Arbeitsgemeinschaft mit aktuellem Thema:

THE LANGLANDS PROGRAM: FROM GLOBAL UNRAMIFIED GEOMETRIC TO LOCAL RAMIFIED ARITHMETIC

Mathematisches Forschungsinstitut Oberwolfach

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Introduction:

The classical Langlands program

The Langlands program emerged as an organizational principle in the theory of automorphic forms. Classically, automorphic forms are functions on symmetric domains G/K where G is a real Lie group and $K \subset G$ a maximal compact subgroup, which are required to be invariant under the action of an arithmetic subgroup $\Gamma \subset G$. The prototypical example is the case of $\mathrm{SL}_2(\mathbb{Z})$ acting on the upper half-space, giving rise to modular forms and Maaßforms. On the space of automorphic forms, one has a large space of symmetries, classically given by differential operators, and Hecke operators. This big space of operators on automorphic forms allows one to extract spectral data. One of the Langlands conjectures predicts that this same spectral data is also seen in (apparently unrelated) arithmetic situations. The prototypical example is the relation between rational modular forms of weight 2 and elliptic curves E over $\mathbb{Q}$, which relates Hecke eigenvalues with the number of $\mathbb{F}_p$-rational points of E.

In the modern formulation, one starts with a reductive group G over $\mathbb{Q}$, and one regards $\mathbb{Q}$ as the function field of the "compact curve" $\overline{\mathrm{Spec}\mathbb{Z}} = \mathrm{Spec}\mathbb{Z} \cup \{\infty\}$. For each place v of this curve, i.e. v is either a prime number p or the archimedean place ∞, one has the completion $\mathbb{Q}_v$ of $\mathbb{Q}$ at v, which are either the p-adic numbers, or the reals $\mathbb{R}$. One can also form the adèles

$\mathbb{A}$ of $\mathbb{Q}$, which is the subring of $\prod_v \mathbb{Q}_v$ given by the condition that almost all components are integral.

An automorphic representation of G is (roughly) an irreducible representation of $G(\mathbb{A})$ that occurs in the space of L^2-functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$. Any irreducible representation π of $G(\mathbb{A})$ decomposes as a (restricted) tensor product

$$\pi = \bigotimes_v \pi_v$$

of irreducible representations π_v of $G(\mathbb{Q}_v)$. The rough statement of the local Langlands conjecture says that for each v, the datum of π_v is equivalent to a representation of the absolute Galois group of $\mathbb{Q}_v$, with values in the Langlands dual group.[1] The rough statement of the global Langlands conjecture is that if π is automorphic, then there is a representation of the absolute Galois group of $\mathbb{Q}$, inducing all these representations of the local absolute Galois groups. Moreover, one should be able to go in the converse direction.

A completely parallel conjecture can be formulated for the function field F of a projective smooth curve over a finite field, in place of $\mathbb{Q}$. Several simplifications occur in this case, the most important being that the space $G(F)\backslash G(\mathbb{A}_F)$ is 0-dimensional, so most analytic aspects of the problem are gone. Notably, many of Langlands' conjectures have been proved in this case by Drinfeld, L. Lafforgue and V. Lafforgue.

The (global, unramified) geometric Langlands program

The geometric Langlands program emerged as a geometric way of looking at Langlands' conjectures in the case of a function field. It is most directly related to the classical picture when looking at the global, everywhere unramified correspondence.

Let C be a smooth projective curve over any field k, and let us continue to denote by F its function field. For any closed point x of C, we write $\mathcal{O}_x$ for the completion of the structure sheaf at x, and F_x for its quotient field. Let $\mathbb{A}_F = (\prod_x \mathcal{O}_x) \otimes F$ be the adèles. If k is a finite field, then everywhere unramified automorphic representations correspond to functions on the double quotient

$$G(F)\backslash G(\mathbb{A}_F)/G(\prod_x \mathcal{O}_x) \ .$$

[1] At least at $v = \infty$, one has to use the Weil group of $\mathbb{R}$; also, there are issues about L-packets etc., ...

The basic observation is that if Bun_G denotes the stack of G-bundles on C, then there is a bijection

$$\mathrm{Bun}_G(k) = G(F)\backslash G(\mathbb{A}_F)/G(\prod_x \mathcal{O}_x)\ .$$

If k is a finite field, then functions on $\mathrm{Bun}_G(k)$ can be geometrized by perverse sheaves on Bun_G: Any perverse sheaf gives a function on k-points by looking at traces of Frobenius on the stalks. The analogue of the Hecke action is given by the action of Hecke correspondences on the stack of G-bundles.

Looking at the other side of the correspondence, everywhere unramified Galois representations are precisely local systems on C (with values in the L-group ${}^LG$ of G). Thus, the geometric Langlands conjecture predicts that for every ${}^LG$-local system E on C, there is a perverse sheaf Aut_E on Bun_G which satisfies a suitable Hecke equivariance property. For $G = \mathrm{GL}_n$, it has been proved by Frenkel, Gaitsgory and Vilonen, following earlier work of Drinfeld, and Laumon.

If k is a finite field, this conjecture implies the global unramified classical Langlands conjecture by passing to the corresponding function on $\mathrm{Bun}_G(k)$.

However, when trying to generalize to ramified representations, it is very difficult to see the arithmetic of supercuspidal representations of $G(\mathbb{F}_p((t)))$, and its relation with irreducible Galois representations of the absolute Galois group of $\mathbb{F}_p((t))$ in this picture. The basic reason is that the geometric picture is automatically compatible with extensions of the base field $k = \mathbb{F}_p$, whereas these arithmetic phenomena are not.

Fargues' conjecture

At his MSRI lecture in December 2014, Fargues stated a most striking conjecture. Recall that in recent work with Fontaine, for any non-archimedean local field E (i.e., E is a finite extension of $\mathbb{F}_p((t))$ or $\mathbb{Q}_p$), he had constructed a certain scheme X_E over E, which behaves like a smooth projective curve over an algebraically closed field, but is not of finite type. This construction was motivated by considerations in p-adic Hodge theory.

Fargues' observation was that if one interprets the *global unramified geometric* Langlands conjecture on this curve, one ends up with a statement that encodes most conjectural properties of the *local ramified arithmetic* Langlands conjecture over E. One critical difference is that the automorphism group of the trivial G-torsor is not the algebraic group G, but the locally profinite group $G(E)$, so (perverse) sheaves on the stack of G-bundles naturally give rise to representations of $G(E)$. One can hope that this makes it possible to adapt methods from the geometric Langlands program to make progress on the local Langlands conjectures, even over p-adic fields.

The goal of this workshop will be to recall the geometric Langlands conjecture (in the "punctual case") and sketch the proof for $GL(2)$, and to formulate Fargues' conjecture.

Talks:

Day 1:

1. **Adic spaces** Give a brief introduction to theory of adic spaces, concentrating on the case of adic spaces over some complete nonarchimedean base field. Moreover, recall the relation to rigid-analytic spaces. Reference: e.g. [12, Section 2] and the references mentioned there.

2. **Geometric class field theory**

   The unramified version:

   (a) State the theorem as follows: pullback along $X \to \mathrm{Pic}(X)$ defines a bijection between character local systems on $\mathrm{Pic}(X)$ and 1-dim local systems on X.

   (b) Prove decent by Deligne's trick using the fact that the fibers of $X^{(d)} \to \mathrm{Pic}^d$ for $d \gg 0$ are simply connected.

   The passage from geometric to classical: Explain why in the abelian case the geometric theory is equivalent to the classical one, using Lang's isogeny.

   The ramified version:

   (a) Explain that if we imitate Deligne's construction, one is led to a geometric formulation of local CFT.

   (b) State the local CFT.

   (c) Explain the connection with Lubin-Tate.

   (d) Deduce the global ramified CFT.

   Reference: [8].

3. **The Fargues-Fontaine curve** Define the Fargues-Fontaine curve $X_E = X_{F,E}$, first as an adic space, and then as a scheme. In the equal characteristic case, emphasize the equation

$$X_{F,E} = \text{“}(\operatorname{Spa} E/\varphi) \times \operatorname{Spa} F\text{''} \ .$$

   State its basic properties: It is a regular noetherian scheme of dimension 1, with all residue fields algebraically closed. Reference: [5], [4].

4. **Perfectoid spaces** Define perfectoid rings and perfectoid spaces. Reference: [12, Sections 5,6], or [13, Lectures 6,7].

   Day 2:

5. **The pro-étale and faithful topology** Discuss the pro-étale and faithful topologies on perfectoid spaces, in particular that they are subcanonical, and that one can glue vector bundles. Reference: [13, Lectures 8,9].

6. **Statement of Galois ⇒ Automorphic in the geometric context**

   The classical case: Explain the statement of Galois to automorphic at the level of functions, and the“faisceaux-fonctions” process.

   Geometric Satake:

   (a) Give the statement (as monoidal categories).

   (b) Describe the commutativity constraint as coming from fusion.

   (c) Outline the proof.

   Statement of global (pointwise) geometric Langlands:

   (a) Notion of Hecke eigen-sheaf (naive version).

   (b) The correct notion of Hecke eigen-sheaf (with multiple points).

   (c) State the existence conjecture.

   References: [9, 11, 6, 7].

7. **Vector bundles on the Fargues-Fontaine curve** State the classification result for vector bundles on the Fargues-Fontaine curve. Moreover, explain the relationship with p-divisible groups, as in [14, Section 5.1, Corollary 6.3.10]. Reference: [5].

8. **Banach-Colmez spaces** The global sections of vector bundles on the Fargues-Fontaine curve are not finite-dimensional in the usual sense, but they are finite-dimensional Banach Spaces *in the sense of Colmez*. Recall these spaces, and their relation to p-divisible groups. Use this to describe the automorphism groups of vector bundles on the Fargues-Fontaine curve.

   Day 3:

9. **The relative Fargues-Fontaine curve** Define the relative Fargues-Fontaine curve $X_{S,E}$ over a perfectoid space S of characteristic p, and discuss the (non-existent) map $\pi : X_{S,E} \to S$, and the relation of sections of π with untilts of S. Reference: [4], [13, Lecture 11].

10. **Beauville-Laszlo uniformization** We will follow Drinfeld-Simpson.

    Existence of B-structures:

    (a) State the theorem on the existence of B-structures, étale locally on the space of parameters.

    (b) Give a proof by reducing to the case of $\mathbb{P}^1$.

    Triviality of the bundle when restricted to the punctured curve:

    (a) State the theorem that the G-bundle on a complete curve becomes trivial when restricted to the punctured curve, fppf locally with respect to the scheme of parameters (comment on when "fppf" can be replaced by "étale").

    (b) Prove by reducing to the case of $GL(2)$.

    Reference: [2].

    Day 4:

11. **Classification of G-bundles** Describe the classification of G-bundles on the Fargues-Fontaine curve in terms of Kottwitz' set $B(G)$ of isocrystals with G-structure. This relates semistable G-bundles with basic elements $b \in B(G)$, and automorphism groups with the inner forms J_b of G. Reference: [3]

12. **Proof of Geometric Langlands for GL(2), first part**

    The classical case:

    (a) Explain the construction of cuspidal automorphic functions based on their Whittaker model.

    (b) Recall the Casselman-Shalika formula for the values of the Whittaker function with specified behavior with respect to Hecke operators.

    The Whittaker model in the geometric case:

    (a) Explain how one would want to mimic the classical construction in geometry, and why this is non-obvious, because one runs into infinite-dimensional objects.

    (b) Explain the truncated version of the construction.

    The Whittaker model in geometry:

    (a) Introduce Laumon's sheaf.

    (b) Explain how it gives a geometric counterpart of the Whittaker function.

    Reference: [7].

13. **Uniformization of** Bun_G Describe the analogue of the Beauville-Laszlo uniformization. In particular, discuss the B_{dR}^+-Grassmannian of [13, Lecture 22]. Use that for any reductive group G, any G-torsor on the Fargues-Fontaine curve becomes trivial after removing a point, cf. [3]. Use this to construct many smooth maps into Bun_G, showing that Bun_G behaves like an Artin stack.

14. **Formulation of Fargues' conjecture** Formulate Fargues' conjecture for discrete L-parameters of the Weil group of K. Reference: [4].

    Day 5:

15. **Relation with the classical local Langlands correspondence** Recall the classical formulation of the local Langlands correspondence, and its extension to "extended pure inner forms" ($= \{J_b, b \in B(G)_{\mathrm{basic}}\}$) by Kaletha, cf. e.g. [10, Conjecture F], and make the link with Fargues' conjecture.

16. **Proof of Geometric Langlands for GL(2), second part**

    Construction of the automorphic sheaf:

    (a) Explain the construction of the sought-for automorphic sheaf from Laumon's sheaf.

    (b) State the key vanishing result.

    (c) Explain that we need to prove the descent statement.

    (d) Explain the trick to prove the descent statement using Euler characteristics.

    The Hecke property:

    (a) Reduce the verification of the Hecke property to the case of the basic Hecke functor.

    (b) Verify the latter on the Whittaker model.

    Reference: [6, 1].

17. **The case of** $\mathbb{G}_m$ Explain how the case $G = \mathbb{G}_m$ of Fargues' conjecture is related to local class field theory (via the Lubin-Tate approach to local class field theory).

18. **Relation with the cohomology of Lubin-Tate spaces** Explain how in the case $G = \mathrm{GL}_n$, the Hecke equivariance property for the simplest cocharacter $\mu = (1, 0, \ldots, 0)$ is related to the description of the cohomology of Lubin-Tate spaces (Carayol's conjecture, proved by Boyer and Harris-Taylor). This generalizes to a relation with the cohomology of moduli spaces of local shtukas.

Remark. Many references (e.g. [4], [5], [13]) listed below are not in their final form, but should be available well before the Arbeitsgemeinschaft. We also advice the speakers to directly contact the authors.

References

[1] Drinfeld, V. G. Two-dimensional l-adic representations of the fundamental group of a curve over a finite field and automorphic forms on GL(2). Amer. J. Math. 105 (1983), no. 1, 85–114.

[2] Drinfeld, V. G.; Simpson, Carlos, B-structures on G-bundles and local triviality. Math. Res. Lett. 2 (1995), no. 6, 823–829

[3] L. Fargues, G-torseurs en théorie de Hodge p-adique, http://webusers.imj-prg.fr/ laurent.fargues/Gtorseurs.pdf.

[4] L. Fargues, Géométrisation de la correspondance de Langlands, in preparation.

[5] L. Fargues, J.-M. Fontaine, Courbes et fibrés vectoriels en théorie de Hodge p-adique, http://webusers.imj-prg.fr/ laurent.fargues/Courbe.pdf

[6] Frenkel, E., Gaitsgory, D., and Vilonen, K. On the geometric Langlands conjecture. J. Amer. Math. Soc. 15 (2002), no. 2, 367–417.

[7] Frenkel, E., Gaitsgory, D., Kazhdan, D., and Vilonen, K. Geometric realization of Whittaker functions and the Langlands conjecture. J. Amer. Math. Soc. 11 (1998), 451–484.

[8] Gaitsgory, D. Informal introduction to geometric Langlands. An introduction to the Langlands program (Jerusalem, 2001), 269–281, Birkhauser Boston, Boston, MA, 2003.

[9] Gaitsgory, D. Outline of the proof of the geometric Langlands conjecture for GL(2), http://www.math.harvard.edu/ gaitsgde/GL/.

[10] T. Kaletha, The local Langlands conjectures for non-quasi-split groups, https://web.math.princeton.edu/ tkaletha/llcnqs.pdf.

[11] Mirković, I.; Vilonen, K. Geometric Langlands duality and representations of algebraic groups over commutative rings. Ann. of Math. (2) 166 (2007), no. 1, 95–143.

[12] P. Scholze, Perfectoid Spaces, Publ. math. de l'IHÉS 116 (2012), no. 1, 245–313.

[13] P. Scholze, J. Weinstein, p-adic geometry, notes from a course at Berkeley, https://math.berkeley.edu/ jared/Math274/ScholzeLectures.pdf

[14] P. Scholze, J. Weinstein, Cambridge Journal of Mathematics 1 (2013), 145–237.

Participation:

The idea of the Arbeitsgemeinschaft is to learn by giving one of the lectures in the program.

If you intend to participate, please send your full name and full postal address to

ag@mfo.de

by December 7th at the latest.

You should also indicate which talk you are willing to give:
First choice: talk no. ...
Second choice: talk no. ...
Third choice: talk no. ...

You will be informed shortly after the deadline if your participation is possible and whether you have been chosen to give one of the lectures.

The Arbeitsgemeinschaft will take place at Mathematisches Forschungsinstitut Oberwolfach, Schwarzwaldstrasse 9-11, 77709 Oberwolfach-Walke, Germany. The institute offers accommodation free of charge to the participants. Travel expenses cannot be reimbursed, except for young participants who may be supported by an NSF grant, see

http://www.mfo.de/for-guest-researchers/apply-for-a-stay/nsf-grant

Further information will be given to the participants after the deadline.