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Arbeitsgemeinschaft: The Geometric Langlands Conjecture

Organised by
Laurent Fargues, Paris
Dennis Gaitsgory, Cambridge MA
Peter Scholze, Bonn
Kari Vilonen, Evanston

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Abstract. The Langlands program is a vast, loosely connected, collection of theorems and conjectures. At quite different ends, there is the geometric Langlands program, which deals with perverse sheaves on the stack of G-bundles on a smooth projective curve, and the local Langlands program over p-adic fields, which deals with the representation theory of p-adic groups. Recently, inspired by applications to p-adic Hodge theory, Fargues and Fontaine have associated with any p-adic field an object that behaves like a smooth projective curve. Fargues then suggested that one can interpret the geometric Langlands conjecture on this curve, to give a new approach towards the local Langlands program over p-adic fields.

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Introduction by the Organisers

The Arbeitsgemeinschaft *The Langlands program: From global unramified geometric to local ramified arithmetic*, organised by Laurent Fargues, Dennis Gaitsgory, Peter Scholze and Kari Vilonen, brought together over 50 students and experts working in different aspects of the Langlands program, algebraic geometry, p-adic Hodge theory and related areas, with a diverse geographic and mathematical background.

The goal of the workshop was to understand the statement of Fargues' conjecture, which builds a bridge between the geometric Langlands conjectures, usually stated in the global and unramified setting, with the more classical 'arithmetic' Langlands conjectures, specifically the (ramified) local Langlands conjecture over

p-adic fields. Thus, the program was split (roughly) in half, with some of the lectures giving an overview of the statement and proof (for GL_2) of the geometric Langlands conjecture, and the other half leading up to the formulation of Fargues' conjecture, including various necessary background talks on perfectoid spaces and the Fargues–Fontaine curve. Because the program was quite dense, various extra discussion sessions were scheduled in the afternoon. This led to an extremely intense and fruitful exchange between researchers from these different areas.

Let us give a brief introduction to these questions. The Langlands program emerged as an organizational principle in the theory of automorphic forms. Classically, automorphic forms are (roughly) functions on symmetric domains G/K where G is a real Lie group and $K \subset G$ a maximal compact subgroup, which are required to be invariant under the action of an arithmetic subgroup $\Gamma \subset G$. The prototypical example is the case of $\mathrm{SL}_2(\mathbb{Z})$ acting on the upper half-space, giving rise to modular forms and Maaßforms. On the space of automorphic forms, one has a large space of symmetries, classically given by differential operators, and Hecke operators. This big space of operators on automorphic forms allows one to extract spectral data. One of the Langlands conjectures predicts that this same spectral data is also seen in (apparently unrelated) arithmetic situations. The prototypical example is the relation between rational modular forms of weight 2 and elliptic curves E over $\mathbb{Q}$, which relates Hecke eigenvalues with the number of $\mathbb{F}_p$-rational points of E.

In the modern formulation, one starts with a reductive group G over $\mathbb{Q}$, and one regards $\mathbb{Q}$ as the function field of the "compact curve" $\overline{\mathrm{Spec}\,\mathbb{Z}} = \mathrm{Spec}\,\mathbb{Z} \cup \{\infty\}$. For each place v of this curve, i.e., v is either a prime number p or the archimedean place ∞, one has the completion $\mathbb{Q}_v$ of $\mathbb{Q}$ at v, which are either the p-adic numbers, or the reals $\mathbb{R}$. One can also form the adèles $\mathbb{A}$ of $\mathbb{Q}$, which is the subring of $\prod_v \mathbb{Q}_v$ given by the condition that almost all components are integral.

An automorphic representation of G is (roughly) an irreducible representation of $G(\mathbb{A})$ that occurs in the space of L^2-functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$. Any irreducible representation π of $G(\mathbb{A})$ decomposes as a (restricted) tensor product

$$\pi = \bigotimes_v \pi_v$$

of irreducible representations π_v of $G(\mathbb{Q}_v)$. The rough statement of the local Langlands conjecture says that for each v, the datum of π_v is equivalent to a representation of the absolute Galois group of $\mathbb{Q}_v$, with values in the Langlands dual group.[1] The rough statement of the global Langlands conjecture is that if π is automorphic with a suitable condition on π_∞, then there is a representation of the absolute Galois group of $\mathbb{Q}$, inducing all these representations of the local absolute Galois groups. Moreover, one should be able to go in the converse direction.

A completely parallel conjecture can be formulated for the function field F of a projective smooth curve over a finite field, in place of $\mathbb{Q}$. Several simplifications

[1] At least at $v = \infty$, one has to use the Weil group of $\mathbb{R}$.

occur in this case, the most important being that the space $G(F)\backslash G(\mathbb{A}_F)$ is 0-dimensional, so most analytic aspects of the problem are gone. Notably, many of Langlands' conjectures have been proved in this case by Drinfeld, L. Lafforgue and V. Lafforgue.

The (global, unramified) geometric Langlands program

The geometric Langlands program emerged as a geometric way of looking at Langlands' conjectures in the case of a function field. It is most directly related to the classical picture when looking at the global, everywhere unramified correspondence.

Let C be a smooth projective curve over any field k, and let us continue to denote by F its function field. For any closed point x of C, we write $\mathcal{O}_x$ for the completion of the structure sheaf at x, and F_x for its quotient field. Let $\mathbb{A}_F = (\prod_x \mathcal{O}_x) \otimes F$ be the adèles. If k is a finite field, then everywhere unramified automorphic representations correspond to functions on the double quotient

$$G(F)\backslash G(\mathbb{A}_F)/G(\prod_x \mathcal{O}_x) \ .$$

The basic observation is that if Bun_G denotes the stack of G-bundles on C, then there is a bijection

$$\mathrm{Bun}_G(k) = G(F)\backslash G(\mathbb{A}_F)/G(\prod_x \mathcal{O}_x) \ .$$

If k is a finite field, then functions on $\mathrm{Bun}_G(k)$ can be geometrized by perverse sheaves on Bun_G: Any perverse sheaf gives a function of k-points by looking at traces of Frobenius on the stalks. The analogue of the Hecke action is given by the action of Hecke correspondences on the stack of G-bundles.

Looking at the other side of the correspondence, everywhere unramified Galois representations are precisely local systems on C (with values in the L-group ${}^LG$ of G). Thus, the geometric Langlands conjecture predicts that for every ${}^LG$-local system E on C, there is a perverse sheaf $\mathrm{Aut}_{\mathrm{E}}$ on Bun_G which satisfies a suitable Hecke equivariance property. For $G = \mathrm{GL}_n$, it has been proved by Frenkel, Gaitsgory and Vilonen, following earlier work of Drinfeld, and Laumon.

If k is a finite field, this conjecture implies the global unramified classical Langlands conjecture by passing to the corresponding function on $\mathrm{Bun}_G(k)$.

However, when trying to generalize to ramified representations, it is very difficult to see the arithmetic of supercuspidal representations of $G(\mathbb{F}_p((t)))$, and its relation with irreducible Galois representations of the absolute Galois group of $\mathbb{F}_p((t))$ in this picture. The basic reason is that the geometric picture is automatically compatible with extensions of the base field $k = \mathbb{F}_p$, whereas these arithmetic phenomena are not.

Fargues' conjecture

At his MSRI lecture in December 2014, Fargues stated a most striking conjecture. In recent work with Fontaine, for any non-archimedean local field K (i.e., K is a finite extension of $\mathbb{F}_p((t))$ or $\mathbb{Q}_p$), he had constructed a certain scheme X_K

over K, which behaves like a smooth projective curve over an algebraically closed field, but is not of finite type. This construction was motivated by considerations in p-adic Hodge theory.

Fargues' observation was that if one interprets the global unramified geometric Langlands conjecture on this curve, one ends up with a statement that encodes most conjectural properties of the local ramified arithmetic Langlands conjecture over K. One critical difference is that the automorphism group of the trivial G-torsor is not the algebraic group G, but the locally profinite group $G(K)$, so (perverse) sheaves on the stack of G-bundles naturally give rise to representations of $G(K)$. One can hope that this makes it possible to adapt methods from the geometric Langlands program to make progress on the local Langlands conjectures.

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Adic Spaces

Torsten Wedhorn

The theory of adic spaces has been developed by Roland Huber ([Hu1], [Hu2], [Hu3], [Hu4]). Important examples of adic spaces are perfectoid spaces, which will also play a central role in this Arbeitsgemeinschaft. The talk was a very short introduction to the theory of adic spaces. Further references are [Co], [Sch1], [SW], [Wd], [CW].

Huber rings and Tate rings.

We will denote by A a *Huber ring* (called *f-adic ring* by Huber). By this we mean that there exists an open subring $A_0 \subseteq A$ (called *ring of definition*) such that the induced topology is the I-adic topology for some finitely generated ideal I of A_0 (called *ideal of definition*). An element $a \in A$ is called *power bounded* (resp. *topological nilpotent*) if $\{ a^n \ ; \ n \geq 1 \}$ is bounded[1] (resp. if $\lim_{n\to\infty} a^n = 0$). Set

$$A^\circ := \{ a \in A \ ; \ a \text{ power bounded}\},$$
$$A^{\circ\circ} := \{ a \in A \ ; \ a \text{ is topologically nilpotent}\}.$$

Then A° is an open subring in A and $A^{\circ\circ}$ is a radical ideal of A°.

A Huber ring A is called *Tate ring* if there exists a topological nipotent unit π in A, which is then called *pseudo uniformizer.* In this case there exists a ring of definition $A_0 \subseteq A$ with $\pi \in A_0$. Moreover, πA_0 is always an ideal of definition and $A = A_0[\frac{1}{\pi}]$. Every open ideal of A contains a pseudo uniformizer and hence is equal to A.

For every continuous ring homomorphism of Huber rings $\varphi \colon A \to B$, $\varphi(\pi)$ is again a topologically nilpotent unit. Hence B is again a Tate ring and there exists a ring of definition $B_0 \subseteq B$ with $\varphi(\pi) \in B_0$, and $\varphi(\pi)B_0$ is an ideal of definition.

Example. Let k be a non-archimedean field (i.e., a complete topological field whose topology is given by a non-trivial absolute value $|\cdot| \colon A \to \mathbb{R}_{\geq 0}$). Then k is a Tate ring, $k^\circ = \mathcal{O}_k$ is the ring of intgers, $k^{\circ\circ}$ its maximal ideal, and any $0 \neq \pi \in k$ with $|\pi| < 1$ is a pseudo uniformizer.

More generally,

$$A := k\langle X_1, \dots, X_n\rangle := \{ \sum a_{\underline{i}} X^{\underline{i}} \in k[\![X_1, \dots, X_n]\!] \ ; \ \lim a_{\underline{i}} = 0 \text{ for } \underline{i} \to \infty\}$$

is a Tate ring, where we can take as ring of definition $A^\circ = \mathcal{O}_k\langle X_1, \dots, X_n\rangle$. The image of π in A is a pseudo uniformizer.

[1] Recall that in a topological ring R a subset $S \subseteq R$ is called *bounded* if for every neighborhood U of 0 in R there exists a neighborhood V of 0 with $\{vs \, ; \, v \in V, s \in S\} \subseteq U$.

The adic spectrum.

A *Huber pair* (A, A^+) (called *affinoid ring* by Huber) consists of a Huber ring A and an open, integrally closed subring $A^+ \subseteq A^\circ$ which is called *ring of integral elements.* Then the adic spectrum of (A, A^+)

$$X := \mathrm{Spa}(A, A^+)$$

is the set of equivalence classes of multiplicatively written continuous valuations $|\cdot|\colon A \to \Gamma \cup \{0\}$, Γ a totally ordered abelian group, such that $|f| \leq 1$ for all $f \in A^+$. Here we call $|\cdot|$ continuous if $\{\, a \in A \ ;\ |a| < \gamma \,\}$ is open in A for all $\gamma \in \Gamma$. For $x \in X$ and $f \in A$ we write $|f(x)|$ instead of $x(f)$.

We endow X with a topology as follows. Let $T \subset A$ be a non-empty finite set such that the ideal $T \cdot A$ generated by T is open in A and let $s \in A$. The corresponding *rational subset* is defined as

$$X(\frac{T}{s}) := \{\, x \in X \ ;\ \forall\, t \in T\colon |t(x)| \leq |s(x)| \neq 0 \,\}.$$

Theorem. *There exists a unique topology on* $X = \mathrm{Spa}(A, A^+)$ *such that the subsets of the form* $X(\frac{T}{s})$ *are a basis of this topology consisting of quasi-compact open subsets. This basis is stable under finite intersections. The topological space* X *is spectral*[2].

We obtain a contravariant functor from the category of Huber pairs[3] to the category of spectral spaces.

Proposition. *Let* (A, A^+) *be a Huber pair. Then* $\mathrm{Spa}(\hat{A}, \hat{A}^+) \to \mathrm{Spa}(A, A^+)$ *is a homeomorphism preserving rational subsets.*

The following result shows that “one has sufficiently many points”.

Proposition. *Suppose that* A *is complete,* $X = \mathrm{Spa}(A, A^+)$*,* $f \in A$*.*
(1) $X = \emptyset \Leftrightarrow A = 0$.
(2) $|f(x)| \neq 0$ *for all* $x \in X$ *if and only if* $f \in A^\times$.
(3) $|f(x)| \leq 1$ *for all* $x \in X$ *if and only if* $f \in A^+$.

The structure presheaf on an adic spectrum

We start by defining localization. Let $A = (A, A^+)$ be a Huber pair and let T and s as above. Then there exists a homomorphism $A \to A\langle\frac{T}{s}\rangle$ of Huber pairs that is universal for homomorphisms $\varphi\colon A \to B$ of Huber pairs, where B is complete, $\varphi(s) \in B^\times$, and $\varphi(t)\varphi(s)^{-1} \in B^+$ for all $t \in T$. Then $A\langle\frac{T}{s}\rangle$ is complete.

Lemma. $\mathrm{Spa}\, A\langle\frac{T}{s}\rangle \to \mathrm{Spa}\, A$ *is an open embedding with image* $X(\frac{T}{s})$*, preserving rational subsets.*

[2] A topological space X is called *spectral* if it is homeomorphic to $\mathrm{Spec}\, R$ for some commutative ring R.

[3] A *homomorphism of Huber pairs* $(A, A^+) \to (B, B^+)$ is a continuous ring homomorphism $\varphi\colon A \to B$ such that $\varphi(A^+) \subseteq B^+$.

Now we define presheaves on the basis of rational subsets by

$$\mathscr{O}_X(X(\frac{T}{s})) := A\langle\frac{T}{s}\rangle,$$
$$\mathscr{O}_X^+(X(\frac{T}{s})) := A\langle\frac{T}{s}\rangle^+.$$

One checks that this is a well defined presheaf. In general $\mathscr{O}_X$ is not a sheaf (see [BV] for several instructive examples). We call the Huber pair A *sheafy* if $\mathscr{O}_X$ is a sheaf (in this case $\mathscr{O}_X^+$ is also a sheaf).

Theorem (Sheafiness)**.** *A complete Huber pair (A, A^+) is sheafy in the following cases.*

(I) *A has the discrete topology.*
(II) *A has a noetherian ring of definition.*
(III) *A is a Tate ring and $A\langle X_1, \ldots, X_n\rangle$ is noetherian for all $n \geq 0$.*
(IV) *A is a Tate ring and for every rational subset $U \subseteq \mathrm{Spa}(A, A^+)$ the ring $\mathscr{O}_X(U)^\circ$ is bounded in $\mathscr{O}_X(U)$.*

Adic Spaces

For $x \in X$, $x\colon A \to \Gamma \cup \{0\}$ induces a valuation v_x on $\mathscr{O}_{X,x}$[4] such that $v_x^{-1}(0)$ is the unique maximal ideal of $\mathscr{O}_{X,x}$. Hence we obtain from a sheafy Huber pair a tuple $(X, \mathscr{O}_X, (v_x)_{x\in X})$ consisting of a topological space X, a sheaf $\mathscr{O}_X$ of complete topological rings on X such that $\mathscr{O}_{X,x}$ is local for all $x \in X$, and a family $(v_x)_x$ of valuations v_x on $\kappa(x)$. Such triples form a category called $\mathcal{V}$.

Proposition. *The contravariant functor* Spa *from the category of complete Huber pairs to the category $\mathcal{V}$ is fully faithful.*

Definition. An *adic space* is an object of $\mathcal{V}$ that is locally isomorphic to $\mathrm{Spa}\, A$ for some sheafy Huber pair A. It is called *analytic* if it is covered by adic spectra of the form $\mathrm{Spa}(A)$, where A is a Tate ring.

Examples

(1) Criterion (I) of the sheafiness theorem allows to construct a fully faithful embedding from the category of all schemes to the category of adic spaces which is locally given by $\mathrm{Spec}(A) \mapsto \mathrm{Spa}(A, A)$, where we endow A with the discrete topology.
(2) Criterion (II) allows to construct a fully faithful embedding ι_2 from the category of locally noetherian formal schemes to the category of adic spaces which is locally given by $\mathrm{Spf}(A) \mapsto \mathrm{Spa}(A, A)$.
(3) Let k be a non-archimedean field. Criterion (III) allows to construct a fully faithful embedding ι_3 from the category of rigid analytic spaces over k to the category of adic spaces which is locally given by $\mathrm{Sp}(A) \mapsto \mathrm{Spa}(A, A^\circ)$.
(4) Criterion (IV) shows that the structure presheaf attached to a perfectoid Huber pair is a sheaf.

[4]Here we form the stalk in the category of rings. It is not a topological ring.

Now let $(k, |\cdot|)$ be a non-archimedean field, $k^\circ = \mathcal{O}_k$ and $\pi \in k^{\circ\circ}$ a pseudo uniformizer. Then $S := \mathrm{Spa}(k^\circ, k^\circ)$ consists of two point: the class η of $|\cdot|$ and the class s of the trivial valuation that sends units in $\mathcal{O}_k$ to 1 and elements in the maximal ideal of $\mathcal{O}_k$ to zero. Then

$$S^0 := \mathrm{Spa}(k, k^\circ) = S(\frac{\{\pi\}}{\pi}) = \{\eta\}$$

is the open subspace consisting of the point η.

Now suppose that $\mathcal{O}_k$ is noetherian (hence a discrete valuation ring). Then Raynaud (in a special case) and Berthelot (in general) have constructed a generic fiber functor $\mathscr{X} \to \mathscr{X}^{\mathrm{rig}}$ from the category $\mathcal{F}_{\mathcal{O}_k}$ of formal schemes that are locally of formally finite type over $\mathcal{O}_k$[5] to the category $\mathcal{R}_k$ of rigid analytic spaces over k. Here $\mathrm{Spf}(\mathcal{O}_k[\![T]\!])$ is sent to the open unit disc over k and $\mathrm{Spf}(\mathcal{O}_k\langle X\rangle)$ is sent to the closed unit disc over k.

The following diagram of functors is 2-commutative

$$\begin{tikzcd}
\mathcal{F}_{\mathcal{O}_k} \arrow[r, hook, "\iota_2"] \arrow[d, "(\ )^{\mathrm{rig}}"'] & (\text{Adic Spaces}/S) \arrow[d, "X \mapsto X \times_S S^0"] \\
\mathcal{R}_k \arrow[r, hook, "\iota_3"] & (\text{Adic Spaces}/S^0).
\end{tikzcd}$$

In other words, in the world of adic spaces $(\ )^{\mathrm{rig}}$ corresponds to passing to the naive generic fibre.

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[5] A formal schemes is said to be *locally of formally finite type over* $\mathcal{O}_k$ if it is locally of the form $\mathrm{Spf}(A)$ for an adic ring A that is a topological quotient of $\mathcal{O}_k[\![T_1, \dots, T_m]\!]\langle X_1, \dots, X_n\rangle$.

Geometric class field theory

Bhargav Bhatt

The primary goal of this talk was to recall a geometric formulation of unramified class field theory in the function field case due to Deligne. We also discussed an extension of this story to the ramified case, and explained how it naturally led to local geometric class field theory (following D. Gaitsgory, S. Raskin, J. Campbell).

1. Unramified class field theory

Let C be a smooth projective geometrically connected curve over $k = \mathbb{F}_q$ with function field K, and let $|C|$ denote the set of closed points. Write $\mathcal{O} = \prod_{c \in |C|} \widehat{\mathcal{O}_{C,c}}$ for the product of all the complete local rings of C and $\mathbb{A} := \prod'_{c \in |C|} \widehat{K_c}$ for the adèle ring of C. The goal of unramified class field theory is to describe the structure of unramified abelian extensions of K; equivalently, we must understand the unramified abelianized absolute Galois group $G_K^{\mathrm{unr,ab}}$ of K. The usual formulation of unramified class field theory identifies this group in terms of $\mathbb{A}$:

Theorem 1. *There is an isomorphism of profinite groups*

$$\Big(\mathbb{G}_m(K)\backslash\mathbb{G}_m(\mathbb{A})/\mathbb{G}_m(\mathcal{O})\Big)^{\widehat{}} \simeq G_K^{\mathrm{unr,ab}},$$

where $(\ )^{\widehat{}}$ *denotes profinite completion, determined by* $(a_c) \mapsto \prod_{c \in |C|} \mathrm{Frob}_c^{\mathrm{ord}_c(a_c)}$.

To reformulate this result geometrically, we adopt the following notation:

Notation 2. Fix a coefficient ring Λ; the choice $\Lambda = \mathbb{Z}_\ell$, for varying primes ℓ, suffices for our purposes. For any topological group H, write $H^\vee := \mathrm{Hom}(H, \Lambda^*)$ for the space of continuous characters of H.

Thus, our task is to first identify $H^\vee$, in geometric terms, for each of the two groups H appearing in Theorem 1, and then reformulate the result as a geometric statement.

The Galois side. Recall that $G_K^{\mathrm{unr,ab}}$ is canonically identified with the étale fundamental group $\pi_1(C)$. But then $\pi_1(C)^\vee$ is naturally identified with the set of isomorphism classes of category $\mathrm{Loc}_1(C)$ of rank 1 local systems on C with coefficients in Λ.

The automorphic side. By Weil's theorem (or the Beauville-Laszlo theorem), the group $\mathbb{G}_m(K)\backslash\mathbb{G}_m(\mathbb{A})/\mathbb{G}_m(\mathcal{O})$ is naturally identified with the Picard group of C; equivalently, this is the set of k-points of the Picard variety $\underline{\mathrm{Pic}}(C)$ of C. To identify the characters of this group geometrically, we need the following notion:

Definition 3. Let G/k be a commutative algebraic group. A *character local system* on G is given by pair (L, ψ) where $L \in \mathrm{Loc}_1(G)$, and $\psi : m^*L \simeq p_1^*L \otimes p_2^*L$ is an isomorphism on $G \times G$ that satisfies the cocycle condition; here $m : G \times G \to G$ is the multiplication map, while the p_i's are the projection maps; write $\mathrm{CharLoc}(G)$ for the category of such data.

Remark 4. One may equivalently describe $\mathrm{CharLoc}(G)$ as the groupoid of homomorphisms $G \to B(\Lambda^*)$ of commutative group stacks, where Λ^* is viewed as a pro-(finite group scheme) over k.

Any local system $L \in \mathrm{Loc}_1(G)$ defines a function $f_L\colon G(k) \to \Lambda^*$ given by the sheaf function correspondence, i.e., by the association $\big(g\colon \mathrm{Spec}(k) \to G\big) \mapsto \mathrm{Trace}(\mathrm{Frob}|g^*L)$. If L lifts to a character local system, one checks that f_L is a homomorphism. Conversely, this process can be often reversed:

Theorem 5. *Let G/k be a connected commutative algebraic group. Then $G(k)^\vee$ is naturally identified with isomorphism classes of objects in* $\mathrm{CharLoc}(G)$.

Proof sketch. As explained above, there is a natural map $a\colon \pi_0(\mathrm{CharLoc}(G)) \to G(k)^\vee$ coming from the sheaf-to-function correspondence. For the inverse, consider the Lang map $L_G\colon G \to G$ given by $g \mapsto \mathrm{Frob}(g)^{-1} \cdot g$. Since G is connected, this map is a finite étale Galois cover with Galois group $G(k)$. Thus, given a character $f\colon G(k) \to \Lambda^*$, we can descend the trivial rank 1 Λ-local system on G along L_G using f to obtain an $L_f \in \mathrm{Loc}_1(G)$; one then shows that this L_f naturally lifts to $\mathrm{CharLoc}(G)$, providing a map $b\colon G(k)^\vee \to \pi_0(\mathrm{CharLoc}(G))$, which is then shown to be an inverse to a. $\square$

The preceding result is not valid for disconnected groups in general, but it is valid for $\mathbb{Z}$. Using this observation, one also checks that the same continues to hold for $\underline{\mathrm{Pic}}(C)$. Thus, we have described $\mathrm{Pic}(C)(k)^\vee$ as the isomorphism classes of objects in $\mathrm{CharLoc}(\underline{\mathrm{Pic}}(C))$.

The geometric formulation. Using the two descriptions above, Theorem 1 follows from:

Theorem 6. *The Abel-Jacobi map $AJ\colon C \to \underline{\mathrm{Pic}}(C)$ induces an equivalence of categories*

$$AJ^*\colon \mathrm{CharLoc}(\underline{\mathrm{Pic}}(C)) \simeq \mathrm{Loc}_1(C).$$

Remark 7. The explicit description of the bijection in Theorem 1 is recovered immediately from the fact that pullbacks commute with taking stalks, and using the following observation: the Abel-Jacobi map AJ carries $c \in C$ to the line bundle $\mathcal{O}_C([c]) \in \mathrm{Pic}(C)$.

The advantage of the geometric formulation in Theorem 6 is twofold: (a) both sides are of a local nature, so the statement can be checked over $k = \overline{k}$, (b) both sides make sense over any field, including $k = \mathbb{C}$. We now sketch a proof of Theorem 6, following Deligne:

Sketch of proof of Theorem 6. We focus on the key assertion: any $L \in \mathrm{Loc}_1(C)$ descends along AJ to some $\mathrm{Aut}_L \in \mathrm{CharLoc}(\underline{\mathrm{Pic}}(C))$. For this, fix an integer $d > 2g-1$, where g is the genus of C. We first construct the degree d component $\mathrm{Aut}_{L,d} \in \mathrm{Loc}_1(\underline{\mathrm{Pic}}^d(C))$ by contemplating the following composition:

$$C^d \xrightarrow{a} [C^d/S_d] \xrightarrow{b} \mathrm{Sym}^d(C) \xrightarrow{c} \underline{\mathrm{Pic}}^d(C),$$

here the first map is the tautological (stacky) quotient by the symmetric group action, the second map is a coarse moduli space, and the last map is the unique one that identifies the composite $C^d \to \underline{\mathrm{Pic}}^d(C)$ as the map sending $(c_1, ..., c_d)$ to $\mathcal{O}_C([c_1] + [c_2] + ... + [c_d])$.

Taking exterior products gives a local system $L^{\boxtimes d} \in \mathrm{Loc}_1(C^d)$. This local system is naturally S_d-equivariant, and thus descends along a to some local system $L^{[d]} \in \mathrm{Loc}_1([C^d/S_d])$.

To descend this local system to $\mathrm{Sym}^d(C)$, we must check that the stabilizers on the stack $[C^d/S_d]$ act trivially on the stalks of $L^{[d]}$. But this is easily seen to be a consequence of L having rank 1: for a free Λ-module M of rank 1 and any integer $n \geq 1$, the natural symmetric group action on $M^{\otimes n}$ is trivial. Thus, $L^{[d]}$ descends along b to some $L^{(d)} \in \mathrm{Loc}_1(\mathrm{Sym}^d(C))$.

To descend $L^{(d)}$ along c, note that c is a projective space bundle by the Riemann-Roch theorem as $d > 2g - 1$. As projective space is simply connected, this implies that c induces an isomorphism on $\pi_1(-)$, and thus *any* local system on $\mathrm{Sym}^d(C)$ descends along c. In particular, $L^{(d)}$ descends to some $\mathrm{Aut}_{L,d} \in \mathrm{Loc}_1(\underline{\mathrm{Pic}}^d(C))$.

Next, we observe the associativity property of exterior products translates to the following compatibility property of this construction: for $d, e > 2g - 1$, if $m\colon \underline{\mathrm{Pic}}^d(C) \times \underline{\mathrm{Pic}}^e(C) \to \underline{\mathrm{Pic}}^{d+e}(C)$ denotes the addition map, then there is a canonical isomorphism

$$m^*\mathrm{Aut}_{L,d+e} \simeq \mathrm{Aut}_{L,d} \boxtimes \mathrm{Aut}_{L,e},$$

and this isomorphism is transitive in d and e in the evident sense. Using this property, one formally constructs $\mathrm{Aut}_{L,d} \in \mathrm{Loc}_1(\underline{\mathrm{Pic}}(C))$ for all values of $d \in \mathbb{Z}$ in such a way that the transitive family of isomorphisms as above hold for all d and e. But this means exactly that the resulting local system $\mathrm{Aut}_L \in \mathrm{Loc}_1(\underline{\mathrm{Pic}}(C))$ is a character local system, and pulls back to $L^{(d)} \in \mathrm{Loc}_1(C^d)$ along the (generalized) Abel-Jacobi map $C^d \to \underline{\mathrm{Pic}}(C)$ considered above; taking $d = 1$ then verifies that Aut_L descends L along AJ^*. □

2. The ramified story

We continue with the notation above, and allow ramification. Thus, fix a non-empty effective divisor D on C with affine complement U, and let $\widehat{C}_{/D}$ be the formal completion of C along D. The goal of ramified class field theory is to understand the Galois group parametrizing abelian extensions of K unramified over U; equivalently, we must understand the category $\mathrm{Loc}_1(U)$. As in the unramified case, the automorphic side will be understood via a certain moduli space of line bundles. More precisely, we study line bundles with "full level structure at D":

Definition 8. Let $\underline{\mathrm{Pic}}_{D_\infty}(C)$ be the space parametrizing line bundles on C with a trivialization over $\widehat{C}_{/D}$.

$\underline{\mathrm{Pic}}_{D_\infty}(C)$ is a pro-algebraic group. Moreover, forgetting the trivialization defines a map

$$\alpha\colon \underline{\mathrm{Pic}}_{D_\infty}(C) \to \underline{\mathrm{Pic}}(C)$$

which is a torsor for $\mathcal{O}^*_{\widehat{C}_{/D}}$, viewed as a pro-algebraic group over k. For any $c \in U$, the line bundle $\mathcal{O}_C([c])$ is canonically trivialized over $\widehat{C}_{/D}$ since $\{c\} \cap D = \emptyset$. Thus, there is an Abel-Jacobi map

$$AJ_U \colon U \to \underline{\mathrm{Pic}}_{D_\infty}(C).$$

The main result of *geometric* ramified class field theory is an exact analogue of Deligne's theorem above:

Theorem 9. *The map AJ_U induces an equivalence of categories*

$$AJ^* \colon \mathrm{CharLoc}(\underline{\mathrm{Pic}}_{D_\infty}(C)) \simeq \mathrm{Loc}_1(U).$$

Remark 10. The usual idele-theoretic formulation can be obtained by inverting the recipe used in the unramified case to go in the reverse direction.

The classical proof of Theorem 9, as presented in [3], relies on the following two facts: (a) the map AJ_U is the (pro-)universal map from U to a smooth connected commutative algebraic group (Rosenlicht), and (b) for any finite abelian group A, any A-torsor $V \to U$ is the pullback of an isogeny $G' \to G$ of smooth connected commutative algebraic groups with kernel A. Below, instead, we proceed by imitating Deligne's argument in the unramified case to reduce to a local statement:

Proof Sketch. Fix an integer $d > 2g - 1$. Then there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Sym}^d(U) \arrow[r, "e"] \arrow[dr, equal] & T|_{\mathrm{Sym}^d(U)} \simeq \mathrm{Sym}^d(U) \times \mathcal{O}^*_{\widehat{C}_{/D}} \arrow[r, "c_U"] \arrow[d, "\alpha_{\mathrm{Sym}^d(U)}"] & T \arrow[r, "b_U"] \arrow[d, "\alpha_{\mathrm{Sym}^d(C)}"] & \underline{\mathrm{Pic}}^d_{D_\infty}(C) \arrow[d, "\alpha"] \\
 & \mathrm{Sym}^d(U) \arrow[r, "c"] & \mathrm{Sym}^d(C) \arrow[r, "b"] & \underline{\mathrm{Pic}}^d(C).
\end{tikzcd}$$

Here both squares are fibre squares, the maps c and c_U are open immersions, all vertical arrows are $\mathcal{O}^*_{\widehat{C}_{/D}}$-torsors, the base change of α along $b \circ c$ is split as any divisor on C supported on U is canonically trivialized over $\widehat{C}_{/D}$, and the map e is the 0-section of the resulting trivial torsor $\alpha_{\mathrm{Sym}^d(U)}$.

We need to check that any $L \in \mathrm{Loc}_1(U)$ descends along AJ_U to a character local system on $\underline{\mathrm{Pic}}_{D_\infty}(C)$. We prove this as in the unramified case. Thus, first, note that the local system $L^{\boxtimes d} \in \mathrm{Loc}_1(U^d)$ descends to some $L^{(d)} \in \mathrm{Loc}_1(\mathrm{Sym}^d(U))$. We must show that $L^{(d)}$ is pulled back along the composite map $\mathrm{Sym}^d(U) \to \underline{\mathrm{Pic}}^d_{D_\infty}(C)$ in the diagram above. By base change, b_U is a projective space bundle; thus, as in the unramified case, it suffices to show that $L^{(d)}$ extends to T. By purity for the fundamental group, it suffices to show the following: there exists a unique character local system M on $\mathcal{O}^*_{\widehat{C}_{/D}}$ such that $L^{(d)} \boxtimes M \in \mathrm{Loc}_1(\mathrm{Sym}^d(U) \times \mathcal{O}^*_{\widehat{C}_{/D}})$ extends across all the codimension 1 points of T lying in $Z := T - T|_{\mathrm{Sym}^d(U)}$. Now, at points of $\mathrm{Sym}^d(C)$ of codimension 1 lying in $\mathrm{Sym}^d(C) - \mathrm{Sym}^d(U)$, at most one point of $\mathrm{Sym}^d(C)$ is allowed to lie in D. Thus, by working locally, we reduce to the case $d = 1$ and $D = \{c\}$ being a single point. In this case, the existence of such an M is ensured by geometric local class field theory, as stated in Theorem 11 below. $\square$

3. The local story

We saw above that geometric ramified class field theory reduces to a local statement; we formulate this local statement somewhat imprecisely and "semi-globally" next, and refer to [1, 2] for precise definitions and statements, including a purely local formulation which makes sense of "punctured formal discs".

Continuing the notation of the previous section, fix a point $c \in |C|$, let $D = [c]$, and $\widehat{C} = \widehat{C}_{/D}$ be the completion of C at c; to avoid a discussion of punctured formal discs, we simply define $\widehat{C}$ to be the spectrum of the formal completion of the local ring of C at c. Let $\widehat{T} \to \widehat{C}$ be the base change of the torsor α considered in the previous section along the map $\widehat{C} \to C \overset{AJ}{\to} \underline{\mathrm{Pic}}(C)$. Thus, one may informally view $\widehat{T}$ as parametrizing points $y \in C$ close to c, together with a trivialization of $\mathcal{O}_C([y])$ at $\widehat{C}$. Over $\widehat{U} = \widehat{C} - \{c\}$, the torsor $\widehat{T}|_{\widehat{U}} \to \widehat{U}$ is canonically trivialized, so we have $\widehat{T}|_{\widehat{U}} \simeq \widehat{U} \times \mathcal{O}^*_{\widehat{C}}$. The main theorem of local geometric class field theory relates local systems on $\widehat{U}$ to character local systems on $\mathcal{O}^*_{\widehat{C}}$ via the torsor $\widehat{T}$. More precisely, we have:

Theorem 11. *There is a canonical equivalence*

$$\mathrm{Loc}_1(\widehat{U})/\mathrm{Loc}_1(\widehat{C}) \simeq \mathrm{CharLoc}(\mathcal{O}^*_{\widehat{C}}).$$

This equivalence is characterized as follows: a local system $L \in \mathrm{Loc}_1(\widehat{U})$ is associated to a character local system $M \in \mathrm{CharLoc}(\mathcal{O}^*_{\widehat{C}})$ if and only if $L \boxtimes M \in \mathrm{Loc}_1(\widehat{T}|_{\widehat{U}})$ extends to a local system on $\widehat{T}$.

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La courbe de Fargues-Fontaine

Pierre Colmez

1. Anneaux de Fontaine

Soient k un corps parfait de caractéristique p, $K_0 = W(k)[\frac{1}{p}]$ le corps de caractéristique 0, complet non ramifié, de corps résiduel k, K une extension finie totalement ramifiée de K, $\overline{K}$ une clôture algébrique de K et C le complété de $\overline{K}$, ce qui fait de C un corps algébriquement clos, complet pour v_p, dont le corps résiduel k_C est une clôture algébrique de k. Soit

$$C^\flat = \{x = (x^{(n)})_{n\in\mathbf{N}},\ (x^{(n+1)})^p = x^{(n)},\ \forall n \in \mathbf{N}\}.$$

On munit $C^\flat$ des lois $+$ et $\cdot$ définies par $x + y = s$ et $xy = t$, avec

$$s^{(n)} = \lim_{k\to+\infty}(x^{(n+k)} + y^{(n+k)})^{p^k}, \quad t^{(n)} = x^{(n)}y^{(n)}$$

Si $x = (x^{(n)}) \in C^\flat$, soit $x^\sharp = x^{(0)}$, et si $x \in C$, on note $x^\flat$ n'importe quel élément de $C^\flat$ tel que $(x^\flat)^\sharp = x$ (et donc $x^\flat$ n'est bien déterminé qu'à $\epsilon^{\mathbf{Z}_p}$ près, où $\epsilon = (1, \zeta_p, \dots)$ et ζ_p est une racine primitive p-ième de l'unité; cela est source de bien des complications).

Théorème 1. *$C^\flat$ est un corps algébriquement clos de caractéristique p, complet pour la valuation $v_{C^\flat}(x) = v_p(x^\sharp)$, de corps résiduel $k_{C^\flat} = k_C$.*

Remarque 2. *La construction $C \mapsto C^\flat$ est une vieille construction de Fontaine, et s'applique à n'importe quelle algèbre munie d'une topologie plus faible que celle définie par la valuation p-adique. Dans la terminologie de Scholze, cette opération s'appelle le basculement (tilting), et $C^\flat$ est le basculé de C en caractéristique p.*

Soit $\mathbf{A}_{\rm inf} = W(\mathcal{O}_{C^\flat})$, l'anneau des vecteurs de Witt à coefficients dans $\mathcal{O}_{C^\flat}$. Si $x \in \mathcal{O}_{C^\flat}$, notons $[x]$ son représentant de Teichmüller. Alors, tout $x \in \mathbf{A}_{\rm inf}$ peut s'écrire, de manière unique, $x = \sum_{k\in\mathbf{N}}[x_k]p^k$, où les x_k sont des éléments arbitraires de $\mathcal{O}_{C^\flat}$. Par fonctorialité, $\mathbf{A}_{\rm inf}$ est muni d'un frobenius φ donné par $\varphi(\sum_{k\in\mathbf{N}}[x_k]p^k) = \sum_{k\in\mathbf{N}}[x_k^p]p^k$. On définit $\theta\colon \mathbf{A}_{\rm inf} \to \mathcal{O}_C$ par

$$\theta(\sum_{k\in\mathbf{N}}[x_k]p^k) = \sum_{k\in\mathbf{N}} p^k x_k^\sharp.$$

Proposition 3. *$\theta\colon \mathbf{A}_{\rm inf} \to \mathcal{O}_C$ est un morphisme surjectif d'anneaux dont le noyau est engendré par $(p - [p^\flat])$.*

Remarque 4. *La proposition précédente montre que l'on peut reconstruire C à partir de $C^\flat$ (i.e. rebasculer en caractéristique 0), en posant $C = W(C^\flat)/(p-[p^\flat])$. Ceci joue un grand rôle dans la construction des diamants de Scholze.*

Soit $\mathbf{B}_{\rm dR}^+ = \varprojlim(\mathbf{A}_{\rm inf}[\frac{1}{p}]/(p - [p^\flat])^k)$. C'est un anneau de valuation discrète de corps résiduel C qui contient $\mathbf{A}_{\rm cris}$, complété de $\mathbf{A}_{\rm inf}[\frac{(p-[p^\flat])^k}{k!},\ k \in \mathbf{N}]$ pour la topologie p-adique. Le frobenius φ s'étend par linéarité et continuité à $\mathbf{A}_{\rm cris}$, et si on note $t = \log[\epsilon] = -\sum_{k\in\mathbf{N}} \frac{(1-[\epsilon])^k}{k+1}$, alors t est une uniformisante de $\mathbf{B}_{\mathrm{dR}}^+$ appartenant à $\mathbf{A}_{\mathrm{cris}}$ et $\varphi(t) = pt$.

L'action de φ s'étend donc au sous-anneau $\mathbf{B}_{\mathrm{cris}} = \mathbf{A}_{\mathrm{cris}}[\frac{1}{t}]$ de $\mathbf{B}_{\mathrm{dR}} = \mathbf{B}_{\mathrm{dR}}^+[\frac{1}{t}]$, et on note $\mathbf{B}_e$ le sous-anneau $\mathbf{B}_{\mathrm{cris}}^{\varphi=1}$. L'inclusion de $\mathbf{B}_e$ dans $\mathbf{B}_{\mathrm{dR}}$ induit alors la *suite exacte fondamentale*:

$$0 \to \mathbf{Q}_p \to \mathbf{B}_e \to \mathbf{B}_{\mathrm{dR}}/\mathbf{B}_{\mathrm{dR}}^+ \to 0.$$

La structure algébrique de l'anneau $\mathbf{B}_e$ est surprenamment simple.

Théorème 5. $\mathbf{B}_e$ *est un anneau principal.*

2. La courbe

Des considérations venant des théorèmes de comparaison entre les cohomologies étale et de de Rham des variétés p-adiques ont conduit Berger à introduire la catégorie des B-paires: une B-paire est une paire (W_e, W_{dR}^+), où W_e est un $\mathbf{B}_e$-module muni d'une action semi-linéaire du groupe de Galois absolu G_K de K, W_{dR}^+ est un sous-$\mathbf{B}_{\mathrm{dR}}^+$-réseau de $\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_e} W_e$ stable par G_K. La catégorie des B-paires contient naturellement la catégorie des représentations p-adiques de G_K.

Sans action de Galois, on tombe sur une catégorie ayant aussi de bonnes propriétés, en particulier une filtration de Harder-Narasimhan, et la question que se sont posée Fargues et Fontaine est: *existe-t-il un objet géométrique qui explique toutes ces belles propriétés*? La réponse est "oui!": on peut considérer la "courbe" $\mathrm{Spec}\,\mathbf{B}_e$ (c'est une courbe un peu spéciale car pas du tout de type fini), que l'on peut compactifier en rajoutant un point ∞; le résultat est une courbe complète

$$X = \mathrm{Proj}\Big(\oplus_{d\in\mathbf{N}} \, (\mathbf{A}_{\mathrm{cris}}[\tfrac{1}{p}])^{\varphi=p^d} \Big),$$

l'anneau $\mathbf{B}_e$ est l'anneau des fonctions régulières sur l'ouvert $X - \{\infty\}$, l'anneau $\mathbf{B}_{\mathrm{dR}}^+$ est le complété de l'anneau local en ∞, et les paires (W_e, W_{dR}) comme ci-dessus sont les fibrés sur X (décrits à la Beauville-Laszlo, en prenant comme recouvrement de X, l'ouvert $\mathrm{Spec}\,\mathbf{B}_e$ et un voisinage infinitésimal du point ∞).

L'histoire ne s'arrête pas là car le fait que les fonctions sur X sont obtenues en prenant les points fixes de φ laisse entendre que X est le quotient d'un espace Y par φ. Le problème est alors: comment définir Y et dans quelle catégorie prendre le quotient? La réponse fait, cette fois, intervenir l'anneau $\mathbf{A}_{\mathrm{inf}}$: pour Y on prend l'espace analytique $\mathrm{Spa}\,\mathbf{A}_{\mathrm{inf}}$ privé des diviseurs $p = 0$ et $[p^\flat] = 0$, et alors l'espace analytique X^{ad} associé à X est le quotient de Y par $\varphi^{\mathbf{Z}}$.

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Perfectoid Spaces

Urs Hartl

This talk gave a brief introduction to perfectoid spaces which were discovered by Peter Scholze [Sch12]. They are adic spaces in the sense of Huber [Hub96]. A crucial feature of perfectoid spaces is the *tilting operation*, which assigns to any perfectoid space X a perfectoid space $X^\flat$ in positive characteristic with same underlying topological space and same étale site as X.

1. Perfectoid Algebras

Throughout we fix a prime number $p \in \mathbb{N}$. For the theory of adic spaces we also refer to the exposition of T. Wedhorn in this Oberwolfach report.

Definition 1. (a) A *complete Tate ring* is a topological ring A for which there is an open subring $A_0 \subset A$, a finitely generated ideal $I \subset A_0$ and an element $\varpi \in A^\times$ such that $\{I^n \colon n \in \mathbb{N}\}$ is a neighborhood basis of 0, the rings A and A_0 are I-adically complete, and $\varpi^n \to 0$ for $n \to \infty$. The element ϖ is called a *pseudo-uniformizer*.

(b) A subset $S \subset A$ is *bounded* if $S \subset \varpi^{-n} A_0$ for some n. This notion does not depend of the choice of A_0 and ϖ. The *set of power bounded elements* is $A^\circ := \big\{ x \in A \colon \{x^n \colon n \in \mathbb{N}\} \text{ is bounded} \big\}$.

(c) A *perfectoid ring* is a complete Tate ring A with A° bounded, such that there is a pseudo-uniformizer $\varpi \in A^\times$ with $\varpi^p | p$ in A° and

$$\Phi \colon A^\circ/(\varpi) \longrightarrow A^\circ/(\varpi^p),\ x \mapsto x^p$$

is an isomorphism. Also this definition does not depend on ϖ.

Examples 2. (a) The p-adic completion $\mathbb{Q}_p^{\mathrm{cycl}}$ of $\mathbb{Q}_p(\sqrt[p^n]{1} : \text{all } n)$ is a perfectoid field of characteristic 0.

(b) The t-adic completion $\mathbb{F}_p(\!(t^{1/p^\infty})\!)$ of $\bigcup_n \mathbb{F}_p(\!(t^{1/p^n})\!)$ is a perfectoid field of characteristic p.

(c) If A is a complete Tate ring with A° bounded and $p = 0$ in A, then A is perfectoid if and only if A is perfect, that is, $\Phi \colon A \to A,\ x \mapsto x^p$ is an isomorphism, [Sch12, Proposition 5.9].

(d) If $A = K$ is a non-archimedean valued field, then K is perfectoid if and only if the valuation is non-discrete, $|p| < 1$ and $\Phi \colon \mathcal{O}_K/(p) \to \mathcal{O}_K/(p),\ x \mapsto x^p$ is surjective, [Sch14, Propositions 6.1.8 and 6.1.9].

We saw in Wedhorn's talk that if we have a Tate ring, then we can form a Huber pair and then take its adic spectrum. However, it is not clear that this gives rise to an adic space because it is not clear that the structure presheaf will be a sheaf. In the perfectoid case things are better.

Theorem 3 ([Sch12, Theorem 6.3(iii)])**.** *Let (A, A^+) be a* Huber pair *(i.e. $A^+ \subset A$ is an integrally closed, open subring with $A^+ \subset A^\circ$) with A perfectoid. Then for* *every rational subset* $U \subset X = \mathrm{Spa}(A, A^+)$ *the ring* $\mathcal{O}_X(U)$ *is perfectoid. In particular* $\mathcal{O}_X(U)^\circ$ *is bounded and this implies that* $\mathcal{O}_X$ *is a sheaf on* X*.*

The proof of Theorem 3 uses

2. Tilting

Definition 4. For a perfectoid ring A the *tilt* is defined as

$$A^\flat := \varprojlim_{x \mapsto x^p} A .$$

We write the elements of $A^\flat$ as $x = (x^{(0)}, x^{(1)}, \ldots)$ with $x^{(n)} = (x^{(n+1)})^p$. The set $A^\flat$ is a ring under the addition $(x^{(n)})_n + (y^{(n)})_n := \big(\lim_{k\to\infty}(x^{(n+k)} + y^{(n+k)})^{p^k}\big)_n$ and the multiplication $(x^{(n)})_n \cdot (y^{(n)})_n := (x^{(n)} \cdot y^{(n)})_n$.

Example 5. $(\mathbb{Q}_p^{\mathrm{cycl}})^\flat = \mathbb{F}_p(\!(t^{1/p^\infty})\!)$ with $t = (1, \varepsilon_1, \varepsilon_2, \ldots) - (1, 1, 1, \ldots)$ and $t^{1/p^n} = (\varepsilon_n, \varepsilon_{n+1}, \ldots) - (1, 1, \ldots)$, where $\varepsilon_1 \neq 1$, $\varepsilon_1^p = 1$, $\varepsilon_{n+1}^p = \varepsilon_n$. The cyclotomic character $\mathbb{Z}_p^\times \xrightarrow{\sim} \mathrm{Aut}^{\mathrm{cont}}_{\mathbb{Q}_p}(\mathbb{Q}_p^{\mathrm{cycl}})$, $a \longmapsto (\varepsilon_n \mapsto \varepsilon_n^{a \bmod p^n})$ induces by functoriality of tilting an action $\mathbb{Z}_p^\times \xrightarrow{\sim} \mathrm{Aut}\big(\mathbb{F}_p(\!(t^{1/p^\infty})\!)\big)$, $a \longmapsto \big(t^{1/p^n} \mapsto (1 + t^{1/p^n})^a - 1\big)$.

Lemma 6 ([Sch14, Lemmas 6.2.2 and 6.2.4])**.**

(a) $A^\flat$ *is a perfectoid ring with* $p = 0$.

(b) $A^{\flat\circ} = \varprojlim_{x\mapsto x^p} A^\circ = \varprojlim_{x\mapsto x^p} A^\circ/(p)$.

(c) *There is a pseudo-uniformizer* $\varpi \in A^\times$ *which has a compatible system of* p^n*-th roots* $\varpi^{1/p^n} \in A$ *for all* n*. We write* $\varpi^\flat = (\varpi, \varpi^{1/p}, \ldots) \in A^\flat$*. Then* $A^\flat = A^{\flat\circ}[\frac{1}{\varpi^\flat}]$.

(d) *The map* $A^\flat \to A$, $(x^{(n)})_n \mapsto x^{(0)}$*, which is denoted* $x \mapsto x^\sharp$*, is multiplicative but not additive in general. It induces an isomorphism of rings* $A^{\flat\circ}/(\varpi^\flat) \xrightarrow{\sim} A^\circ/(\varpi)$ *for* $\varpi = (\varpi^\flat)^\sharp$.

(e) *For a fixed perfectoid ring* A *with tilt* $A^\flat$*, the assignment* $A^+ \longmapsto A^{+\flat} := \varprojlim_{x\mapsto x^p} A^+$ *yields a bijection*

$$\{A^+ \colon (A, A^+) \text{ is a Huber pair}\} \xrightarrow{\sim} \{A^{\flat+} \colon (A^\flat, A^{\flat+}) \text{ is a Huber pair}\} .$$

Theorem 7 ([Sch12, Theorem 6.3(i),(ii)])**.**

(a) *There is a homeomorphism* $X = \mathrm{Spa}(A, A^+) \xrightarrow{\sim} X^\flat = \mathrm{Spa}(A^\flat, A^{+\flat})$ *sending a valuation* $x = |\,.\,|_x$ *on* A *to the valuation* $x^\flat = |\,.\,|_{x^\flat}$ *on* $A^\flat$ *with* $|f|_{x^\flat} := |f^\sharp|_x$ *for* $f \in A^\flat$*. It preserves rational subsets, that is, maps a rational subset* $U \subset X$ *homeomorphically onto the rational subset* $U^\flat \subset X^\flat$*.*

(b) *If* $U \subset X$ *is a rational subset then* $\big(\mathcal{O}_X(U), \mathcal{O}_X^+(U)\big)$ *is perfectoid with tilt* $\big(\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}^+_{X^\flat}(U^\flat)\big)$.

3. Perfectoid Spaces — The Étale Site

Definition 8. A *perfectoid space* is an adic space covered by $\mathrm{Spa}(A, A^+)$ with A perfectoid.

Theorem 7 implies that tilting glues to give a functor $X \longmapsto X^\flat$ on perfectoid spaces X.

Theorem 9. *Let A be a perfectoid ring with tilt $A^\flat$. Then*

(a) *any finite étale A-algebra B is perfectoid,* [Sch12, Theorem 7.9],
(b) *the functor $B \longmapsto B^\flat$ is an equivalence between*
- *perfectoid A-algebras and perfectoid $A^\flat$-algebras,* [Sch12, Theorem 5.2],
- *finite étale A-algebras and finite étale $A^\flat$-algebras,* [Sch12, Thm. 5.25].

Definition 10. A morphism of perfectoid spaces $f\colon Y \to X$ is

(a) *finite étale* if for every open affinoid $U = \mathrm{Spa}(A, A^+) \subset Y$ the preimage $f^{-1}U = \mathrm{Spa}(B, B^+)$ is affinoid with B a finite étale A-algebra and B^+ being the integral closure of A^+ in B.
(b) *étale* if for every point $y \in Y$ there is an open neighborhood $V \subset Y$ of y, an open subset $U \subset X$ with $f(V) \subset U$, and a commutative diagram

$$\begin{tikzcd} V \arrow[rr, hook, "\text{open}"] \arrow[dr, "f"'] & & W \arrow[dl, "\text{finite étale}"] \\ & U & \end{tikzcd}$$

Remark 11. Since all perfectoid rings are reduced, it does not make sense to define étale maps with the infinitesimal lifting property. However, for perfectoid spaces (and rigid analytic spaces) the above definition yields the correct theory of étale maps, although this definition would be false for schemes.

Proposition 12. (a) *(Finite) étale morphism of perfectoid spaces are stable under composition and base change (and in particular fiber products exist, which is not true for general adic spaces),* [Sch12, Lemma 7.3 and Corollary 7.8].
(b) *Étale morphisms of perfectoid spaces are open,* [Sch12, Corollary 7.8].
(c) *If $f\colon Z \to Y$ and $g\colon Y \to X$ are morphisms of perfectoid spaces with g and $g \circ f$ étale, then f is étale,* [Sch14, Proposition 7.5.2].
(d) *A morphism $f\colon Y \to X$ between perfectoid spaces is étale if and only if its tilt $f^\flat\colon Y^\flat \to X^\flat$ is étale,* [Sch12, Proposition 6.17 and Theorem 5.25].

Proof. Via (d) statements (a), (b) and (c) are transfered to characteristic p, where they are reduced to results of Huber [Hub96]. □

Definition 13. The *étale site* $X_{\text{ét}}$ of a perfectoid space X is the category of perfectoid spaces étale over X with topological coverings.

Corollary 14. $X_{\text{ét}} \cong X^\flat_{\text{ét}}$, $Y \mapsto Y^\flat$.

Remark 15. The *philosophy of tilting* is that for a perfectoid space X all topological information like the underlying topological space, or the étale site $X_{\text{ét}}$ can be recovered from $X^\flat$. However, if X is in addition an adic space over $\mathbb{Q}_p$ the structure morphism $X \to \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ is forgotten under tilting. To remedy this, Scholze introduced diamonds; see the talk of M. Morrow on "Relative Fargues-Fontaine Curves" in this Oberwolfach report.

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The pro-étale and the v-topology

Eugen Hellmann

We introduce the *pro-étale* and the *v-topology* on the category of perfectoid spaces, following [2, Lectures 8, 9 and 15].

Definition: (i) A morphism $\mathrm{Spa}(A_\infty, A_\infty^+) \to \mathrm{Spa}(A, A^+)$ of affinoid perfectoid spaces is called *affinoid pro-étale* if

$$(A_\infty, A_\infty^+) = \varpi\text{-adic completion of } \varinjlim (A_i, A_i^+),$$

where $\varpi \in A$ is a pseudo-uniformizer and (A_i, A_i^+) is a filtered system of affinoid perfectoid spaces such that $\mathrm{Spa}(A_i, A_i^+) \to \mathrm{Spa}(A, A^+)$ is étale.
(ii) A morphism of perfectoid spaces is called *pro-étale* if it is locally on source and target an affinoid pro-étale morphism.

We point out that the notion of being pro-étale is local in the analytic topology. Moreover, compositions of pro-étale morphisms are pro-étale and the base change of a pro-étale morphism is pro-étale. However, it is not true that pro-étale morphisms are open in general: for example the inclusion of a point in a pro-finite set is pro-étale but usually not open.

We define (slightly sloppy) the *pro-étale topology* to be the (pre-)topology on the category of perfectoid spaces whose coverings are generated by the open coverings and the surjective affinoid pro-étale morphisms.

Proposition: *The structure sheaf $X \mapsto \mathcal{O}_X(X)$ is a sheaf for the pro-étale topology.*

Proof: Let $X = \mathrm{Spa}(A, A^+)$ be an affinoid perfectoid space and let $\varpi \in A$ be a pseudo-uniformizer. By standard arguments it is enough to show that $Y \mapsto (\mathcal{O}_Y^+/\varpi)^a$ is a sheaf for the pro-étale topology on the category of perfectoid spaces over X. For affinoid pro-étale covers of $Y = \mathrm{Spa}(B, B^+)$ the sheaf property is easily deduced from the fact that

$$H^i_{\text{ét}}(Y, \mathcal{O}_Y^+/\varpi) \overset{a}{=} \begin{cases} B^+/\varpi & i = 0 \\ 0 & i > 0. \end{cases}$$

Here the superscript a indicates that this an *almost equality*, i.e. an equality in the category of *almost A^+/ϖ-modules.*

The following is a direct consequence of the sheaf property of $\mathcal{O}_X$.

Corollary: *The pro-étale topology is subcanonical, i.e. for every perfectoid space X the functor h_X represented by X is a sheaf for the pro-étale topology.*

The property of being pro-étale is not local in the pro-étale topology. In fact a morphism $f\colon Y \to X$ of perfectoid spaces becomes pro-étale over some pro-étale cover $X' \to X$ if and only if it is *locally quasi pro-finite*, i.e. if for every geometric point $\mathrm{Spa}(C, C^+) \to X$ with C algebraically closed the fiber product

$$Y \times_X \mathrm{Spa}(C, C^+)$$

is pro-étale over $\mathrm{Spa}(C, C^+)$.

For future use we give the definition of a diamond.

Definition: A *diamond* is a sheaf $\mathcal{F}$ for the pro-étale topology on the category (Perf) of perfectoid spaces of characteristic p such that there exists a perfectoid space Y and a surjective morphism $h_Y \to \mathcal{F}$ of pro-étale sheaves that is relatively representable and locally quasi pro-finite.

We point out that the above implies that the structure sheaf is still a sheaf in the topology generated by the locally quasi pro-finite covers. This class of coverings does not involve any flatness assumptions. This motivates the following definition of the v-topology (called the *faithful topology* in [2, Lecture 15]).

Definition: The *v-topology* is the (pre-)topology whose coverings are generated by the open coverings and the surjective morphisms $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$.

Theorem: *The structure sheaf is a sheaf for the v-topology.*

Proof: Given a surjective morphism $X' = \mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+) = X$ of affinoid perfectoid spaces we need to show that the complex

$$0 \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{F}(X') \longrightarrow \mathcal{F}(X' \times_X X') \longrightarrow \dots$$

is (almost) exact, where $\mathcal{F} = \mathcal{O}_X^+/\varpi$. We split this claim into two parts:

(i) $X' \to X$ is a w-localization in the sense of [1]

(ii) X' is arbitrary, but X is w-local.

Recall that a spectral space X is called *w-local* if every connected component has a unique closed point and if the set of closed points X^c is closed in X. In this case the composition of the canonical maps

$$X^c \longrightarrow X \longrightarrow \pi_0(X)$$

is a homeomorphism. Moreover, given an affinoid (perfectoid) space X there exists a w-local space X^Z with a morphism $X^Z \to X$ that is universal for morphisms from

w-local spaces to X. By construction the space X^Z is again affionid (perfectoid) and the morphism $X^Z \to X$ is pro-étale. This implies that the claim is true in case (i).
We are left to prove (ii). This is the content of the following surprising lemma.

Lemma: *Let $X = \mathrm{Spa}(A, A^+)$ be an affinoid w-local perfectoid space. Let $\varpi \in A$ be a pseudo-uniformizer and let $f\colon Y = \mathrm{Spa}(B, B^+) \to X$ be a morphism of affinoid perfectoid spaces. Then B^+/ϖ is flat over A^+/ϖ (and even faithfully flat if f is surjective).*

Proof: Consider the diagram

$$Y \xrightarrow{f} X \xrightarrow{g} T = \pi_0(X)$$

and write $\mathcal{A} = g_*(\mathcal{O}_X^+/\varpi)$ and $\mathcal{M} = (g \circ f)_*(\mathcal{O}_Y^+/\varpi)$. Then $B^+/\varpi = \Gamma(T, \mathcal{M})$ is flat over $A^+/\varpi = \Gamma(T, \mathcal{A})$ if and only if for all $y \in T$ the stalk $\mathcal{M}_y$ is flat over $\mathcal{A}_y$. But since X is w-local a point $y \in T$ is just a closed point $\mathrm{Spa}(K, K^+)$ of X and we need to show that $B_y^+/\varpi = \mathcal{M}_y$ is flat over $K^+/\varpi = \mathcal{A}_y$. Now K^+ is a valuation ring and hence flatness over K^+ is equivalent to ϖ-torsion freeness. The claim follows from the fact that B_y^+ is ϖ-torsion free.

Corollary: *The v-topology is subcanonical.*

Finally we show that we can glue vector bundles in the v-topology:

Theorem: *The groupoid of vector bundles is a stack for the v-topology.*

This is the content of [2, Lemma 20.2.2]. One needs to show that vector bundles with a descend datum can be descended along surjective morphisms

$$\mathrm{Spa}(B, B^+) \longrightarrow \mathrm{Spa}(A, A^+).$$

First one treats the case where A is a field. Then the general case is dealt with by an approximation argument.

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Statement of Galois to Automorphic in the geometric context

Tsao-Hsien Chen

In the talk I will explain the statement of Geometric Langlands correspondence for a general reductive group. The main references are [1, 2, 3, 4, 5].

Let $\mathbb{F}_q$ be a finite field and X be a smooth complete curve over $\mathbb{F}_q$. Let K be the function field of X. We denote by $\pi_1(X)$ the étale fundamental group of X. For each closed point $x \in |X|$ we denote by $\mathbb{O}_x$ the completed local ring at x and K_x its field of fraction. We denote by $\mathbb{A}_K$ the ring of adèles, and $\mathbb{O}_K$ the ring of integral adèles. We fixe a prime ℓ not equal to the characteristic of $\mathbb{F}_q$.

In the first part of my talk I will explain the statement of the classical unramified Langlands correspondence for GL_n. Let

$$A_n := \mathrm{Funct}_{cusp}(\mathrm{GL}_n(K)\backslash \mathrm{GL}_n(\mathbb{A}_K)/\mathrm{GL}_n(\mathbb{O}_K), \bar{\mathbb{Q}}_\ell)$$

be the space of cuspidal unramified automorphic forms on GL_n and H_x the local Hecke algebra corresponding to a closed point $x \in |X|$, that is, the space of compactly supported $\mathrm{GL}(\mathbb{O}_x)$-biinvariant functions on $\mathrm{GL}_n(K_x)$. The basic facts are

(1) there is an action $H_x \times A_n \to A_n, (T, f) \mapsto T(f)$

(2) the Satake isomorphism $H_x \simeq K^0(\mathrm{Rep}(\mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)) \otimes_{\mathbb{Z}} \bar{\mathbb{Q}}_\ell$.

The classical unramified Langlands correspondence for GL_n says that to every geometrically irreducible ℓ-adic representation $\sigma\colon \pi_1(X) \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$ there corresponds to a (non-zero) cuspidal unramified automorphic form $f_\sigma \in A_n$ satisfying the Hecke eigenproperty with respect to σ. The last property means that for each $V \in \mathrm{Rep}(\mathrm{GL}_n(\bar{\mathbb{Q}}_\ell))$ we have

$$T_V(f_\sigma) = \mathrm{Tr}(\sigma(\mathrm{Frob}_x)|V) f_\sigma,$$

here $T_V \in H_x$ is the image of V under the Satake isomorphism and $\mathrm{Frob_x}$ is the Frobenius conjugacy class in $\pi_1(X)$ corresponding to $x \in |X|$.

In the second part of my talk, I will recall some basic facts about affine Grassmannians for a general reductive group and state the geometric Satake equivalence. Let k be an algebraically closed field. Let G be a reductive group over k and let $T \subset G$ be a maximal torus. Let $LG = G(k((t)))$ and $L^+G = G(k[[t]])$ be the corresponding loop group and positive loop group. The affine Grassmannian for G is the fpqc quotient

$$\mathrm{Gr}_G := LG/L^+G.$$

It is an ind-projective scheme. The positive loop group L^+G acts naturally on Gr_G and the assignment

$$\lambda \in X_*^+(T) \mapsto \mathcal{O}_\lambda := LG^+ \cdot t^\lambda$$

defines a bijection between the set of dominant co-weights and the set of LG^+-orbits on Gr_G. Here $\mathcal{O}_\lambda$ is the LG^+-orbit through the image of $t^\lambda \in T(k((t)))$ inside Gr_G. Let $\mathrm{Sat}_G := \mathrm{Perv}_{L^+G}(\mathrm{Gr}_G)$ be the category of L^+G-equivariant perverse sheaves on Gr_G. There is a convolution product $^*\colon \mathrm{Sat}_G \times \mathrm{Sat}_G \to \mathrm{Sat}_G$ on Sat_G. Define

$$\mathbb{H} := \bigoplus_{i\in\mathbb{Z}} R^i\Gamma(\mathrm{Gr}_G, -)\colon \mathrm{Sat}_G \to \mathrm{Vect},$$

where Vect is the category of vector spaces over $\bar{\mathbb{Q}}_\ell$. Let $(\mathrm{Rep}(\widehat{G}), \otimes)$ be the tensor category of finite dimensional representations of the dual group $\widehat{G}$ over $\bar{\mathbb{Q}}_\ell$. The geometric Satake equivalence is the following assertion.

Theorem 1. *1) The pair* $(\mathrm{Sat}_G, ^*)$ *admits a unique structure of a tensor category such that the functor* $\mathbb{H}$ *is symmetric monoidal.*
2) There is an equivalence of tensor categories

$$\mathcal{S}\colon (\mathrm{Sat}_G, ^*) \simeq (\mathrm{Rep}(\widehat{G}), \otimes)$$

such that $\mathrm{For} \circ \mathcal{S} \simeq \mathbb{H}$, *where* For *is the forgetful functor on* $(\mathrm{Rep}(\widehat{G}), \otimes)$.

Finally, I will explain how to use the geometric Satake equivalence to introduce the notion of a Hecke eigensheaf and state a version of the unramified geometric Langlands correspondence for a general reductive group. To this end, let Hecke be the Hecke stack which classifies quadruples $(\mathcal{E}_1, \mathcal{E}_2, x, \beta)$, where $\mathcal{E}_i$ is a G-bundle on X, $x \in X$ and $\beta \colon \mathcal{E}_1|_{X-x} \simeq \mathcal{E}_2|_{X-x}$ is an isomorphism between $\mathcal{E}_1$ and $\mathcal{E}_2$ away from x. Let Bun_G be the moduli stack of G-bundles on X. We have natural maps

$$p_i \colon \mathrm{Hecke} \to \mathrm{Bun}_G,$$
$$\pi \colon \mathrm{Hecke} \to X$$

where $p_i(\mathcal{E}_1, \mathcal{E}_2, x, \beta) = \mathcal{E}_i$ and $\pi(\mathcal{E}_1, \mathcal{E}_2, x, \beta) = x$. The fibers of the map

$$p_2 \times \pi \colon \mathrm{Hecke} \to \mathrm{Bun}_G \times X$$

are (non-canonically) isomorphic to the affine Grassmannian and for each $V \in \mathrm{Rep}(\widehat{G})$ there exists a perverse sheaf $\mathrm{IC}_V^{\mathrm{Hk}}$ on Hecke such that its restriction to each fiber of $p_2 \times \pi$ is isomorphic to $\mathrm{IC}_V := \mathcal{S}^{-1}(V)$.

Denote by $D(\mathrm{Bun}_G)$ (resp. $D(\mathrm{Bun}_G \times X)$) the bounded derived category of ℓ-adic sheaves on Bun_G (resp. $\mathrm{Bun}_G \times X$). Define the Hecke functor

$$\mathrm{Hk} \colon \mathrm{Rep}(\widehat{G}) \times D(\mathrm{Bun}_G) \to D(\mathrm{Bun}_G \times X)$$

by the formula $\mathrm{Hk}(V, \mathcal{F}) = (p_2 \times \pi)_!(p_1^* \mathcal{F} \otimes \mathrm{IC}_V^{\mathrm{Hk}})$.

Let E be a $\widehat{G}$-local system on X, viewed as a tensor functor $E : \mathrm{Rep}(\widehat{G}) \to \mathrm{Loc}(X), V \mapsto E^V$.

Definition 2. A Hecke eigensheaf with eigenvalue E is a perverse sheaf $\mathcal{F}$ on Bun_G together with isomorphisms $\alpha_V : \mathrm{Hk}(V, \mathcal{F}) \simeq \mathcal{F} \boxtimes E^V$, for all $V \in \mathrm{Rep}(\widehat{G})$, that are compatible with the symmetric tensor structure on $\mathrm{Rep}(\widehat{G})$.

We are now ready to state a version of the geometric Langlands correspondence.

Conjecture 3. *To every irreducible $\widehat{G}$-local system E, there exists a non-zero Hecke eigensheaf $\mathcal{F}$ with eigenvalue E such that its restriction to each irreducible component of* Bun_G *is an irreducible perverse sheaf.*

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Vector bundles on the Fargues-Fontaine curve

Gabriel Dospinescu

1. The Fargues-Fontaine curve

Let E be a local field with residue field $\mathbf{F}_q$ of characteristic p and let F be a perfectoid and algebraically closed extension of $\mathbf{F}_q$. Finally, let π be a uniformizer of E and ϖ a pseudo-uniformizer of F.

One associates to this data an adic space

$$Y^{\mathrm{ad}} = \mathrm{Spa}(W_{\mathcal{O}_E}(\mathcal{O}_F)) \setminus V(\pi[\varpi])$$

endowed with a Frobenius φ, acting properly and discontinuously and giving rise to an adic space $X^{\mathrm{ad}} = Y^{\mathrm{ad}}/\varphi^{\mathbf{Z}}$ over E. We also obtain a scheme $X = \mathrm{Proj}(P)$ over E, where P is the graded E-algebra

$$P = \bigoplus_{d \geq 0} \mathcal{O}(Y^{\mathrm{ad}})^{\varphi = \pi^d}.$$

Even though P depends on π, X does not. Moreover, if E' is a finite extension of E and X' is the curve attached to E' and F, there is a canonical isomorphism $X' \simeq X \otimes_E E'$.

We fix once and for all a point $\infty \in |X|$ and let C be its residue field (an algebraically closed and complete extension of E) and $i \colon \{\infty\} \to X$ the natural inclusion. X is not of finite type over E, but we have the following deep result of Fargues and Fontaine [3] ($|X|$ is the set of closed points of X).

Theorem 1. (a) *X is a noetherian, regular scheme of dimension* 1.

(b) *$X \setminus \{\infty\}$ is the spectrum of a principal ideal domain.*

(c) *For each rational function $f \in E(X)^*$ we have*

$$\sum_{x \in |X|} v_x(f) = 0.$$

The previous theorem yields the existence of a degree map $\deg \colon \mathrm{Pic}(X) \to \mathbf{Z}$ which allows us to define the degree of any vector bundle on X (using its determinant). Moreover, the theorem shows that $\mathrm{Pic}^0(X) = 0$ and so the degree map $\deg \colon \mathrm{Pic}(X) \to \mathbf{Z}$ is an isomorphism. One checks that a map of vector bundles of the same rank and degree which is a generic isomorphism is actually an isomorphism, thus the theory of Harder-Narasimhan filtrations for vector bundles on X applies (in particular the notions of semi-stable bundle and slope of a bundle make sense).

2. Construction of vector bundles

Let $L = \widehat{E^{\mathrm{nr}}}$ and let $\varphi\mathrm{Mod}_L$ be the category of L-isocrystals (i.e. finite dimensional L-vector spaces with a bijective semi-linear Frobenius). By the Dieudonné-Manin theorem, the category $\varphi\mathrm{Mod}_L$ is semi-simple, with simple objects parametrized by their slope, which is a rational number λ.

By construction Y^{ad} lives over $\mathrm{Spa}(L)$, yielding a functor from $\varphi\mathrm{Mod}_L$ to the category $\mathrm{Bun}_{X^{\mathrm{ad}}}$ of vector bundles on X^{ad} (sending an isocrystal D to the bundle $(D \times Y^{\mathrm{ad}})/\varphi^{\mathbf{Z}}$). This category has good properties since X^{ad} is strongly noetherian, by a theorem of Kedlaya. Using GAGA (which follows by combining work of Fargues-Fontaine [3], Hartl-Pink [6] and Kedlaya-Liu [7]), we obtain a functor

$$\varphi\mathrm{Mod}_L \to \mathrm{Bun}_X, \quad D \mapsto \mathcal{E}(D).$$

Explicitly, $\mathcal{E}(D)$ is the quasi-coherent sheaf attached to the graded P-module

$$\bigoplus_{d \geq 0} (D \otimes_L \mathcal{O}(Y^{\mathrm{ad}}))^{\varphi = \pi^d}.$$

In particular, we obtain a vector bundle $\mathcal{O}(\lambda) = \mathcal{O}_X(\lambda)$ for each $\lambda \in \mathbf{Q}$, namely $\mathcal{O}(\lambda) = \mathcal{E}(D)$ for $D \in \varphi\mathrm{Mod}_L$ simple of slope $-\lambda$ (note the minus sign!).

Remark 2. If $\lambda \in \mathbf{Z}$, then $\mathcal{O}(\lambda)$ is the line bundle attached to the graded P-module $P[\lambda]$ (this really is a line bundle since P is generated by its degree 1 homogeneous elements, a theorem of Fargues and Fontaine). Moreover, one proves that there is an isomorphism $\mathcal{O}(\lambda) \simeq \mathcal{O}(\lambda \cdot \infty)$ (where $\mathcal{O}(\lambda \cdot \infty)$ is the usual line bundle attached to the divisor $\lambda \cdot \infty$), thus $\deg \mathcal{O}(\lambda) = \lambda$ and so $\lambda \mapsto \mathcal{O}(\lambda)$ is the inverse of the isomorphism $\mathrm{Pic}(X) \simeq \mathbf{Z}$ induced by the degree map.

The result (due again to Fargues and Fontaine [3]) we want to discuss in this talk is the following:

Theorem 3 (classification theorem)**.** *The functor $D \mapsto \mathcal{E}(D)$ is essentially surjective. In other words every vector bundle on X is isomorphic to $\mathcal{O}(\lambda_1) \oplus \ldots \oplus \mathcal{O}(\lambda_n)$ for some $\lambda_1, \ldots, \lambda_n \in \mathbf{Q}$, uniquely determined up to permutation.*

We stress that the functor is very far from being fully faithful: $\mathrm{Hom}(\mathcal{O}, \mathcal{O}(1))$ is infinite dimensional over E, while the corresponding Hom on the level of isocrystals is 0.

3. Cohomology of $\mathcal{O}(\lambda)$

Let λ be an integer. Fargues and Fontaine prove that $H^0(\mathcal{O}(\lambda)) = \mathcal{O}(Y^{\mathrm{ad}})^{\varphi = \pi^\lambda}$ and that this is 0 if $\lambda < 0$. The fundamental exact sequence of p-adic Hodge theory translates into

$$0 \to E \to H^0(X \setminus \{\infty\}, \mathcal{O}) \to \mathrm{Frac}(\widehat{\mathcal{O}_{X,\infty}})/\widehat{\mathcal{O}_{X,\infty}} \to 0$$

and this is easily seen to be equivalent to the following

Theorem 4. *We have $H^0(\mathcal{O}) = E$ and $H^1(\mathcal{O}) = 0$.*

Suppose now that $\lambda \notin \mathbf{Z}$ and write $\lambda = \frac{d}{h}$ with $d, h \in \mathbf{Z}$ relatively prime and $h > 0$. Let E_h be the unramified extension of degree h of E and write X_h for the curve attached to E_h and F. We have a finite étale covering $\pi_h \colon X_h \to X$ induced by $X_h \simeq X \otimes_E E_h$. Using the explicit form of the simple isocrystal of slope $-\lambda$, one checks that

$$\mathcal{O}_X(\lambda) = \pi_{h*}(\mathcal{O}_{X_h}(d)).$$

In particular $H^i(\mathcal{O}_X(\lambda)) = H^i(X_h, \mathcal{O}_{X_h}(d))$ for all i, and $\mathcal{O}_X(\lambda)$ is a vector bundle of rank h and degree d, thus of slope λ. Moreover, we have $\pi_h^*\mathcal{O}_X(\lambda) \simeq \mathcal{O}_{X_h}(d)^{\oplus h}$, hence $\mathcal{O}_X(\lambda)$ is semistable.

Using these results and the easily checked fact that $\mathcal{O}(\lambda)\otimes\mathcal{O}(\mu)$ is a direct sum of finitely many copies of $\mathcal{O}(\lambda+\mu)$, one obtains:

Theorem 5. *We have*

(a) $H^0(\mathcal{O}(\lambda)) = 0$ *for* $\lambda < 0$ *and* $H^1(\mathcal{O}(\lambda)) = 0$ *for* $\lambda \geq 0$.

(b) $\mathrm{Hom}(\mathcal{O}(\lambda), \mathcal{O}(\mu)) = 0$ *for* $\mu < \lambda$ *and* $\mathrm{Ext}^1(\mathcal{O}(\lambda), \mathcal{O}(\mu)) = 0$ *for* $\mu \geq \lambda$.

Remark 6. (a) When nonzero the spaces $H^0(O(\lambda))$ and $H^1(O(\lambda))$ are huge (with the exception of $H^0(\mathcal{O}) = E$), for instance $H^1(\mathcal{O}(-1)) \simeq C/E$ and $H^0(\mathcal{O}(1))$ lives in an exact sequence

$$0 \to E \to H^0(\mathcal{O}(1)) \to C \to 0.$$

(b) In particular, there is a huge amount of extensions

$$0 \to \mathcal{O}(-1) \to \mathcal{E} \to \mathcal{O}(1) \to 0$$

and theorems 3 and 5 imply that for any such extension we have $H^0(\mathcal{E}) \neq 0$. This is fairly hard to prove directly. It can be reformulated in the following innocent-looking statement: for all $a, b \in \widehat{\mathcal{O}_{X,\infty}}$ there are $x, y \in H^0(\mathcal{O}(1))$ not both 0 such that $v_\infty(ax+by) \geq 2$ (this can be proved using the theory of Banach-Colmez spaces).

(c) More generally, theorems 3 and 5 imply that any nonzero vector bundle of degree 0 has (nonzero) global sections, that semi-stable vector bundles of slope 0 are trivial and more generally semi-stable vector bundles of slope λ are isomorphic to $O(\lambda)^{\oplus n}$ for some n.

(d) Conversely, in order to prove theorem 3 it suffices to show that any nonzero bundle of degree 0 has (nonzero) global sections. Indeed, one first proves by induction on the rank that semi-stable bundles of slope 0 are trivial[1]. One deduces by twisting and Galois descent that semi-stable bundles of slope λ are isomorphic to $\mathcal{O}(\lambda)^{\oplus n}$. Combining this with theorem 5 yields theorem 3 (and the non-canonical splitting of the Harder-Narasimhan filtration of bundles on X).

In order to prove that bundles of degree 0 have global sections, Fargues and Fontaine use periods of p-divisible groups. We will sketch the argument in the next section.

[1] If $\mathcal{E}$ is such a bundle, by our assumption $H^0(\mathcal{E}) \neq 0$, so there is an embedding $\mathcal{O} \to \mathcal{E}$. The Harder-Narasimhan formalism implies that $\mathcal{E}/\mathcal{O}$ is a semi-stable bundle of slope 0, thus trivial by induction and so $\mathcal{E}$ is trivial since $\mathrm{Ext}^1(\mathcal{O}, \mathcal{O}) = 0$.

4. Link with p-divisible groups

From now on we assume for simplicity that $E = \mathbf{Q}_p$.

Fix a p-divisible group H over $\overline{\mathbf{F}}_p$ and let $D \in \varphi\mathrm{Mod}_L$ be its rational covariant Dieudonné module. Let $\mathcal{E} = \mathcal{E}(D) \otimes \mathcal{O}(1)$. We will use the deformation theory of p-divisible groups to construct many minuscule modifications of $\mathcal{E}$ and give a practical criterion for their triviality.

Let $\mathcal{M}$ be the rigid space over L, generic fibre of the formal scheme over $\mathcal{O}_L$ classifying deformations by quasi-isogenies of H ($\mathcal{M}$ is the Rapoport-Zink space [9] attached to H). A C-point of $\mathcal{M}$ is a (class of isomorphism of a) pair (G, ρ), where G is a p-divisible group over $\mathcal{O}_C$ and ρ is a quasi-isogeny between $H \otimes_{\overline{\mathbf{F}}_q} \mathcal{O}_C/p$ and $G \otimes_{\mathcal{O}_C} \mathcal{O}_C/p$ (we fixed a section of the natural map $\mathcal{O}_C/p \to \overline{\mathbf{F}}_q$).

We have an étale morphism of rigid spaces, the period map $\pi_{GM} \colon \mathcal{M} \to \mathcal{F}$, where $\mathcal{F}$ is the rigid space over L classifying $d = \dim H$-dimensional quotients of D. On C-points π_{GM} is described as follows: let $(G, \rho) \in \mathcal{M}(C)$, then Grothendieck-Messing theory implies that the quasi-isogeny ρ induces an isomorphism between $D \otimes C$ and[2] $\mathrm{Lie}(E(G))[1/p]$. Since $\mathrm{Lie}(G)[1/p]$ is a d-dimensional quotient of $\mathrm{Lie}(E(G))[1/p]$, we obtain a C-point of $\mathcal{F}$, which is by definition $\pi_{GM}(G, \rho)$.

Let $x = [D \otimes C \to W] \in \mathcal{F}(C)$. Recall that $i \colon \{\infty\} \to X$ is the natural inclusion and $\mathcal{E} = \mathcal{E}(D) \otimes \mathcal{O}(1)$. We have a canonical isomorphism $i^*\mathcal{E} = D \otimes C$, which defines a surjection $\mathcal{E} \to i_*(D \otimes C)$ and thus a surjection $\mathcal{E} \to i_*(W)$. Call $\mathcal{E}(x)$ the kernel of this map. It is a vector bundle on X sitting in an exact sequence

$$0 \to \mathcal{E}(x) \to \mathcal{E} \to i_* W \to 0.$$

Definition 7. A degree d (minuscule) modification of a vector bundle $\mathcal{E}$ on X is the kernel of a surjection $\mathcal{E} \to i_* W$, where W is a d-dimensional C-vector space.

The previous construction gives a map $x \mapsto \mathcal{E}(x)$ from $\mathcal{F}(C)$ to the set of degree d modifications of $\mathcal{E}$. The following theorem is a "sheafy" version of the p-adic comparison theorem for p-divisible groups:

Theorem 8. *For any $x = (G, \rho)$ in the image of π_{GM} the vector bundle $\mathcal{E}(x)$ is trivial. More precisely, there is a canonical isomorphism $\mathcal{E}(x) = V_p(G) \otimes_{\mathbf{Q}_p} \mathcal{O}_X$.*

The converse of the previous theorem holds, by results of Fargues [4], Faltings [2] and Scholze-Weinstein [10], giving a description of the image of the period map in terms of modifications of vector bundles on X.

We sketch the proof of the previous theorem: letting $B = \mathcal{O}(Y^{\mathrm{ad}})$, Dieudonné and Grothendieck-Messing theory give a map $V_p(G) \to (D \otimes B)^{\varphi = p}$ such that the induced map $V_p(G) \otimes B[1/t] \to D \otimes B[1/t]$ is an isomorphism. On the other hand we have $H^0(\mathcal{E}) = (D \otimes B)^{\varphi = p}$, thus we obtain a map $V_p(G) \otimes \mathcal{O}_X \to \mathcal{E}$, which factors through $\mathcal{E}(x)$ again by general considerations, yielding a generic isomorphism $V_p(G) \otimes \mathcal{O}_X \to \mathcal{E}(x)$. Since the two vector bundles have the same rank (namely $\dim D = \mathrm{ht}(H) = \dim V_p(G)$) and the same degree (an easy computation

[2] $\mathrm{Lie}(E(G))$ is the Lie algebra of the universal vector extension $E(G)$ of G

shows that $\mathcal{E}$ has degree $d = \dim H$, thus $\mathcal{E}(x)$ has degree 0), this map is thus an isomorphism, yielding the theorem.

Corollary 9. *If π_{GM} is surjective, then all degree d modifications of $\mathcal{E}$ are trivial vector bundles.*

An important example is the following: let H be a formal p-divisible group of dimension 1 and height $n \geq 1$ over $\overline{\mathbf{F}}_p$, then $\mathcal{M}$ is the Lubin-Tate space, $\mathcal{F}$ is the projective space over D (thus isomorphic to $\mathbf{P}^{n-1}$), and a theorem of Laffaille [8] and Gross-Hopkins [5] shows that π_{GM} is surjective. Moreover $\mathcal{E} = \mathcal{O}(\frac{1}{n})$. The previous corollary thus yields the first part of the following crucial result:

Theorem 10. *Let $n \geq 1$.*
(a) *Any degree 1 modification of $\mathcal{O}(\frac{1}{n})$ is trivial (i.e. isomorphic to $\mathcal{O}^n$).*
(b) *Any degree 1 modification of $\mathcal{O}^n$ either has nonzero global sections or is isomorphic to $\mathcal{O}(-\frac{1}{n})$.*

Part (b) of theorem 10 follows by combining the previous corollary with a theorem of Drinfeld [1], giving the image of the period map for the Drinfeld space. This space appears naturally as follows: if $\mathcal{E}$ is a degree 1 modification of $\mathcal{O}^n$, arising as kernel of a map $\mathcal{O}^n \to i_*C$, then saying that $\mathcal{E}$ has no nonzero global sections is precisely saying that the n elements of C that define the map $\mathcal{O}^n \to i_*C$ are linearly independent over $\mathbf{Q}_p$, i.e. they define a point in the Drinfeld space $\Omega(C) \subset \mathbf{P}^{n-1}(C)$.

Theorem 10 implies theorem 3 rather formally. Since the technical details are a bit painful, I will just explain why any extension

$$0 \to \mathcal{O}(-1) \to \mathcal{E} \to \mathcal{O}(1) \to 0$$

has (nonzero) global sections. This already contains the key ideas in the proof.

Assume that $H^0(\mathcal{E}) = 0$. Let $\mathcal{F}$ be the kernel of the map $\mathcal{E} \to \mathcal{O}(1) \to i_*C$, where $\mathcal{O}(1) \to i_*C$ is the natural map (with kernel $\mathcal{O}$). Then we have an exact sequence $0 \to \mathcal{O}(-1) \to \mathcal{F} \to \mathcal{O} \to 0$. Embed $\mathcal{O}(-1) \to \mathcal{O}$ (as $\mathcal{O}$-modules) and consider the resulting push-out extension, which is an extension of $\mathcal{O}$ by itself, thus trivial (since $\mathrm{Ext}^1(\mathcal{O}, \mathcal{O}) = 0$). It follows that $\mathcal{F}$ embeds (as $\mathcal{O}$-module) in $\mathcal{O}^2$ and so it is a degree -1 modification of $\mathcal{O}^2$, without nonzero global sections (as $H^0(\mathcal{E}) = 0$ and $\mathcal{F}$ embeds into $\mathcal{E}$). By theorem 10 we obtain $\mathcal{F} \simeq \mathcal{O}(-\frac{1}{2})$, thus we obtain an exact sequence

$$0 \to \mathcal{O}(-\frac{1}{2}) \to \mathcal{E} \to i_*C \to 0.$$

Taking duals and using theorem 10 again, we deduce that $\mathcal{E}^\vee$ is trivial, thus $\mathcal{E}$ is trivial, contradicting the equality $H^0(\mathcal{E}) = 0$. This finishes the proof.

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Banach-Colmez spaces

Arthur-Cesar Le Bras

Banach-Colmez spaces were introduced by Colmez in [1] (under the name "*Espaces de Banach de Dimension Finie*") almost fifteen years ago to give a new proof of the conjecture "weakly admissible implies admissible" in p-adic Hodge theory. The goal of the talk was to show why they are important and ubiquitous.

Let C be the completion of an algebraic closure of $\mathbf{Q}_p$. Let Perf_C be the category of perfectoid spaces over C, and $\mathrm{Perf}_{C,\text{proét}}$ be the big pro-étale site of C (the above category endowed with the pro-étale topology). We will look at presheaves on the category Perf_C with values in the category of $\mathbf{Q}_p$-topological vector spaces, which are sheaves on $\mathrm{Perf}_{C,\text{proét}}$ when viewed simply as presheaves of $\mathbf{Q}_p$-vector spaces. Such a functor $\mathcal{F}$ is called a *Banach sheaf* when $\mathcal{F}(X)$ is a Banach space for all affinoid perfectoid X. Here are two simple examples of Banach sheaves : the constant sheaf $\underline{V}$, for any finite dimensional $\mathbf{Q}_p$-vector space V ; the sheaf $W \otimes_C \mathcal{O}$, for any finite dimensional C-vector space W.

The following definition looks a bit different from Colmez's one, but is actually equivalent.

Definition 1. An *effective Banach-Colmez space* is a Banach sheaf $\mathcal{F}'$ which is an extension[1]

$$0 \to \underline{V} \to \mathcal{F}' \to W \otimes_C \mathcal{O} \to 0,$$

V (resp. W) being a finite dimensional $\mathbf{Q}_p$-vector space (resp. a finite dimensional C-vector space). A *Banach-Colmez space* is a Banach sheaf $\mathcal{F}$ which is a quotient

$$0 \to \underline{V'} \to \mathcal{F}' \to \mathcal{F} \to 0,$$

where $\mathcal{F}'$ is an effective Banach-Colmez space and V' a finite dimensional $\mathbf{Q}_p$-vector space. The category of Banach-Colmez spaces will be denoted $\mathcal{BC}$.

[1] By definition, a sequence of Banach sheaves is said to be *exact* if it is so as a sequence of sheaves of $\mathbf{Q}_p$-vectors spaces on $\mathrm{Perf}_{C,\text{proét}}$.

To any presentation of a Banach-Colmez space as in the definition, we associate two integers : its *dimension* $\dim_C W$ and its *height* $\dim_{\mathbf{Q}_p} V - \dim_{\mathbf{Q}_p} V'$.

The definition of the category of Banach-Colmez spaces may look a bit strange, but Colmez proved the following difficult theorem ([1]).

Theorem 2. *The category $\mathcal{BC}$ is an abelian category. The functor $\mathcal{F} \mapsto \mathcal{F}(C)$ is exact and conservative on $\mathcal{BC}$.*

Moreover, the functions dimension and height do not depend on the presentation and define two additive functions on $\mathcal{BC}$.

At this point it is not clear that there exist interesting examples of Banach-Colmez spaces apart the obvious ones. To construct geometrically such examples, one can use p-divisible groups, as was observed by Fargues ([2]). Let G be a p-divisible group over $\mathcal{O}_C$. Its *universal cover* $\tilde{G}$ is the sheaf which associates to any perfectoid algebra R over C the $\mathbf{Q}_p$-vector space

$$\tilde{G}(R) = \varprojlim_{\times p} \varprojlim_{k} \varinjlim_{n} G[p^n](R^\circ/p^k).$$

This sheaf is representable by a perfectoid space over C. For example if $G = \mathbf{Q}_p/\mathbf{Z}_p$, $\tilde{G} = \underline{\mathbf{Q}_p}$.[2] In general, one has an exact sequence of pro-étale sheaves

$$0 \to \underline{V(G)} \to \tilde{G} \xrightarrow{\log_G} \mathrm{Lie}(G)[p^{-1}] \otimes \mathcal{O} \to 0, \tag{1}$$

$V(G)$ being the rational Tate module of G. As moreover $\tilde{G}(R)$ is a Banach space for any perfectoid C-algebra R, this exact sequence shows that universal covers of p-divisible groups are examples of effective Banach-Colmez spaces ! Actually, one can prove the following result

Theorem 3. *Universal covers of p-divisible groups over $\mathcal{O}_C$ are Banach-Colmez spaces and any Banach-Colmez space is the quotient of the universal cover of a p-divisible group by the Banach-Colmez space $\underline{V}$ associated to some finite dimensional $\mathbf{Q}_p$-vector space V.*

This result has two consequences. The first one is the

Corollary 4. *Banach-Colmez spaces are diamonds over* $\mathrm{Spa}(C^\flat)$.[3]

The deep results of Fargues [2] and Scholze-Weinstein [4] on p-divisible groups allow to describe universal covers in terms of p-adic Hodge theory. The second consequence of the theorem is thus that one can get many explicit examples of Banach-Colmez spaces by playing with Fontaine rings. Here is an example : for any $\lambda = d/h \in \mathbf{Q}$, $\lambda \geq 0$, the functor $\mathbf{U}_\lambda \colon R \mapsto B^+_{\mathrm{cris}}(R^\circ)^{\varphi^h = p^d}$ is a Banach-Colmez space. For instance, $\mathbf{U}_1 = \tilde{\mu}_{p^\infty}$ and the exact sequence (1) for $G = \mu_{p^\infty}$ evaluated on C becomes identified with the famous exact sequence

$$0 \to \mathbf{Q}_p.t \to (B^+_{\mathrm{cris}})^{\varphi = p} \xrightarrow{\theta} C \to 0.$$

[2]This sheaf is representable by the perfectoid space $\mathrm{Spa}(\mathcal{C}^0(\mathbf{Q}_p, C), \mathcal{C}^0(\mathbf{Q}_p, \mathcal{O}_C))$.

[3]Here we implicitely identify $\mathrm{Perf}_{C,\mathrm{pro\acute{e}t}}$ with $\mathrm{Perf}_{C^\flat,\mathrm{pro\acute{e}t}}$, using Scholze's equivalence.

To completely elucidate the nature and the structure of the category $\mathcal{BC}$, we now turn to the relation with the Fargues-Fontaine curve X (for $E = \mathbf{Q}_p$, $F = C^\flat$). Let

$$\mathrm{Coh}^{0,-}(X) = \{\mathcal{F} \in D(X), H^i(\mathcal{F}) = 0 \text{ for } i \neq -1, 0; H^{-1}(\mathcal{F}) < 0, H^0(\mathcal{F}) \geq 0\},$$

where $D(X)$ is the bounded derived category of the abelian category $\mathrm{Coh}(X)$ of coherent sheaves on X, and where for $\mathcal{E} \in \mathrm{Coh}(X)$, the notation $\mathcal{E} \geq 0$ (resp. $\mathcal{E} < 0$) means that all the slopes of $\mathcal{E}$ are non negative (resp. negative). This full subcategory of $D(X)$ is actually an abelian category (this is a consequence of the general theory of *tilting* and *torsion pairs*), and is endowed with a degree function $\deg^{0,-}$ and a rank function $\mathrm{rk}^{0,-}$.

For any perfectoid space S over $C^\flat$, there exists a relative version X_S of the curve (for $S = \mathrm{Spa}(C^\flat)$, this is just the usual Fargues-Fontaine curve X). Although there is *no* morphism of adic spaces $X_S \to S$, one has a morphism of sites τ from $\mathrm{Perf}_{C^\flat,\text{proét}}$ to the big pro-étale site of X. In particular, one can associate to any complex of coherent sheaves $\mathcal{F}$ on X a sheaf $R^j\tau_*\mathcal{F}$ on $\mathrm{Perf}_{C^\flat,\text{proét}}$, for any $j \geq 0$.

Theorem 5. *The functor $R^0\tau_*$ induces an equivalence of categories*

$$\mathrm{Coh}^{0,-}(X) \simeq \mathcal{BC}.$$

Under this equivalence the functions $\deg^{0,-}$ *and* $-\mathrm{ht}$ *(resp.* $\mathrm{rk}^{0,-}$ *and* $\dim$*) correspond to each other.*

For example, $R^0\tau_*$ sends $\mathcal{O}_X$ to $\underline{\mathbf{Q}_p}$, $i_{\infty,*}C$ to $\mathcal{O}$, and $\mathcal{O}_X(-1)[1]$ to $\mathcal{O}/\mathbf{Q}_p$. This result gives a precise meaning to the idea that all Banach-Colmez spaces can be obtained by using H^0 and H^1 of coherent sheaves on the Fargues-Fontaine curve. It also shows that the category $\mathcal{BC}$ only depends on $C^\flat$.

Using this result and the corollary 4, one can show that automorphism groups of vector bundles on X are diamonds : see [3, Prop. 2.5]. For example, $\mathrm{Aut}(\mathcal{O}_X^n) = \underline{GL_n(\mathbf{Q}_p)}$ (and not the algebraic group GL_n!). This point is important for Fargues's conjecture.

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The relative Fargues–Fontaine curve

Matthew Morrow

There are two primary goals of this talk:

(1) Define Y_S and the relative Fargues–Fontaine curve $X_S = Y_S/\varphi^{\mathbb{Z}}$ for an arbitrary perfectoid space S over $\mathbb{F}_p$. These will be adic spaces over $\operatorname{Spa}\mathbb{Q}_p := \operatorname{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ which, in the special case $S = \operatorname{Spa}(\mathbb{C}_p^\flat, \mathcal{O}_{\mathbb{C}_p}^\flat)$, reduce to the adic spaces Y^{ad} and X^{ad} which appeared in Colmez' talk.

(2) Relate Y_S to untiltings of S and describe how the formula
$$\text{“}Y_S = S \times \operatorname{Spa}\mathbb{Q}_p\text{”}$$
can be made precise using diamonds.

We mention that, by picking an auxiliary local (or perfectoid) field E, one may more generally construct $Y_{S,E}$ and $X_{S,E}$; in this talk we are implicitly restricting entirely to the case $E = \mathbb{Q}_p$.

For further details and references we refer the reader primarily to Caraiani–Scholze [1, §3.3] and Fargues [3, §1.1–1.3] [4, §1.1–1.4].

1. Constructing Y_S and X_S

1.1. **Case of affinoid perfectoid** S**.** We begin by constructing Y_S and X_S in the case that $S := \operatorname{Spa}(R, R^+)$ is affinoid perfectoid over $\mathbb{F}_p$; fix a pseudo-uniformiser $\pi \in R$. Set $\mathbb{A} := W(R^+)$, which is equipped with the $\langle p, [\pi]\rangle$-adic topology, and define a preadic $\operatorname{Spa}\mathbb{Q}_p$-space
$$Y_{(R,R^+)} := \operatorname{Spa}(\mathbb{A}, \mathbb{A}) \setminus V(p[\pi]).$$
Concretely, a point of $Y_{(R,R^+)}$ is a continuous absolute value $|\cdot|\colon \mathbb{A} \to \Gamma \cup \{0\}$ which satisfies $|a| \le 1$, for all $a \in \mathbb{A}$, and $|p[\pi]| \neq 0$; it follows from this latter condition that the vanishing ideal of $|\cdot|$ is not open in $\mathbb{A}$, i.e., $Y_{(R,R^+)}$ is an *analytic* preadic space, and that moreover the *radius function*
$$\delta\colon Y_{(R,R^+)} \longrightarrow (0,1), \quad (|\cdot|, \Gamma) \mapsto p^{-\sup\{r/s \in \mathbb{Q}_{\ge 0} : |[\pi]|^r \ge |p|^s\}}$$
(“the closest point to $|p|$ on the positive real line spanned by $|[\pi]|$”) is a well-defined, continuous map. We may therefore introduce, for any closed interval $I \subset (0,1)$, the associated *annulus*
$$Y_{(R,R^+)} \overset{\text{open}}{\supseteq} Y^I_{(R,R^+)} := \text{the interior of the preimage } \delta^{-1}(I),$$
which can be shown, in the case that $I = [p^{-r/s}, p^{-r'/s'}]$ for $r, s, r', s' \in \mathbb{N}$, to be the rational subdomain of $\operatorname{Spa}(\mathbb{A}, \mathbb{A})$ consisting of those points $|\cdot|$ for which $|[\pi]|^r \le |p|^s$ and $|[\pi]|^{r'} \ge |p|^{s'}$. Clearly therefore $Y_{(R,R^+)}$ is the filtered increasing union, over all closed intervals $I \subset (0,1)$, of the associated annuli.

It can be shown that $Y_{(R,R^+)}$ is sheafy, i.e., an adic space. To do this one picks a perfectoid field $E/\mathbb{Q}_p$ and checks that $Y^I_{(R,R^+)} \times_{\operatorname{Spa}\mathbb{Q}_p} \operatorname{Spa} E$ is affinoid perfectoid, hence sheafy by Scholze or Kedlaya–Liu. In other words $Y^I_{(R,R^+)}$ is *preperfectoid*, and hence is also sheafy; see [2, §2.2] for further details and references. It then follows immediately from the description of $Y_{(R,R^+)}$ as a union of annuli that it is also sheafy.

1.2. **The quotient by the Frobenius.** The usual Witt vector Frobenius φ on $\mathbb{A}$ induces a Frobenius action φ on $Y_{(R,R^+)}$ which satisfies $\delta(\varphi(y)) = \delta(y)^{1/p}$ for all $y \in Y_{(R,R^+)}$. It follows that this latter action is proper and totally discontinuous, whence

$$X_{(R,R^+)} := Y_{(R,R^+)}/\varphi^{\mathbb{Z}}$$

is a well-defined adic space over $\operatorname{Spa}\mathbb{Q}_p$ and $Y_{(R,R^+)} \to X_{(R,R^+)}$ is an open quotient map. Moreover, if $I = [a,b] \subset (0,1)$ is an interval satisfying $b^p < a \leq b < a^{1/p}$, then $Y^I_{(R,R^+)}$ is disjoint from $\varphi^n(Y^I_{(R,R^+)})$ for all $0 \neq n \in \mathbb{Z}$, and so this quotient map sends $Y^I_{(R,R^+)}$ isomorphically to an open subspace of $X_{(R,R^+)}$. In short, sufficiently thin annuli provide an explicit affinoid open cover of $X_{(R,R^+)}$.

1.3. **The case of general S.** For any closed interval $I \subset (0,1)$ and suitable elements $f_1, \ldots, f_n, g \in R^+$, it is not hard to check that there is a natural identification between

$$Y^I_{(R,R^+)}\left\langle \tfrac{[f_1],\ldots,[f_n]}{[g]} \right\rangle \qquad \text{and} \qquad Y^I_{(R\langle \frac{f_1,\ldots,f_n}{g}\rangle, R\langle \frac{f_1,\ldots,f_n}{g}\rangle^+)},$$

where the left is a rational subdomain of $Y^I_{(R,R^+)}$ and the right is $Y^I_{(-,-)}$ of a localisation of the pair (R,R^+). It is therefore straightforward to glue along rational subdomains in order to define Y_S and the *relative Fargues–Fontaine curve* $X_S := Y_S/\varphi^{\mathbb{Z}}$ for an arbitrary perfectoid space S over $\mathbb{F}_p$.

1.4. **The map θ.** In the case in which S is the tilt $S^{\sharp\flat}$ of some fixed perfectoid space $S^\sharp$ over $\operatorname{Spa}\mathbb{Q}_p$, there is an induced closed immersion $\theta\colon S^\sharp \hookrightarrow Y_S$ which is locally given by Fontaine's map $\theta\colon W(R^+) \to R^{\sharp+}$ arising from the universal property of Witt vectors. Remarkably, the composition $S^\sharp \overset{\theta}{\hookrightarrow} Y_S \to X_S$ is still a closed embedding: indeed, we may assume that $S = \operatorname{Spa}(R,R^+)$ and $S^\sharp = \operatorname{Spa}(R^\sharp, R^{\sharp+})$ are affinoid perfectoid, in which case the kernel of Fontaine's map is generated by a *degree one primitive element*, i.e., an element $\xi \in \mathbb{A}$ of the form $\xi = [\pi] + pu$ where $\pi \in R$ is a pseudo-uniformiser and $u \in \mathbb{A}^\times$; it follows easily that the closed immersion $\theta\colon \operatorname{Spa}(R^\sharp, R^{\sharp+}) \hookrightarrow Y_{(R,R^+)}$ factors through the annulus associated to the interval $[p^{-1}, p^{-1}]$, which as explained in 1.2 maps isomorphically to an open subspace of $X_{(R,R^+)}$.

2. Diamonds and untilting

If X is an analytic adic space over $\operatorname{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$, then there is an associated presheaf

$$X^\diamond\colon \operatorname{Perf}_{\mathbb{F}_p} \longrightarrow \text{Sets}, \qquad T \mapsto \text{untilts over } X \text{ of } T,$$

where the right side is more precisely defined to the set of pairs, up to the obvious notion of an isomorphism of pairs, $(T^\sharp, \iota)$ where $T^\sharp$ is a perfectoid space over X and $\iota\colon T^{\sharp\flat} \overset{\simeq}{\to} T$. If X is itself a perfectoid space, then the equivalence of categories between perfectoid spaces over $X^\flat$ and perfectoid spaces over X implies that $X^\diamond$

canonically identifies with the representable presheaf $\mathrm{Hom}(-, X^\flat)$; as a special case, if X is a perfectoid space over $\mathbb{F}_p$ then $X^\diamond$ identifies with $\mathrm{Hom}(-, X)$.

An important result (though not strictly necessary for the talk) is that $X^\diamond$ is a sheaf for the pro-étale topology on $\mathrm{Perf}_{\mathbb{F}_p}$, and even a diamond (recall from Hellmann's talk that diamonds are a full subcategory – informally the pro-étale quotients of representable objects – of pro-étale sheaves on $\mathrm{Perf}_{\mathbb{F}_p}$). Informally, this is proved by picking a perfectoid cover $\{U_i\}_i$ of X in the pro-étale topology and then noting that $X^\diamond$ is a pro-étale quotient of $\bigsqcup_i U_i^\diamond = \bigsqcup_i \mathrm{Hom}(-, U_i^\flat)$.

We may now state the two main results of the talk; let S be a perfectoid space over $\mathbb{F}_p$. Firstly, there is a natural isomorphism of diamonds (equivalently, of pro-étale sheaves on $\mathrm{Perf}_{\mathbb{F}_p}$)

$$Y_S^\diamond \cong S^\diamond \times \mathrm{Spa}\,\mathbb{Q}_p^\diamond,$$

which gives a precise meaning to the sense in which Y_S is the product of S and $\mathrm{Spa}\,\mathbb{Q}_p$. Secondly, the following four collections are in canonical bijection with one another:

(I) Sections of the projection $Y_S^\diamond \to S^\diamond$.
(II) Maps $S^\diamond \to \mathrm{Spa}\,\mathbb{Q}_p^\diamond$.
(III) Untilts in characteristic zero (i.e., over $\mathrm{Spa}\,\mathbb{Q}_p$) of S.
(IV) Closed immersions into Y_S defined locally by a degree one primitive element.

Concerning proofs, we restrict ourselves here to the briefest sketch. The isomorphism in the product formula is given, for each test object $T \in \mathrm{Perf}_{\mathbb{F}_p}$, by

$$\mathrm{Hom}(T, S) \times \mathrm{Spa}\,\mathbb{Q}_p^\diamond(T) \longrightarrow Y_S^\diamond(T), \qquad (f, (T^\sharp, \iota)) \mapsto (T^\sharp, \iota),$$

where the $T^\sharp$ on the right is viewed as a perfectoid space over Y_S via the composition

$$T^\sharp \overset{\theta}{\hookrightarrow} Y_{T^{\sharp\flat}} \overset{\iota}{\cong} Y_T \xrightarrow{f} Y_S.$$

This is shown to be a bijection using the universal nature of Fontaine's map. Meanwhile, (I) and (II) trivially correspond since $Y_S^\diamond \cong S^\diamond \times \mathrm{Spa}\,\mathbb{Q}_p$; secondly, (II) and (III) correspond by the Yoneda Lemma; thirdly, (III) and (IV) correspond thanks to the converse of an assertion in 1.4, namely that each degree one primitive element $\xi \in \mathbb{A}$ gives rise to an untilt $\mathbb{A}/\xi[\frac{1}{p}]$ of R.

The two main results of the previous paragraph have obvious analogues in which Y_S is replaced by X_S, untilts are taken modulo Frobenius equivalence, and $S^\diamond$ is replaced by $S^\diamond/\varphi^{\mathbb{Z}}$, though these were unfortunately not covered in the talk.

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Beauville-Laszlo uniformization

Yakov Varshavsky

Abstract: Let k be an algebraically closed field, G be a split reductive group over k, and X a connected smooth projective curve over k. The goal of my talk is to show that the moduli stack $\mathrm{Bun}_G(X)$ of G-bundles on X is uniformized by the affine Grassmanian Gr_G.

We follow a beautiful work of Drinfeld–Simpson (see [1]).

I. Existence of B-structures. Let $B \subset G$ be a Borel subgroup. Our first goal is to show that a G-bundle over X admits a B-structure, étale locally on the space of parameters. More precisely, let S be a scheme over k, and let $\mathcal{F}$ be a G-bundle over $X \times S$. Then we have the following result.

Theorem 1. There exists an étale surjective morphism $S' \to S$ such that the G-bundle $\mathcal{F} \times_S S'$ over $X \times S'$ has a B-structure.

Remark. By a theorem of Steinberg, we have $H^1(k(X), G) = 1$, therefore every G-bundle over X is generically trivial. Thus, for $S = \mathrm{Spec}\, k$, our Theorem 1 follows from the valuative criterion.

Strategy of the proof. Consider the moduli scheme $M_{\mathcal{F}}$ classifying B-structures of $\mathcal{F}$, and its open (and closed) subscheme $M_{\mathcal{F}}^+ \subset M_{\mathcal{F}}$ classifying B-structures "all of its degrees" are sufficiently small. It suffices to show that the projection $M_{\mathcal{F}}^+ \to S$ is smooth and surjective.

The smoothness assertion follows from the deformation theory and the Riemann-Roch theorem. To show the surjectivity, we can assume that $S = \mathrm{Spec}\, k$. Then, by the remark above, we can assume that $\mathcal{F}$ is trivial, hence $X = \mathbb{P}^1$, and G is semisimple and simply connected. In this case, the assertion follows from the fact that the substack of trivial G-bundles in $\mathrm{Bun}_G(\mathbb{P}^1)$ is open and dense.

II. Triviality of the bundle, restricted to the punctured curve. Assume now that G is semisimple. Our second goal is to show that the G-bundle on X becomes trivial when restricted to the punctured curve, fppf locally with respect to the scheme of parameters (and "fppf" can be replaced by "étale" in most cases). More precisely, let $x \in X$ be a closed point, set $X^0 := X - x$, and let S and $\mathcal{F}$ be as above. Then we have the following result.

Theorem 2. There exists a faithfully flat morphism of finite presentation $S' \to S$ such that the restriction $\mathcal{F} \times_S S'|_{X^0 \times S'}$ is a trivial G-bundle. Moreover, $S' \to S$ can be chosen to be étale, if the cardinality of $\pi_1(G)$ is invertible in k.

Remark. Theorem 2 is equivalent to the assertion that the natural projection $\mathrm{Gr}_G \to \mathrm{Bun}_G(X)$ from the affine Grassmannian of G to the moduli stack of G-bundles on X is surjective in the fppf (or étale) topology.

Strategy of the proof. The assertion is local in S, so we can assume that S is affine. By Theorem 1, we can assume that $\mathcal{F}$ comes from a B-bundle. Moreover, using the fact that X^0 and S are affine, we can assume that $\mathcal{F}$ comes from a T-bundle $\mathcal{E}$, where $T \subset B$ is a maximal torus.

Since the assertion is tautological, if $\mathcal{E}$ is trivial, it remains to show that for every two T-bundles $\mathcal{E}$ and $\mathcal{E}'$ over $X \times S$, the induced G-bundles $G \overset{T}{\times} \mathcal{E}$ and $G \overset{T}{\times} \mathcal{E}'$ have isomorphic restrictions to $X^0 \times S$ after we pass to a fppf (or étale) cover of S.

Replacing S by a finite abelian covering $S' \to S$, whose order is a power of $|\pi_1(G)|$, we can assume that G is simply connected. In this case, we can assume that $\mathcal{E}$ and $\mathcal{E}'$ differ by a simple root, thus reducing to the "GL(2)-case". Namely, it suffices to show that if $\mathcal{E}$ and $\mathcal{E}'$ are rank two vector bundles over $X \times S$ such that $\det \mathcal{E} \cong \det \mathcal{E}'$, then the restrictions of $\mathcal{E}$ and $\mathcal{E}'$ to $X^0 \times S$ become isomorphic after we pass to a Zariski cover of S.

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The Classification of G-bundles

Michael Rapoport

We want to generalize the classification of vector bundles on Fargues-Fontaine curves to G-bundles (recovering the old results when $G = \mathrm{GL}_n$).

1. Background

1.1. **Notation.** We fix the notation for this talk: let

- E be a local field (of characteristic 0 or p),
- ϖ_E the uniformizer,
- $\mathbb{F}_q$ the residue field,
- $\breve{E}$ the completion of the maximal unramified extension, and
- F an algebraically closed perfectoid field of characteristic p.
- $X = X_{E,F}$ the corresponding Fargues-Fontaine curve.

1.2. **Classical G-bundles.**

Definition 1. Let G be a connected linear algebraic group over E. A *G-bundle* on X can be defined in either of the following two ways:

(i) ("internal") A principal homogeneous space $\mathcal{T}$ under G on X which is locally trivial for the (étale or fppf) topology (a *G-torsor*).

(ii) ("external") An exact faithful E-linear $\otimes-$functor $\mathcal{V}\colon \mathrm{Rep}_E\, G \to \mathrm{Vect}_X$.

(Note, however, that Vect_X is not an abelian category, it is only an exact category.)

Example 2. Why are the two definitions equivalent? We sketch one direction. Given a G-torsor $\mathcal{T}$, we can define the functor

$$\mathcal{V}_{\mathcal{T}}((V,\rho)) = \mathcal{T} \times^{G,\rho} V.$$

Definition 3. We denote by $|\operatorname{Bun}_G|$ the set of isomorphism classes of G-bundles on X.

Example 4. If $G = \mathrm{GL}_n$ then $\operatorname{Bun}_G = \operatorname{Vect}_{X,n}$.

1.3. **The classification of** Vect_X**.** We have a functor

$$\mathcal{E}\colon \varphi - \operatorname{Mod}_{\breve{E}} \to \operatorname{Vect}_X$$

sending

$$(N,\varphi) \mapsto \bigoplus_{d\geq 0} \left(B^+_{E,F} \otimes_{\breve{E}} N\right)^{\varphi=\varpi_E^d}.$$

An equivalent way, using GAGA, is to take the quotient of $N \times_{\breve{E}} Y_E$ with φ acting diagonally on Y_E and on N.

Theorem 5. *The functor $\mathcal{E}$ is a faithful exact and E-linear $\otimes$-functor, which is essentially surjective (but not fully faithful, see Warning 6).*

It also induces an equivalence of categories

$$\text{(isoclinic } \varphi\text{-isocrystals)} \leftrightarrow \text{(semi-stable vector bundles)}$$

and a bijection of sets of isomorphism classes

$$|\varphi - \operatorname{Mod}_{\breve{E}}| = |\operatorname{Vect}_X|.$$

Warning 6. The functor is *not* fully faithful because $\operatorname{End}(\mathrm{Triv} \oplus \mathrm{Triv}(1))$ is $E \oplus E$ in the category of isocrystals but a “Banach-Colmez-like object” $\begin{pmatrix} E & BC \\ & E \end{pmatrix}$ in the category of vector bundles.

This theorem is what we want to generalize, from vector bundles to G-bundles.

2. G-isocrystals (following Kottwitz)

In this section we survey some results in [3].

2.1. **The definition.**

Definition 7. Let G be a connected linear algebraic group over E. A *G-isocrystal* can be defined in either of the following two ways:

(i) (external) An exact faithful E-linear $\otimes$-functor

$$N\colon \operatorname{Rep}_E G \to \varphi - \operatorname{Mod}_{\breve{E}}.$$

(ii) (internal) An element $b \in G(\breve{E})$. These form a category via

$$\operatorname{Hom}(b,b') = \{g \in G(\breve{E}) \mid gb\sigma(g)^{-1} = b'\}.$$

We denote by $B(G)$ the set of G-isocrystals up to isomorphism.

Example 8. Why are the internal and external versions equivalent? Given $b \in G(\breve{E})$, we can associate the functor N_b defined by

$$N_b((V,\rho)) = (V \otimes_E \breve{E}, \rho(b) \circ (\mathrm{Id} \otimes \sigma))$$

Essential surjectivity follows from Steinberg's theorem.

Example 9. For $G = \mathrm{GL}_n$, the classical isocrystal description of an element $b \in G(\breve{E})$ is $(\breve{E}^{\oplus n}, b \circ \sigma)$.

2.2. **The Newton and Kottwitz invariants.** Let G be reductive. We construct two invariants associated to G-isocrystals.

The Newton Invariant. Let $b \in G(\breve{E})$. Then we can associate to b a homomorphism

$$\nu_b \colon \mathbb{D}_{\breve{E}} \to G_{\breve{E}}$$

where $\mathbb{D}$ is the split pro-torus over E with $X^*(\mathbb{D}) = \mathbb{Q}$. This homomorphism ν_b is characterized by the property that for all (V, ρ), the morphism

$$\rho \circ \nu_b \colon \mathbb{D}_{\breve{E}} \to \mathrm{GL}(V_{\breve{E}})$$

has induced $\mathbb{Q}$-grading on $V_{\breve{E}}$ equal to the slope grading of $(V_{\breve{E}}, \rho(b)\sigma)$.

The set $X_*(G)_{\mathbb{Q}}$ of homomorphisms $\mathbb{D}_{\breve{E}} \to G_{\breve{E}}$ has an action of $G(\breve{E})$, and we set

$$X_*(G)_{\mathbb{Q}}/G = \mathrm{Hom}_{\breve{E}}(\mathbb{D}_{\breve{E}}, G_{\breve{E}})/G(\breve{E}).$$

There is an action of σ on $X_*(G)_{\mathbb{Q}}/G$, and one can show that the image of ν_b in $(X_*(G)_{\mathbb{Q}}/G)^\sigma$ only depends on $[b]$, thus inducing a well-defined map

$$\nu \colon B(G) \to (X_*(G)_{\mathbb{Q}}/G)^\sigma. \tag{1}$$

This is the *Newton invariant.*

Example 10. If G is quasi-split, say with Borel B, maximal torus $T \subset B$, and maximal split torus $A \subset T \subset B$, then the right side of (1) can be identified with $X_*(A)^+_{\mathbb{Q}}$.

Remark 11. There is also an internal definition of the Newton invariant. Given b, there exists b' with $b \sim b'$ such that for $s \gg 0$

$$(b'\sigma)^s = s\nu_{b'}(\varpi_E) \cdot \sigma^s,$$

with the equality taking place in $G(\breve{E}) \rtimes \langle \sigma \rangle$. This characterizes $\nu_{[b]} = \nu_{[b']}$ (since ν is supposed to be defined on isomorphism classes).

The Kottwitz invariant. Consider

$$\pi_1(G) = X_*(T)/X_*(T_{sc}).$$

This is independent of the choice of the maximal torus, and is canonically and functorially associated to G, and admits an action of $\Gamma := \mathrm{Gal}(\bar{E}/E)$. The Kottwitz invariant is described in terms of this fundamental group, as a map into the set of coinvariants,

$$\kappa \colon B(G) \to \pi_1(G)_\Gamma.$$

This is not so easy to define, but we will try to give some feeling for it. Roughly $G(\breve{E})$ is similar to a loop group LG (although this isn't quite right) and $\pi_0(LG) =$

$\pi_1(G)_I$, where I denotes the inertia subgroup of Γ. Now κ is defined so as to make the following diagram commutative,

$$\begin{tikzcd} B(G) \arrow[r, dotted] & \pi_1(G)_\Gamma \\ G(\breve{E}) \arrow[u] \arrow[r] & \pi_1(G)_I \arrow[u] \end{tikzcd}$$

Theorem 12. *The map $B(G) \to (X_*(G)_{\mathbb{Q}}/G)^\sigma \times \pi_1(G)_\Gamma$ is injective.*

The description of the image is not easy, but in the quasi-split case it is fairly easy to describe it.

Example 13. Let $G = \mathrm{GL}_n$. Then $X_*(A)^+_{\mathbb{Q}} = (\mathbb{Q}^n)_+$ and $\pi_1(G)_\Gamma = \mathbb{Z}$. In this case the first component of the map gives the slopes of the Newton polygon, and the second component gives the endpoint of the Newton polygon. So in this case the first component determines the second, since the endpoint can be determined from the slopes via the formula

$$(\lambda_i) \in (\mathbb{Q}^n)_+ \mapsto \sum \lambda_i.$$

In this case, the image can be characterized as the tuples whose break points are integers.

Example 14. Let $G = T$, a torus. Then $X(A)^+_{\mathbb{Q}} = X_*(T)_\Gamma \otimes \mathbb{Q}$. (There are no positivity conditions because there are no roots.) The second component is $\pi_1(T)_\Gamma = X_*(T)_\Gamma$. In this case the second component determines the first, via

$$\gamma \in X_*(T)_\Gamma \to X_*(T)_\Gamma \otimes \mathbb{Q}.$$

In general, the first component determines the second up to torsion, i.e., its image in $\pi_1(G)_\Gamma \otimes \mathbb{Q}$

2.3. **More structure to $B(G)$.** *The basic subset.* There is an analogue of the semistable/isoclinic set.

Definition 15. Let

$$B(G)_{\text{basic}} = \{[b] \mid \nu_b \text{ is a central homomorphism}\},$$
$$B(G)^0_{\text{basic}} = \{[b] \mid \nu_b \text{ trivial}\}$$

Example 16. For $G = \mathrm{GL}_n$, basic means isoclinic; the second subset is the analogue of the unit root isocrystals.

The basic set forms a section to the Kottwitz invariant. In other words, κ induces *bijections*

$$B(G)_{\text{basic}} \to \pi_1(G)_\Gamma$$
$$B(G)^0_{\text{basic}} \to \pi_1(G)_{\Gamma,\text{tors}} \cong H^1(E, G).$$

In this sense $B(G)$ is an extension of Galois cohomology.

The automorphism group. Another piece of structure is the automorphism group. For $b \in G(\breve{E})$, we can associate a group functor on E-algebras

$$J_b(R) := \{g \in G(\breve{E} \otimes R) \mid gb\sigma(g)^{-1} = b\}.$$

This functor is representable by a reductive group over E. Then $J_b(E) = \mathrm{Aut}(b)$. If G is quasisplit, then J_b is an inner form of a Levi subgroup.

Further facts.

- An element $b \in G(\breve{E})$ is basic if and only if J_b is an inner form of G.
- If $Z(G)$ is connected, then *every* inner form comes from some basic b.
- If G is quasisplit, then $B(G)$ can be described in terms $B(M)_{\mathrm{basic}}$ for standard Levi subgroups $M \subset G$.
- $B(G)$ is a partially ordered set such that only finitely many elements are smaller than a given one. The basic elements are the minimal elements.

3. G-bundles on the Fargues-Fontaine Curve (following Fargues)

In this section we survey results from [1].

3.1. **The main result.** We want to define a functor

$$\mathcal{E}_G \colon G - \text{isocrystals} \to \mathrm{Bun}_G .$$

There are again two definitions.

(i) (external) Given a G-isocrystal $\mathrm{Rep}_E\, G \to \varphi - \mathrm{Mod}_{\breve{E}}$ in the external sense, composing with $\mathcal{E}$ gives

$$\mathrm{Rep}_E\, G \to \varphi - \mathrm{Mod}_{\breve{E}} \xrightarrow{\mathcal{E}} \mathrm{Vect}_X .$$

This is a G-bundle in the external sense.

(ii) (internal) Given $b \in G(\breve{E})$, form $G_{\breve{E}} \times_{\breve{E}} Y_E/\varphi^{\mathbb{Z}}$ with φ acting diagonally by φ on Y_E and by $g \mapsto b\sigma(g)$ on $G_{\breve{E}}$. The G-torsor structure on $G_{\breve{E}} \times_{\breve{E}} Y_E$ from the right descends to a G-torsor structure on the quotient. Here we are using implicitly GAGA.

Theorem 17. *Assume that* $\mathrm{char}\, E = 0$. *Then the functor* $\mathcal{E}_G$ *is faithful and induces a bijection*

$$B(G) \to |\,\mathrm{Bun}_G\,|.$$

Furthermore, $\mathcal{E}_G$ *induces an equivalence of categories between* $B(G)_{\mathrm{basic}}$ *and the category of semi-stable* G*-bundles.*

Here we are using the following definition.

Definition 18. A G-bundle $\mathcal{T}$ is *semi-stable* if

(i) (half-external) $\mathcal{T}((\mathrm{Lie}\, G, \mathrm{Ad}))$ is a semi-stable vector bundle.

(ii) (external) $\mathcal{T}((V, \rho))$ is a semi-stable vector bundle if ρ is homogeneous.

(We are using here that the tensor product of semistable vector bundles is semistable, as follows from their classification.)

(iii) (INTERNAL, IF G IS QUASI-SPLIT) We first introduce some notation. Let $P \subset G$ be a parabolic subgroup. Let A_P be the split part of the center of M_P. We have dually A'_P, the split part of the cocenter of M_P. Then the map

$$A_P \to A'_P$$

is an isogeny, identifying the rational cocharacter groups. Let $\mathcal{T}$ be a G-bundle and suppose $\mathcal{T}_P$ is a P-structure on $\mathcal{T}$. Then we define the *slope* cocharacter $\mu(\mathcal{T}_P) \in X_*(A_P)_{\mathbb{Q}}$ which is characterized by the property

$$\langle \mu(\mathcal{T}_P), \lambda \rangle = \deg \lambda_*(\mathcal{T}_P) \quad \text{for all } \lambda \in X^*(A'_P),$$

where $\lambda_*(\mathcal{T}_P)$ is the line bundle obtained by push-out.

Now $\mathcal{T}$ is said to be semi-stable if and only if for all parabolic subgroups P and all P-structures $\mathcal{T}_P$ on $\mathcal{T}$, we have

$$\langle \mu(\mathcal{T}_P), \alpha \rangle \leq 0, \qquad \text{for all roots } \alpha \text{ occurring in } \operatorname{Lie} N_P.$$

3.2. **The two invariants.** How are the two invariants expressed in terms of the corresponding G-bundles?

Newton invariant. By descent, we may assume that G is quasi-split, with $A \subset T \subset B$ as before. Let $\mathcal{T}$ be a G-bundle. The *Harder-Narasimhan reduction theorem* says that there exists a unique pair $(P, \mathcal{T}_P)$ with P a standard parabolic subgroup and $\mathcal{T}_P$ a P-structure such that

(1) $\mathcal{T}_P \times^P M_P$ is a semistable M_P-bundle, and
(2) $\mu(\mathcal{T}_P) \in X_*(A_P)^{++}_{\mathbb{Q}}$.

The inclusion $A_P \subset A$ gives a map $X_*(A_P)^{++}_{\mathbb{Q}} \to X_*(A)^{+}_{\mathbb{Q}}$, sending $\mu(\mathcal{T}_P)$ to $\nu_{\mathcal{T}} \in X_*(A)^{+}_{\mathbb{Q}}$.

Proposition 19. *There is the identity in* $(X_*(G)_{\mathbb{Q}}/G)^{\sigma}$,

$$[\nu_b] = [-\nu_{\mathcal{T}(b)}].$$

Why the minus sign? It came up already in Dospinescu's talk: a minus sign was taken to get compatibility of endomorphism algebras.

Kottwitz invariant. We know that

$$|\operatorname{Bun}_G| = H^1_{\text{ét}}(X, G).$$

Fargues defines a G-equivariant Chern class

$$c_1^G \colon H^1_{\text{ét}}(X, G) \to \pi_1(G)_\Gamma.$$

Proposition 20. *There is the identity*

$$\kappa(b) = c_1^G(\mathcal{E}_G(b)).$$

Conjecture 21. *Theorem 17 also holds when E is of characteristic p.*

The reason for the restriction on E in Theorem 17 is the following key step in the proof. In the book of Fargues and Fontaine [2], they construct equivalences of various categories of φ-isocrystals with the category of vector bundles. One of these functors is not exact. When one wants to apply the external definition of

G-bundles, one needs an exact functor; this uses the fact that in characteristic 0 the representation theory is semisimple.

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Proof of Geometric Langlands for $GL(2)$, Part One

Stefan Patrikis

1. Statement of the theorem

Let X be a smooth projective geometrically connected curve over a finite field $k = \mathbb{F}_q$ of residue characteristic p. Let $F = k(X)$ denote the function field of X. The aim of this talk is, following [3], to begin to explain the construction of cuspidal automorphic representations associated to everywhere unramified, geometrically irreducible, Galois representations

$$\sigma\colon\ \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell).$$

We begin by making this goal precise. Let $|X|$ denote the set of closed points of X, and for all $x \in |X|$, let $\mathcal{O}_x$ be the complete local ring of X at x, with fraction field F_x, and set $G_x = \mathrm{GL}_n(F_x)$, $K_x = \mathrm{GL}_n(\mathcal{O}_x)$, and $\mathcal{H}_x = \mathcal{H}(G_x, K_x)$, the ($\ell$-adic) spherical Hecke algebra at x. Let fr_x denote a geometric frobenius element at x, and let γ_x denote the semi-simple conjugacy class in $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ associated to the semi-simple part of $\sigma(fr_x)$. Let $\mathbf{A}_F$ denote the ring of adèles of F, and let $\mathcal{O}$ be the subring $\prod_{x\in|X|}\mathcal{O}_x$. We begin with the most classical formulation of unramified Geometric Langlands:

Theorem 1 (Drinfeld ([2]), Lafforgue ([7]), Frenkel, Gaitsgory, Kazhdan, Vilonen ([3], [4], [5]))**.** *For each everywhere unramified, geometrically irreducible, Galois representation $\sigma\colon\ \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$, or equivalently for each geometrically irreducible local system $E = E_\sigma$ on X, there is a non-zero unramified automorphic form*

$$f_\sigma\colon\ \mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbf{A}_F)/\mathrm{GL}_n(\mathcal{O}) \to \overline{\mathbb{Q}}_\ell,$$

related to σ as follows: for all $x \in |X|$, under the Satake isomorphism $\mathcal{H}_x \xrightarrow{\sim} \mathrm{R}(\mathrm{GL}_n)$ between the spherical Hecke algebra and the representation ring (over $\overline{\mathbb{Q}}_\ell$) $\mathrm{R}(\mathrm{GL}_n)$ of the algebraic group GL_n, the eigencharacter χ_{γ_x} of $\mathcal{H}_x$ acting on the line $\overline{\mathbb{Q}}_\ell \cdot f_\sigma$ corresponds to the character

$$\begin{aligned}\mathrm{R}(\mathrm{GL}_n) &\to \overline{\mathbb{Q}}_\ell\\ [V] &\mapsto \mathrm{tr}(\sigma(\gamma_x)|V).\end{aligned}$$

In the geometric setting there is a stronger formulation, which also opens up avenues for proving this theorem that are not available in the arithmetic (number field) setting. Let Bun_n denote the moduli stack over k of rank n vector bundles on X; it is a smooth algebraic stack. A fundamental observation of Weil is that we can identify the double coset space $\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbf{A}_F)/\mathrm{GL}_n(\mathcal{O}) \to \overline{\mathbb{Q}}_\ell$ with the set of isomorphism classes $|\mathrm{Bun}_n(k)|$ (in fact, there is even an equivalence of groupoids). A stronger form of unramified geometric Langlands then asserts:

Theorem 2 (Drinfeld ([2]), Frenkel, Gaitsgory, Kazhdan, Vilonen ([3], [4], [5]))**.** *There is a perverse Hecke eigensheaf* Aut_E *on* Bun_n *whose trace function*

$$\mathrm{tr}(\mathrm{Aut}_E)\colon |\mathrm{Bun}_n(k)| \to \overline{\mathbb{Q}}_\ell$$

is under Weil's equivalence identified with the automorphic form f_σ *of Theorem 1.*

This talk will begin the construction of a candidate for the automorphic sheaf Aut_E.

2. Classical motivation: cusp forms from Whittaker models via the Fourier expansion

A simple classical idea underlies the first steps in the construction of Aut_E. Fix a non-trivial character $\Psi\colon F\backslash\mathbf{A}_F \to \overline{\mathbb{Q}}_\ell^\times$. Let T, B, and N be the subgroups of diagonal, upper triangular, and unipotent upper triangular matrices in GL_n. The character Ψ induces a character, which we also denote Ψ, of $N(\mathbf{A}_F)$ via $(n_{i,j}) \mapsto \sum_i \Psi(n_{i,i+1})$. When $n = 2$, any cuspidal automorphic form φ is easily seen to admit a Fourier expansion

$$\varphi(g) = \sum_{\gamma\in F^\times} W_{\varphi,\Psi}\left(\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}\cdot g\right),$$

where $W_{\varphi,\Psi}$ denotes the *Whittaker function*

$$W_{\varphi,\Psi}(g) = \int_{N(F)\backslash N(\mathbf{A}_F)} \varphi(ng)\Psi(n)^{-1}dn;$$

to see this, one simply takes an abelian Fourier expansion for the function $x \mapsto \varphi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\cdot g\right)$ on $F\backslash\mathbf{A}_F$. With more endurance, one establishes the following equivalence:

Proposition 3 (Theorem 5.9 of [9])**.** *Let* $P_1 \subset \mathrm{GL}_n$ *denote the mirabolic subgroup of matrices of the form* $\begin{pmatrix}A & B\\ 0 & 1\end{pmatrix}$*, where* A *is an* $(n-1)\times(n-1)$ *block matrix. Then there is a* $\mathrm{GL}_n(\mathbf{A}_F)$*-equivariant (for right translation) isomorphism*

$$\Phi\colon \mathcal{C}^\infty(\mathrm{GL}_n(\mathbf{A}_F))^{(N(\mathbf{A}_F),\Psi)} \xrightarrow{\sim} \mathcal{C}^\infty(P_1(F)\backslash \mathrm{GL}_n(\mathbf{A}_F))_{\mathrm{cusp}}$$

between smooth functions W *satisfying* $W(ng) = \Psi(n)W(g)$ *for all* $n \in N(\mathbf{A}_F)$ *and* $g \in \mathrm{GL}_n(\mathbf{A}_F)$*, and smooth cuspidal functions left-invariant under* $P_1(F)$*.*

Our everywhere unramified Galois representation σ easily leads to the construction of an element of the space $\mathcal{C}^\infty(\mathrm{GL}_n(\mathbf{A}_F))^{(N(\mathbf{A}_F),\Psi)}$. Namely, for all $x \in |X|$, there is (see [1]) a unique unramified Whittaker function $W_{\gamma_x} : G_x \to \overline{\mathbb{Q}}_\ell$ satisfying $W_{\gamma_x}(1) = 1$, $W_{\gamma_x}(ngk) = \Psi_x(n)W_{\gamma_x}(g)$ for all $n \in N(F_x)$, $g \in G_x$, and $k \in K_x$, and $h \cdot W_{\gamma_x} = \chi_{\gamma_x}(h)W_{\gamma_x}$ for all $h \in \mathcal{H}_x$. Assembling these local constructions, we produce

$$W_\sigma = \prod_{x\in|X|} W_{\gamma_x} \colon \mathrm{GL}_n(\mathbf{A}_F) \to \overline{\mathbb{Q}}_\ell$$

and its corresponding cuspidal function

$$f'_\sigma(g) := \Phi(W_\sigma)(g) = \sum_{\gamma\in N_{n-1}(F)\backslash \mathrm{GL}_{n-1}(F)} W_\sigma\left(\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}\cdot g\right).$$

To prove Theorem 1, one has to show that f'_σ is (left) $\mathrm{GL}_n(F)$-invariant, rather than merely $P_1(F)$-invariant. The first step is to realize the function f'_σ as the trace function of a sheaf (complex) on a new moduli space $\mathrm{Bun}'_n \to \mathrm{Bun}_n$.

3. From functions to sheaves

In what follows, it is convenient to work with moduli spaces whose sheaves define trace functions not on $\mathrm{GL}_n(\mathbf{A}_F)$ and its subgroups, but on a twisted (Zariski-locally isomorphic) form of GL_n over the curve X. We will, following [3], write GL_n^J, P_1^J, etc. for these twisted forms; the constructions, especially of W_σ and f'_σ, of the previous section carry through for these twisted versions. The domain of $f'_\sigma \colon P_1^J(F)\backslash \mathrm{GL}_n^J(\mathbf{A}_F)/\mathrm{GL}_n^J(\mathcal{O}) \to \overline{\mathbb{Q}}_\ell$ is naturally identified with the k-points of the moduli space $\mathrm{Bun}_n^{\prime,rat}$ of pairs $(L, \Omega_F^{\otimes n-1} \hookrightarrow L_F)$, where $L \in \mathrm{Bun}_n$ and $\Omega_F^{\otimes n-1} \hookrightarrow L_F$ is a *generic* embedding of coherent sheaves (we do not make this notion precise in this abstract).

The basic geometric problem faced here is that $\mathrm{Bun}_n^{\prime,rat}$ is not represented by an ind-scheme or algebraic stack. In order to make sense of it geometrically, one has to do algebraic geometry in a setting in which functors like $\mathrm{Ran}_X = \mathrm{colim}_I X^I$, the (non-filtered!) colimit taken over all finite sets, are "spaces." This is in fact possible (see eg [6]), but in this talk, as in [3], we take a more classical approach. Thus we begin by defining Bun'_n to be the moduli of pairs $(L, \Omega^{n-1} \hookrightarrow L)$, where now the embedding $\Omega^{n-1} \hookrightarrow L$ is everywhere, rather than merely generically, regular. There is then an identification

$$|\mathrm{Bun}'_n(k)| \cong P_1^J(F)\backslash P^J(\mathbf{A}_F)^+/P^J(\mathcal{O}),$$

where P is the standard $(n-1,1)$ parabolic, and $P^J(\mathbf{A}_F)^+ = \prod'_{x\in|X|} P^J(F_x)^+$, with $P^J(F_x)^+ = \left\{\begin{pmatrix}a & b\\ 0 & d\end{pmatrix} \in P^J(F_x) : |d| \leq 1\right\}$. We can now state the main theorem of the talk:

Theorem 4 ([3]). *There is an object* $\mathrm{Aut}'_E \in D_c^b(\mathrm{Bun}'_n; \overline{\mathbb{Q}}_\ell)$ *such that*

$$\mathrm{tr}(\mathrm{Aut}'_E) = f'_\sigma|_{\mathrm{Bun}'_n(k)}.$$

We conclude with some remarks on the proof of the theorem. The essential step is geometrizing the Whittaker function W_σ, or rather its restriction to a subset of $N(F)\backslash \mathrm{GL}_n(\mathbf{A}_F)/\mathrm{GL}_n(\mathcal{O})$ that is the k-points of an algebraic stack $\nu\colon \mathcal{Q} \to \mathrm{Bun}'_n$ over Bun'_n such that $\nu_!$ corresponds to the Fourier expansion Φ of Proposition 3: an appropriate $\mathcal{Q}$ gives

$$\mathcal{Q}(k) = N(F)\backslash B(\mathbf{A}_F)^+/B(\mathcal{O}),$$

where $B(F_x)^+ = N(F_x)\cdot(T(F_x)\cap \mathrm{Mat}_n(\mathcal{O}_x))$. A basic observation that allows geometrization of $W_\sigma|_{\mathcal{Q}(k)}$ is that the unramified Whittaker function W_{γ_x} is determined by its restriction to $T(F_x)$, and there it is, modulo center, supported on $T(F_x)\cap \mathrm{Mat}_n(\mathcal{O}_x)$, with values explicitly given by the Shintani-Casselman-Shalika formula:

Theorem 5 (Shintani ([10]), Casselman-Shalika ([1]))**.** *If $\lambda \in X_\bullet(T)$ is not a dominant weight, then $W_{\gamma_x}(\lambda(\varpi_x)) = 0$. If λ is dominant, then*

$$W_{\gamma_x}(\lambda(\varpi_x)) = q_x^{-\langle\lambda,\rho\rangle}\,\mathrm{tr}(\gamma_x|V(\lambda)),$$

where V_λ denotes the irreducible representation of (the dual group) GL_n of highest weight λ.

Laumon ([8]) exploited this observation to geometrize a version of the unramified Whittaker function on a moduli space of torsion coherent sheaves on X, and used this construction to give a conjectural construction of the sheaf Aut'_E. Frenkel, Gaitsgory, Kazhdan, and Vilonen proved Laumon's conjecture to establish Theorem 4 above.

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Beauville–Laszlo Uniformization for the Fargues–Fontaine curve

Peter Scholze

Let X be a smooth projective curve over a field k, G/k a reductive group, and $x \in X(k)$ a rational point. In this situation, one has the stack of G-bundles Bun_G on X, as well as the affine Grassmannian Gr_G associated with G. Modifying the trivial G-bundle at the chosen point x defines the Beauville–Laszlo uniformization map

$$\mathrm{Gr}_G \to \mathrm{Bun}_G \ ,$$

which is a flat cover if G is semisimple by Drinfeld–Simpson (see the exposition of Y. Varshavsky in this report). The goal of this talk was to define the similar objects in the context of the Fargues–Fontaine curve.

For simplicity, we fix the field $E = \mathbb{Q}_p$. First, we define the analogue of the affine Grassmannian. In this context, this affine Grassmannian parametrizes B_{dR}^+-lattices in B_{dR}^n (if $G = \mathrm{GL}_n$), where B_{dR}^+ is the complete discrete valuation ring defined by Fontaine in p-adic Hodge theory. It appears as the complete local ring of a closed point on the Fargues–Fontaine curve.

To define this in families, one has to associate a ring $B_{\mathrm{dR}}^+(R)$ for any perfectoid Tate ring R. This is defined as the ξ-adic completion of $W(R^{\flat\circ})[\frac{1}{[\varpi^\flat]}]$, where $R^\flat$ is the tilt of R, $\varpi^\flat \in R^\flat$ is a pseudo-uniformizer, and as usual ξ generates the kernel of $\theta\colon W(R^{\flat\circ}) \to R^\circ$. One also sets $B_{\mathrm{dR}}(R) = B_{\mathrm{dR}}^+(R)[\xi^{-1}]$. If $R = \mathbb{C}_p$, $B_{\mathrm{dR}}^+(R)$ is a complete discrete valuation ring with residue field $\mathbb{C}_p$, and so is abstractly isomorphic to $\mathbb{C}_p[[\xi]]$. However, there is no good choice of such an isomorphism, and in families such an isomorphism fails to exist.

Now, if G is a reductive group over $\mathbb{Q}_p$, one can define Gr_G as the functor on all perfectoid Tate rings R over $\mathbb{Q}_p$, sending R to the set of all G-torsors over $\operatorname{Spec} B_{\mathrm{dR}}^+(R)$ with a trivialization over $\operatorname{Spec} B_{\mathrm{dR}}(R)$. Note that by definition of $\mathrm{Spa}(\mathbb{Q}_p)^\diamond$, one can reinterpret the input data as a pair of an affinoid perfectoid space S of characteristic p, together with a map $S \to \mathrm{Spa}(\mathbb{Q}_p)^\diamond$. Under this translation, Gr_G becomes an ind-diamond with a structure map to $\mathrm{Spa}(\mathbb{Q}_p)^\diamond$.

On the other hand, one can define the stack of G-bundles Bun_G. This takes any perfectoid space S of characteristic p to the groupoid of G-bundles on the relative Fargues–Fontaine curve X_S. An important feature, which differs drastically from the case of a usual smooth projective curve, is that the automorphisms of the trivial G-torsor are not the algebraic group G, but the p-adic group $G(\mathbb{Q}_p)$. We briefly discussed that there should be a notion of smooth morphism between diamonds, and a corresponding notion of an Artin stack; Bun_G should be an example of an Artin stack.

Finally, as in the classical situation, one can define the Beauville–Laszlo uniformization map

$$\mathrm{Gr}_G \to \mathrm{Bun}_G \ ,$$

which is surjective (even if G is not semisimple) by a theorem of Fargues (see [1]). The Beauville–Laszlo map can be used to construct interesting maps into Bun_G. For example, for $G = \mathrm{GL}_2$, one gets a map $(\mathbb{P}^1_{\mathbb{Q}_p})^\diamond \to \mathrm{Bun}_G$; this sends the

$\mathbb{Q}_p$-rational points $\mathbb{P}^1(\mathbb{Q}_p)$ to the vector bundle $\mathcal{O} \oplus \mathcal{O}(1)$, and the complement $\mathbb{P}^1_{\mathbb{Q}_p} \setminus \mathbb{P}^1(\mathbb{Q}_p)$, known as Drinfeld's upper half-plane, to the vector bundle $\mathcal{O}(1/2)$. This demonstrates the highly transcendent nature of the geometry of Bun_G.

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The relationship with the classical local Langlands correspondence

Ana Caraiani

In this talk, I recalled the local Langlands conjecture in its refined form and its extension to non-quasi-split groups via extended pure inner forms, as in Conjecture F of [4]. This form of the conjecture is necessary to connect Fargues' geometrization conjecture [3] to the local Langlands correspondence.

Let E be a local field of characteristic 0 and G a connected reductive group defined over E. The local Langlands correspondence proposes to understand irreducible admissible representations of $G(E)$ in terms of arithmetic data. As a consequence of the Langlands classification, it is enough to understand tempered representations $\Pi_{\mathrm{temp}}(G)$, which are supposed to be matched with *tempered L-parameters.*

Let L_E be the local Langlands group of E: the Weil group W_E if E is archimedean and $W_E \times SU_2(\mathbb{R})$ if E is non-archimedean. In the non-archimedean case, this is a form of the so-called Weil-Deligne group of E and was introduced to account for representations such as the Steinberg. Let $\widehat{G}$ be the connected complex Langlands dual group of G, which has a natural action of $\Gamma := \mathrm{Gal}(\bar{E}/E)$, and define the *L-group* of G to be ${}^LG := \widehat{G} \rtimes \Gamma$. Note that ${}^LG$ only depends on the quasi-split inner form of G.

Let $\Phi_{\mathrm{temp}}(G)$ be the set of $\widehat{G}$-conjugacy classes of tempered admissible L-homomorphisms

$$\phi\colon L_E \to {}^LG.$$

The requirement that ϕ be an L-homomorphism is the requirement that it commutes with the map $W_E \to \Gamma$. The admissibility condition says that ϕ is continuous and sends elements of W_E to semisimple elements of ${}^LG$. The temperedness condition says that the image of ϕ projects to a bounded subset of $\widehat{G}$.

Conjecture 1. *There exists a map*

$$LL\colon \Pi_{\mathrm{temp}}(G) \to \Phi_{\mathrm{temp}}(G)$$

with finite fibers $\Pi_\phi(G) := LL^{-1}(\phi)$*, which are called* L-packets.

The map LL should have several additional nice properties: we should understand its image, it should be compatible with the Satake isomorphism in the unramified case and with parabolic induction etc. For the precise statement, see [2].

Remark. (1) For E a p-adic field and $G = GL_n$, it is a theorem of Harris-Taylor and Henniart that the desired map LL exists and is a bijection. In particular, in this case every L-packet consists of one element. This is very far from being true in general.
(2) If $E = \mathbb{R}$ and $G = SL_2$, the discrete series representations with the same infinitesimal character give examples of L-packets of size 2.

From now on we will be interested in finding a way to address individual representations in a given L-packet $\Pi_\phi(G)$. The original motivation for this is global: in order to understand the spectral content of the stable trace formula and in order to compute multiplicities of automorphic representations (see [4]).

If G is a quasi-split group, we have the following parametrization of tempered L-packets, known as the *refined local Langlands conjecture.* Let

$$S_\phi := \{ \ g \in \widehat{G} \mid g\phi g^{-1} = \phi \ \}.$$

Note that we always have $Z(\widehat{G})^\Gamma \subset S_\phi$ and that the connected component of the identity S_ϕ° in S_ϕ is a reductive group. Set $\bar{S}_\phi := S_\phi / Z(\widehat{G})^\Gamma$ and let $\pi_0(\bar{S}_\phi)$ be its group of connected components. Since we are in the quasi-split case, we can and do choose $\mathfrak{w} = (B, \psi)$, a Whittaker datum for G.

Conjecture 2. (1) *There exists an injective map*

$$\iota_{\mathfrak{w}} \colon \Pi_\phi(G) \hookrightarrow \mathrm{Irr}\left(\pi_0(\bar{S}_\phi)\right),$$

which is a bijection if E is p-adic.
(2) *There is a unique $\mathfrak{w}$-generic constituent of $\Pi_\phi(G)$, which corresponds to the trivial representation of $\pi_0(\bar{S}_\phi)$ under $\iota_{\mathfrak{w}}$.*
(3) *If E is p-adic, the bijection can be reinterpreted as a "perfect pairing"*

$$\langle \ , \ \rangle \colon \Pi_\phi(G) \times \pi_0(\bar{S}_\phi) \to \mathbb{C}$$

which should satisfy certain endoscopic character identities. For the precise statement, see Section 1.4 of [4].

Observe that the map $\iota_{\mathfrak{w}}$ does depend on the choice of Whitakker datum $\mathfrak{w}$, by the second part of the conjecture. If G is no longer quasi-split, one can no longer formulate the conjecture as above. See Section 2 of [4] for a detailed discussion on why all three statements above become problematic.

If G is no longer quasi-split, the fundamental idea is to extend the local Langlands conjecture by treating several (or even all) inner forms at once [1, 5]. In the archimedean case, the problem is completely solved by work of Adams, Barbasch and Vogan [1]. Assume that E is non-archimedean. Let G^* be the quasi-split inner form of G. Let $B(G^*)$ be Kottwitz's set of isocrystals with G^*-structure, which coincides with the set of σ-conjugacy classes in $G^*(\breve{E})$, where $\breve{E}$ is the completion of the maximal unramified extension of E and σ is the lift of Frobenius. Kottwitz has defined a map

$$\kappa \colon B(G^*) \to X^*(Z(\widehat{G}^*)^\Gamma).$$

Note that $X^*(Z(\widehat{G}^*))$ is canonically isomorphic to the algebraic fundamental group $\pi_1(G^*)$ as a Γ-module, so $\pi_1(G^*)_\Gamma$ is used in some formulations.

Let $B(G^*)_{\mathrm{bas}} \subset B(G^*)$ be the subset of *basic* elements. On this subset, we have a bijection $\kappa\colon B(G^*)_{\mathrm{bas}} \xrightarrow{\sim} X^*(Z(\widehat{G}^*)^\Gamma)$. Given $b \in B(G^*)_{\mathrm{bas}}$, we can obtain an inner form J_b of G^*: in fact, by choosing a representative of b in $G^*(\breve{E})$ we get a pair (b, ξ) with $\xi\colon G^* \to J_b$ an inner twist. The pair (ξ, b) is called an *extended pure inner twist* and J_b is an *extended pure inner form* of G^*.

Assume, for simplicity, that ϕ is a *discrete L-parameter*, which is defined by the requirement that $S_\phi/Z(\widehat{G}^*)^\Gamma$ be a finite group.

Conjecture 3. (1) *There is a commutative diagram*

$$\begin{tikzcd} \bigsqcup_{(b,\xi)} \Pi_\phi\left((b,\xi)\right) \arrow[r, "\iota_{\mathfrak{w}},\sim"] \arrow[d] & \mathrm{Irr}\left(S_\phi\right) \arrow[d] \\ B(G^*)_{\mathrm{bas}} \arrow[r, "\kappa,\sim"] & X^*(Z(\widehat{G}^*)^\Gamma). \end{tikzcd}$$

The (b,ξ) run over all extended pure inner twists and $\Pi_\phi\left((b,\xi)\right)$ is the L-packet for the corresponding extended pure inner form. The left vertical map is given by $(b,\xi) \mapsto b \in B(G^*)$, whereas the right vertical map is induced by the natural restriction.

(2) *There is a unique $\mathfrak{w}$-generic constituent of $\Pi_\phi\left((1, id)\right)$, which corresponds to the trivial representation of S_ϕ.*

(3) *Analogues of the endoscopic character identities of Conjecture 2 hold. For the precise statement, see Conjecture F of* [4].

Remark. (1) The theory of extended pure inner forms is consistent with and generalizes Vogan's theory of *pure inner forms*, which are parametrized by $H^1(\Gamma, G^*)$ [5]. There is a natural injection $H^1(\Gamma, G^*) \hookrightarrow B(G^*)$, whose image under κ can be identified with $\pi_0(Z(\widehat{G}^*)^\Gamma)^\vee$. This is the torsion part of $X^*(Z(\widehat{G})^\Gamma) \simeq \pi_1(G)_\Gamma$. By Hilbert's Theorem 90, the theory of pure inner forms does not reach any non-trivial inner form of $G^* = GL_n$.

(2) When $Z(G^*)$ is connected, the map $B(G^*) \to H^1(\Gamma, G^*_{\mathrm{ad}})$ is surjective, so the theory of extended pure inner forms covers all inner forms of G^*.

In the case when ϕ is a discrete parameter for $G^* = GL_n$, we have $S_\phi = Z(\widehat{G}^*)^\Gamma = \mathbb{G}_m$, so the right vertical map is the identity on $\mathbb{Z}$. The set $B(GL_n)_{\mathrm{bas}}$ can be identified with $\mathbb{Z}$ as well. All the L-packets have size 1. Note that the L-packet for a fixed inner form appears infinitely many times on top of the diagram: in this case, the inner form only depends on the image under the natural surjection $B(GL_n)_{\mathrm{bas}} = \mathbb{Z} \twoheadrightarrow H^1(\Gamma, PGL_n) = \mathbb{Z}/n\mathbb{Z}$.

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Formulation of the conjecture

Laurent Fargues

Let E be either a finite degree extension of $\mathbb{Q}_p$ or $\mathbb{F}_q((\pi))$. We note $\mathbb{F}_q = \mathcal{O}_E/\pi$. Let G be a quasi-split reductive group over E. To this datum is associated a stack Bun_G on $\mathrm{Perf}_{\mathbb{F}_q}$, the catgory of $\mathbb{F}_q$-perfectoid spaces equipped with the pro-étale topology. In fact, if $S \in \mathrm{Perf}_{\mathbb{F}_q}$ then one can consider the relative curve

$$X_S = Y_S/\varphi^{\mathbb{Z}}$$

as an E-adic space. One has

$$\mathrm{Bun}_G(S) = \{G\text{-bundles over } X_S\}.$$

This stack is in some sense a "perfectoid stack" (by analogy with the notion of an algebraic stack); we can put some perfectoid charts on it.

Let $\breve{E}$ be the completion of the maximal unramified extension of E and σ its Frobenius. Recall Kottwitz set

$$B(G) = G(\breve{E})/\sigma\text{-conjugacy}.$$

According to the main theorem of [1] there is an identification

$$B(G) = \big|\, \mathrm{Bun}_{G,\overline{\mathbb{F}}_q} \big|.$$

The stack Bun_G has a nice Harder-Narasimhan stratification for which the semi-stable locus is open. There is moreover a decomposition into open/closed substacks

$$\mathrm{Bun}_G = \coprod_{\alpha \in \pi_1(G)_\Gamma} \mathrm{Bun}_G^{\alpha},$$

where $\Gamma := \mathrm{Gal}(\bar{E}/E)$, given by the Kottwitz invariant

$$\kappa\colon B(G) \longrightarrow \pi_1(G)_\Gamma.$$

For each $\alpha \in \pi_1(G)_\Gamma$ the open subset

$$|\, \mathrm{Bun}_{G,\overline{\mathbb{F}}_q}^{\alpha,ss} \,|$$

is one point given by $\kappa\colon B(G)_{\mathrm{basic}} \xrightarrow{\sim} \pi_1(G)_\Gamma$. The residual gerbe at this point is given by

$$\left[\mathrm{Spa}(\overline{\mathbb{F}}_q)/\underline{J_b(E)}\right] \xrightarrow{\sim} \mathrm{Bun}_{G,\overline{\mathbb{F}}_q}^{\alpha,ss}.$$

for b basic with $\kappa(b)=\alpha$.

Fix $T \subset B$ in G. For each $\mu \in X_*(T)^+/\Gamma$ there is a Hecke diagram

$$\begin{tikzcd} & \mathrm{Hecke}^{\leq\mu} \arrow[dl, "\overleftarrow{h}"'] \arrow[dr, "\overrightarrow{h}"] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \times \mathrm{Spa}(E)^\diamond. \end{tikzcd}$$

The right morphism if a locally trivial fibration in $\mathrm{Gr}^{B_{\mathrm{dR}},\leq\mu}/E^\diamond$, a closed Schubert cell in Scholze's B_{dR}-affine Grassmanian. Conjecturaly one can construct an intersection cohomology complex IC_μ on $\mathrm{Gr}^{B_{\mathrm{dR}},\leq\mu}$ and transfer it to $\mathrm{Hecke}^{\leq\mu}$ via $\overrightarrow{h}$. Given a complex of sheaves $\mathcal{F}$ on Bun_G one can then construct its Hecke transform

$$\overrightarrow{h}_!\big(\overleftarrow{h}^*\mathcal{F} \otimes IC_\mu\big).$$

The conjecture is now the following.

Conjecture. *Fix a Whittaker datum for G. Let $\varphi : W_E \longrightarrow {}^LG$ be a discrete Langlands parameter where ${}^LG = \widehat{G} \rtimes W_E$ with $\widehat{G}$ the $\overline{\mathbb{Q}}_\ell$-Langlands dual, $\ell \neq p$ fixed. Here discrete means $S_\varphi/Z(\widehat{G})^\Gamma$ is finite where $S_\varphi := \{g \in \widehat{G} \mid g\varphi g^{-1} = \varphi\}$. We then conjecture there exists a "perverse" $\overline{\mathbb{Q}}_\ell$-Weil sheaf $\mathcal{F}_\varphi$ on $\mathrm{Bun}_{G,\overline{\mathbb{F}}_q}$ equipped with an action of S_φ such that:*

(1) *For each $\alpha \in \pi_1(G)_\Gamma$ the action of $Z(\widehat{G})^\Gamma$ on $\mathcal{F}_{\varphi|\mathrm{Bun}_G^\alpha}$ is given by α via $\pi_1(G)_\Gamma = X^*(Z(\widehat{G})^\Gamma)$.*

(2) *For each $b \in G(\breve{E})$ basic if $x_b\colon \left[\mathrm{Spa}(\overline{\mathbb{F}}_q)/\underline{J_b(E)}\right] \hookrightarrow \mathrm{Bun}_{G,\overline{\mathbb{F}}_q}$ then*

$$x_b^*\mathcal{F}_\varphi = \bigoplus_{\substack{\rho \in \widehat{S_\varphi} \\ \rho_{|Z(\widehat{G})^\Gamma} = \kappa(b)}} \rho \otimes \pi_{\varphi,b,\rho}$$

where $\{\pi_{\varphi,b,\rho}\}_\rho$ is an L-packet for a local Langlands correspondence for the extended pure inner form J_b of G. Moreover $\pi_{\varphi,1,1}$ is generic with respect to the chosen Whittaker datum.

(3) *If φ is moreover cuspidal i.e. the 1-cocyle $I_E \xrightarrow{\varphi} {}^LG \twoheadrightarrow \widehat{G}$ has finite image, then*

$$\mathcal{F}_\varphi = j_! j^* \mathcal{F}_\varphi$$

with j the inclusion of the semi-stable locus.

(4) *For each $\mu \in X_*(T)^+/\Gamma$*

$$\overrightarrow{h}_!\big(\overleftarrow{h}^*\mathcal{F} \otimes IC_\mu\big) \simeq \mathcal{F}_\varphi \boxtimes r_\mu \circ \varphi$$

that is to say $\mathcal{F}_\varphi$ is an Hecke eigensheaf with eigenvalue the Weil local system $r_\mu \circ \varphi$ on $\mathrm{Spa}(\breve{E})^\diamond$

(5) *There is a "naive" character sheaf property: for each $\delta \in G(E)$, x_δ is defined over $\mathbb{F}_q$ and the action of the corresponding Frobenius coming from the Weil structure on $x_\delta^* \mathcal{F}_\varphi$ is given by $\delta \in J_\delta(E)$*

(6) *There is a local/global compatibility between $\mathcal{F}_\varphi$ and the Caraiani-Scholze sheaf constructed in* [2] *for Hodge type Shimura varieties.*

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Proof of Geometric Langlands for GL(2), second part

Jochen Heinloth

As in the first talk by S. Patrikis on the proof of geometric Langlands for GL(2) we fix a smooth, geometrically irreducible projective curve $X/\mathbb{F}_q$. The aim of the talk is to explain how Frenkel, Gaitsgory and Vilonen [3] proved that for every geometrically irreducible local system E of rank n on X there exists a Hecke-eigensheaf Aut_E on Bun_n. To clarify the basic strategy, we will restrict to the case $n = 2$ for most of the talk. This case is due to Drinfeld [1] but we will try to rephrase the proof in the formulation of [3].

An exceptionally clear and much more detailed exposition can be found in Laumon's Bourbaki talk on this work [8]

1. The starting point

To begin, let us recall the case $n = 1$ form Bhargav Bhatt's talk: Given a 1-dimensional local system L on X we already knew the values of the potential automorphic function f_L on divisors. A geometric incarnation of this function was given by the symmetric power $L^{(d)}$ on the symmetric power $X^{(d)}$ of the curve, classifying effective divisors of degree d on X:

$$\begin{tikzcd} X^{(d)} \arrow[r, equal] \arrow[d, "AJ"] & \langle \mathcal{O} \hookrightarrow \mathcal{L} \mid \mathcal{L} \in \mathrm{Pic}^d \rangle . \\ \mathrm{Pic}^d & \end{tikzcd}$$

(Here and in the following we will often denote by $\langle$objects$\rangle$ the algebraic stack classifying the objects inside of the brackets.)

As for $d > 2g-2$ the Abel-Jacobi map AJ is a projective bundle and projective space is simply connected any local system on $X^{(d)}$ descends to Pic^d. This procedure constructed Aut_L.

In higher rank the strategy is similar: Again a candidate for the automorphic function is known and can this time be interpreted as the trace function of a sheaf

Aut'_E on

$$\begin{tikzcd} \mathrm{Bun}'_2 \arrow[r, equal] \arrow[d, "\text{forget}"] & \langle \Omega_C \hookrightarrow \mathcal{E} | \mathcal{E} \in \mathrm{Bun}_2 \rangle \ . \\ \mathrm{Bun}_2^d & \end{tikzcd}$$

The main point of the talk is thus to sketch that

1. Aut'_E is an (irreducible) perverse sheaf (this is crucial for proving 2.).
2. Aut'_E descends to a (perverse) sheaf Aut_E on Bun_2.
3. Aut_E is a Hecke eigensheaf. (Of course, we know this already for its trace function.)

Again the forgetful map is a bundle over a large open subset of Bun_2. As it is not too hard to keep track of the problem that this only true on an open subset, we will ignore this issue for the purpose of this talk.

2. Reminder on the construction of the candidate sheaf Aut'_E

We briefly need to recall that Aut'_E was constructed through the following diagram

$$\begin{tikzcd} & \langle (\mathcal{J} \subset \mathcal{E}) |^{\mathcal{J} \in \mathrm{Ext}^1(\mathcal{O},\Omega)}_{\mathcal{E} \in \mathrm{Bun}_2} \rangle \arrow[r, "\mathrm{ext} \times \mathrm{quot}"] \arrow[dl, "\tilde{\nu}"] & \mathbb{A}^1 \times \mathrm{Coh}_0 \\ \mathrm{Bun}'_2 \arrow[r, equal] & \langle \Omega_C \hookrightarrow \mathcal{E} | \mathcal{E} \in \mathrm{Bun}_2 \rangle & \end{tikzcd}$$

as

$$\mathrm{Aut}'_E := \tilde{\nu}_!(\mathrm{ext}^* AS \times \mathrm{quot}^*(\mathcal{L}_E))$$

where AS is the Artin-Schreier sheaf on $\mathbb{A}^1$ and $\mathcal{L}_E$ is Laumon's sheaf on the stack of torsion sheaves on X, which was denoted by Coh_0. Recall that the construction of $\mathcal{L}_E$ was closely related to the symmetric products of X via:

$$\begin{tikzcd} \widetilde{\mathrm{Coh}}_0^d \arrow[r, equal] \arrow[d, "\pi"] & \langle \mathcal{T}_1 \subset \cdots \subset \mathcal{T}_d \rangle \arrow[r, "\mathrm{gr}"] & \prod Coh_0^1 \arrow[r] & X^d \arrow[d] \\ \mathrm{Coh}_0^d \arrow[rrr, "\mathrm{supp}"] & & & X^{(d)} \end{tikzcd}$$

as $\mathcal{L}_{E|\,\mathrm{Coh}_0^d} := (\mathbb{R}\pi_* \, \mathrm{gr}^* \, E^{\boxtimes d})^{S_d}$, where the S_d action comes from the fact that π is a small map, which is generically an S_d covering. Alternatively $\mathcal{L}_E$ can be described, on Coh_0^d, as the middle extension of $E^{(d)}$ from the substack of torsion sheaves supported at d distinct points.

3. Perversity of Aut'_E

The trick to show that Aut'_E is perverse is to reinterpret the above construction as a Fourier transformation, which is known to preserve perversity [7]. The reinterpretation due to Laumon comes from noting that $\langle \Omega_C \hookrightarrow \mathcal{E} | \mathcal{E} \in \mathrm{Bun}_2 \rangle$ is an open substack of the stack of extensions $\mathrm{Ext}(\_\_, \Omega)$ over the stack Coh_1 of coherent sheaves of rank 1. By Serre duality this is dual to $\mathrm{Hom}(\mathcal{O}, \_\_)$ which contains the injective homomorphisms $j\colon \mathrm{Hom}^{\mathrm{inj}}(\mathcal{O}, \_\_) \hookrightarrow \mathrm{Hom}(\mathcal{O}, \_\_)$, which in turn map to Coh_0.

Unraveling the definitions one finds that Aut'_E is the Fourier transform of $j_!(\mathrm{quot}^* \mathcal{L}_E)$. Thus one needs to show that in this case $j_!(\mathrm{quot}^* \mathcal{L}_E) = j_{!*}(\mathrm{quot}^* \mathcal{L}_E)$.

In general for equivariant sheaves defined on the complement of the zero section of a vector bundle this property is equivalent to the vanishing of the cohomology of the corresponding sheaf on the associated projective bundle.

In the talk we explained how this is connected to the vanishing conjecture of [3], proven in [4] and sketched Deligne's argument in the rank 2 case from [1]. Thus we find that Aut'_E is indeed perverse and irreducible if E is.

4. The descent argument

As an irreducible perverse sheaf is the middle extension of some local system on some subset, we could prove descent of Aut'_E as in the rank 1 case if we only knew that this open subset of Bun'_2 can be chosen as the preimage of an open subset of Bun_2. The first trick is to observe that:

Lemma 1. *A perverse sheaf F on a smooth variety is a local system if and only if the Euler-characteristic of its stalks is constant.*

So to show that Aut'_E descends we only need to show that the Euler characteristic of the stalks of Aut'_E are constant along the fibers of the forgetful map to Bun_2.

In turn this property is independent of E if one manages to rewrite the construction of Aut'_E in a way that uses only push forwards along projective maps. This can be achieved using Drinfeld's compactification (called $\overline{\mathcal{Q}}$ in [3]).

Thus one is reduced to showing the property for a single local system E_0, e.g. one could take E_0 trivial. In [3] the authors use that it would also suffice to find one local system for which the corresponding automorphic representation is known to exist and argue that such a system can be constructed using cyclic base change.

An alternative approach is given in Gaitsgory's thesis by comparing Aut'_E to Eisenstein series in case $E = \oplus_{i=1}^n L_i$ is a generic direct sum of rank 1 local systems. Note that in this case the argument $j_! = \mathbb{R}j_*$ for Aut'_E no longer holds, but looking at the vanishing result one finds that the Euler characteristics of the two complexes still agree, which in the end turns out to suffice.

Also note that as a consequence, the constancy of the Euler characteristics also holds for the trivial local system which is a non-trivial geometric statement, which by constructibility can then be transferred to characteristic 0.

5. The Hecke property

As little time was left, we briefly explained how the Hecke property for GL_n needs only be checked on the first Hecke operator, if one knew that the Hecke invariance under the basic Hecke operator satisfies a symmetry property with respect to iterations. The argument is an application of Springer theory.

To show the Hecke property of Aut_E for the first Hecke operator one does a direct calculation using that Laumon's sheaf $\mathcal{L}_E$ already satisfies a Hecke property on Coh_0.

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The case of $\mathbb{G}_m$

Ulrich Görtz

In this talk, we prove Fargues's conjecture [1] §4 in the case of $G = GL_1$, using local class field theory in the form of Lubin-Tate theory. The crucial point is the Hecke eigensheaf property. We closely follow loc. cit., §9.1.

We fix a finite extension $E/\mathbb{Q}_p$, and a uniformizer $\pi \in E$. Let q denote the cardinality of the residue class field of E.

For $G = GL_1$, the stack of G-bundles on the Fargues-Fontaine curve X is just the Picard stack Pic of line bundles. It decomposes as

$$\mathrm{Pic} = \coprod_{d \in \mathbb{Z}} \mathrm{Pic}^d,$$

according to the degree d of a line bundle. In this case, the semi-stable locus is equal to the whole stack Pic.

The choice of the uniformizer π gives rise to a line bundle $\mathcal{O}(1)$ on X, hence line bundles $\mathcal{O}(d)$ for all $d \in \mathbb{Z}$, and these induce isomorphisms

$$[\mathrm{Spa}(\mathbb{F}_q)/\underline{E}^\times] \cong \mathrm{Pic}^d\,.$$

From this point of view, the universal object over Pic^d is an $\underline{E}^\times$-torsor on $\mathrm{Spa}(\mathbb{F}_q)$ which we denote by $\mathscr{T}_d$.

Now fix a cocharacter μ of GL_1, say $\mu(z) = z^k$. We restrict the Hecke diagram

$$\mathrm{Pic} \xleftarrow{\overleftarrow{h}} \mathrm{Hecke}^{\leq\mu} \xrightarrow{\overrightarrow{h}} \mathrm{Pic} \times \mathrm{Div}^1_X,$$

where $\mathrm{Div}^1_X = \mathrm{Spa}(E)^\diamond/\varphi^{\mathbb{Z}}$, to $\mathrm{Pic}^d \times \mathrm{Div}^1_X$ and obtain

$$\mathrm{Pic}^{d+k} \xleftarrow{\overleftarrow{h}} \mathrm{Hecke}^{\leq\mu,d} \xrightarrow{\overrightarrow{h}} \mathrm{Pic}^d \times \mathrm{Div}^1_X,$$

(with $\mathrm{Hecke}^{\leq\mu,d} := \overrightarrow{h}^{-1}(\mathrm{Pic}^d \times \mathrm{Div}^1_X)$). In this diagram, the morphism $\overrightarrow{h}$ is an isomorphism: For a lattice in a *one-dimensional* vector space over a discretely valued field (such as B_{dR}), there exists a unique lattice in relative position k. Hence given a line bundle on X, there is a unique modification along a given Cartier divisor on X with a fixed relative position.

Now fix a Lubin-Tate formal $\mathcal{O}_E$-module attached to π, let $E(1)$ denote its rational Tate module, $E(k) = E(1)^{\otimes k}$. This is a pro-étale local system on $\mathrm{Spa}(E)^\diamond$ (and hence on Div^1_X). Denote by $\mathcal{LT}_d$ the corresponding $\underline{E}^\times$-torsor.

The key property that will allow us to define a Hecke eigensheaf on Pic is the following

Proposition 1. *There is a natural isomorphism*

$$\overleftarrow{h}^*\mathscr{T}_{d+k} \cong \overrightarrow{h}^*(\mathscr{T}_d) \times^{\underline{E}^\times} \mathcal{LT}_{-k}$$

of $\underline{E}^\times$-torsors on $\mathrm{Hecke}^{\leq\mu,d}$.

Proof. The main ingredient is the fundamental exact sequence in p-adic Hodge theory. In fact, seen as a sequence of sheaves over $\mathrm{Spa}(E)^\diamond$, it allows us to see $\mathcal{O}(d+k)$ as a modification of $\mathcal{O}(d)$ (of relative position $k \geq 0$, say) of line bundles on the relative curve (over some perfectoid space S over $\mathrm{Spa}(E)^\diamond$), induced from a section of $\mathcal{O}(k)$ which "comes from" $E(k)^\diamond$. Comparing this modification with the universal modification over the Hecke stack, pulled back to S, gives the proposition. See [1] Prop. 9.1 (2). $\square$

Now define a Weil-$\underline{E}^\times$-torsor on $\mathrm{Pic}_{\overline{\mathbb{F}}_q}$ as follows:

- $\mathscr{F}|_{\mathrm{Pic}^d_{\overline{\mathbb{F}}_q}} = \mathscr{T}_d$
- The Weil descent datum on $\mathrm{Pic}^d_{\overline{\mathbb{F}}_q}$ is given by multiplying the canonical Weil descent datum (which we have since $\mathscr{T}_d$ is defined over $\mathbb{F}_q$) by π^{-d}.

Now consider a continuous character $\varphi\colon W_E \to \overline{\mathbb{Q}}_\ell^\times$, i.e., a (discrete, cuspidal) Langlands parameter for GL_1. We denote by $\mathrm{Art}\colon E^\times \to W_E^{\mathrm{ab}}$ the Artin map for E (normalized so that uniformizers correspond to lifts of Frobenius), and let $\chi\colon E^\times \to \overline{\mathbb{Q}}_\ell^\times$ be the composition $\chi := \varphi \circ \mathrm{Art}$.

We define

$$\mathscr{F}_\varphi := \mathscr{F} \times^{\underline{E}^\times,\chi} \underline{\overline{\mathbb{Q}}_\ell},$$

a Weil ℓ-adic sheaf on Pic, and claim that this sheaf satisfies the conditions in Fargues's conjecture. Note that in this case, there is a unique action of $S_\varphi(= \overline{\mathbb{Q}}_\ell^\times)$ such that part (1) of the conjecture is satisfied. Furthermore, the cuspidality condition (2) is clear since the semi-stable locus is all of Pic, and the compatiblity with the local Langlands correspondence is clear by the definitions. It remains to show the Hecke eigensheaf property and the character sheaf property. The latter is checked easily (see [1] §9.1).

The Hecke eigensheaf property follows from the above proposition together with Lubin-Tate theory. Namely, denoting by $\chi_{\mathcal{LT}}\colon W_E^{\mathrm{ab}} \to \mathcal{O}_E^\times \subset E^\times$ the Lubin-Tate character, we have that

(1) If $\sigma = \mathrm{Art}(\pi)$, then $\mathrm{Art} \circ \chi_{\mathcal{LT}}(\sigma) = 1$,
(2) if σ lies in the image of the inertia subgroup of W_E in W_E^{ab}, then $\mathrm{Art} \circ \chi_{\mathcal{LT}}(\sigma) = \sigma^{-1}$.

In particular, part (2) implies that for the restriction to the inertia group $I_E \subset W_E$, we have

$$(\chi \circ \chi_{\mathcal{LT}})^{-k}|_{I_E} = r_\mu \circ \varphi|_{I_E}$$

(recall that $\mu(z) = z^k$). Using this and the above proposition, we obtain

$$\overrightarrow{h}_!(\overleftarrow{h}^*\mathscr{F}_\varphi)|_{\mathrm{Pic}^d_{\overline{\mathbb{F}}_q} \times \mathrm{Div}^1_{\overline{\mathbb{F}}_q}} = \mathscr{F}_\varphi|_{\mathrm{Pic}^d_{\overline{\mathbb{F}}_q}} \boxtimes r_\mu \circ \varphi|_{I_E},$$

where we regard $r_\mu \circ \varphi|_{I_E}$ as an ℓ-adic local system on $\mathrm{Div}^1 = \mathrm{Spa}(E)^\diamond/\varphi^{\mathbb{Z}}$. (Cf. [2] 17.3). It is easy to check using (1) above that with the Weil sheaf structure defined on $\mathscr{F}$ above and the canonical Weil sheaf structure on $r_\mu \circ \varphi|_{I_E}$ this is an isomorphism of Weil sheaves, as desired.

Along similar lines, the conjecture can be proved for arbitrary tori; see [1] §9.2.

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Relation with the cohomology of Lubin-Tate spaces

Jared Weinstein

The goal of this talk is to confirm a prediction of Fargues' conjecture in the case of the group GL_n. We will see that the existence of the Hecke eigensheaf implies that the cohomology of *Lubin-Tate space* realizes the local Langlands and local Jacquet-Langlands correspondences simultaneously.

Define the following data:

- $G = \mathrm{GL}_n / \mathbf{Q}_p$,
- $\mu(z) = \mathrm{diag}(z, 1, \ldots, 1)$,
- $b = \begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ p^{-1} & & & \end{pmatrix} \in G(\breve{\mathbf{Q}}_p)$,
- $J_b(\mathbf{Q}_p) = D^*$ where $D/\mathbf{Q}_p$ is a division algebra of invariant $1/n$.

1. The Hecke stack

We have a Hecke stack

$$\begin{tikzcd} & \mathrm{Hecke}^{\mu} \arrow[dl, "h^{\leftarrow}"'] \arrow[dr, "h^{\rightarrow}"] & \\ \mathrm{Bun}_{G,\overline{\mathbf{F}}_p} & & \mathrm{Bun}_{G,\overline{\mathbf{F}}_p} \times \mathrm{Spa}\,\mathbf{Q}_p^{\diamond} \end{tikzcd}$$

where Hecke^{μ} is the following sheaf on Perf:

$$\mathrm{Hecke}^{\mu}(S) = \left\{ (\mathcal{E}, \mathcal{E}', S^{\#}, u) \colon \begin{array}{c} \mathcal{E}, \mathcal{E}' = G\text{-bundles} \\ S^{\#} = \text{ untilt } \leftrightarrow i\colon D \hookrightarrow X_S \\ u\colon \mathcal{E} \overset{\leq \mu}{\dashrightarrow} \mathcal{E}' \text{ such that} \\ \mathrm{Coker}\,\mu \text{ is supported on } D \end{array} \right\}$$

We could (and usually would) write $\mathrm{Hecke}^{\leq\mu}$ but in this case there's no difference because μ is minuscule. The modification will be

$$0 \to \mathcal{E} \to \mathcal{E}' \to i_* W \to 0$$

where W is a locally free $S^{\#}$-module of rank 1.

The functor Hecke^{μ} will not be representable by a perfectoid space, because $\mathcal{E}$ and $\mathcal{E}'$ generally have many automorphisms. We can address this issue by rigidifying $\mathcal{E}$ and $\mathcal{E}'$, and that is how the Lubin-Tate tower shows up.

2. Rigidification: the Lubin-Tate tower at infinite level

Let $y_1\colon \operatorname{Spa}\overline{\mathbf{F}}_p \to \operatorname{Bun}_{G,\overline{\mathbf{F}}_p}$ and $y_b\colon \operatorname{Spa}\overline{\mathbf{F}}_p \to \operatorname{Bun}_{G,\overline{\mathbf{F}}_p}$ be two points. (We pass to the algebraic closure because we do not want to keep track of the Weil descent datum right now; one can always go back to this later.) We define a sheaf $\mathcal{M}_\infty$ on Perf by the cartesian diagram

$$\begin{tikzcd} \mathcal{M}_\infty \arrow[r] \arrow[d] & \mathrm{Hecke}^\mu \arrow[d, "h^{\leftarrow}\times h_0^{\rightarrow}"] \\ \operatorname{Spa}\overline{\mathbf{F}}_p \arrow[r, "y_1\times y_b"] & \operatorname{Bun}_{G,\overline{\mathbf{F}}_p}\times \operatorname{Bun}_{G,\overline{\mathbf{F}}_p} \end{tikzcd}$$

where $h_0^{\rightarrow} = p_1 \circ h^{\rightarrow}$. Since y_1 corresponds to the trivial bundle, $\mathcal{M}_\infty$ parametrizes modifications of the form

$$0 \to \mathcal{O}_X^{\oplus n} \xrightarrow{u} \mathcal{O}_X(1/n) \to i_*W \to 0.$$

Note that the only thing that varies here is u.

Theorem 1 (Scholze-Weinstein [1])**.** *Let $H_0/\overline{\mathbf{F}}_p$ be the p-divisible group which is connected of dimension 1 and height n (exactly the one corresponding to the isocrystal b).*

(1) *For a perfectoid $\mathbf{Q}_p$-algebra (R,R^+), we have*

$$\mathcal{M}_\infty(R,R^+) = \left\{ (H,\iota,\alpha)\colon \begin{array}{c} H = p\text{-div group } / \ R^+ \\ \alpha = \text{quasi-isog.}\colon H\otimes_{R^+} R^+/p \sim H_0 \otimes_{\overline{\mathbf{F}}_p} R^+/p \\ \iota\colon T_pH\otimes \mathbf{Q}_p \cong \mathbf{Q}_p^{\oplus n} \end{array} \right\}$$

This has an action of $\mathrm{GL}_n(\mathbf{Q}_p)\times J_b(\mathbf{Q}_p)$, with $\mathrm{GL}_n(\mathbf{Q}_p)$ acting on ι and $J_b(\mathbf{Q}_p)$ acting on α.

(2) *$\mathcal{M}_\infty$ is a preperfectoid space.*

Remark 2. The $\mathrm{GL}_n(\mathbf{Q}_p)\times J_b(\mathbf{Q}_p)$-action is also clear from the description of $\mathcal{M}_\infty$ as parametrizing extensions

$$0 \to \mathcal{O}_X^{\oplus n} \xrightarrow{u} \mathcal{O}_X(1/n) \to i_*W \to 0.$$

because $\mathrm{GL}_n(\mathbf{Q}_p)$ is automorphism group of $\mathcal{E}_1 = \mathcal{O}_X^n$ and $J_b(\mathbf{Q}_p)$ is automorphism group of $\mathcal{E}_b = \mathcal{O}_X(1/n)$.

Remark 3. $\mathcal{M}_\infty$ comes equipped with a map to $\mathbf{Q}_p$ because it's fibered over Hecke^μ, which comes equipped with a map to $\operatorname{Spa}\mathbf{Q}_p^\diamond$.

Proof Sketch. How do we parametrize these morphisms u? Well, u is a map of vector bundles $\mathcal{O}_X^n \to \mathcal{O}_X(1/n)$, which is the same as giving n global sections of $\mathcal{O}_X(1/n)$. So that gives a map

$$\mathcal{M}_\infty \to H^0(X,\mathcal{O}(1/n))^{\oplus n}.$$

(For clarity, we spell out that $H^0(X, \mathcal{O}(1/n))^{\oplus n}$ is the sheaf that assigns to $S \in \mathrm{Perf}_{\mathbf{F}_p}$ the set of n-tuples of sections in $H^0(X_S, \mathcal{O}(1/n))$.) As was explained in Le Bras' talk the sheaf $H^0(X, \mathcal{O}(1/n))$ is the same as $\widetilde{H}$, the universal cover of any lift $H/W(\overline{\mathbf{F}}_p)$ of H_0. (We have $H_0 \leftrightarrow b \leftrightarrow \mathcal{E}_b$, and the general theorem is that $H^0(X, \mathcal{E}_b) = \widetilde{H}$). The universal cover $\widetilde{H}$ is preperfectoid. Scholze-Weinstein [1] proves that $\mathcal{M}_\infty \to \tilde{H}$ is a locally closed embedding, from which it follows that $\mathcal{M}_\infty$ is a preperfectoid space.

□

Remark 4. In fact $\mathcal{M}_\infty$ is a perfectoid space. One sees this by using the *determinant map* from $\mathcal{M}_\infty$ onto the Lubin-Tate space for $\mathrm{GL}_1/\mathbf{Q}_p$, which is $\mathbf{Z}$ copies of $\mathrm{Spa}\,\mathbf{Q}_p^{\mathrm{cycl}}$. Thus $\mathcal{M}_\infty$ comes equipped with a map to a perfectoid field.

3. Another Rigidification

We just related the Hecke stack to a perfectoid space at infinite level, by rigidifying both $\mathcal{E}$ and $\mathcal{E}'$. Perhaps this is overkill. What if we rigidify at just one vector bundle and not the other? Suppose we just fix $\mathcal{E}' = \mathcal{O}_X(1/n)$. Then we are considering the moduli problem which assigns to a test object S of Perf the set of modifications

$$0 \to \mathcal{E} \to \mathcal{O}_{X_S}(1/n) \to i_*W \to 0.$$

Giving such a modification just amounts to specifying W. More precisely, one has to choose an untilt $S^\sharp$ of S, and then a rank 1 quotient of the fiber of the rank n vector bundle $\mathcal{O}_{X_S}(1/n)$ at $S^\sharp$. But the fiber of $\mathcal{O}_{X_S}(1/n)$ at $S^\sharp$ can be identified with $(S^\sharp)^n$. So our moduli problem is really just $\mathbf{P}^{n-1,\diamond}_{\breve{\mathbf{Q}}_p}$.

To understand what $\mathcal{E}$ is, we note that $\mathcal{O}_{X_S}(1/n)$ has rank n and degree 1 at all geometric points of S, while i_*W has rank 0 and degree 1. By the additivity of rank and degree, we deduce that $\mathcal{E}$ has rank n and degree 0 at all geometric points of S. We also know that $\mathcal{O}_{X_S}(1/n)$ is semistable. So what could a slope of $\mathcal{E}$ be? There cannot be a slope $> 1/n$ by the semistability of $\mathcal{O}_X(1/n)$. However, any other positive slope would have a larger denominator, hence larger rank. So we conclude that $\mathcal{E}$ must be semistable of slope 0.

Remark 5. This is a really special feature of the Lubin-Tate situation.

It's proven in Kedlaya-Liu [2] that semistable vector bundles of slope 0 can be trivialized over a pro-étale cover. That is: $\underline{\mathrm{Isom}}(\mathcal{E}, \mathcal{O}_{X_S}^{\oplus n})$ is a pro-étale $G(\mathbf{Q}_p)$-torsor over S. These are classified by the stack $[\mathrm{Spa}\,\overline{\mathbf{F}}_p/\mathrm{GL}_n(\mathbf{Q}_p)]$. We have thus constructed a morphism

$$r\colon \mathbf{P}^{n-1,\diamond}_{\breve{\mathbf{Q}}_p} \to [\mathrm{Spa}\,\overline{\mathbf{F}}_p/\mathrm{GL}_n(\mathbf{Q}_p)].$$

Thus we get a diagram

(1)
$$\begin{tikzcd} & & {[\operatorname{Spa}\overline{\mathbf{F}}_p/\operatorname{GL}_n(\mathbf{Q}_p)]} \arrow[d] \\ \mathbf{P}^{n-1,\diamond}_{\breve{\mathbf{Q}}_p} \arrow[r] \arrow[urr, "r"] & \mathrm{Hecke}^\mu \arrow[r, "h^{\leftarrow}"] & \mathrm{Bun}_{G,\overline{\mathbf{F}}_p} \end{tikzcd}$$

The map

$$r\colon \mathbf{P}^{n-1,\diamond}_{\breve{\mathbf{Q}}_p} \to [\operatorname{Spa}\overline{\mathbf{F}}_p/\operatorname{GL}_n(\mathbf{Q}_p)]$$

corresponds by definition to a $\operatorname{GL}_n(\mathbf{Q}_p)$-torsor on $\mathbf{P}^{n-1,\diamond}_{\breve{\mathbf{Q}}_p}$. Unraveling the constructions, this torsor is none other than $\mathcal{M}_\infty$. The map to $\mathbf{P}^{n-1,\diamond}_{\breve{\mathbf{Q}}_p}$ factors through some finite layer, i.e., we have a diagram

$$\begin{tikzcd} \mathcal{M}^\diamond_\infty \arrow[rr] \arrow[dr] & & \mathbf{P}^{n-1,\diamond}_{\breve{\mathbf{Q}}_p} \\ & \mathcal{M}^\diamond_K \arrow[ur] & \end{tikzcd}$$

where $K \subset \operatorname{GL}_n(\mathbf{Q}_p)$ is a compact open subgroup.

In order to match things up with the Hecke correspondence, we now base change to $\mathbf{Q}_p$ (because one of the maps of Hecke^μ is to $\mathrm{Bun}_{G,\overline{\mathbf{F}}_p} \times (\operatorname{Spa}\mathbf{Q}_p)^\diamond$).

$$[\operatorname{Spa}\breve{\mathbf{Q}}_p^\diamond/J_b(\mathbf{Q}_p)] \xrightarrow{(x_b,1)} \mathrm{Bun}_{G,\overline{\mathbf{F}}_p} \times \operatorname{Spa}\mathbf{Q}_p^\diamond.$$

We have a commutative diagram

$$\begin{tikzcd} {[\mathbf{P}^{n-1,\diamond}_{\breve{\mathbf{Q}}_p}/J_b(\mathbf{Q}_p)]} \arrow[r, "i"] \arrow[d, "j"] & \mathrm{Hecke}^\mu \arrow[d, "h^{\rightarrow}"] \\ {[\operatorname{Spa}\breve{\mathbf{Q}}_p^\diamond/J_b(\mathbf{Q}_p)]} \arrow[r, "{(x_b,1)}"] & \mathrm{Bun}_{G,\overline{\mathbf{F}}_p} \times \operatorname{Spa}\mathbf{Q}_p^\diamond \end{tikzcd}$$

(We have written down this diagram before without modding out be J_b on the left side.) The map $i\colon [\mathbf{P}^{n-1,\diamond}_{\breve{\mathbf{Q}}_p}/J_b(\mathbf{Q}_p)] \to \mathrm{Hecke}^\mu$ is an open embedding. Indeed, as Peter mentioned in his talk, there is a theorem that the semistable locus Bun_G^{ss} of Bun_G is given by

$$\coprod_{b\text{ basic}} \left[\operatorname{Spa}\overline{\mathbf{F}}_p/J_b(\mathbf{Q}_p)\right] = \mathrm{Bun}_G^{ss}$$

and i is a base change of this map.

To summarize, we have the commutative diagram

$$\begin{tikzcd}
 & & {[\operatorname{Spa}\overline{\mathbf{F}}_p/\operatorname{GL}_n(\mathbf{Q}_p)]} \arrow[d, "x_1"] \\
{[\mathbf{P}^{n-1,\diamond}_{\breve{\mathbf{Q}}_p}/J_b(\mathbf{Q}_p)]} \arrow[urr, "r"] \arrow[r, "i"] \arrow[d, "j"] & \operatorname{Hecke}^{\mu} \arrow[r, "h^{\leftarrow}"] \arrow[d, "h^{\rightarrow}"] & \operatorname{Bun}_G \\
{[\operatorname{Spa}\breve{\mathbf{Q}}_p^{\diamond}/J_b(\mathbf{Q}_p)]} \arrow[r, "{(x_b,1)}"] & \operatorname{Bun}_{G,\overline{\mathbf{F}}_p}\times\operatorname{Spa}\mathbf{Q}_p^{\diamond} &
\end{tikzcd}$$

4. Fargues' conjecture

Let $\varphi\colon W_{\mathbf{Q}_p}\to\operatorname{GL}_n(\overline{\mathbf{Q}}_\ell)$ be a discrete Weil parameter. What does Fargues' conjecture say in this case? (The situation here is a little simplified by the fact that S_φ is trivial.) It predicts that there exists $\mathcal{F}_\varphi$ on $\operatorname{Bun}_{G,\overline{\mathbf{F}}_p}$ such that (up to shifts and twists)

- We have

(2) $$h^{\rightarrow}_{!}h^{\leftarrow *}\mathcal{F}_\varphi=\mathcal{F}_\varphi\boxtimes\varphi.$$

  (This is simpler than in general because the IC sheaf is constant up to shifts and twists, and also it is unnecessary to write r_μ because it is the standard representation of GL_n.)
- We have $x_1^*\mathcal{F}_\varphi=\pi$ and $x_b^*\mathcal{F}_\varphi=\rho$ where π and ρ correspond to φ under the local Langlands correspondence.

Consequences of the conjecture. Pulling back (2) through $(x_b,1)^*$ gives

(3) $$(x_b,1)^*h^{\rightarrow}_{!}h^{\leftarrow *}\mathcal{F}_\varphi=(x_b,1)^*\mathcal{F}_\varphi\boxtimes\varphi.$$

On the right side we get $\rho\otimes\varphi$ by the second property of the sheaf $\mathcal{F}_\varphi$. On the left side, first apply proper base change to j from the earlier diagram

$$\begin{tikzcd}
{[\mathbf{P}^{n-1,\diamond}_{\breve{\mathbf{Q}}_p}/J_b(\mathbf{Q}_p)]} \arrow[r, "i"] \arrow[d, "j"] & \operatorname{Hecke}^{\mu} \arrow[d, "h^{\rightarrow}"] \\
{[\operatorname{Spa}\breve{\mathbf{Q}}_p^{\diamond}/J_b(\mathbf{Q}_p)]} \arrow[r, "{(x_b,1)}"] & \operatorname{Bun}_{G,\overline{\mathbf{F}}_p}\times\operatorname{Spa}\mathbf{Q}_p^{\diamond}
\end{tikzcd}$$

to deduce that

(4) $$\rho\otimes\varphi=(x_b,1)^*h^{\rightarrow}_{!}h^{\leftarrow *}\mathcal{F}_\varphi=j_!i^*h^{\leftarrow *}\mathcal{F}_\varphi.$$

Now we use the top part of the diagram

$$\begin{tikzcd} & & & {[\mathrm{Spa}\,\overline{\mathbf{F}}_p/\mathrm{GL}_n(\mathbf{Q}_p)]} \arrow[d, "x_1"] \\ {[\mathbf{P}^{n-1,\diamond}_{\breve{\mathbf{Q}}_p}/J_b(\mathbf{Q}_p)]} \arrow[rrru, "r"] \arrow[r, "i"] & \mathrm{Hecke}^{\mu} \arrow[rr, "h^{\leftarrow}"] & & \mathrm{Bun}_G \end{tikzcd}$$

to deduce that

$$j_! i^* h^{\leftarrow *}\mathcal{F}_\varphi = j_! r^* x_1^* \mathcal{F}_\varphi.$$

Then the second property of $\mathcal{F}_\varphi$, i.e., $x_1^*\mathcal{F} = \pi$, implies that this is $j_! r^*\pi$, so combining this with (4) gives

$$\rho \otimes \varphi = j_! r^* \pi.$$

Recall that r corresponds to the ($J_b(\mathbf{Q}_p)$-equivariant) $\mathrm{GL}_n(\mathbf{Q}_p)$-torsor $\mathcal{M}^\diamond_\infty$ on $\mathbf{P}^{n-1,\diamond}_{\breve{\mathbf{Q}}_p}$.

Now we apply $j_!$ to get

$$\rho \otimes \varphi = H^*_c(\mathbf{P}^{n-1}_{\mathbf{C}_p}, r^*\pi). \tag{5}$$

Here we have base-changed to $\mathbf{C}_p$ and gotten rid of the J_b quotient at the cost of remembering the action of Galois and J_b. So the above isomorphism is equivariant for the action of $J_b(\mathbf{Q}_p) \times W_{\mathbf{Q}_p}$.)

We ignored shifts and twists; if you keep track of them then (assuming that π is cuspidal) you get

$$\boxed{\rho \otimes \varphi = H^{n-1}_c(\mathcal{M}_\infty, \mathbf{Q}_\ell)[\pi^\vee](\frac{1-n}{2}).} \tag{6}$$

This is a very deep theorem of Harris-Taylor. How did we get from (5) to (6)? The Hochschild-Serre spectral sequence for the fibration

$$\begin{tikzcd} \mathcal{M}_{\infty,\mathbf{C}_p} \arrow[d] \\ \mathbf{P}^{n-1}_{\mathbf{C}_p} \arrow[d] \\ \mathbf{C}_p \end{tikzcd}$$

converges as

$$H_i(\mathrm{GL}_n(\mathbf{Q}_p), H^j_c(\mathcal{M}_{\infty,\mathbf{C}_p}, \overline{\mathbf{Q}}_\ell) \otimes \pi) \Rightarrow H^{-i+j}(\mathbf{P}^{n-1}_{\mathbf{C}_p}, r^*\pi).$$

In the supercuspidal case there is no higher group cohomology, so you take the invariants in this tensor product, which gives what we claim.

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Reporter: Johannes Anschütz

Participants

Johannes Anschütz
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 288
69120 Heidelberg
GERMANY

Prof. Dr. Joseph Bernstein
School of Mathematical Sciences
Tel Aviv University
Ramat Aviv
Tel Aviv 69978
ISRAEL

Dr. Bhargav Bhatt
Department of Mathematics
University of Michigan
530 Church Street
Ann Arbor, MI 48109-1043
UNITED STATES

Dr. Alexis Bouthier
Einstein Institute of Mathematics
The Hebrew University of Jerusalem
Edmond J. Safra Campus
Givat Ram
Jerusalem 91904
ISRAEL

Christopher Justin Campbell
Department of Mathematics
Harvard University
Science Center
One Oxford Street
Cambridge MA 02138-2901
UNITED STATES

Dr. Ana Caraiani
Department of Mathematics
Princeton University
710 Fine Hall
Washington Road
Princeton NJ 08544-1000
UNITED STATES

Charlotte Chan
Department of Mathematics
University of Michigan
530 Church Street
Ann Arbor, MI 48109-1043
UNITED STATES

Dr. Tsao-Hsien Chen
Department of Mathematics
Northwestern University
Lunt Hall
2033 Sheridan Road
Evanston, IL 60208-2730
UNITED STATES

Dr. Przemyslaw Chojecki
Mathematical Institute
Oxford University
Andrew Wiles Bldg.
Woodstock Road
Oxford OX2 6GG
UNITED KINGDOM

Prof. Dr. Pierre Colmez
Institut de Mathématiques de Jussieu,
CNRS
Université de Paris VI
Theorie des Nombres, Case 247
4, Place Jussieu
75252 Paris Cedex 05
FRANCE

Prof. Dr. Jean-Francois Dat
Institut de Mathématiques de Jussieu
Université de Paris VI
Case 247
4, Place Jussieu
75252 Paris Cedex 05
FRANCE

Prof. Dr. Christopher Deninger
Mathematisches Institut
Universität Münster
Einsteinstrasse 62
48149 Münster
GERMANY

Dr. Galyna Dobrovolska
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY

Dr. Gabriel Dospinescu
Département de Mathématiques
École Normale Superieure de Lyon
46, Allée d'Italie
69364 Lyon Cedex 07
FRANCE

Prof. Dr. Laurent Fargues
Institut de Mathématiques de Jussieu
Université de Paris VI
Case 247
4, Place Jussieu
75252 Paris Cedex 05
FRANCE

Tony Feng
2727 Midtown Court
Apartment # 5
Palo Alto CA 94303
UNITED STATES

Jessica Fintzen
Department of Mathematics
Harvard University
Science Center
One Oxford Street
Cambridge MA 02138-2901
UNITED STATES

Prof. Dr. Jean-Marc Fontaine
Laboratoire de Mathématiques
Université Paris Sud (Paris XI)
Batiment 425
91405 Orsay Cedex
FRANCE

Dragos Fratila
Institut de Mathématiques
Université de Strasbourg
7, rue René Descartes
67084 Strasbourg Cedex
FRANCE

Ildar M. Gaisin
Institut de Mathématiques
Université Pierre et Marie Curie
Tour 46, 5eme etage
4 place Jussieu
75252 Paris Cedex 05
FRANCE

Prof. Dr. Dennis Gaitsgory
Department of Mathematics
Harvard University
Science Center
One Oxford Street
Cambridge MA 02138-2901
UNITED STATES

Prof. Dr. Ulrich Görtz
Institut für Experimentelle Mathematik
Universität Duisburg-Essen
45117 Essen
GERMANY

Dr. Sam Gunningham
Department of Mathematics
The University of Texas at Austin
1 University Station C1200
Austin, TX 78712-1202
UNITED STATES

Dr. Paul Hamacher
Zentrum Mathematik - M 11
Technische Universität München
Boltzmannstrasse 3
85748 Garching bei München
GERMANY

Prof. Dr. Urs Hartl
Mathematisches Institut
Universität Münster
Einsteinstrasse 62
48149 Münster
GERMANY

Jochen Heinloth
Fachbereich Mathematik
Universität Duisburg-Essen
Universitätsstrasse 3
45117 Essen
GERMANY

Dr. Eugen Hellmann
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Valentin Hernandez
Institut de Mathématiques de Jussieu
Université de Paris VI
Case 247
4, Place Jussieu
75252 Paris Cedex 05
FRANCE

Prof. Dr. Bruno Klingler
Equipe Formes Automorphes
Université Paris - Diderot
Croisement Avenue de France / Rue
Alice
75013 Paris Cedex
FRANCE

Prof. Dr. Yakov Kremnitzer
Mathematical Institute
Oxford University
Andrew Wiles Bldg.
Woodstock Road
Oxford OX2 6GG
UNITED KINGDOM

Prof. Dr. Eike Lau
Fakultät EIM - Elektrotechnik,
Informatik und Mathematik
Universität Paderborn
Warburger Strasse 100
33098 Paderborn
GERMANY

Dr. Arthur-César Le Bras
École Normale Superieure
D. M. A.
45, Rue d'Ulm
75230 Paris Cedex 05
FRANCE

Prof. Dr. Ruochuan Liu
Beijing International Center for
Mathematical Research
No. 5 Yiheyuan Road Haidian District
Beijing 100 871
CHINA

Dr. Yifeng Liu
Department of Mathematics
Northwestern University
Lunt Hall
2033 Sheridan Road
Evanston, IL 60208-2730
UNITED STATES

Judith Ludwig
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Keerthi Madapusi Pera
Department of Mathematics
The University of Chicago
5734 South University Ave.
Chicago, IL 60637-1514
UNITED STATES

Dr. Michael McBreen
Department of Mathematics
Massachusetts Institute of Technology
77 Massachusetts Avenue
Cambridge, MA 02139-4307
UNITED STATES

Prof. Dr. Sophie Morel
Department of Mathematics
Princeton University
Fine Hall
Princeton, NJ 08544-1000
UNITED STATES

Dr. Matthew Morrow
Hausdorff Center for Mathematics
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Wieslawa Niziol
Dept. de Mathématiques, U.M.P.A.
École Normale Superieure de Lyon
46, Allée d'Italie
69364 Lyon Cedex 07
FRANCE

Prof. Dr. Vytautas Paskunas
Fakultät für Mathematik
Universität Duisburg-Essen
45117 Essen
GERMANY

Stefan Patrikis
Department of Mathematics
University of Utah
155 South 1400 East
Salt Lake City, UT 84112-0090
UNITED STATES

Prof. Dr. Vincent Pilloni
Mathématiques
École Normale Supérieure de Lyon
46, Allée d'Italie
69364 Lyon Cedex 07
FRANCE

Prof. Dr. Michael Rapoport
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Dr. Sam Raskin
Department of Mathematics
Massachusetts Institute of Technology
77 Massachusetts Avenue
Cambridge, MA 02139-4307
UNITED STATES

Dr. Nick Rozenblyum
Department of Mathematics
The University of Chicago
5734 South University Avenue
Chicago, IL 60637-1514
UNITED STATES

Prof. Dr. Peter Schneider
Mathematisches Institut
Universität Münster
Einsteinstrasse 62
48149 Münster
GERMANY

Prof. Dr. Peter Scholze
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Benoit Stroh
Département de Mathématiques
LAGA - CNRS: UMR 7539
Université Paris Nord (Paris XIII)
99, Avenue J.-B. Clement
93430 Villetaneuse Cedex
FRANCE

Dr. Yakov Varshavsky
Einstein Institute of Mathematics
The Hebrew University
Givat Ram
Jerusalem 91904
ISRAEL

Prof. Dr. Eva Viehmann
Zentrum Mathematik - M 11
Technische Universität München
Boltzmannstrasse 3
85748 Garching b. München
GERMANY

Prof. Dr. Kari Vilonen
Department of Mathematics
Northwestern University
Lunt Hall
2033 Sheridan Road
Evanston, IL 60208-2730
UNITED STATES

Prof. Dr. Torsten Wedhorn
Fachbereich Mathematik
Technische Universität Darmstadt
Schlossgartenstrasse 7
64289 Darmstadt
GERMANY

Prof. Dr. Jared Weinstein
Department of Mathematics & Statistics
Boston University
111 Cummington Mall
Boston, MA 02215
UNITED STATES

Prof. Dr. Liang Xiao
Department of Mathematics
University of Connecticut
Storrs, CT 06269-3009
UNITED STATES

Cong Xue
Mathématiques
Université de Paris-Sud
Batiment 430
91405 Orsay Cedex
FRANCE

Dr. Ting Xue
Department of Mathematics & Statistics
University of Melbourne
Melbourne VIC 3010
AUSTRALIA

Rong Zhou
Department of Mathematics
Harvard University
Science Center
One Oxford Street
Cambridge MA 02138-2901
UNITED STATES

Prof. Dr. Xinwen Zhu
Department of Mathematics
California Institute of Technology
1200 E. California Blvd.
Pasadena, CA 91125
UNITED STATES

Yihang Zhu
Department of Mathematics
Harvard University
Science Center
One Oxford Street
Cambridge MA 02138-2901
UNITED STATES