Arbeitsgemeinschaft mit aktuellem Thema:

Diophantine Approximation, Fractal Geometry and Dynamics

Mathematisches Forschungsinstitut Oberwolfach

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Organizers:

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Introduction:

In 1983 W. M. Schmidt [56] formulated a conjecture about the existence of points in the Euclidean plane that are simultaneously badly approximable with weights $(\frac{1}{3}, \frac{2}{3})$ and $(\frac{2}{3}, \frac{1}{3})$, that is $\mathbf{Bad}(\frac{1}{3}, \frac{2}{3}) \cap \mathbf{Bad}(\frac{2}{3}, \frac{1}{3}) \neq \emptyset$. Should this intersection be empty, it would immediately prove a well known conjecture of Littlewood from the 1930s. Weighted badly approximable points are characterised by (non-)proximity by rational points in a metric 'skewed' by some weights of approximation, say, i and j where $i, j \geq 0$, $i + j = 1$, which are 'attached' to coordinate directions. The central goal of this workshop is to expose its participants to recent breathtaking development regarding Schmidt's conjecture stemming from its proof by Badziahin, Pollington and Velani in 2011. We shall delve in the details of the proof of the conjecture given in a subsequent work of Jinpeng An (2013) and also discuss the solution to Davenport's problem, which boils down to the study of the intersections of $\mathbf{Bad}(i,j)$ with planar curves. More generally we shall study the recent work of Beresnevich (2015) on badly approximable points on manifolds. There also have been a burst in the development of new techniques, namely variants of Schmidt's game and Generalised Cantor sets constructions, which will also be studied in some depth. The workshop will also have a discussion of the broader area of metric Diophantine approximation: theorems of Khintchine and Jarník, Ubiquity and Mass Transference, introduce the notions of Dirichlet improvable and singular points, dynamical aspects of Diophantine approximation and the landmark results of Kleinbock and Margulis.

Talks:

1. **Background: one-dimensional Diophantine approximation**
   This talk will introduce the metric theory of Diophantine approximation. Give Dirichlet's theorem and the definition of **Bad** (the set of badly approximable numbers). Explain what **Bad** means in term of continued fractions. State Khintchine's theorem [38] and the Duffin-Schaeffer conjecture [31]. Give examples. The notes [15] should suffice to cover all of the above, but also see [23, 55] and the survey [11]. Staying in the one-dimensional setting, introduce ubiquity. State the ubiquity lemma [12, Theorem 9] for the ambient measure; sketch its proof, see [12, Appendix I]. State what ubiquity means for rational points in $[0,1]$ (this is essentially [15, Theorem 1.3]). Prove Khintchine's theorem using ubiquity technique and deduce that **Bad** is null.

2. **Jarník, Besicovitch and Mass Transference**
   State Jarník's theorem and the Jarník-Besicovitch theorem [15, §3.2,§3.3]. How are they related? Recall the definition of the Hausdorff measure and dimension as appropriate. Introduce the Mass Transference Principle (MTP) [16] and show (using MTP) that Khintchine's theorem implies Jarník's theorem (divergence), while Dirichlet's theorem implies the Jarník-Besicovitch theorem (the lower bound) [16]. Outline the proof of MTP: explain the relationship between the ambient measure statement and 'packing/covering' (Lemma 5 in [16]) and how this is used to construct a Cantor set; define a measure on the Cantor set

and explain the property it should satisfy in order to use the Mass Distribution Principle.

3. **Dirichlet and Bad in higher dimensions**
Present Minkowski's theorems for convex bodies and for systems of linear forms. Deduce Dirichlet's theorem for simultaneous approximations by rational numbers and for systems of linear forms. Introduce badly approximable systems of linear forms. For the above see the notes [15] and Schmidt's book [55, Ch. 2]. Show the existence of algebraic badly approximable points [55, Theorem 4A]. Deduce the weighted analogue of Dirichlet (from Minkowski) and introduce weighted **Bad** [39], [52] (which in dimension 2 will be denoted by $\mathbf{Bad}(i,j)$). Think of this: can the argument used in [55, Theorem 4A] be adapted to show that $\mathbf{Bad}(i,j) \neq \emptyset$? Describe the conjectures of Schmidt and Littlewood and their relationship [6, §2]. Explain the following transference principle (the like of Khintchine's transference) that a matrix is badly approximable if and only if so is its transpose (including for weighted approximations) [55, §IV.5], [23, Ch. 5], [47]. Explain the implications of the transference for the conjectures of Littlewood and Schmidt [6, Appendix], [10, Appendix A].

4. **Diophantine approximation and dynamics**
Introduce the convergence of lattices, bounded sets of lattices, divergent sequences of lattices and Mahler's compactness theorem [46], [24, Ch. 5]. Explain Dani's correspondence for badly approximable matrices [28] and Kleinbock's generalisation [39]. Give the dynamical interpretation of Littlewood's conjecture [32, Proposition 11.1] and [59]. Introduce Dirichlet improvable and singular points and give their dynamical interpretation [26, 27, 42, 58]. Introduce very well/multiplicatively very well approximable points and give their dynamical interpretation [40].

5. **Quantitative non-divergence in the space of lattices**
Introduce (C,α)-good functions and give their properties and examples [40, §3]. Introduce the norm of a lattice [40, §5] (explain what the wedge product is, see e.g. [55]). State and explain the main result of Kleinbock and Margulis on quantitative non-divergence [40, Theorem 5.2]. Outline how it was used to solve the Baker-Sprinžuk conjecture (Corollary 2.2, Proposition 2.3 and Theorem 5.4 from [40]). You may also wish to mention its Khintchine type generalisation [19].

6. **Schmidt's games and symmetric Bad**
Introduce Schmidt's games and winning sets. Explain the properties of winning sets regarding their cardinality/dimension and intersections (some arguments for the properties can be given). Give/sketch a proof that the set $\mathbf{Bad}_n$ of (symmetric) badly approximable points is winning. See [55, Ch. III], [53], [54], and see [15, §7] for the one-dimensional case.

7. **Weighted Bad and inhomogeneous theory**
Demonstrate the main ideas of the proof from [52] that $\mathbf{Bad}(i,j)$ has full dimension and then how to extend this to the inhomogeneous setting following [17, §3]. The one dimensional Cantor set construction given in [15, §7] may be useful for understanding. Find an oversight in [52]. Describe the generalisations of the results on $\mathbf{Bad}(i,j)$ to higher dimensions obtained in [44] and [43], explaining the role of the Simplex Lemma.

8. **$\mathbf{Bad}(i,j)$ wins on a fibre: rooted trees and the strategy**
Now we have all the tools to understand the proof of Schmidt conjecture. Discuss the main result of [1] on Schmidt's conjecture and in what way it is different to the main result in [6]. Set up the terminology of rooted trees, regular subtrees and explain Proposition 2.1 from [1] (including ideas of its proof). Describe the winning strategy, the role of rooted trees in this strategy and the role of Proposition 3.1 [1, §3].

9. **$\mathbf{Bad}(i,j)$ wins on a fibre: partitioning of intervals**
Describe the proof of Proposition 3.1 from [1] in as much detail as possible. This is the key statement that enables the winning strategy (described in the previous talk). If possible, explain the difference between the partitionings used in [1] and [6].

10. **Playing Schmidt's game on fractals**
Introduce the notion of absolutely friendly measures and show how Schmidt's game can be played on their support and try to explain the required modifications to show that $\mathbf{Bad}_n$ is winning on the support of such a measure. For demonstration purposes use the Cantor ternary set in $[0,1]$. There are many paper on this topic, see for example [21] and [35] (see also [41] and [44]). Explain the contribution of [49] to Schmidt's conjecture, and in particular that of [49, Appendix B].

11. **Bad**(i, j) **is winning in** $\mathbb{R}^2$
Describe the work [2] of An on **Bad**(i, j) being Schmidt's winning in $\mathbb{R}^2$ as far as possible highlighting its difference to winning on a fibre [1]. Pay attention to colouring of rooted trees and changes in the winning strategy.

12. **Strong, absolute and hyperplane winning**
In recent years a few other type of games have been suggested, starting with McMullen's strong and absolute winning [45]. Explain McMullen's games and winning sets for the games, also the notions of hyperplane absolute winning [22]. Describe properties of winning sets for different games and the relationships between the different types of winning sets, advantages or disadvantages. Describe the contribution of [22] regarding symmetric **Bad** in $\mathbb{R}^n$ and that of [50] regarding **Bad**(i, j).

13. **Potential winning and weighted Bad in higher dimensions**
Describe hyperplane potential winning sets introduced in [36] and how they are related to hyperplane absolute winning sets. Describe the recent work [37] on the winning property of **Bad**$(\mathbf{r})$ in $\mathbb{R}^d$ (for a certain type of weights $\mathbf{r} = (r_1, \ldots, r_d)$). Outline the key ideas of the proof and the role of the potential winning game. If possible, explain the obstacles for proving that any **Bad**$(\mathbf{r})$ is winning.

14. **Generalised Cantor sets**
Describe the Generalised Cantor sets introduced in [7], how these lead to Cantor-rich sets as introduced in [10] and the properties of Cantor-rich sets. Describe the Cantor winning sets of [5] and their properties (sticking to dimension 1). Outline the abstract framework of [5] by demonstrating how it can be implemented in, for example, $\mathbb{R}^2$. Explore potential relations to winning sets.

15. **What could be 'badly approximable' in multiplicative sense?**
Recall Littlewood's conjecture and any known results (we suggest to look at least at [32], [25] and [51]). Also you may wish to describe the multiplicative analogue of Khintchine (Gallagher's theorem) and/ or recent generalisation for fibres [13], [14]. Describe the MAD conjectures of [7] and the results of [7] and [4]. Outline the proof given in either [4] or [7] highlighting the role of generalised Cantor sets.

16. **Davenport's problem: an overview**
    Describe Davenport's problem [8] and the original results of Davenport [29]. (Emphasise that **Bad** on manifolds in of measure zero [9].) Describe the results on the problem obtained to date for curves, lines and manifolds [8], [3] and [10]. Compare them and outline the 'type' of techniques [8] and [3] develop/inherit. Describe the role of fibering in [10]. Describe the application for real number badly approximable by algebraic numbers [10, §2.3 & Appendix B]. (You may also wish to mention Wirsing's conjecture [60] and the theorem of Davenport and Schmidt [30], see also [20].) Summarise the various conjectures made in [8], [3] and [10] (also see [15, Conjecture 7.1]).

17. **Badly approximable points on manifolds: Part 1**
    Recall the goal (which is the proof of Theorem 2 in [10]) and explain that this will be archived using Cantor-rich sets (described in Talk 13). Emphasize that the proofs relies on the linear forms definition of badly approximable points [10, Lemma 1(iii)]. Show how $\mathbf{Bad}(\mathbf{r})$ is encoded using bounded orbits of certain lattices [10, Lemma 2], and then explain how an appropriate Cantor set is defined (this is the first paragraph of the proof of Proposition 3 in [10]). Hence introduce 'dangerous' intervals [10, §4] and how their length and numbers are 'controlled' (Propositions 1 and 2). Mention the role of the quantitative non-divergence of Kleinbock and Margulis from [19] within Proposition 2.

18. **Badly approximable points on manifolds: Part 2**
    This continues the previous talk. Explain the construction of a Cantor set - Proposition 3 from [10]. State Proposition 3 and go through its proof in as many details as it is practical, showing how different counting results (Propositions 1 and 2 and Lemma 6) feed into the final counting estimates. Explain how considering dangerous intervals on a relatively short interval results in counting lattice points in a subgroup of smaller rank. State and explain Lemma 6 which allows to estimate the number of dangerous 'paths' in the Cantor set construction.

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