which leads to a different form of Liouville's equation for which measures with polynomial densities are guaranteed to exist.

This lifting was developed in [3] and used to design a hierarchy of semidefinite-programming problems providing an asymptotically optimal sequence of rational feedback controllers for infinite-horizon discounted optimal control problems.

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Pure Koopmanism

Kari Küster

Given a dynamical system $\varphi : K \to K$, the idea of switching to a Koopman system consists of going from the state space K to an observable space $\mathcal{F} \subseteq \{f : K \to \mathbb{C}\}$ and from the dynamics φ to the *Koopman operator* $T_\varphi : \mathcal{F} \to \mathcal{F}$, $f \mapsto f \circ \varphi$. In the following, we choose $\mathcal{F} := C(K)$ for K compact and φ continuous. The linear system $(C(K); T_\varphi)$ preserves all information about the underlying dynamical system $(K; \varphi)$ in the following sense (see [1, Th. 4.13 and Cor. 4.15] or [3, Th. 2.3.5]).

Theorem 1. *Two dynamical systems $(K; \varphi)$ and $(L; \psi)$ are isomorphic (i.e., there is a homeomorphism $\theta : K \to L$ such that $\psi \circ \theta = \theta \circ \varphi$) if and only if the corresponding Koopman systems $(C(K); T_\varphi)$ and $(C(L); T_\psi)$ are isomorphic (i.e. there is an C^*-algebra isomorphism $T : C(L) \to C(K)$ such that $T \circ T_\psi = T_\varphi \circ T$).*

This makes it promising to investigate in detail the interplay between a dynamical system and its Koopman system and to find "translations" of properties of the dynamical system to properties of the Koopman system and vice versa.

Before giving examples, we recall the basic properties of the spectrum $\sigma(T_\varphi)$ of the Koopman operator.

Theorem 2. *(i) The spectral radius is $r(T_\varphi) = 1$, hence $\sigma(T_\varphi)$ is contained in the unit disk D.*
(ii) If T_φ be bijective, then $\sigma(T_\varphi)$ is a cyclic closed subset of the unit circle $\mathbb{T}$.
(iii) If T_φ is not bijective, then $\sigma(T_\varphi) = \{0\} \cup M$ for $M \subseteq \mathbb{T}$ cyclic and closed or $\sigma(T_\varphi) = D$.

This reveals that there are not so many possibilities for the spectrum $\sigma(T_\varphi)$ and we therefore turn our attention to the point spectrum $P\sigma(T_\varphi)$.

Theorem 3. *(i)* $1 \in P\sigma(T_\varphi)$.
(ii) If $\lambda \in P\sigma(T_\varphi)$, *then* $\lambda^k \in P\sigma(T_\varphi)$ *for every* $k \in \mathbb{N}_0$.
(iii) $P\sigma(T_\varphi)$ *is cyclic, that is, for every* $\lambda = |\lambda|e^{i\varphi_\lambda} \in P\sigma(T_\varphi)$ *also* $|\lambda|e^{i\varphi_\lambda k} \in P\sigma(T_\varphi)$ *for all* $k \in \mathbb{Z}$.
(iv) If K *is metrizable, then* $P\sigma(T_\varphi) \cap \mathbb{T}$ *is countable.*

We give an example how to obtain information on the Koopman system from the dynamical system (see [3, Th 3.0.2]).

Example 4. Take $X := \left(\mathbb{C}^k; \|\cdot\|\right)$ and a matrix $A \in \mathbb{C}^{k\times k}$ with $\|A\| \leq 1$ for the induced operator norm. For the closed unit ball U in X, consider the restriction $A\big|_U \colon U \to U$, and the associated Koopman operator $T_A : C(U) \to C(U), \quad f \mapsto f \circ A\big|_U$. How is the spectrum of the corresponding Koopman operator determined by the spectrum of the dynamics A?

Theorem 5. *(see [3, Th. 3.1.5 and Th. 3.1.14])*

(i) If $\sigma(A) \subseteq \mathbb{T}$, *then* $P\sigma(T_A)$ *is the group* $\langle\sigma(A)\rangle$ *generated by* $\sigma(A)$.
(ii) If there is $\lambda \in \sigma(A)$ *such that* $0 < |\lambda| < 1$ *and* $\sigma(A) \cap \mathbb{T} \neq \emptyset$, *then* $D^\circ \cup \langle\sigma(A) \cap \mathbb{T}\rangle \subseteq P\sigma(T_A)$.
(iii) If there is $\lambda \in \sigma(A)$ *such that* $0 < |\lambda| < 1$ *and* $\sigma(A) \cap \mathbb{T} = \emptyset$, *then* $P\sigma(T_A) = D^\circ \cup \{1\}$.

Remark 6. Even though the system $(U; A\big|_U)$ is itself linear it makes sense to investigate its Koopman linearization, since $(U; A\big|_U)$ can be isomorphic to non-linear dynamical systems $(K; \varphi)$, e.g., via linearized stability analysis. Since the point spectrum is an isomorphy invariant, we have $P\sigma(T_A) = P\sigma(T_\varphi)$.

Next we draw conclusions from the Koopman system to its underlying dynamics.

Example 7. Consider $C(K)_r := \overline{lin}\{f \in C(K) : T_\varphi f = \lambda f \text{ for some } \lambda \in \mathbb{T}\}$. Since $C(K)_r$ is a commutative C^*-subalgebra of $C(K)$, there is some compact space L such that $C(K)_r \cong C(L)$ by the Gelfand-Naimark theorem. This yields the following partitioning of the state space.

Theorem 8. *(see [3, Th. 4.2.3])) There is a disjoint decomposition of the state space as* $K = \dot{\bigcup}_{s\in L} K_s$, *where* $K_s := \psi^{-1}(s)$ *for a surjective continuous mapping* $\psi : K \to L$. *Moreover, for all* $s \in L$ *exists some* $t_s \in L$ *such that* $\varphi(K_s) \subseteq K_{t_s}$ *and* $K_{t_{s_1}} \neq K_{t_{s_2}}$ *for* $s_1 \neq s_2$.

For a root of unity contained in the spectrum of the Koopman operator, we even obtain a cyclic behavior of the dynamics.

Theorem 9. *(see [3, Th. 3.2.4]) If* $\{1, \zeta_k, \zeta_k^2, ..., \zeta_k^{k-1}\} \subseteq P\sigma(T_\varphi)$ *and* $\dim fix\,(T_\varphi) = 1$, *then* K *is a union of* k *pairwise disjoint, open and closed sets* $K_1, ..., K_k$ *and* $\varphi(K_1) \subseteq K_2$, $\varphi(K_2) \subseteq K_3$, ..., $\varphi(K_k) \subseteq K_1$.

In some cases, the space $C(K)_r$ plays a particular important role.

Definition 10. If $(K;\varphi)$ is a dynamical system with corresponding Koopman system $(\mathcal{F};T_\varphi)$ we say that there exists a *Jacobs-de-Leeuw-Glicksberg splitting* if $\mathcal{F}=\mathcal{F}_s\oplus\mathcal{F}_r$ with *stable part*

$$\mathcal{F}_s := \{f\in\mathcal{F}\,\exists\,(n_i)_{i\in\mathbb{N}}\subseteq\mathbb{N} \text{ with density 1 such that } T_\varphi^{n_i}f \overset{i\to\infty}{\longrightarrow} 0 \text{ weakly}\}$$

and *reversible part* $\mathcal{F}_r := \overline{lin}\{f\in\mathcal{F}:\,\exists\,\lambda\in\mathbb{T} \text{ such that } T_\varphi f=\lambda f\}$.

Theorem 11. *(see e.g. [2, Th. I.1.11 and Th. II.4.1]) If T_φ has relatively weakly compact orbits, then it has a Jacobs-de Leeuw-Glicksberg splitting.*

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Approximation of the Koopman-Operator eigenvalues of Trajectories

Uwe Küster

For a linear or nonlinear operator $\varphi : K \longrightarrow K$ on a compact topological space K, the Koopman-operator or composition operator is given by $T_\varphi : \mathcal{F} \longrightarrow \mathcal{F}$ defined by $T_\varphi(f) = f\circ\varphi$ forall $f\in\mathcal{F}$. The so-called observables $\mathcal{F}$ are a closed linear space of continuous functions on the state space K with the necessary stability property $f\circ\varphi\in\mathcal{F}$ for all $f\in\mathcal{F}$. $\mathcal{F}$ might be the complete space of continuous functions on the state space on the one hand or might be generated by a single element f together with the stability property on the other hand. If the space of observables increases, the Koopman operator on the inclosing space induces the Koopman operator on the enclosed space. The Koopman operator is bounded, has a spectrum and as subset the point spectrum of the eigenvalues and eigenvectors associated to these. The eigenvectors are elements of the observables $\mathcal{F}$. Eigenpairs on an enclosed space are eigenpairs on the inclosing space. This approach enables applying the concept of the spectrum and the spectral decomposition of a linear operator to the analysis of nonlinear operators. Koopman theory is embedded in ergodic theory of functional analysis. For the whole theoretical framework see the book [1] and as introduction for the subject [7]. Reasoning the significance of the theory in physical and technical application fields is done in [2] and with regard to computational fluid dynamics in [6] and [8]. The generality of the approach can be understood as potential for gaining insight in the results of complex numerical

simulations as done by discretized Navier-Stokes equations (by any discretization technique) including turbulent flows or the even more complicated systems for weather forecast. In the case of the discretized partial diffferential equations the state space is a compact part of the space of discretized functions or parameters describing these, for example coefficients in spectral elements. It turns out, that it might be enough to analyse the values of the observables on the states belonging to one or a few trajectories to describe some spectral properties of the operator. This makes numerical analysis feasible. Important is an appropriate problem dependent selection of observables. The numerically spectrum as an approximation of the subset of the spectrum depends on this selection. If the selection allows, then the product of two numerically given eigenvalues is also an eigenvalue, unless the product of the eigenvector observables disappears. The generality of the Koopman approach enables for the analysis of dynamical systems where the state space K has no evident algebraic structure as for agent based systems, which can be used to describe emergence in complex systems based on simple rules. Unlike the state space the observables have an algebraic structure. The Koopman operator gives a linear view on all these nonlinear phenomena by its simple construction and reflects spectral attributes as eigenvalues and eigenvectors in the space of observables on the state space which can be used to decompose the state space in different domains as levelsets of eigenfunctions (see [2]). The operator has not to be known explicitly but its effect of iterating on the trajectories starting at some states seen by observables. We assume a finite set of observables given as a vector $h \in \oplus_{i=1}^{i_{max}} \mathcal{F}$. Starting with the state q_0 we get n iterated states $q_k = \varphi^k q_0$ with the observed vectorial values $g_k = h(q_k)$. Reordering these as a Hankel type matrix $G_{0:n-p,\ 0:p}$ for some $p \leq n$ of lines, which are shifted element by element, we get

(1)
$$\begin{bmatrix} h(q_0) & (T_\varphi\ h)(q_0) & \dots & \left(T_\varphi^p\ h\right)(q_0) \\ h(q_1) & (T_\varphi\ h)(q_1) & \dots & \left(T_\varphi^p\ h\right)(q_1) \\ h(q_2) & (T_\varphi\ h)(q_2) & \dots & \left(T_\varphi^p\ h\right)(q_2) \\ & \vdots & & \\ h(q_{n-p}) & (T_\varphi\ h)(q_{n-p}) & \dots & \left(T_\varphi^p\ h\right)(q_n) \end{bmatrix} = \begin{bmatrix} g_0 & g_1 & \cdots & g_{0+p} \\ g_1 & g_2 & \cdots & g_{1+p} \\ g_2 & g_3 & \cdots & g_{2+p} \\ & \vdots & & \\ g_{n-p} & g_{n-p+1} & \cdots & g_n \end{bmatrix}$$

If there exist a normalized vector c such that $G_{0:n-p,\ 0:p}c \approx 0$ we can extract approximated values $\widetilde{g}_{0:n}$ by applying a $(n+1)\times(n+1)$ projektion Q as $\widetilde{G}_{0:n,\ 0:0} = G_{0:n,\ 0:0}(I-Q)$ from the right, which depends on c. This leads to a decomposition of the approximating values

$$\tilde{g}_k = \sum_{l=1}^{p} v_l\ \lambda_l^k \forall\ k = 0, \cdots, n \tag{2}$$

where the complex values λ_l are the roots of the polynom $\lambda \mapsto c(\lambda)$ of degree p given by the cofficient vector c, assuming here that they are unique. The vectors v_l are the so-called Koopman modes and have the same dimension as h. This decomposition is the result of the so called Dynamic modes Decomposition of [5] for the special case $p = n$. Preferable is a smaller number of modes p. Multiplying the

Hankelmatrix of the approximating values from the left by a polynom coefficient vector w_i of degree $p-1$ having all λ_l as roots except for $l=i$, we select a single element $v_i w_i(\lambda_i)$ meaning that we have a Koopman eigenvector of the approximating sequence composed linearly on the values of h along the trajectory. Any linear combination of the components of the vector v_i is a Koopman eigenvector for the approximating eigenvalue λ_i. The vector c can be the vector minimizing the Rayleigh quotient $\frac{<H(q_0)\ c,\ c>}{<c,c>}$ for the matrix $H(q_0) = G^T_{0:n-p,\ 0:p} G_{0:n-p,\ 0:p}$. Summing up matrices of this type for some different initial states q_0, which are not part of the same trajectory, it will be possible to extent the approxiative decomposition for identical values λ_l on these different trajectories as long as $G_{0:n-p,\ 0:p} c \approx 0$. The error can be described explicitely. By this way ensembles of trajectories can be analysed by a common decomposition.

The composition allows for the extraction of approximative Koopman eigenfunctions on the set of trajectories and may help for a deeper numerical investigation of the properties of the behaviour of the operator on these trajectories. The role in sense of a physical interpretation of the eigenfunctions depends on the observables and might be not self-evident. For eigenvalues λ of modulus 1 the norm of the vector consisting on related Koopman eigenfunctions will remain constant, only the complex phase of this vector will vary with the number of iterations. A special well known case is the mean value along the trajectory, which is surely invariant for physically relevant cases. This is shown by von Neumanns mean ergodic theorem for measure preserving operators under very general conditions (see [1] page 136).

Understanding nonlinear operators as given by a variety of applications in different areas by there Koopman operator counterpart gives the chance of recognizing structures in the state space motivated by a spectral decomposition.

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An invitation to evolution semigroups and linearized Euler operator

Yuri Latushkin

We survey results on the evolution semigroups and Koopman operators, and demonstrate how they can be applied to the study of the essential spectrum of the operator obtained by linearizing the two- or three-dimensional Euler equations of ideal incompressible fluids about a steady state solution. The results on the evolution semigroups are taken from [1], while the applications to the Euler equations follow [2, 5–8].

The evolution semigroup is a semigroups of weighted Koopman operators of the type $(E^t u)(x) = B^t(\varphi^{-t}x)u(\varphi^{-t}x)$ where u is a vector valued function on a set X with values in a Banach space, φ^t is a flow on X, and B^t is a cocycle over the flow with values in the set of bounded linear operators in the Banach space, that is, $B^{t+s}(x) = B^s(\varphi^t x)B^t(x)$, $x \in X$. A typical example of the evolution semigroup is the push forward semigroup $E^t u = D\varphi^t \circ u \circ \varphi^{-t}$, where $D\varphi^t$ is the differential of the flow φ^t on the smooth manifold generated by the vector field u_0; the infinitesimal generator of this semigroup is the Lie bracket $Lu = -(u_0 \cdot \nabla)u + (u \cdot \nabla)u_0$.

Of particular interest is the evolution semigroup associated with the *bicharacteristic amplitude system (BAS)* for the Euler equations linearized about its steady state solution u_0. The linearized equation is of the form $u_t = -(u_0 \cdot \nabla)u - (u \cdot \nabla)u_0 - \nabla p$ where p is the pressure while the BAS is a finite dimensional system obtained by inserting into this equation a fast oscillating solution. Using some abstract isomorphism theorems for operator algebras we prove that the Fredholm spectrum of the propagator of the linearized Euler equation is related to the spectrum of the generator of the evolution semigroup associated with the BAS.

One of the main set of results on the evolution semigroups is the construction of a sequence of norm one functions u_n such that the norm of $(L - \lambda)u_n$ approaches zero as $n \to \infty$; the sequence is called the *approximate eigenfunction* and λ is called the *approximate eigenvalue* of the generator L of the evolution semigroup. This general construction is applied and modified for the case of the linearized two dimensional Euler equations. It is then used to prove that the spectrum of the linearized operator on a Sobolev space is a vertical band whose width is determined by the top Lyapunov exponent of the flow generated by u_0 and the smoothness of the space.

Another set of results relates the spectral radius of the evolution operator E, $(Eu)(x) = B(\varphi^{-1}x)u(\varphi^{-1}x)$, $x \in X$, acting in the space of continuous $\mathbb{R}^d$ vector valued functions on a compact X and induced by a homeomorphism φ and a continuous function B with values in the set of $(d \times d)$ matrices. Fix any ergodic φ-invariant measure $\nu \in \mathrm{Erg}(\varphi)$ on X. By the (Oseledets) Multiplicative Ergodic Theorem, for ν-almost all $x \in X$ there exist Lyapunov exponents $\lambda_\nu^{(1)} \geq \cdots \geq \lambda_\nu^{(d)}$ and a B-invariant decomposition $\mathbb{R}^d = \oplus_{j=1}^d W_\nu^{(j)}(x)$ such that

$$\lambda_\nu^{(j)} = \lim_{n\to\infty} \frac{1}{n} \log \|B^n(x)w\|_{\mathbb{R}^d} \text{ for each } w \in W_\nu^{(j)}(x),\ j = 1, \ldots, d.$$

Here, $B^n(x) = B(\varphi^{n-1}x) \cdot \ldots \cdot B(\varphi x)B(x)$ is the cocycle over φ^n, $n = 1, 2, \ldots$.

The following formula holds for the spectral radius of the operator E, see [1] and references therein:

$$\log \operatorname{sprad}(E) = \lim_{n\to\infty} \max_{x\in X} \|B^n(x)\|_{\mathbb{R}^{d\times d}} = \sup_{\nu\in \operatorname{Erg}(\varphi)} \lambda_\nu^{(1)}. \tag{1}$$

This formula can be applied in computing the joint spectral radius $\rho(A_1, \ldots, A_m)$ of given $(d \times d)$ matrices $A_1, \ldots, A_m$ defined by the formula

$$\rho(A_1, \ldots, A_m) = \lim_{n\to\infty} \max_{1\le j_0,\ldots,j_{n-1}\le m} \|A_{j_0} \cdot \ldots \cdot A_{j_{n-1}}\|_{\mathbb{R}^{d\times d}}.$$

Let (X, φ) be the two sided full topological Markov shift with m symbols, that is, X is the set of all two sided sequences $x = (j_k)_{k\in\mathbb{Z}}$ with the entries $j_k \in \{1, \ldots, m\}$ and $\varphi : (j_k) \mapsto (j_{k+1})$. We define a (constant on the cylinders $C_j = \{x = (j_k)_{k\in\mathbb{Z}} : j_0 = j\}$ and continuous on X) function B with values in the set of $(d \times d)$ matrices by the formula $B(x) = A_{j_0}$ where $x = (\ldots, j_{-1}, j_0, j_1, \ldots)$. Then $B^n(x) = B(\varphi^{n-1}x) \cdot \ldots \cdot B(x)$ depends only on the entries $j_0, \ldots, j_{n-1}$ of the sequence x and $\max_{1\le j_0,\ldots,j_{n-1}\le m} \|A_{j_0} \cdot \ldots \cdot A_{j_{n-1}}\|_{\mathbb{R}^{d\times d}} = \max_{x\in X} \|B^n(x)\|_{\mathbb{R}^{d\times d}}$. Formula (1) now yields the following relation:

$$\rho(A_1, \ldots, A_m) = \sup_{\nu\in \operatorname{Erg}(\varphi)} \lambda_\nu^{(1)}.$$

For interesting connections of the Multiplicative Ergodic Theorem and joint spectral radii one should consult the important papers by I. Morris [3, 4].

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Analysis of dissipative nonlinear systems using the eigenfunctions of the Koopman operator

Alexandre Mauroy
(joint work with Igor Mezić)

In this talk we investigated the interplay between the spectral properties of a semigroup of Koopman operators and the stability properties of the underlying nonlinear system. This result provides a new approach to global stability analysis which mirrors the spectral stability analysis of linear systems.

Koopman eigenvalues and global stability. We consider a nonlinear dynamical system described by a flow $\varphi : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$. The semigroup of Koopman operators $U_\varphi^t : \mathcal{F} \to \mathcal{F}$, $t > 0$, is defined by the composition $U_\varphi^t f(x) = f \circ \varphi(t, x)$ for all observable $f : \mathbb{R}^n \to \mathbb{C}$, $f \in \mathcal{F}$ [2].

A Koopman eigenfunction $\phi_\lambda \in \mathcal{F}$ satisfies $U_\varphi^t \phi_\lambda = e^{\lambda t}\phi_\lambda$ for all $t > 0$, and $\lambda \in \mathbb{C}$ is the corresponding Koopman eigenvalue. If the flow is induced by a vector field $\dot{x} = F(x)$, with $F \in C^1$, then the Koopman eigenfunction $\phi_\lambda \in \mathcal{F} \subseteq C^1$ satisfies the eigenvalue equation $F \cdot \nabla \phi_\lambda = \lambda \phi_\lambda$.

It can be shown that if the flow φ admits an attractor $\Gamma \subset X \subseteq \mathbb{R}^n$, then the eigenfunctions $\phi_\lambda \in \mathcal{F} \subseteq C^0(X)$, with $\Re\{\lambda\} < 0$, satisfy $\phi_\lambda(x) = 0$ for all $x \in \Gamma$. Moreover, we have the following result.

Theorem 1. *Suppose that $X \subset \mathbb{R}^n$ is a forward invariant compact set and that the semigroup of Koopman operators U_φ^t admits an eigenfunction $\phi_\lambda \in \mathcal{F} \subseteq C^0(X)$ with the eigenvalue $\Re\{\lambda\} < 0$. Then the zero level set*

$$M_0 = \{x \in X | \phi_\lambda(x) = 0\}$$

is forward invariant under φ and globally asymptotically stable.

Hyperbolic attractors and numerical methods. Theorem 1 can be used to obtain necessary and sufficient (spectral) conditions for global stability of hyperbolic attractors.

Theorem 2 (Hyperbolic fixed point)**.** *Let $X \subset \mathbb{R}^n$ be a connected, forward invariant, compact set and let φ be the flow induced by $\dot{x} = F(x)$, with $F \in C^2(X)$. Assume that there exists $x^* \in X$ such that $F(x^*) = x^*$ and so that $\partial F/\partial x(x^*)$ has N distinct eigenvalues with strictly negative real part. Then, the fixed point x^* is globally asymptotically stable in X if and only if the semigroup of Koopman operators U_φ^t has n distinct eigenfunctions $\phi_{\lambda_k} \in C^1(X)$, with $\Re\{\lambda_k\} < 0$ and $\nabla \phi_{\lambda_k}(x^*) \neq 0$. Moreover, the eigenvalues λ_k are the eigenvalues of $\partial F/\partial x(x^*)$.*

For a given vector field, the Koopman eigenfunctions ϕ_{λ_k} can be computed on a finite basis of polynomials, as an approximate (finite-dimensional) solution $\tilde{\phi}_{\lambda_k}$ of the eigenvalue equation $F(x) \cdot \nabla \phi_\lambda = \lambda \phi_\lambda$. According to Theorem 2, one can therefore obtain a systematic spectral method for global stability analysis of nonlinear systems. Moreover, the approximate eigenfunctions provide candidate

Lyapunov functions $\mathcal{V}(x) = \left(\sum_{k=1}^{n} |\tilde{\phi}_{\lambda_k}(x)|^p\right)^{1/p}$, $p \geq 1$, that can be used to estimate the basin of attraction of the fixed point.

Similar results and numerical methods can be derived in the case of stable hyperbolic limit cycles. See [3] for more details.

Differential positivity. The last part of the talk briefly focused on differential positivity [1]. This is a joint work with F. Forni and R. Sepulchre. Let $\mathcal{X}$ be a smooth manifold and consider a flow $\varphi : \mathbb{R}^+ \times \mathcal{X} \to \mathcal{X}$ and its differential $\partial\varphi(t,x) : \mathcal{T}_x\mathcal{X} \to \mathcal{T}_{\varphi(x)}\mathcal{X}$, where $\mathcal{T}_x\mathcal{X}$ denotes the tangent space at $x \in \mathcal{X}$. The flow is differentially positive with respect to a cone field $\mathcal{K}(x) \subset \mathcal{T}_x\mathcal{X}$ if $\partial\varphi(t,x)\mathcal{K}(x) \subseteq \mathcal{K}(\varphi(t,x))$ for all $x \in \mathcal{X}$, $t > 0$.

In joint work with A. Sootla, it was shown in [5] that a flow is differentially positive if and only if the associated semigroup of Koopman operators is positive with respect to the cone of functions $\mathcal{H}_\mathcal{K} = \{f \in C^1(\mathcal{X}) \,|\, \partial f(x) \in \mathcal{K}^*(x)\}$, where $\mathcal{K}^*(x)$ is the dual cone field of $\mathcal{K}(x)$. The cone $\mathcal{H}_\mathcal{K}$ can be derived from Koopman eigenfunctions ϕ_{λ_k}, and the cone field $\mathcal{K}(x)$ can be obtained through the differentials $\partial\phi_{\lambda_k}$. It follows that the property of differential positivity is expressed in terms of properties of Koopman eigenfunctions, a result which yields converse results for differential positivity. In particular, stable hyperbolic nodes and limit cycles are always differentially positive in their basin of attraction [4].

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The Koopman Operator Formalism

Igor Mezić

Driven by success in operator-based framework in quantum theory, Bernard Koopman proposed in his 1931 paper [1] to treat classical mechanics in a similar way, using the spectral properties of an operator associated with dynamical system evolution. It was only in the 1990's that potential for wider applications of the operator-theoretic approach has been realized [2, 3]. In this century the trend of applications of this approach has continued, as summarized in [4]. This is partially due to the fact that strong connections have been made between the spectral properties of the Koopman operator for dissipative systems and the geometry of the state space. In fact, the hallmark of the work on the operator-theoretic approach

in the last two decades is the linkage between geometrical properties of dynamical systems - whose study has been advocated and strongly developed by Poincaré and followers - with the geometrical properties of the level sets of Koopman eigenfunctions [3, 5]. The operator-theoretic approach has been shown capable of detecting objects of key importance in geometric study, such as invariant sets, but doing so globally, as opposed to locally as in the geometric approach. It also provides an opportunity for study of high-dimensional evolution equations in terms of dynamical systems concepts [6, 7] via a spectral decomposition, and links with associated numerical methods for such evolution equations [8].

An idea that permeates applied mathematics, theoretical physics and engineering is that of the expansion of a possibly complicated function of space and time into an infinite sum of simpler components, some of which are then retained - using criteria for "goodness" of approximation - for a simplified description of the problem. The most common such examples are the Taylor and Fourier expansions (or decompositions), with the more recent addition of wavelet decompositions that are inherently multi-scale, and the expansion into Proper Orthogonal Modes, called the Proper Orthogonal Decomposition (POD). An alternative concept has emerged in [6], in which the author studied the problem of decomposing evolution of a field from the perspective of operator theory. The idea is to provide a decomposition that is based on the projection onto eigenfunctions of a linear operator - the Koopman, or composition operator - associated with the dynamical evolution of the underlying field. The approach is not based on the concept of closeness of the projection to the full dynamics of the field, but on decomposition into dynamically relevant modes. In the case of time-evolving processes these modes have the property that they represent "collective" properties of evolution, where a spatial shape is multiplied by a time-dependent function of form $\exp(\lambda t)$ for complex λ which is an eigenvalue of the Koopman operator.[1] In fact, the resulting decomposition in the case of a linear system is the eigenvalue decomposition. Thus, by applying it, we achieve continuity in treatment of linear and nonlinear dynamics. The resulting modes - that were named Koopman modes in [7] - are not necessarily orthogonal.

Consider a general dynamical system

$$\dot{\mathbf{z}} = \boldsymbol{F}(\mathbf{z}), \tag{1}$$

defined on a state-space A (i.e. $\mathbf{z} \in A$), where $\mathbf{z}$ is a state (possibly infinite-dimensional) and $\boldsymbol{F}$ is a possibly nonlinear operator. We assume that A and N are compact metric spaces with the associated Borel σ-algebras $\mathcal{B}$ and $\mathcal{C}$ and measures ν and ρ.

Denote by $S^t(\mathbf{z}_0)$ the position at time t of trajectory of (1) that starts at time 0 at point $\mathbf{z}_0$ (provided solutions exist and are unique), for some value of parameter vector $\boldsymbol{p}$.

[1] Time dependence can be more complicated in the case of degenerate eigenvalues.

Denote by $\boldsymbol{g}$ an arbitrary observable from A to observation space $\mathbb{O}$. For a fluid dynamics problem, an example is vorticity at a point, in which case $\mathbb{O} = \mathbb{R}^3$. The value of this observable $\boldsymbol{g}$ that the system trajectory starting from $\mathbf{z}_0$ at time 0 sees at time t is

$$\boldsymbol{g}(t, \mathbf{z}_0) = \boldsymbol{g}(S^t(\mathbf{z}_0)). \tag{2}$$

Note that the space of observables $\boldsymbol{g}$ is a vector space. The family of operators U^t, acting on the space of observables parametrized by time t is defined by

$$U^t \boldsymbol{g}(\mathbf{z}_0) = \boldsymbol{g}(S^t(\mathbf{z}_0)). \tag{3}$$

Thus, for a fixed time τ, U^τ maps the vector-valued observable $\boldsymbol{g}(\mathbf{z}_0)$ to $\boldsymbol{g}(\tau, \mathbf{z}_0)$. We call the family of operators U^t the Koopman family associated with the the continuous-time system (1). Since the Koopman operator is linear, the approach we undertake is spectral. The Koopman family has the following spectral expansion for a large class of dynamical systems with an attractor supporting an ergodic invariant measure (see [6] for the discrete time version):

$$U^t \boldsymbol{g}(\mathbf{z}, \boldsymbol{x}) = \boldsymbol{g}^*(\boldsymbol{x}) + \sum_{j=1}^{k} \exp(\lambda_j t)\phi_j(\mathbf{z})\boldsymbol{s}_j(\boldsymbol{x}) + \int_{-\infty}^{\infty} \exp(i2\pi\alpha t) dE(\alpha)(\boldsymbol{g}(\mathbf{z}, \boldsymbol{x})), \tag{4}$$

where $\boldsymbol{g}^*(\boldsymbol{x})$ is the time-average at a "field" point $\boldsymbol{x}$, $\phi_j(\mathbf{z})$ is the Koopman operator eigenfunction associated with eigenvalue λ_j, $\boldsymbol{s}_j(\boldsymbol{x})$ is the Koopman mode - the projection of the observable $\boldsymbol{g}$ on the eigenspace associated with λ_j, and E is a complex continuous spectral measure on L^2, which in the last part of the expansion represents contribution from continuous part of the spectrum [9]. The expansion (4), discovered in the discrete-time case in [6], is known as the Koopman Mode Decomposition (KMD) [7].

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Dissipative Dynamics, Spaces of Observables for the Associated Koopman Operator, and the GLA Theorem

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(joint work with Igor Mezić)

Let (X, Φ) be a dynamical system with state space X and flow map $\Phi : T \times X \to X$, where T is a group or semigroup representing time, and let $\mathcal{F}$ be a vector space of functions (the space of observables) whose domain is X. The Koopman operator $U_\Phi^t : T \times \mathcal{F} \to \mathcal{F}$ is defined by composition[1] with an observable

$$U_\Phi^t f(\cdot) = f \circ \Phi(t, \cdot), \qquad (t \in T, f \in \mathcal{F}). \tag{1}$$

It is easy to see that this operator is linear which immediately puts the tools of functional analysis at our disposal. Thus we can analyze a *nonlinear* problem with *linear* tools.

Spectral properties of the Koopman operator – and its eigenfunctions in particular – play an important role in the analysis of nonlinear dynamical systems [1]. However, such spectral characteristics are highly dependent on the choice of the space $\mathcal{F}$. Immediately, a few important questions arise. What space of observables $\mathcal{F}$ do we choose? To what extent are the properties of Φ encoded by the operator U_Φ? And, how do we construct eigenfunctions of the operator?

These questions are fully settled in the classical setting of measure preserving automorphisms. If $(X, \mathcal{B}(X), \mu, \Phi)$ is a measure preserving dynamical system, where $\Phi : X \to X$ is an automorphism and μ is a finite Borel measure, the nicest and most natural choice for $\mathcal{F}$ is the Hilbert space $L^2(X, \mu)$. With this choice, $U_\Phi : L^2(\mu) \to L^2(\mu)$ is unitary and by the Spectral Theorem has an integral representation $U_\Phi = \int_{-\pi}^{\pi} e^{it} dE_\Phi(t)$. In this case, the properties of Φ are encoded fully in the spectral properties of U_Φ. Eigenfunctions can be constructed from certain generalized Fourier averages. That the limit of a function

$$f_\omega^* = \lim_{n \to \infty} n^{-1} \sum_{k=0}^{n-1} e^{-i\omega k} U_\Phi^k f, \qquad (f \in L^2(\mu)), \tag{2}$$

is an element of the eigenspace $\ker(e^{i\omega} I - U_\Phi)$ can be easily deduced from the continuity of U_Φ. The existence of this limit in the L^2-norm for all $f \in L^2(\mu)$ is guaranteed by von Neumann's [2] or Yosida's [5] ergodic theorem. Pointwise convergence (μ-a.e. in X) is due to Birkhoff's ergodic theorem ($\omega = 0$) [2] or Wiener-Wintner's ergodic theorem ($\omega \neq 0$) [4].

For dissipative dynamics, the situation is not so clear cut. Consider the simplest example of dissipative dynamics, $x' = \lambda x$, for $x \in \mathbb{R}$ and $0 < \lambda < 1$. This system

[1] For this reason, the Koopman operator is also called a composition operator in some contexts

has a trivial invariant measure, δ_0, supported at the fixed point. Choosing the space of observables as $L^2(\delta_0)$, as we would in the non-dissipative case, gives no information off the attractor since $L^2(\delta_0)$ is isomorphic to $\mathbb{C}$ and we have no eigenfunctions other than the constant ones. At the other end, we can choose the space of observables to be $C(K)$ for some compact K. It turns out that the spectrum is too large; there are simple examples where the point spectrum is all of the unit disc. So what is the appropriate space of observables so that the operator is spectral and we have an analog of (2)?

We give a construction of a natural $\mathcal{F}$ for $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ having a simple[2] hyperbolic attractor $\mathcal{A}$ and where Φ is at least C^2. The space constructed is in essence a generalization of the Hardy space $H^2(\mathbb{D})$ on the open unit disc. The construction relies on four basic facts. (i) If ϕ and ψ are eigenfunctions of the operator and $\phi\psi$ exists in $\mathcal{F}$ and is nonzero, then the product $\phi\psi$ is also an eigenfunction of the operator [1]. (ii) Topological conjugacies preserve point spectra. In particular, if $\Phi : X \to X$ and $\Psi : Y \to Y$ are topologically conjugate via the homeomorphism $h : X \to Y$ and ψ is an eigenfunction of U_Ψ, then $\phi = \psi \circ h$ is an eigenfunction of U_Φ at the same eigenvalue [1]. (iii) The set of eigenfunctions is closed under complex-conjugation. (iv) Under certain technical conditions, for any positively invariant compact set K in the basin of attraction and containing the attractor, there is a topologically conjugacy $h : \mathbb{R}^d \to \mathbb{R}^d$ between Φ and its linearization A around the attractor which reduces to the identity on the attractor [3].

Let K be a compact, Φ-positively-invariant set in the basin of attraction of $\mathcal{A}$ which contains $\mathcal{A}$. Let $h : K \to \mathbb{R}^d$ be the topological conjugacy given by (iv). We build $\mathcal{F}$ in the following steps. (1) Find the *principle* eigenfunctions for the linearized system. The principle eigenfunctions are those associated with the generator A that are bounded on the compact, A-positively-invariant set $L = h(K)$ in the basin of attraction. (2) Generate a unital $*$-algebra from the principle eigenfunctions. The elements of this have an expansion into eigenfunctions due to (i) and (iii) above. (3) Represent the algebra as a set of polynomials over a normed commutative ring and equip it with a polynomial norm. The indeterminants are the coordinates determined by the off-attractor principle eigenfunctions. The ring is the portion of the algebra that is only a function on the attractor. It is a subset of $L^2(\mathcal{A}, \mu)$, where μ is a measure supported on the attractor $\mathcal{A}$ and preserved by A (or equivalently Φ since the two systems are identical on $\mathcal{A}$). An element of the algebra has a polynomial representation as $g(\theta, Y) = \sum_{|n| \leq N} f_n(\theta) Y^n$, where f_n is a bounded $L^2(\mu)$ function on the attractor, θ represents the coordinates on the attractor, and $Y = Y_1 \cdots Y_d$ are the determinants defined by the off-attractor principle eigenfunctions. The polynomial norm is

$$\|g\| = \sum_{n=0}^{N} \|f_n(\cdot)\|^2_{L^2(\mathcal{A},\mu)} \,.$$

[2]Simple in this case means a fixed point or limit cycle in $\mathbb{R}^d$.

As pointed out above, if the attractor is a fixed point then $L^2(\mathcal{A},\mu) = L^2(\{0\},\delta_0) \cong \mathbb{C}$ and the above polynomial norm reduces to the regular polynomial norm. (4) Complete the algebra under this norm, denoted by $\overline{G}$, and pullback to a space of functions defined on K by using the topological conjugacy h. The space of observables $\mathcal{F}$ has the form

$$\mathcal{F} = \overline{G} \circ h := \{g \circ h : g \in \overline{G}\}.$$

The form that $\mathcal{F}$ takes depends on both the algebra and the form that the conjugacy takes.

As a simple example, take $\Phi(x) = Ax + G(x) : \mathbb{R}^d \to \mathbb{R}^d$, where A is a diagonalizable matrix, $G = O(|x|^2)$, and assume there is an asymptotically stable fixed point at the origin. Let (λ_i, v_i) be the eigenpairs of A and $(\overline{\lambda}_i, w_i)$ the eigenpairs of A^* normalized such that $\langle v_i, w_j\rangle = \delta_{ij}$. The principle eigenfunctions of U_A are $\phi_i(x) = \langle x, w_i\rangle$. The indeterminants are $Y_i = \phi_i / \|\phi_i\|_{L,\infty}$. The completion of the generated unital $*$-algebra under the polynomial norm is the space of powers series with square summable coefficients

$$\overline{G} = \left\{ \sum_{n\in\mathbb{N}_0^d} c_n (Y_1^{n_1} \cdots Y_d^{n_d}) : n = (n_1,\ldots,n_d) \in \mathbb{N}_0^d, \sum_{n\in\mathbb{N}_0^d} |c_n|^2 < \infty \right\}$$

To get $\mathcal{F}$ for U_Φ we use the topological conjugacy.

By the construction, the spectrum of $U_\Phi : \mathcal{F} \to \mathcal{F}$ is restricted to isolated circles centered at the origin and additionally U_Φ is a contraction. When this is the case, we have an analog of (2) which constructs eigenfunctions from time averages. Instead of a generalized Fourier average, we now have a generalized Laplace average. For $f \in \mathcal{F}$ and λ in the point spectrum,

$$f_\lambda^* = \lim_{n\to\infty} n^{-1} \sum_{k=0}^{n-1} \lambda^{-k} U^k (I - P_{\Omega(\lambda)}) f \tag{3}$$

is an element of $\ker(\lambda I - U_\Phi)$, where $P_{\Omega(\lambda)} : \mathcal{F} \to \mathcal{F}$ is a projection onto $\operatorname{span}\{\ker(\omega I - U_\Phi) : |\omega| > |\lambda|\}$ which commutes with U_Φ. The convergence is with respect to the norm and the proof is an application of Yosida's mean ergodic theorem [5]. We sketch the argument. Since the spectrum is restricted to isolated circles, the projection $I - P_\Omega$ is bounded and deletes the part of the spectrum of U_Φ with modulus greater than $|\lambda|$. It follows that λ is a peripheral eigenvalue of $U_\Phi(I - P_{\Omega(\lambda)})$, hence $\left\{(\lambda^{-1} U_\Phi (I - P_{\Omega(\lambda)}))^k : k \in \mathbb{N}\right\}$ is a power bounded sequence. The mean ergodic theorem gives that the Cesàro means of the sequence converge in the strong topology to a $\lambda^{-1}U_\Phi(I - P_{\Omega(\lambda)})$-invariant function and, furthermore, the limit satisfies $f_\lambda^* = (I - P_{\Omega(\lambda)}) f_\lambda^*$. As of yet, the question of pointwise convergence remains open.

In practice, the polynomial norm on this space is hard to compute since we must first know the topological conjugacy and then get the polynomial representation of the observable. Are there other spaces that have easier to compute norms and such that (3) holds?

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Moment problems, maximal entropy and exponential transforms

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(joint work with Marko Budišić)

Maximum entropy has emerged as a natural additional assumption in numerous inverse problems, way beyond its original statistical mechanics motivation. The philosophical essay by Jaynes [6] contains convincing arguments for adopting max entropy on a large scale. However, the method has its limitations and pitfalls. We address one of these in the present abstract, and propose a remedy, based on recent joint work with M. Budišić [2].

We start by exploring a simple example in one variable. Assume n is a fixed degree and (a, b) is an interval on the real line, bounded or not. Given the moment data $\gamma_0, ..., \gamma_n$, we seek a positive measure μ carried by the closure of (a, b) satisfying

$$\gamma_k = \int x^k d\mu(x), \quad 0 \leq k < n,$$

and

$$\gamma_n \geq \int x^n d\mu.$$

The max entropy method proposes to search μ of the from $d\mu(x) = \exp(\lambda_0 + \lambda_1 x + ... + \lambda_n x^n)dx$, assuming that the integrability condition

$$\int_a^b \exp(\lambda_0 + \lambda_1 x + ... + \lambda_n x^n)dx < \infty$$

is assured by the choice of the parity and sign of the leading term.

The proper choice of the parameters λ_k is made by imposing the optimality (maximum entropy) condition:

$$\sup \left\{ \lambda_0\gamma_0 + ... + \lambda_n\gamma_n - \int_a^b \exp(\lambda_0 + \lambda_1 x + ... + \lambda_n x^n)dx \right\} \tag{1}$$

where the supremum is taken over all admissible (i.e. integrable exponential) tuples $\lambda = (\lambda_0, ..., \lambda_n)$. Let us similarly denote $\gamma = (\gamma_0, ..., \gamma_n)$ and $\mathbf{x} = (1, x, x^2, ..., x^n)$.

The starting point of our discussion is the observation that the functional

$$L(\lambda) = \lambda \cdot \gamma - \int_a^b \exp[\lambda \cdot \mathbf{x}]dx,$$

is concave. Moreover, the inner critical points of the functional are given by the vanishing gradient conditions:

$$\frac{\partial L}{\partial \lambda_j} = \gamma_j - \int_a^b x^j \exp[\lambda \cdot \mathbf{x}] dx = 0,$$

which solve in a very elegant way the original moment problem.

The difficulty related to the above method lies in the complicated structure of the set of admissible multipliers. For instance, in the unbounded support case $[0, \infty)$ it is possible that the extremal value in problem (1) is attained on the boundary of the set of admissible multipliers. For a detailed analysis of the inherent pathologies see [4, 8].

Even worse, starting with the moment data $\gamma_0 = 1, \gamma_j = 0,\ 1 \leq j < p$, derived from Dirac mass at $x = 0$, we immediately see that no exponential weight density will match them.

One way to avoid such complications is to rely on Markov's original idea of treating moment problems with a bounded weight (the L-moment problems). Specifically, given a finite positive measure μ on the real line one proves with standard techniques of function theory of a complex variable that there exists an integrable function $\xi : \mathbf{R} \longrightarrow [0, 1]$, satisfying the asymptotic equivalence

$$1 + \int_{\mathbf{R}} \frac{d\mu(t)}{t - z} \sim \exp(\int \frac{\xi(s) ds}{s - z}), \quad \Im z > 0,\ z \mapsto \infty,$$

see for details the appendix in the monograph [7].

Denote by $\gamma_j(\nu)$ the j-th power moment of a measure ν. Note that at the level of generating functions of moments we obtain following Markov a formal series relation

$$1 - \sum_{k=0}^{n} \frac{\gamma_k(\mu)}{z^{k+1}} \equiv \exp[-\sum_{j=0}^{n} \frac{\gamma_j(\xi ds)}{z^{j+1}}] \quad mod(\frac{1}{z^{n+2}}).$$

This gives universal polynomial dependence relations

$$\gamma_k(\mu) = P_k(\gamma_0(\xi ds), ..., \gamma_k(\xi ds)),$$

and

$$\gamma_k(\xi ds) = Q_k((\gamma_0(\mu), ..., \gamma_k(\mu)).$$

Our recent works [2] aim at generalizing this classical exponential transform to several variables, with emphasis on solving truncated moment problems of singular measures, where the max entropy method fails. Quite specifically, let $\Gamma \subset \mathbf{R}^d$ be a closed, solid, acute convex cone and μ be a finite positive measure supported by its polar cone Γ^*. We consider the analytic extension of the Fantappiè transform:

$$\Phi(-z, y) = \int_{\Gamma^*} \frac{d\mu(x)}{-z + x \cdot y}, \quad y \in \Gamma,\ \Im z > 0.$$

Note the positivity property

$$\frac{1}{-\overline{z} + x \cdot y} - \frac{1}{-z + x \cdot y} = \frac{\overline{z} - z}{|z - x \cdot y|^2},$$

that is

$$(y \in \Gamma,\ \Im z > 0) \quad \Rightarrow \quad \Im\Phi(-z, y) > 0.$$

By adapting a classical observation of Verblunsky [11] we obtain the following multiplicative representation of the Fantappiè transform.

Theorem. *Let μ be a finite positive measure supported on the cone Γ^*. For every $y \in \Gamma$ there exists a phase function $\xi_y \in L^1([0,\infty), dt)$, $0 \le \xi_y \le 1$, measurably depending on y, such that*

$$1 + \int_\Gamma \frac{d\mu(x)}{x \cdot y - z} = \exp \int_0^\infty \frac{\xi_y(t)dt}{t - z}, \quad \Im z > 0. \tag{2}$$

Moreover, if $\int_{\Gamma^*} |x|^n d\mu(x) < \infty$ for some $n \in \mathbf{N}$, then $\int_0^\infty t^n \xi_y(t)dt < \infty$ for all $y \in \Gamma$.

By combining this observation with the explicit form of the reproducing kernel in the Hardy space of the tube domain over Γ, we obtain a versatile technique of solving the original moment sequence via regularization and the max entropy method. Once the phase function ξ is well approximated, Radon transform inversion, or simply Hilbert transform inversion in one variable, contribute to the recovery of the original measure μ.

The two articles [2] contain numerous illustrations of the method, theoretical in nD, and numerical in 1D.

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Connections between Koopman and Dynamic Mode Decomposition

Clarence W. Rowley
(joint work with Matthew O. Williams and Ioannis G. Kevrekidis)

We present a method for determining finite-dimensional approximations of the Koopman operator, directly from data, without the need for a model of the underlying dynamical system. We show that dynamic mode decomposition, a technique developed in the fluid mechanics community, may be used to find such approximations, and we illustrate the approach with a number of examples.

Dynamic mode decomposition (DMD) is a method for approximating dynamics from data. In the original formulation, introduced in [2,3], one collects snapshots of measurements $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_{M+1}$ (vectors in $\mathbb{R}^N$), taken at equally spaced times, and one assumes they are linearly related, as

$$\boldsymbol{y}_{m+1} = \boldsymbol{A}\boldsymbol{y}_m, \qquad m = 1, \ldots, M,$$

for some (unknown) $N \times N$ matrix $\boldsymbol{A}$. One then uses a Krylov subspace algorithm (considering the subspace spanned by $\{\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_{M+1}\}$) to approximate eigenvalues and eigenvectors of $\boldsymbol{A}$, without explicit knowledge of $\boldsymbol{A}$. These are called DMD eigenvalues and eigenvectors, and they provide information about the dynamics of the underlying system.

Here, we consider a more general definition, given in [4], in which one collects pairs of snapshots $(\boldsymbol{y}_m, \boldsymbol{y}_m^\#)$, for $m = 1, \ldots, M$, such that $\boldsymbol{y}_m^\#$ is the measurement one timestep after $\boldsymbol{y}_m$ (but the $\boldsymbol{y}_m$ need not be at different times). One then seeks a matrix $\boldsymbol{A}$ such that

$$\boldsymbol{y}_m^\# = \boldsymbol{A}\boldsymbol{y}_m, \qquad m = 1, \ldots, M.$$

It is clear that this formulation is a generalization of the original, with $\boldsymbol{y}_m^\# = \boldsymbol{y}_{m+1}$. Stacking these snapshots into matrices

$$\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{y}_1 & \cdots & \boldsymbol{y}_M \end{bmatrix}, \qquad \boldsymbol{Y}^\# = \begin{bmatrix} \boldsymbol{y}_1^\# & \cdots & \boldsymbol{y}_M^\# \end{bmatrix},$$

one defines the DMD modes and eigenvalues as eigenvectors and eigenvalues of the matrix $\boldsymbol{A} = \boldsymbol{Y}^\#\boldsymbol{Y}^+$, where $\boldsymbol{Y}^+$ denotes the pseudo-inverse of $\boldsymbol{Y}$. It is shown in [4] that the resulting eigenvalues are identical to those obtained by the original DMD algorithm, and the DMD eigenvectors are nearly the same (more precisely, the DMD eigenvectors from the original algorithm are projections of the eigenvectors of $\boldsymbol{A}$ onto the subspace spanned by $\{\boldsymbol{y}_m\}$).

The connection with the Koopman operator was first established in [1], and described more explicitly in [4, §4.1]. Consider a discrete-time dynamical system evolving on a set X, according to the map $x \mapsto T(x)$. Here, we will take X to be a measure space and define the Koopman operator $\mathcal{K} : L^2(X) \to L^2(X)$ by $\mathcal{K}g = g \circ T$. Suppose we measure the system using a vector of *observables* $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_N)$, where each $\psi_n \in L^2(X)$: that is, if $x \in X$ is the state, then $\boldsymbol{\psi}(x)$ is a vector of measurements of the state (suppose the ψ_n are continuous functions, so that pointwise evaluation is well defined). Let us sample the system at states $x_1, \ldots, x_M$, and at the images of these points, $T(x_1), \ldots, T(x_M)$, but suppose

that we do not have access to the full states x_m, but only to the measurements $\boldsymbol{y}_m = \boldsymbol{\psi}(x_m)$, $\boldsymbol{y}_m^{\#} = \boldsymbol{\psi}(T(x_m))$. One can then perform DMD on this dataset: one constructs the matrices $\boldsymbol{Y}, \boldsymbol{Y}^{\#}$ as before, and defines $\boldsymbol{A} = \boldsymbol{Y}^{\#}\boldsymbol{Y}^{+}$. We then have the following result:

Theorem 1. *Suppose φ is an eigenfunction of $\mathcal{K}$ with eigenvalue λ, and suppose φ lies in the span of the observables $\{\psi_1, \ldots, \psi_N\}$, so that*

$$\varphi = \bar{w}_1\psi_1 + \cdots + \bar{w}_N\psi_N = \boldsymbol{w}^*\boldsymbol{\psi}$$

for some constants $(w_1, \ldots, w_N) = \boldsymbol{w} \in \mathbb{C}^N$. Suppose further that $\boldsymbol{w}$ lies in the range of the data matrix $\boldsymbol{Y}$. Then λ is an eigenvalue of $\boldsymbol{A} = \boldsymbol{Y}^{\#}\boldsymbol{Y}^{+}$, and $\boldsymbol{w}$ is a left eigenvector of $\boldsymbol{A}$: that is, $\boldsymbol{w}^*\boldsymbol{A} = \lambda\boldsymbol{w}^*$.

Thus, under the conditions of the theorem, Koopman eigenvalues are DMD eigenvalues, and the corresponding Koopman eigenfunction may be found from the left eigenvectors of the DMD matrix $\boldsymbol{A}$.

The connection between DMD and the Koopman operator is explored further in [5], which shows that the DMD matrix may be viewed as an approximation of the Koopman operator, using a spectral collocation method, with basis functions given by the observables ψ_n. These basis functions may be given explicitly, as described above, or even implicitly, using a kernel function, as described in [6].

In the talk, a number of examples are shown, including approximating Koopman eigenfunctions on separate basins of attraction in the Duffing equation, finding almost-invariant sets in the double gyre, and finding low-dimensional descriptions of a one-dimensional PDE, the Fitzhugh-Nagumo equation.

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