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# Mini-Workshop: Spaces and Moduli Spaces of Riemannian Metrics

Organised by
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Abstract. The mini-workshop focused on central questions and new results concerning spaces and moduli spaces of Riemannian metrics with lower or upper curvature bounds on open and closed manifolds and, moreover, related themes from Anosov geometry. These are all described in detail below. The event brought together young and senior researchers working about (moduli) spaces of negative and nonnegative sectional, nonnegative Ricci and positive scalar curvature as well as Anosov metrics, and the talks and discussions brought about many new and inspiring research problems to pursue.

*Mathematics Subject Classification (2010):* 53C20, 58D27 (primary), 53C23, 57R19, 58D1.

## Introduction by the Organisers

The MFO mini-workshop "Spaces and Moduli Spaces of Riemannian Metrics", organised by Tom Farrell (Beijing) and Wilderich Tuschmann (Karlsruhe), was held January 8-14, 2017. The meeting was attended by 16 participants, ranging from first year graduate students to senior researchers. The purpose of the meeting was to relate and study new developments concerning spaces and moduli spaces of Riemannian metrics with lower or upper curvature bounds on open and closed manifolds and, moreover, related themes from Anosov geometry.

The meeting was organised around nine one-hour and two 30 minute talks which were accompanied by problem and discussion sections, thus leaving also plenty of time between and after these for further informal exchange.

After a brief introduction by the organisers, the workshop started off with a talk by David Wraith who presented his recent work about spaces and moduli spaces of positive and non-negative scalar curvature metrics. The afternoon session began with a talk by Mark Walsh on his work with Boris Botvinnik and David Wraith on the observer moduli space of positive Ricci curvature metrics.

On Tuesday, Andrey Gogolev introduced the audience to the moduli problem for smooth conjugacy of Anosov diffeomorphisms and flows and the moduli problem for isometries of negatively curved metrics. In the afternoon, Mauricio Bustamante explained his work on nonconnectedness of the space of Anosov metrics on a high dimensional manifold.

On Wednesday, Anand Dessai discussed in his talk different approaches of how to detect components of the space or moduli space of metrics with lower curvature bounds. Igor Belegradek described joint work with Jing Hu and Taras Banakh on spaces of nonnegatively curved surfaces, in which he also addressed fundamental desirable properties of topologies on (moduli) spaces of metrics in general.

On Thursday, Igor Belegradek explained his joint work with Tom Farrell and Vitali Kapovitch on higher homotopy groups of spaces of nonnegatively curved metrics, paving thus also the ground for a related talk by Jiang Yi. Then Boris Botvinnik gave a talk about his work with David Wraith on the topology of the space of Ricci-positive metrics. In the afternoon, Jiang Yi presented and explained her joint work with Mauricio Bustamante and Tom Farrell on involutions on pseudoisotopy spaces and spaces of metrics.

On Friday, Su Yang described joint work with Matthias Kreck about the classification of closed seven-manifolds with infinite cyclic fundamental group and its possible relations to the existence of metrics with positive or non-negative curvature. Then Thomas Schick gave a concluding talk about the topology of positive scalar curvature metrics. Besides describing the use and limitations of index methods, which he illustrated by his joint work with Diarmuid Crowley and Wolfgang Steimle, he also raised several questions concerning higher secondary index theory of the Dirac operator whose solutions would yield many interesting new results in the field.

We wish to thank all of the participants for their strong commitment and dedication which made every single discussion and problem session lively, intense and inspiring. In fact, from them we received ourselves many positive reactions to structuring the workshop in a way that gave enough time to explore each individual talk in great detail. Particular thanks go in addition to Andrey Gogolev for collecting a list of open problems and questions which arose from all our joint discussions, and to the meeting's reporter Jan-Bernhard Kordaß, who also assisted in many further ways, for his prudent and professional work.

Last but not least it is our pleasure to thank the Institute for the invitation to organise this event, and all the staff for their excellent hospitality.

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## Mini-Workshop: Spaces and Moduli Spaces of Riemannian Metrics

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# Abstracts

## Non-negative versus positive scalar curvature

David J. Wraith

Our motivating question is as follows: from the point-of-view of (moduli) spaces of metrics, what is the difference between positive and non-negative scalar (or Ricci) curvature? Our main result below (Theorem A) describes a situation when a non-negative scalar curvature metric on a closed spin manifold has no harmonic spinors. This has implications for the index theory of Dirac operators, and in turn leads to topological consequences for (moduli) spaces of non-negative scalar curvature metrics. Some of these applications appear below as Theorems C, D and E. An in depth treatment of this material can be found in [14].

Our set-up is as follows. We suppose that $(X^{4k}, g)$ is Riemannian spin manifold with non-empty boundary $(Y^{4k-1}, g')$, and assume that $g$ takes the form $dt^2 + g'$ in a neighbourhood of $Y$. We consider the Atiyah-Singer (half) Dirac operator $D^+(X, g)$ acting on the space of positive spinors over $X$, and the induced Dirac operator $D(Y, g')$ on $Y$. We restrict the domain of $D^+(X, g)$ to the subspace of positive spinors for which the restriction to the boundary belongs to the span of the negative eigenspaces of $D(Y, g')$. The index of $D^+(X, g)$ is then defined to be

$$\operatorname{ind} D^+(X, g) := \dim\ker D^+(X, g) - \dim\operatorname{coker} D^+(X, g).$$

Recall that a harmonic spinor on $(Y, g')$ is a section belonging to $\ker D(Y, g')$, and $(Y, g')$ is said to have *no harmonic spinors* if this kernel is trivial.

Much of our understanding about the structure of spaces of metrics with various (positive) curvature restrictions arises from the index theory of Dirac operators. The following result follows by examining the classical Atiyah-Patodi-Singer index formula for manifolds with boundary ([2] Theorem 4.2), and is foundational among index-theoretic results about spaces of metrics:

**Theorem.** *For a smooth path of metrics $g_t$ on $X$ (products near the boundary),* $\operatorname{ind} D^+(X, g_t)$ *is constant provided the induced metrics on $Y$ have no harmonic spinors.*

One way to guarantee that no harmonic spinors are encountered along such a path is if each $g_t$ has positive scalar curvature. This follows from the classical Schrödinger-Lichnerowicz theorem (see for example [12] page 160):

**Theorem.** *(Schrödinger-Lichnerowicz) If $(M, g)$ is a closed Riemannian spin manifold with either positive scalar curvature or non-negative scalar curvature which is positive at some point, then $M$ admits no harmonic spinors.*

**Corollary.** *(Of above two theorems.) If $(Y, g')$ has positive scalar curvature, then* $\operatorname{ind} D^+(X, g)$ *depends only on the path-component of the space of positive scalar curvature metrics containing $g'$. (So different indices mean different path-components.)*

We now turn our attention to paths of non-negative scalar curvature metrics on closed spin manifolds. Our first main result is

**Theorem A.** *If $(M, g)$ is a closed Riemannian spin manifold with positive scalar curvature and $\bar{g}$ is any metric with non-negative scalar curvature in the same path-component of non-negative scalar curvature metrics as $g$, then $(M, \bar{g})$ admits no harmonic spinors.*

Theorem A suggests that from the point-of-view of index theory, there might be little difference between working with metrics of positive scalar curvature and metrics of non-negative scalar curvature, provided the relevant path-component of non-negative scalar curvature metrics contains a positive scalar curvature metric.

It follows from the Scrödinger-Lichnerowicz theorem that in order to prove Theorem A, the situation we really need to understand is the scalar flat case. Two key results here are:

**Theorem.** *([12] II.8.10) On a closed spin manifold with identically vanishing scalar curvature, every harmonic spinor is globally parallel.*

**Theorem.** *([8] Proposition 3.2) The existence of a non-trivial parallel spinor forces the metric to be Ricci flat.*

Theorem A now follows easily from

**Theorem B.** *Let $M$ be a closed spin manifold and suppose $g_t$, $t \in [0, T]$, is a smooth path of non-negative scalar curvature metrics. If $g_0$ admits a parallel spinor (and so is Ricci-flat), then $g_t$ is Ricci-flat for all $t \in [0, T]$. If furthermore $\pi_1(M) = 0$, then $g_t$ also admits a parallel spinor for all $t \in [0, T]$.*

We remark that the proof of Theorem B is straightforward except in the case where $M$ has infinite fundamental group. The proof relies on results of Dai, Wang and Wei [7], (specifically Theorems 3.4 and 4.2), and recent results of Ammann, Kröncke, Weiss and Witt ([1]) concerning holonomy and parallel spinors.

Turning our attention now to the implications of Theorem A for (moduli) spaces of metrics, we recall that for a closed Riemannian spin manifold $(M^{4k-1}, g)$ with $k \geq 2$, positive scalar curvature and vanishing real Pontrjagin classes, Kreck and Stolz [11] define an invariant $s(M, g) \in \mathbb{Q}$ which is an invariant of the path-component of positive scalar curvature metrics containing $g$. If in addition $H^1(M; \mathbb{Z}_2) = 0$, $|s|$ is an invariant of the path-component containing $[g]$ in the moduli space of positive scalar curvature metrics. The $s$-invariant has been a key tool in proving many disconnectedness results for moduli spaces, see for example [11], [13], [6]. It follows using Theorem A that if two path-components in the moduli space of positive scalar curvature metrics are distinguished by $s$, then the corresponding sets of non-negative scalar curvature metrics are also disjoint. This observation then allows us to generalize a positive scalar curvature moduli space result of Kreck-Stolz (see [11] Corollary 2.15):

**Theorem C.** *Given any $M$ with $H^1(M; \mathbb{Z}_2) = 0$ for which $s(M, g)$ is defined, the moduli space of non-negative scalar curvature metrics on $M$ has infinitely many path-components.*

We can also obtain results about spaces of non-negative Ricci curvature metrics:

**Theorem D.** *If $K^4$ denotes a K3 surface and $\Sigma^{4n-1}$, is any homotopy $(4n-1)$-sphere $(n \geq 2)$ which bounds a parallelisable manifold, then $\Sigma \times K^4$ has infinitely many path-components of non-negative Ricci curvature metrics.*

To the best of the author's knowledge, Theorem D is the first result about the topology of spaces of non-negative Ricci metrics.

Following on from Theorem D we can obtain non-compact examples (as pointed out to the author by W. Tuschmann):

**Theorem E.** *With the notation of Theorem D, $\Sigma \times K^4 \times \mathbb{R}$ has infinitely many path-components of complete Ricci non-negative metrics.*

(Note that by the Cheeger-Gromoll splitting theorem [3], the examples in Theorem E do not admit any complete metrics of positive Ricci curvature.)

There are many other results in the literature about (moduli) spaces of positive scalar curvature metrics which rely on the invertibility of Dirac operators, and which could be generalized to non-negative scalar curvature using the above ideas. In this regard we mention work of Hitchin ([9] Theorem 4.7), Crowley and Schick ([4]), Crowley, Schick and Steimle ([5]), and Hanke, Schick and Steimle ([10]).

An open question (raised by Thomas Schick) is to what extent results presented above extend to the more general case of Dirac operators twisted by flat bundles?

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## The Observer Moduli Space of Positive Ricci Curvature Metrics

Mark Walsh
(joint work with Boris Botvinnik, David J. Wraith)

In recent years, there have been great efforts made to better understand the topology of moduli spaces of Riemannian metrics of positive scalar curvature on a smooth compact (usually spin) manifold; see [3, 2, 1, 7]. Apart from results of Kreck and Stolz in [8] and Wraith in [12] concerning path-connectivity, we know very little about the corresponding moduli spaces of positive Ricci curvature metrics. Whether or not there is any non-triviality in the higher homotopy groups of such spaces is still an open question. Our contribution here is to consider this question for a related space: the observer moduli space of positive Ricci metrics, on the sphere. Roughly, we show that the higher homotopy groups of this space have lots of non-trivial elements. The proof is based on that of an analogous theorem by Botvinnik, Hanke, Schick and Walsh for the observer moduli space of positive *scalar* curvature metrics; see [1]. Both proofs rely heavily on work of Farrell, Hsiang, Hatcher and Götte; see [4] and [6]. Techniques for constructing families of metrics are also required. In the scalar curvature case, this means a family version of the Gromov-Lawson surgery technique from [5], described in [11], a technique which permits the detection of non-triviality for manifolds besides the sphere. Unsurprisingly, the Ricci curvature case requires a more delicate construction, based on a gluing theorem of Perelman.

Letting $M$ denote a smooth, connected, closed manifold of dimension $n$, we denote by $\mathcal{R}(M)$, the space of all Riemannian metrics on $M$ with its usual $C^\infty$ topology. Contained inside are the subspaces $\mathcal{R}^{s>0}(M)$ and $\mathcal{R}^{\mathrm{Ric}>0}(M)$ of positive scalar and positive Ricci curvature metrics on $M$. We fix an arbitrary base point $x_0 \in M$ and let $\mathrm{Diff}_{x_0}(M)$ denote the subgroup of $\mathrm{Diff}(M)$ consisting of diffeomorphisms which fix $x_0$ and for which the derivative map $d\phi_{x_0} : T_{x_0}M \to T_{x_0}M$ is the identity map. It is easy to observe that since $M$ is connected, $\mathrm{Diff}_{x_0}(M)$ acts freely on $\mathcal{R}(M)$ by pull-back. We then define the *observer moduli space of Riemannian metrics on* $M$ as the quotient space

$$\mathcal{M}_{x_0}(M) := \mathcal{R}(M)/\operatorname{Diff}_{x_0}(M).$$

By restricting the action of $\mathrm{Diff}_{x_0}(M)$ to the subspaces $\mathcal{R}^{s>0}(M)$ and $\mathcal{R}^{\mathrm{Ric}>0}(M)$ in $\mathcal{R}(M)$, we obtain the *observer moduli spaces of positive scalar and positive Ricci curvature metrics*. Respectively, these are denoted $\mathcal{M}_{x_0}^{s>0}(M)$ and $\mathcal{M}_{x_0}^{\mathrm{Ric}>0}(M)$. Our main result deals with the case when $M$ is the sphere $S^n$. Denoting by $g_0$, the standard round metric on $S^n$, the main result can be stated as follows.

**Main Theorem.** *For any $k \in \mathbb{N}$, there is an integer $N(k)$ such that for all odd $n > N(k)$, the group $\pi_i(\mathcal{M}_{x_0}^{\mathrm{Ric}>0}(S^n), [g_0])$ is non-trivial when $i \leq 4k$ and $i \equiv 0$* mod 4.

The observer moduli space is obtained as the quotient of a free action. As $\mathcal{R}(M)$ is contractible, it follows that $\mathcal{M}_{x_0}(M)$ is a classifying space for $\mathrm{Diff}_{x_0}(M)$ and we write $\mathrm{BDiff}_{x_0}(M) = \mathcal{M}_{x_0}(M)$. Recall that isomorphism classes of principal

$\mathrm{Diff}_{x_0}(M)$-bundles over a space $X$ are in one to one correspondence with homotopy classes of maps $X \to \mathrm{BDiff}_{x_0}(M)$. This correspondence is achieved by using such maps to pull back the universal bundle $\mathcal{R}(M) \to \mathcal{M}_{x_0}(M)$. Associated to this bundle is the universal $M$-bundle with total space $\mathcal{R}(M) \times_{\mathrm{Diff}_{x_0}} M$ defined as the quotient of $\mathcal{R}(M) \times M$ by the action $\phi.(h, x) = ((\phi^{-1})^* h, \phi(x))$. Thus, given a space $X$ and a map $f : X \to \mathcal{M}_{x_0}(M)$, we obtain a commutative diagram:

$$\begin{array}{ccc} E_f & \longrightarrow & \mathcal{R}(M) \times_{\mathrm{Diff}_{x_0}(M)} M \\ \downarrow & & \downarrow \\ X & \xrightarrow{\ f\ } & \mathcal{M}_{x_0}(M) \end{array}$$

where the bundle $E_f \to M$ is the pull-back bundle arising from the map $f$. This is a bundle with fibre $M$ and it is of course isomorphic to any bundle obtained by a map which is homotopy equivalent to $f$. There is however, a more refined structure which we can associate to such a bundle. The total space $\mathcal{R}(M) \times_{\mathrm{Diff}_{x_0}(M)} M$ admits a "universal fibre metric." To each point $[h, x] \in \mathcal{R}(M) \times_{\mathrm{Diff}_{x_0}(M)} M$, there is a well-defined inner product associated with the tangent space to the fibre at that point; see page 61 of [10]. This fibre inner product varies smoothly over $\mathcal{R}(M) \times_{\mathrm{Diff}_{x_0}(M)} M$ and pulls back to a continuous fibrewise family of Riemannian metrics on $E_f$. More precisely, each fibre of the bundle $E_f \to X$, already diffeomorphic to $M$, is equipped with a Riemannian metric. Moreover, varying the map $f$ by a homotopy changes the fibrewise metric structure of the bundle. With a little work we can establish a one to one correspondence between maps $X \to \mathcal{M}_{x_0}(M)$ and families of metrics on $M$ which are parameterised by $X$.

Supposing $X$ is the sphere $S^i$, we consider the homomorphism of homotopy groups

$$\pi_i(\mathcal{M}_{x_0}^{\mathrm{Ric}>0}(M), [g_0]) \longrightarrow \pi_i(\mathcal{M}_{x_0}(M), [g_0]),$$

induced by the inclusion $\mathcal{M}_{x_0}^{\mathrm{Ric}>0}(M) \subset \mathcal{M}_{x_0}(M)$. Let $f : S^i \to \mathcal{M}_{x_0}(M)$ represent a non-trivial element of $\pi_i(\mathcal{M}_{x_0}(M), [g_0])$. This element determines (and is determined by) a fibrewise family of metrics on $E_f$. Thus, it is possible to lift this element of $\pi_i(\mathcal{M}_{x_0}(M), [g_0])$ to an element of $\pi_i(\mathcal{M}_{x_0}^{\mathrm{Ric}>0}(M), [g_0])$, provided we can construct a fibrewise family of positive Ricci curvature metrics on the corresponding sphere-bundle, $E_f \to S^i$. It is now that we recall the work of Farrell and Hsiang. In [4], they show that when $k, N(k), n$ and $i$ are as in the hypotheses of the Main Theorem above,

$$\pi_i(\mathrm{BDiff}_{x_0}(S^n)) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } i \equiv 0 \mod 4, \\ 0 & \text{otherwise.} \end{cases}$$

As $\mathrm{BDiff}_{x_0}(S^n) = \mathcal{M}_{x_0}(S^n)$, this means that for appropriate $i$, we now have lots of non-trivial elements in the groups $\pi_i(\mathcal{M}_{x_0}(S^n), [g_0])$ to work with. This also explains the hypotheses of the main theorem.

Our next task is to realise the associated $S^n$ bundles in such a way that we can equip these bundles with fibrewise families of positive Ricci curvature metrics.

Following work of Hatcher, we know that every element of $\pi_i(\mathcal{M}_{x_0}(S^n), [g_0]) \otimes \mathbb{Q}$ determines a specific $S^n$ bundle over $S^i$ which can be built in the following way. These "Hatcher bundles" are described by Götte in [6]. Roughly, we start with a trivial bundle $E = S^n \times S^i \to S^i$, and consider each fibre sphere, $S^n$, as a pair of hemispherical disks $D^n_- \cup D^n_+$. Thus $E$ decomposes into a pair of spaces $E_-$ and $E_+$, by restricting fibres to the hemispheres $D^n_-$ and $D^n_+$. Working for now on the space $E_-$, we decompose each fibre disk, $D^n_-$, as $D^n_- = D^{p+1} \times D^q$ and in turn as $D^n_- = (S^p \times [r_0, 1]) \times D^q \cup D^{p+1}(r_0) \times D^q$, where $p + q + 1 = n$, $r_0 \in (0, 1)$ and $D^{p+1}(r_0) \times D^q$ is a smaller version of the original disk (with radius $r_0$) surrounded by an annular region $(S^p \times [r_0, 1]) \times D^q$. These two pieces share a common piece of boundary, $S^p \times \{r_0\} \times D^q$. For sufficiently large $p$ and $q$ (and hence $n$), representatives of certain non-trivial elements $\lambda \in \pi_i(O(p+1))$ (those in the kernel of the $J$-homomorphism) can be adjusted and extended to give rise to a family of smooth embeddings $\Lambda_t : S^p \times D^q \to S^p \times D^q$, parameterised by $t \in S^i$. For each $t$, the corresponding fibre disk, $D^n_-$, may be reassembled using the map $\Lambda_t$ to glue along the inner boundary piece. This reassembly goes through continuously for all $t \in S^i$ and we denote this new bundle $E_-(\lambda) \to S^i$. Moreover, the maps $\Lambda_t$ are constructed so that the hemispheres $D^n_+$ can be reattached, via the identity map in each case, to obtain an $S^n$ bundle over $S^i$, namely $E(\lambda) = E_-(\lambda) \cup E_+$. Interestingly, this bundle is homeomorphic to but not diffeomorphic to, the trivial $S^n$ bundle over $S^i$; see [6].

It remains to construct a fibrewise family of positive Ricci curvature metrics. This is possible due to a powerful gluing theorem of Perelman, stated in [9], which states that two positive Ricci curvature manifolds with isometric boundaries may be glued together along the boundary to obtain a smooth positive Ricci metric, provided each of the normal curvatures (with outward normal) at one boundary is greater than the negative of the corresponding normal curvature at the other boundary. Importantly, the construction in this theorem can be shown to work continuously for compact families of metrics. Thus, we equip the total spaces $E_-(\lambda)$ and $E_+$ with fibrewise families of positive Ricci curvature metrics which are individually (though not canonically) isometric to a standard product of hemisphere metrics on $D^{p+1} \times D^q$. We then perform a delicate smoothing so as to ensure that the hypotheses of Perelman's Theorem are met on each fibre. As the theorem goes through for compact families, we can then use it to glue, in a fibrewise sense, the metric families on $E_-(\lambda)$ and $E_+$ to obtain a fibrewise family of positive Ricci curvature metrics on any Hatcher bundle associated to an element in $\pi_i(\mathcal{M}_{x_0}(S^n)) \otimes \mathbb{Q}$. In particular, we see that the homomorphism of rational homotopy groups, induced by the inclusion $\mathcal{M}^{\mathrm{Ric}>0}_{x_0}(M) \subset \mathcal{M}_{x_0}(M)$, is actually a surjection.

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## Moduli of smooth conjugacy for Anosov dynamical systems

Andrey Gogolev

Recall that given a compact smooth Riemannian manifold $M$ an *Anosov diffeomorphism* $f$ is a diffeomorphism that preserves a continuous splitting $TN = E^s \oplus E^u$, uniformly contracts the stable subbundle $E^s$ and uniformly expands the unstable subbundle $E^u$.

An Anosov diffeomorphism is called *conformal* if the stable quasi-conformal distortion

$$K^s(x,n) = \frac{\max\{\|Df^n(v)\| : v \in E^s, \|v\| = 1\}}{\min\{\|Df^n(v)\| : v \in E^s, \|v\| = 1\}}$$

and analogously defined unstable distortion $K^u(x,n)$ are uniformly bounded in $x \in M$ and $n \in \mathbb{Z}$.

In this talk we exhibited two moduli problems: the moduli problem for smooth conjugacy of Anosov diffeomorphisms and flows and the moduli problem for isometries of negatively curved metrics.

### 1. Moduli

If two Anosov diffeomorphisms are smoothly (or just $C^1$) conjugate then the Jordan normal forms of the differentials at corresponding periodic points must coincide. Hence the Jordan normal form provide a countable set of moduli. This set of moduli is called *periodic data*. When the dimension of $N$ is 2, de la Llave-Marco-Moriyón proved that periodic data is a complete invariant of smooth conjugacy.

Despite many partial results in higher dimension it is still an open question whether periodic data moduli is a complete invariant in higher dimensions.

Analogously, given a negatively curved Riemannian metric on a manifold $M$, each free homotopy class of closed curves contains a unique minimizing geodesic whose length is an isometry invariant. The collection of lengths of closed geodesic in all free homotopy classes is called *marked length spectra*. When the dimension of $M$ is 2, Otal and Croke (independently) proved that marked length spectrum is a complete invariant of isometry. Despite many partial results in higher dimension it is still an open question whether marked length spectrum is a complete invariant in higher dimensions.

In the talk we have explained how a major step in the Otal-Croke theorem can be deduced from the de la Llave-Marco-Mariyón technique by relating the marked length spectrum to periodic eigenvalue data of the geodesic flow.

## 2. The Dictionary

Further we discussed the following vague dictionary.

| Geometry | Dynamics |
|---|---|
| Hyperbolic metric on $M$ | Conformal Anosov diffeomorphism of $N$ |
| Negatively curved metric $g$ on $M$ | Anosov diffeomorphism $f$ of $N$ |
| The space $\mathcal{MET}^{sec<0}(M)$ | The space $\mathcal{X}_f$ of Anosov diffeomorphism homotopic to $f$ |
| Pullback of a negatively curved metric $g$ by a diffeomorphism $h\colon M \to M$ | Conjugation of Anosov diffeomorphism $f$ by a diffeomorphism $h\colon N \to N$ |

The similarity is confirmed by various results and conjectures. For example, the analogue of Mostow rigidity is the following result of Kalinin and Sadovskaya, which is based on work of Benoist and Labourie.

**Theorem** (Kalinin-Sadovskaya)**.** *Let $f$ be a transitive Anosov diffeomorphism of a compact manifold $N$ which is conformal on the stable and unstable distributions. Suppose that both distributions have dimension at least three. Then $f$ is smoothly conjugate to an affine Anosov automorphism of a flat Riemannian manifold.*

## Nonconnectedness of the space of Anosov metrics on a high dimensional manifold

Mauricio Bustamante

A Riemannian metric $g$ on a smooth $n$-dimensional manifold $M$ is said to be an *Anosov* metric if its geodesic flow $\varphi_t : SM \to SM$ on the unit tangent bundle $SM$ of $M$ is of Anosov type. This means that there is a $\varphi_t$-invariant splitting of $TSM$ into three subbundles

$$TSM = E^s \oplus E^u \oplus X$$

such that vectors in the *stable* (*unstable*) subbundle $E^s$ ($E^u$) are exponentially contracted (expanded) in positive time; and $X$ corresponds to the line bundle generated by the flow lines.

Assume that $M$ is a closed smooth manifold and let $\mathrm{MET}^{<0}(M)$ and $\mathrm{MET}^{\mathcal{A}}(M)$ denote the space (with the smooth topology) of all Riemannian metrics on $M$ with negative sectional curvature and with geodesic flow of Anosov type, respectively.

Anosov [1] showed that every Riemannian metric of negative sectional curvature is an Anosov metric, in other words there is an inclusion

$$\mathrm{MET}^{<0}(M) \subset \mathrm{MET}^{\mathcal{A}}(M).$$

However, R. Gulliver [3] has proved that there exist Anosov metrics that have some positive sectional curvature. This motivates the following question: from the point of view of (moduli) spaces of Riemannian metrics, how different are $\mathrm{MET}^{<0}(M)$ and $\mathrm{MET}^{\mathcal{A}}(M)$? Are they homotopy equivalent?

When $\mathrm{MET}^{<0}(M) \neq \emptyset$, Farrell and Ontaneda [2] have constructed a self-diffeomorphism $\varphi : M \to M$ of $M$ which is supported on a tubular neighborhood of a closed geodesic of $M$, and has the property that $g$ and $\varphi^* g$ cannot be joined by a path of negatively curved metrics. To obtain such "exotic" objects, they make use of deep work of Waldhausen, Igusa and Hatcher relating stable pseudoisotopy theory to algebraic $K$-theory.

We show that Farrell and Ontaneda's method can be generalized to prove the following theorem.

**Theorem 1.** *Let $(M^n, g)$ be a closed negatively curved manifold. Then there exists a diffeomorphism $\varphi \in$ Diff(M) such that $\varphi^* g$ and $g$ can't be joined by a path in $MET^{\mathcal{A}}(M)$, provided $\dim M > 9$.*

**Questions and open problems**:

1. Let $M$ be a closed surface of genus $\geq 2$. Ricci flow techniques can be used to show that $\mathrm{MET}^{<0}(M)$ deformation retracts onto the space of hyperbolic metrics on $M$, which is known to be contractible. Is $\mathrm{MET}^{\mathcal{A}}(M)$ a contractible space?
2. Is the inclusion $\mathrm{MET}^{<0}(M) \subset \mathrm{MET}^{\mathcal{A}}(M)$ a (weak) homotopy equivalence?

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## Disconnected moduli spaces for lower curvature bounds

Anand Dessai

Let $M$ be a closed smooth connected manifold, $\mathcal{R}^{scal>0}(M)$ the space of metrics of positive scalar curvature (psc), $\mathcal{M}^{scal>0}(M)$ its moduli space and use corresponding notations for other curvature conditions like $Ric > 0$, $sec > 0$, $sec \geq 0$ etc. In this talk we focus on the following

**Basic Question:** How can one detect components of the space of metrics / moduli space?

We describe three approaches and some applications. The first approach goes back to work of Gromov and Lawson. Let $M$ be a spin manifold of dimension $4n-1$ which is the boundary of two Riemannian spin manifolds $(W_i, g_{W_i})$, $i = 0, 1$, for which the metrics are of product form near the boundary and induce metrics $g_i$ on $M$. Then one can glue the bordisms together to obtain a closed $4n$-dimensional spin manifold $X$ for which the index of the Dirac operator $\hat{A}(X)$ vanishes if the metrics $g_0$ and $g_1$ belong to the same component of $\mathcal{R}^{scal>0}(M)$.

As an application consider a homotopy sphere $\Sigma$ of dimension $4n-1 \geq 7$ which is the boundary of a parallelizable manifold. Then there are constants $c, \tilde{c} \in \mathbb{Z}$, $c \neq 0$, such that for any $l \in \mathbb{Z}$ there exists a spin manifold $W_l$, defined via plumbing, of signature $l \cdot c + \tilde{c}$ and with boundary $\Sigma$. Carr constructed psc-metrics on $W_l$ which are products near the boundary. It follows from Novikov's additivity property of the signature and the construction of the $W_l$ that the gluing construction above for different $l$ yields manifolds $X$ with non-vanishing $\hat{A}$-genus. Hence, the induced metrics $g_l$ on $\Sigma$ belong to different components of $\mathcal{R}^{scal>0}(\Sigma)$. Moreover, since the signature is not sensitive to a change of gluing via an orientation preserving diffeomorphism of $M$ the components remain different after passing to the moduli space $\mathcal{M}^{scal>0}(\Sigma)$ (for details see [4, 6, 10], cf. [7] for related important work).

This result has been extended into two directions. On the one hand Wraith [11] constructed metrics of positive Ricci curvature on the boundary $\Sigma$ which extend to psc-metrics on $W_l$ being a product near the boundary and concluded that the moduli space of metrics of positive Ricci curvature $\mathcal{M}^{Ric>0}(\Sigma)$ has infinitely many components. On the other hand, as pointed out in Lawson-Michelson [10, p. 329], the psc-argument extends almost immediately via a boundary connected sum construction to show that for *any* closed spin psc-manifold $M$ of dimension $4n-1 \geq$

7 the space of psc-metrics on $M$ has infinitely many components. Moreover, this remains true after passing to its moduli space $\mathcal{M}^{scal>0}(M)$.

In view of these results one may ask

**Question 1:** Does $\mathcal{M}^{Ric>0}(M)$ have infinitely many components for a closed spin manifold $M$ of dimension $4n-1 \geq 7$ and of positive Ricci curvature?

**Question 2:** Does the moduli space of metrics of nonnegative sectional curvature $\mathcal{M}^{sec\geq 0}(\Sigma)$ have infinitely many components for a nonnegatively curved homotopy sphere $\Sigma$ of dimension $4n-1 \geq 7$?

The second approach goes back to work of Kreck and Stolz [9] and utilizes the Atiyah-Patodi-Singer (APS) index theorem [1, 2]. Let $(W, g_W)$ be a $4n$-dimensional Riemannian spin manifold with boundary $M = \partial W$ for which $g_W$ is a product near the boundary and let $g_M$ be the induced metric on $M$. Kreck and Stolz used an Eells-Kuiper-type combination of the APS-formulas for the signature and Dirac operator to derive a rationally valued invariant $s(M, g_M)$ of the Riemannian boundary. For this approach to work it is necessary to impose certain conditions on the topology of $M$ which are for example satisfied if the rational total Pontrjagin class of $M$ is trivial. In this situation $s(M, g_M)$ is a locally constant function on $\mathcal{R}^{scal>0}(M)$. Moreover, if $M$ admits an infinite family of psc-metrics with pairwise different s-invariants then $\mathcal{M}^{scal>0}(M)$ has infinitely many components.

Kreck and Stolz [9] used their invariant to exhibit a seven-dimensional Aloff-Wallach space $M$ with disconnected moduli space $\mathcal{M}^{sec>0}(M)$ and 7-dimensional Witten manifolds $N$ for which the moduli space $\mathcal{M}^{Ric>0}(N)$ has infinitely many components. As observed in [8] $N$ also carries infinitely many submersion metrics which in addition belong to different components of the moduli space of metrics of nonnegative sectional curvature. More recently, it was shown in joint work with Klaus and Tuschmann [5] that in each dimension $4k-1 \geq 7$ there are infinitely many generalized Witten manifolds $N$, pairwise non-homotopic, for which the moduli spaces $\mathcal{M}^{sec\geq 0}(N)$ and $\mathcal{M}^{Ric>0}(N)$ both have infinitely many components.

We like to remark that the Gromov-Lawson and Kreck-Stolz invariants in combination with work of Grove and Ziller on cohomogeneity one manifolds can be used to exhibit many other interesting families of manifolds for which the moduli spaces $\mathcal{M}^{sec\geq 0}$ and $\mathcal{M}^{Ric>0}$ both have infinitely many components, including the 7-dimensional homotopy spheres constructed by Milnor in the 1950s.

The two approaches above are confined to dimension $4k-1$. The third approach addresses this issue. It is based on reduced eta-invariants and the APS-index theorem for Dirac operators twisted with flat bundles. As before let $(W, g_W)$ be a Riemannian spin manifold with boundary $M = \partial W$ for which $g_W$ is a product near the boundary and let $g_M$ be the induced metric on $M$. Suppose $\dim M \geq 5$, $\pi_1(M) \neq 0$ and $E \to M$ is a flat bundle which extends over $W$. Suppose $g_M$ has psc. As pointed out in [2] the index of the Dirac operator of $W$ twisted with the reduced bundle $\tilde{E}$, which essentially equals the reduced eta invariant, depends only on the component of $g_M$ in $\mathcal{R}^{scal>0}(M)$. This approach was first used by Botvinnik and Gilkey [3] in combination with bordism arguments to prove that in *any* dimension $n \geq 5$ there are manifolds $M$ for which the moduli space $\mathcal{M}^{scal>0}(M)$ has infinitely many components.

It would be interesting to see whether reduced eta invariants can be used to exhibit non-simply connected manifolds in dimension $\neq 4n-1$ for which the moduli space of metrics of nonnegative *sectional* curvature has infinitely many components.

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## Spaces of nonnegatively curved surfaces

Igor Belegradek

(joint work with Jing Hu, Taras Banakh)

Let $V$ be a connected smooth manifold and let $\mathfrak{R}_{\geq 0}(V)$ be the set of complete $C^\infty$ Riemannian metrics on $V$ of nonnegative sectional curvature, which we abbreviate to $K \geq 0$. The present work is motivated by the following questions:

- What is the "right" topology on $\mathfrak{R}_{\geq 0}(V)$?
- What is the "right" notion of the corresponding moduli space?

Tentatively, we could think of the moduli space $\mathfrak{M}_{\geq 0}(V)$ as the quotient space of $\mathfrak{R}_{\geq 0}(V)$ by the standard pushforward action of $D(V)$, a "large" subgroup of Diff $V$. To explain what "right" could mean here is a "wish list" of properties we might want:

- $\mathfrak{R}_{\geq 0}(V)$ is metrizable and $\mathfrak{M}_{\geq 0}(V)$ is locally metrizable at "most" points.
- Paths in $\mathfrak{M}_{\geq 0}(V)$ can be lifted to paths in $\mathfrak{R}_{\geq 0}(V)$. This is true by Ebin's slice theorem if $V$ is compact and $\mathfrak{R}_{\geq 0}(V)$ is given the $C^\infty$ topology.

- The quotient map $\mathfrak{R}_{\geq 0}(V) \to \mathfrak{M}_{\geq 0}(V)$ is a fiber bundle. This holds when $V$ is compact and $D(V)$ is the group of diffeomorphisms that fixes one tangent space.
- "Obviously continuous" families in $\mathfrak{R}_{\geq 0}(V)$ are continuous; e.g., the family of surfaces of revolution $z = t(x^2 + y^2)$, where $t \in [0, 1]$, is continuous in the compact-open topology but is not continuous in the uniform topology.
- The convergence in $\mathfrak{M}_{\geq 0}(V)$ mimics the non-collapsing pointed Gromov-Hausdorff convergence, perhaps with some extra regularity, such as $C^{r,\alpha}$, so that various precompactness results apply.

In the present work we study the case when $V$ is a simply-connected surface, i.e. $S^2$ or $\mathbb{C}$. While such $V$ can be treated with purely 2-dimensional techniques, the answers and challenges provide an idea of what might occur for a general $V$.

To discuss the surfaces simultaneously we set $M_0 = \mathbb{C}$ and $M_1 = S^2$ and fix a complete metric $g_\kappa$ of constant curvature $\kappa \in \{0, 1\}$ on $M_\kappa$. Let $D(M_\kappa)$ be the group of orientation-preserving diffeomorphisms of $M_\kappa$ that fix the points $0, 1 \in \mathbb{C}$ if $\kappa = 0$ and $0, 1, \infty$ if $\kappa = 1$.

By the uniformization theorem combined with a classical result of Blanc-Fiala for the case $\kappa = 0$ any metric $g \in \mathfrak{R}_{\geq 0}(M_\kappa)$ can be written uniquely as $g = \phi_* e^{-2u} g_\kappa$ where $\phi \in D(M_\kappa)$ and $u \in C^\infty(M_\kappa)$. Thus the map $(u, \phi) \to \phi_* e^{-2u} g_\kappa$ is a bijection. Is it a homeomorphism? To make sense of the question one has to specify the topology on the domain and the codomain. Results of Earle and Schatz on smooth dependence of solutions of Beltrami equation on the dilatation show that the map is indeed a homeomorphism if we vary $g$, $u$ and $\phi_*$ in the $C^{r,\alpha}$ topology, where $\alpha \in (0, 1)$ and $k$ is any nonnegative integer or $\infty$, see [1, 2]; we expect this to fail when $\alpha = 0$.

Let $\mathcal{O}_\kappa$ be the subset of $C^\infty(M_\kappa)$ consisting of functions $u$ such that $e^{-2u} g_\kappa$ is in $\mathfrak{R}_{\geq 0}(M_\kappa)$. Nonnegativity of the curvature is equivalent to the inequality $\Delta_{g_\kappa} u \geq -\kappa$ where $\Delta_{g_\kappa}$ is the $g_\kappa$-Laplacian. If $\kappa = 1$, the inequality gives a complete description of $\mathcal{O}_1$, but the case $\kappa = 0$ is more subtle because completeness of $e^{-2u} g_0$ imposes further restrictions on $u$. The main result of [1] characterizes such $u$ as a subharmonic function on $\mathbb{C}$ satisfying a certain growth condition. This allows one to show that $\mathcal{O}_\kappa$ is a closed convex subset of $C^\infty(M_\kappa)$.

In [3] we combine various 2-dimensional results with techniques of infinite dimensional topology to determine the homeomorphism types of $\mathcal{O}_\kappa$ and $D(M_\kappa)$ and in particular prove the following.

**Theorem 1.** *$\mathfrak{R}_{\geq 0}(M_\kappa)$ equipped with the $C^{r,\alpha}$ topology is homeomorphic to*

(1) *$\ell^2$ if $r = \infty$,*
(2) *$\Sigma^\omega$ if $r$ is finite.*

Here $\ell^2$ is the separable Hilbert space, $\Sigma$ is the linear span of the standard Hilbert cube in $\ell^2$, and $\Sigma^\omega$ is the product of countably many copies of $\Sigma$. Since the space $\Sigma^\omega$ may be unfamiliar to the reader, let us mention that it is a locally convex linear space which is a countable union of nowhere dense sets, so $\Sigma^\omega$ is not completely metrizable. If $\Omega$ is either $\ell^2$ or $\Sigma^\omega$, then it has the following properties:

(a) $\Omega$ is not $\sigma$-compact, and in particular, not locally compact.
(b) Any two open homotopy equivalent subsets of $\Omega$ are homeomorphic.
(c) The complement to any compact subset of $\Omega$ is contractible.
(d) Any homeomorphism of two compact subsets of $\Omega$ extends to a homeomorphism of $\Omega$.

The conclusion of Theorem 1 also holds for $\mathfrak{R}_{>0}(M_\kappa)$, the subspace of $\mathfrak{R}_{\geq 0}(M_\kappa)$ consisting of positively curved metrics.

We noted in [1] that the quotient space $\mathfrak{R}_{\geq 0}(\mathbb{C})/\operatorname{Diff}\mathbb{C}$ is not Hausdorff at the point $g_0$ in the $C^{r,\alpha}$ topology. By contrast, the quotient space $\mathfrak{R}_{\geq 0}(M_\kappa)/D(M_\kappa)$ is Hausdorff, and also contractible via the deformation $e^{-2tu}g_\kappa$, $t \in [0,1]$.

The above methods deliver optimal results when $\mathfrak{M}_{\geq 0}(M_\kappa)$ is given the $C^{r,\alpha}$ topology, but it is perhaps more natural to equip the space with the Gromov-Hausdorff topology. This requires a different set of tools which work best when $M_\kappa = S^2$.

Let us think of a metric $g \in \mathfrak{R}_{\geq 0}(S^2)$ via the Alexandrov realization theorem, which says that $g$ is isometric to the boundary of a convex body $C_g$ in $\mathbb{R}^3$. Uniqueness of $C_g$ up to a rigid motion is due to Cohn-Vossen (and for less regular metrics to Pogorelov). Regularity of $\partial C_g$ was understood by Pogorelov and Nirenberg, who showed $\partial C_g$ is $C^\infty$ near the points of $K > 0$. Under the assumption $K \geq 0$ there are examples where $\partial C_g$ is not even $C^3$. The stability theorem of Volkov shows that if two metrics $g, h \in \mathfrak{M}_{\geq 0}(S^2)$ are Gromov-Hausdorff close, then $C_g$, $C_h$ can be made Hausdorff close after a rigid motion.

Let **cb** be the set of convex bodies in $\mathbb{R}^3$ that have center of mass at the origin and $C^\infty$ boundary of $K > 0$. We equip **cb** with the Hausdorff metric. The above considerations easily imply that

**Proposition 2.** *$\mathfrak{M}_{>0}(S^2)$ equipped with the Gromov-Hausdorff topology is the quotient space of* **cb** *by the standard $O(3)$-action.*

If $B$ is the unit ball in $\mathbb{R}^3$, then the convex combination $tB + (1-t)C$, where $t \in [0,1]$ and $C \in$ **cb** in an $O(3)$-equivariant deformation of **cb** to the point, and we conclude

**Theorem 3.** *$\mathfrak{M}_{>0}(S^2)$ is contractible in the Gromov-Hausdorff topology.*

The fact that **cb** is closed under the convex combination with the unit ball is known, see [4] and references therein.

The author believes that the methods of [3] permit to find the homeomorphism type of **cb**, which then (via the slice theorem) would lead to understanding of the local structure of $\mathfrak{M}_{>0}(S^2)$ with the Gromov-Hausdorff topology.

To some extent the same strategy works for $g \in \mathfrak{M}_{\geq 0}(S^2)$. Here one encounters the difficulty that $\partial C_g$ need not be $C^\infty$, but one could use Schneider's regularization in order to instantly and $O(3)$-equivariantly push the space of 3-dimensional convex bodies into the subspace of $C^\infty$ bodies, and then apply the convex combination as above to show that $\mathfrak{M}_{\geq 0}(S^2)$ is contractible in the Gromov-Hausdorff topology.

The Alexandrov realization theorem also holds for metrics in $\mathfrak{M}_{\geq 0}(\mathbb{R}^2)$, and so do the regularity results, but the uniqueness is true only after fixing the tangent cone, and the Volkov stability theorem does not seem be in the literature. We hope to address these issues in future work.

The reader may be wondering why we never attempted to use geometric flows to understand the homotopy type of $\mathfrak{M}_{\geq 0}(M_\kappa)$. While this might be eventually doable, a basic difficulty is to prove the continuous dependence on the initial data in the weak topology (e.g. Gromov-Hausdorff or $C^\alpha$) which we believe is still an open problem.

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## Higher homotopy groups of spaces of nonnegatively curved metrics


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(joint work with F. Thomas Farrell, Vitali Kapovitch)


Henceforth "smooth" means $C^\infty$, all manifolds are smooth, and any set of smooth maps, such as diffeomorphisms, embeddings, pseudoisotopies, or Riemannian metrics, is equipped with the smooth compact-open topology. The phrase "nonnegative sectional curvature" is abbreviated to $K \geq 0$.

Let $\mathfrak{R}_{\geq 0}(V)$ be the space of complete smooth Riemannian metrics of $K \geq 0$ on a connected manifold $V$. The group $\mathrm{Diff}\, V$ acts on $\mathfrak{R}_{\geq 0}(V)$ by pushforward. Let $\mathfrak{M}_{\geq 0}(V)$ be the associated *moduli space*, the quotient space of $\mathfrak{R}_{\geq 0}(V)$ by the above $\mathrm{Diff}\, V$-action.

Open complete manifolds of $K \geq 0$ enjoy a rich structure theory, e.g., the soul construction of Cheeger and Gromoll takes as the input a basepoint of a complete manifold $V$ of $K \geq 0$ and outputs a compact totally convex submanifold without boundary, called a *soul*, such that $V$ is diffeomorphic to the interior of a tubular neighborhood of the soul.

We call a connected open manifold *indecomposable* if it admits a complete metric of $K \geq 0$ such that the normal sphere bundle to a soul has no section. A result of Yim implies that if $V$ is indecomposable, then the soul of any metric in $\mathfrak{R}_{\geq 0}(V)$ in unique, i.e., independent of a basepoint, and moreover, it is shown in [1, 2] that the souls of metrics in any path-connected subset of $\mathfrak{R}_{\geq 0}(V)$ are ambiently isotopic. This was used to find many open manifolds $V$ for which $\mathfrak{M}_{\geq 0}(V)$ is not path-connected, or even has infinitely many path-components, see [1, 2, 3, 4]. Here is a new geometric ingredient for the present work:

**Theorem 1.** *If $V$ is indecomposable, then any continuous change in $g \in \mathfrak{R}_{\geq 0}(V)$ results in a smooth change of the soul of $g$.*

This is not obvious because the soul construction involves asymptotic geometry which is not preserved when the metric is varied in the smooth compact-open topology. To see how this result gives rise to a new topological invariant of metrics $g$ we fix an arbitrary metric $h \in \mathfrak{R}_{\geq 0}(V)$ with soul $S_h$ of normal injectivity radius $i_h$. Set $\rho(s) = \frac{s}{s+1}$ and let $N_h$ be the $\rho(i_h)$-neighborhood of $S_h$. Let $\theta_h$ be the orbit map of a metric $h \in \mathfrak{R}_{\geq 0}(V)$ under the pushforward action of $\operatorname{Diff} V$. Consider the following commutative diagram.

$$\begin{array}{ccccccccc}
 & & * \simeq \operatorname{Diff}(V, \operatorname{rel} N_h) & \longrightarrow & \operatorname{Diff} V & \xrightarrow{\theta_h} & \mathfrak{R}_{\geq 0}(V) & & \\
 & & & & \downarrow & & \downarrow{\scriptstyle \delta} & & \\
\Omega\mathcal{X}(N_h, V) & \xrightarrow{\Omega f} & \operatorname{Diff} N_h & \longrightarrow & \operatorname{Emb}(N_h, V) & \xrightarrow{q} & \mathcal{X}(N_h, V) & \xrightarrow{f} & B \operatorname{Diff} N_h
\end{array}$$

The map $q$ taking an embedding to its image is a principal bundle, and $f$ denotes its classifying map. The bottom row is the corresponding fiber sequence. The leftmost vertical arrow is given by restricting to $N_h$, which is a fiber bundle due to the parametrized isotopy extension theorem. Its fiber over the inclusion $\operatorname{Diff}(V, \operatorname{rel} N_h)$ is contractible by the Alexander trick towards infinity. Theorem 1 implies that the map $\delta$ taking $g$ to the closed $\rho(i_g)$-neighborhood of $S_g$ is continuous.

Let $\pi_k(\theta_h)$ be the homomorphism induced by $\theta_h$ on the $k$th homotopy groups based at the identity map of $V$, and similarly, let $\pi_k(q)$, $\pi_k(f)$, $\pi_k(\Omega f)$ be the induced maps of homotopy groups based at inclusions. With these notations the commutativity of the diagram implies that $\operatorname{Im} \pi_k(q)$ is a quotient of a subgroup of $\operatorname{Im} \pi_k(\theta_h)$.

Since the bottom row of the diagram is a fiber sequence we get isomorphisms $\operatorname{Im} \pi_k(q) \cong \ker \pi_k(f) \cong \ker \pi_{k-1}(\Omega f)$ for each $k \geq 1$. Fix a collar neighborhood of $\partial N$, and consider the inclusion $\iota_N \colon P(\partial N) \to \operatorname{Diff} N$ that extends a pseudoisotopy on the collar neighborhood of $\partial N$ in $N$ by the identity outside the neighborhood. One can identify the homomorphisms $\pi_{k-1}(\Omega f)$ and $\pi_{k-1}(\iota_N)$ for each $k \geq 2$, where $\pi_{k-1}(\iota_N)$ is the map induced by $\iota_N$ on the $(k-1)$th homotopy group with identity maps as the basepoints. In summary, we get

**Theorem 2.** *Let $N$ be a compact manifold with indecomposable interior. Then for every $h \in \mathfrak{R}_{\geq 0}(\operatorname{Int} N)$ and each $k \geq 2$, the group $\ker \pi_{k-1}(\iota_N)$ is a quotient of a subgroup of $\pi_k(\mathfrak{R}_{\geq 0}(\operatorname{Int} N), h)$.*

Prior to this result there has been no tool to detect nontrivial higher homotopy groups of $\mathfrak{R}_{\geq 0}(V)$.

We make a systematic study of $\ker \pi_*(\iota_N)$ and find a number of manifolds for which $\ker \pi_*(\iota_N)$ is infinite and $\operatorname{Int} N$ admits a complete metric of $K \geq 0$. Here is a sample of what we can do:

**Theorem 3.** *Let $U$ be the total space of one of the following vector bundles:*

(1) *the tangent bundle to $S^{2d}$, $CP^d$, $HP^d$, $d \geq 2$, and the Cayley plane,*

(2) *the Hopf* $\mathbb{R}^4$ *or* $\mathbb{R}^3$ *bundle over* $HP^d$, $d \geq 1$,
(3) *any linear* $\mathbb{R}^4$ *bundle over* $S^4$ *with nonzero Euler class,*
(4) *any nontrivial* $\mathbb{R}^3$ *bundle over* $S^4$,
(5) *the product of any bundle in* (1), (2), (3), (4) *and any closed manifold of* $K \geq 0$ *and nonzero Euler characteristic.*

*Then there exists* $m$ *such that every path-component of* $\mathfrak{R}_{\geq 0}(U \times S^m)$ *has some nonzero rational homotopy group.*

The fact that each $U$ in Theorem 3 admits a complete metric of $K \geq 0$ is well-known. Other computations are surely possible. In fact we are yet to find $N$ with indecomposable interior and such that $\iota_N$ is injective on all homotopy groups; the latter does happen when $N$ is the $n$-disk.

In the present work we are unable to compute $m$ in Theorem 3 (this deficiency has apparently been fixed, see Jiang Yi's talk in this volume). The smallest $k \geq 1$ for which we know that $\mathfrak{R}_{\geq 0}(U \times S^m)$ is nonzero is $k = 7$, which occurs when $U$ is the total space of a nontrivial $\mathbb{R}^3$ bundle over $S^4$.

We do not yet know how to detect nontriviality of $\pi_k\, \mathfrak{M}_{\geq 0}(V)$, $k \geq 1$. The nonzero elements in $\pi_k\, \mathfrak{R}_{\geq 0}(U \times S^m)$ given by Theorem 3 lie in the kernel of the $\pi_k$-homomorphism induced by the quotient map onto the moduli space.

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## On the topology of the space of Ricci-positive metrics


Boris Botvinnik

(joint work with David J. Wraith)


**Index-difference map.** Let $M$ be a closed compact spin manifold, and $\mathcal{R}(M)$ be the space of Riemannian metrics on $M$ equipped with $C^\infty$-topology. We denote by $\mathcal{R}^{s>0}(M)$ and $\mathcal{R}^{\mathrm{Ric}>0}(M)$ the subspaces of metrics with positive scalar and with positive Ricci curvature respectively.

Recall Hitchin's construction of the *index difference* map. Ignoring some technical details, the definition is as follows. Fix a basepoint $g_0 \in \mathcal{R}^{s>0}(M)$, where $\dim M = d$, so that for any other psc-metric $g$ there is the path of metrics $g_t = (1-t) \cdot g_0 + t \cdot g$ for $t \in [0, 1]$. Then there is an associated path of Dirac operators in the space $\mathbf{Fred}^d$ of Clifford$^d$-linear self-adjoint odd Fredholm operators on a Hilbert space, and it starts and ends in the subspace of invertible operators, which is contractible. Since the space $\mathbf{Fred}^d$ represents the real $K$-theory functor $KO^{-d}(-)$, we obtain an element

$$\mathrm{inddiff}_{g_0}(g) \in KO^{-d}([0,1],\{0,1\}) = KO^{-d-1} = KO_{d+1}.$$

This construction generalizes to families, and gives a well-defined homotopy class of maps

$$\mathrm{inddiff}_{g_0} : \mathcal{R}^{s>0}(M) \longrightarrow \Omega^{\infty+d+1}KO \tag{1}$$

to the infinite loop space which represents real $K$-theory. In particular, there is an isomorphism $\pi_q\Omega^{\infty+d+1}KO \cong KO_{d+1+q}$.

**Theorem 1.** (Botvinnik, Ebert, Randal-Williams [1]) *Let $M$ be a spin manifold with $\dim M \geq 6$. Assume $M$ admits a psc-metric, and $g_0 \in \mathcal{R}^+(M)$ is a base point. Then the homomorphism $(\mathrm{inddiff}_{g_0})_* : \pi_q\mathcal{R}^{s>0}(M) \longrightarrow KO_{d+1+q}$ induced by the map $\mathrm{inddiff}_{g_0}$ is non-trivial provided the target group $KO_{d+1+q}$ is non-trivial.*

**Main result.** It is known that the space $\mathcal{R}^{\mathrm{Ric}>0}(M)$ of metrics with positive Ricci curvature has non-trivial topology, [3]. In particular, the space $\mathcal{R}^{\mathrm{Ric}>0}(M)$ has many path-components for some particular manifolds $M$.

Assume a manifold $M$ admits a metric with positive Ricci curvature and consider a natural inclusion map $\iota : \mathcal{R}^{\mathrm{Ric}>0}(M) \to \mathcal{R}^{s>0}(M)$. We denote by $\mathsf{inddiff}_{g_0}$ the composition

$$\mathsf{inddiff}_{g_0} : \mathcal{R}^{\mathrm{Ric}>0}(M) \xrightarrow{\iota} \mathcal{R}^{s>0}(M) \xrightarrow{\mathrm{inddiff}_{g_0}} \Omega^{n+1}KO.$$

Here is our main result:

**Theorem A.** *For given $\ell \geq 1$ and even integer $d \geq 6$, there exists a spin manifold $W$, $\dim W = d$, together with a metric $g_0 \in \mathcal{R}^{\mathrm{Ric}>0}(W)$, such that the map $\mathsf{inddiff}_{g_0} : \mathcal{R}^{\mathrm{Ric}>0}(W) \to \Omega^{\infty+d+1}KO$ induces a non-trivial homomorphism between the homotopy groups $(\mathsf{inddiff}_{g_0})_* : \pi_q\mathcal{R}^{\mathrm{Ric}>0}(W) \to \pi_{q+d+1}KO$ provided the group $\pi_{q+d+1}KO$ is non-trivial and $q \leq \ell$.*

We next introduce some notations. Let $h_0$ denote the round metric of radius 1. For a smooth manifold $X^d$ with boundary $S^{d-1}$, we define $\mathcal{R}(X)_{h_0}$ to be the space of all Riemannian metrics on $X$ which are a product $dt^2+h_0$ near the boundary and restrict to the metric $h_0$ on $\partial X$. Then we define $\mathcal{R}^{s>0}(X)_{h_0} = \mathcal{R}(X)_{h_0} \cap \mathcal{R}^{s>0}(X)$.

Now we define a subspace of Riemannian metrics $\mathcal{R}(X)^*_{h_0}$ with *spherical boundary condition* which is relevant only for manifolds with spherical boundary. Namely, we require that a metric $g \in \mathcal{R}(X)^*_{h_0}$ restricts to $h_0$ at the boundary, and that there is a collar neighbourhood of the boundary in which $g$ takes the form

$$dr^2 + R^2 \sin^2(r/R) ds^2_{d-1}$$

for $r \geq r_0 := R\sin^{-1}(1/R)$, with $R \in [1,\infty)$. Then the metric in this collar neighbourhood is round with constant sectional curvature $1/R^2$, $r$ is the (inward) normal parameter to the boundary, and the boundary itself corresponds to $r = r_0$. We then define the following subspaces of metrics:

$$\mathcal{R}^{s>0}(X)^*_{h_0} = \mathcal{R}(X)^*_{h_0} \cap \mathcal{R}^{s>0}(X), \quad \mathcal{R}^{\mathrm{Ric}>0}(X)^*_{h_0} = \mathcal{R}(X)^*_{h_0} \cap \mathcal{R}^{\mathrm{Ric}>0}(X).$$

Let $(W, g_0)$ be a manifold as in Theorem A. We assume further that the metric $g_0$ is such that there exists a base point $x_0 \in W$ together with an open geodesic disk $D_\epsilon(x_0)$ such that the manifold $\bar{W} := W \setminus D_\epsilon(x_0)$ has a boundary $\partial \bar{W} = S^{d-1}$ with the standard round metric $h_0$. We can always arrange for this to be the case.

We need a sequence of particular manifolds each equipped with a Ricci-positive metric. Let

$$W_k^{2n} := (S^n \times S^n)^{\sharp k},$$

where $x_0 \in W_k^{2n}$, and $\bar{W}_k^{2n} := W_k^{2n} \setminus D_\epsilon(x_0)$ with $\partial \bar{W}_k^{2n} = S^{2n-1}$, $k = 1, 2, \ldots$. We also let $\bar{W}_0 = D^{2n}$. The manifolds $W_k^{2n}$ are the ones we need to prove Theorem A. The following geometrical fact plays a crucial role to prove Theorem A.

**Theorem B.** *For each $k = 1, 2, \ldots$, there exists a metric $g_k \in \mathcal{R}^{\mathrm{Ric}>0}(\bar{W}_k^{2n})^*_{h_0}$.*

The proof of Theorem B builds on results of Sha and Yang, [2].

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## Involution on pseudoisotopy spaces and the space of nonnegatively curved metrics


Yi Jiang

(joint work with Mauricio Bustamante, F. Thomas Farrell)


In this talk, we present a continuation of the work [1] of Belegradek, Farrell and Kapovitch. Let $\mathfrak{R}_{K\geq 0}(V)$ denote the space of complete Riemannian metrics of nonnegative sectional curvature on a connected manifold $V$, equipped with the smooth compact-open topology. It is shown in [1] that for many open manifolds $V$ the space $\mathfrak{R}_{K\geq 0}(V)$ has nontrivial rational higher homotopy groups. For example, when $U$ is the total space of the tangent bundle to the $2d$ dimensional sphere $S^{2d}$ for $d \geq 2$, they find explicit integers $i \geq 2$ such that there are sufficiently large integers $m$ for which

$$\pi_i \mathfrak{R}_{K\geq 0}(U \times S^m) \otimes \mathbb{Q} \neq 0.$$

A left question is to specify these $m$ for given integers $i$.

By [1, Lemmas 5.2 and 9.4], to answer this question, reduces to studying *the canonical involution on pseudoisotopy spaces* $P(M)$ when $M$ is the total space of the sphere bundle associated to $U$. (For instance, when $U$ is the total space of the tangent bundle to $S^{2d}$, the manifold $M$ is the total space of the unit tangent bundle to $S^{2d}$.) In recent joint work with Bustamante and Farrell [5], we obtain some answer for the question in the cases when $U$ are the total spaces of the tangent bundles to $S^{2d}$ and $\mathbb{C}P^2$ because we are able to compute the positive and negative eigenvector spaces of the involution on the rational homotopy groups of the corresponding pseudoisotopy spaces $P(M)$.

A pseudoisotopy space of a closed smooth manifold $M$ is defined to be the group of self-diffeomorphisms of $M \times [0,1]$ which are the identity on $M \times 0$ equipped with the smooth compact-open topology and the canonical involution $\tau : P(M) \longrightarrow P(M)$ is given by

$$\tau(f) = (id_M \times r) \circ f \circ (id_M \times r) \circ ((f|_{M\times 1})^{-1} \times id_{[0,1]}),$$

where $r : [0,1] \longrightarrow [0,1]$ is defined as $r(t) = 1 - t$. In order to figure out the involution $\tau$ on the rational homotopy groups of the pseudoisotopy space $P(M)$, an expected strategy is to relate it to the calculation of the canonical geometric involution on the rational $S^1$-equivariant homology $H_*^{S^1}(LM;\mathbb{Q})$ of the free loop space $LM$ of $M$ as the calculation of the latter involution is tractable by the work [10] and [7].

The connection between these two involutions has been expected by some previous work, c.f. [3, Theorem 4.1]. A lot of work has been done on this connection, but there are still some gaps. One crucial medium for this connection is Waldhausen's K-theory $A(M)$ of the manifold $M$ which is a topological space whose rational homotopy group $\pi_* A(M) \otimes \mathbb{Q}$ can be equipped with two involutions: one is defined by Vogell [11] and the other one is given by Burghelea and Fiedorowicz [4]. On one hand, the involution $\tau$ on the pseudoisotopy space $P(M)$ is related to Vogell's involution on $\pi_* A(M) \otimes \mathbb{Q}$ by the work of Waldhausen and Vogell (c.f. [12] and [11]). On the other hand, the connection between Burghelea-Fiedorowicz's involution on $\pi_* A(M) \otimes \mathbb{Q}$ and the geometric involution on the $H_*^{S^1}(LM;\mathbb{Q})$ has been given by the work [8] and [9]. (For the connection between $\pi_* A(M) \otimes \mathbb{Q}$ and $H_*^{S^1}(LM;\mathbb{Q})$ without involution, c.f. [2] and [6]). Our main result to fill in the gap in this connection is to prove that Vogell's and Burghelea-Fiedorowicz's involutions coincide, so this completes the connection between the involution $\tau$ on $\pi_* P(M) \otimes \mathbb{Q}$ and the canonical geometric involution on $H_*^{S^1}(LM;\mathbb{Q})$.

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## Recognizing certain 7-manifolds with $\pi_1 = \mathbb{Z}$

Yang Su

There is a current research interest in recognizing manifolds in terms of their algebraic-geometric properties. The celebrated Poincaré conjecture is the prototype of this kind of study. For the recent study of manifolds detected by their algebraic-topological invariants see e. g. [1]. In this talk I will describe a project joint with M. Kreck on detecting the 7-manifold $S^1 \times \mathbb{C}\mathrm{P}^3$, and the Dold manifold $S^1 \times_\tau \mathbb{C}\mathrm{P}^3$, where $\tau\colon \mathbb{C}\mathrm{P}^3 \to \mathbb{C}\mathrm{P}^3$ denotes the complex conjugation.

The following theorem is a combination of the Browder-Levine's fibration theorem [2] and computations with homological algebra.

**Theorem 1.** *Let $M^7$ be an orientable smooth closed 7-manifold with $\pi_1(M) \cong \mathbb{Z}$, $\pi_2(M) \cong \mathbb{Z}$ and $H_3(\widetilde{M}) = 0$. Then $M$ is a smooth fiber bundle over $S^1$ with fiber a simply-connected 6-manifold $N^6$ such that $H_2(N) \cong \mathbb{Z}$ and $H_3(N) = 0$.*

By the classification of simply-connected 6-manifolds [3], the fiber $N$ here is determined by $H^*(N)$ and $p_1(N)$. Hence the diffeomorphism classification of $M$ is determined by $H^*(M)$, $p_1(M)$ and the mapping class group of $N$.

The mapping class group of the relevant 6-manifolds is essentially determined by [4]. In the case $N = \mathbb{C}\mathrm{P}^3$ we have $MCG^+(\mathbb{C}\mathrm{P}^3) = \mathbb{Z}/4$ and $MCG(\mathbb{C}\mathrm{P}^3) = \mathbb{Z}/4 \rtimes \mathbb{Z}/2$. Especially we have an enumeration of the 7-manifolds with the given algebraic data.

**Theorem 2.** *Let $M$ be a smooth closed oriented spin 7-manifold $M^7$ such that $\pi_1(M) \cong \mathbb{Z}$, $\pi_2(M) \cong \mathbb{Z}$, $H_3(\widetilde{M}) = 0$. Then $H^*(M) \cong \mathbb{Z}$ for $* = 0, 1, \cdots, 7$. Let $x \in H^2(M)$ and $y \in H^4(M)$ be generators. If $x^2 = \pm y$ and $p_1(M) = \pm 4y$. Then $M$ is diffeomorphic to $S^1 \times \mathbb{C}\mathrm{P}^3 \sharp \Sigma_i^7$, where $\Sigma_i \in \Theta_7 \cong \mathbb{Z}/28$ is a homotopy sphere with $i = 0, 1, 2, 3$.*

But we don't know an invariant which can detect the connected sum factor of exotic spheres. In [5] a generalized Eells-Kuiper invariant was defined for any spin 7-manifold. But when applied to these manifolds, the invariant is identically zero.

So here we would like to ask if it is possible to characterize the standard $S^1 \times \mathbb{C}\mathrm{P}^3$ and $S^1 \times_\tau \mathbb{C}\mathrm{P}^3$ with their metric properties.

*Question 1.* Let $M$ be a smooth fiber bundle over $S^1$ with fiber $\mathbb{C}\mathrm{P}^3$. If $M$ admits a nonnegatively curved metric, then is $M$ diffeomorphic to $S^1 \times \mathbb{C}\mathrm{P}^3$ or $S^1 \times_\tau \mathbb{C}\mathrm{P}^3$?

*Question 2.* Let $M$ be a smooth fiber bundle over $S^1$ with fiber $\mathbb{C}\mathrm{P}^3$. If $M$ admits positively curved metrics on each fiber, varying smoothly, then is $M$ diffeomorphic to $S^1 \times \mathbb{C}\mathrm{P}^3$ or $S^1 \times_\tau \mathbb{C}\mathrm{P}^3$?

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## The topology of positive scalar curvature

Thomas Schick

In recent years we have learned a lot about the topology of spaces of metrics of positive scalar curvature and of related spaces.

Among those is one consequence of recent joint work with Diarmuid Crowley and Wolfgang Steimle [3]:

**Theorem 1.** *Let $M$ be a closed spin manifold of dimension $m \geq 6$. Let $R(M)$ be any space of Riemanninan metrics, which is invariant under the pullback action of the diffeomorphism group, and which embeds into $Riem^+(M)$, the space of metrics of positive scalar curvature. Examples for $R(M)$ could be the space of metrics with positive Ricci curvature or with positive sectional curvature. Assume that $g_0 \in R(M)$, i.e. $R(M)$ is not empty.*

*Then, $R(M)$ is topologically highly non-trivial. More precisely, whenever $k \geq 0$ is such that $k + m \equiv 0, 1 \pmod 8$, $\pi_k(R(M), g_0)$ is non-trivial. More precisely, if $k \geq 1$ there is a split epimorphism of $\pi_k(R(M), g_0)$ onto the two-element group.*

Actually, the assumptions can be weakened slightly:

**Theorem 2.** *In the previous theorem, the condition that $R(M)$ embeds into $Riem^+(M)$ can be weakened: it suffices that $R(M)$ embeds into $Riem^{\mathrm{inv}}(M)$, the space of metrics on $M$ such that the associated Dirac operator (using the spin structure and the metric) is invertible, i.e. does not admit a harmonic spinor, with the same consequences.*

This applies e.g. if $R(M)$ is the space of metrics of non-negative sectional or Ricci or even scalar curvature, provided $M$ does not admit any Ricci flat metrics (e.g. if $M$ is a sphere).

**Remark.** David Wraith [6] has extended the methods to the space of metrics with non-positive scalar curvature even on manifolds which do admit Ricci-flat metrics. Based on a new holonomy result of Ammann, Kroencke, Weiss, and Witt [1], he obtains the same result if $g_0$ is not itself Ricci flat.

The method of proof for all results presented is the same: one constructs an element in $\pi_k(\mathrm{Diff}(D^m, \partial), id)$, the homotopy groups of the group of diffeomorphisms of the disk which are the identity in a neighborhood of the boundary. One embeds into the diffeomorphism group of $M$ by embedding $D^m$ into $M$ and extending the diffeomorphisms trivially. Finally, one obtains elements in $\pi_k(R(M), g_0)$ by pulling back $g_0$ with the family of diffeomorphisms.

The non-triviality of the resulting homotopy class of metrics is checked by a (topological) index calculation of the relative family index of the resulting family of Dirac operators. In the case at hand, this index takes values in $\mathbb{Z}/2\mathbb{Z}$. If the family could be contracted within the space $\mathit{Riem}^{\mathrm{inv}}(M)$, then the Schrödinger-Lichnerowicz formula [5] implies that this relative index is zero. But it is not!

**Remark.** The method just described, by its very nature, gives information about the space $R(M)$, but not about the moduli space $R(M)/\mathrm{Diff}(M)$.

Improvements of the methods could be based on *higher* secondary index theory of the Dirac operator; the corresponding invariants take values in $KO_*(C^*\pi_1 M)$, the real K-theory of the group $C^*$-algebra of the fundamental group. The above $\mathbb{Z}/2\mathbb{Z}$ indeed equals $KO_*(\mathbb{R})$ for $* = 1, 2 \pmod 8$.

This leads to quite a number of deep questions:

(1) The method uses the Dirac operator and its index, which is only available for spin manifolds (and with some tricks and restrictions sometimes if only the universal covering of $M$ is spin).

An almost completely unsolved question is: what can be said about the spaces $R(M)$, in particular the space $\mathit{Riem}^+(M)$, if $M$ is a simply-connected non-spin manifold, or more generally a manifold whose universal covering does not admit a spin structure?

In very low dimensions, the minimal hypersurface method of Schoen and Yau [4] is available. The general case seems to lack any idea for a promising method.

(2) The constructions are *local* in nature (changes only within a small disk) but have *global* consequences. Do these global effects persist if we pass to non-compact $M$? Obviously, here one has to decide on the appropriate context and has to make restrictions on the geometries to be considered.

In a different direction, are there additional global phenomena in the space $\mathit{Riem}^+(M)$? And how can they be distinguished from those coming from the above construction? Perhaps with higher index theory?

(3) Connected to the previous question is the question, how different the spaces $Riem^+(M)$ and $Riem^{\mathrm{inv}}(M)$ are. However, we know that higher index theory gives additional information, so that the truely relevant space is $Riem^{\mathrm{inv}_\infty}(M)$, the space of metrics where the Dirac operator twisted with the Mishchenko bundle (with fiber $C^*\pi_1 M$) is invertible. So, the question is whether $Riem^+(M) \to Riem^{\mathrm{inv}_\infty}(M)$ is a homotopy equivalence.

Note that the usual index techniques to understand such spaces do not seem to be able to distinguish them. The only extra tool which seems available at the moment again is the minimal hypersurface method.

(4) Higher index theory gives maps from $\pi_k(Riem^+(M))$ to $KO_{k+m+1}(C^*\pi_1(M))$.

How much is this map a surjection? The main result of [3] implies that the $\mathbb{Z}/2\mathbb{Z}$-summands coming from $KO_*(\mathbb{R})$ are in the image. Fundamental work of Botvinnik, Ebert, and Randal-Williams shows the same for all of $KO_*(\mathbb{R})$; but the general picture seems completely unclear.

Concerning question (3), we present an example of a 5-dimensional manifold with two different components of $Riem^+(M)$ which cannot be distinguished using Dirac operator methods and (this is work in progress with Zhizhang Xie) lie in the same component of $Riem^{\mathrm{inv}_\infty}(M)$. The same example shows that the map from question 4 is not an isomorphism.

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## Discussion session summary and open questions

The following is a brief summary of some of the questions, which arose during the discussion sessions collected by Andrey Gogolev.

We utilize the following abbreviations of participant names: **BB** Boris Botvinnik, **IB** Igor Belegradek, **MB** Mauricio Bustamante, **AD** Anand Dessai, **TF** F. Thomas Farrell, **AG** Andrej Gogolev, **TS** Thomas Schick, **WT** Wilderich Tuschmann, **DW** David Wraith, **MW** Mark Walsh, **SY** Su Yang.

## Day 1

**TF:** By work of Farrell-Hsiang $\pi_{4k-1}(\mathrm{Diff}(\mathbb{D}^n, \partial)) \otimes \mathbb{Q} \simeq \mathbb{Q}$ (or $\mathbb{Q}^2$, depending on parity of $n$) can be put into $\pi_{4k-1}(\mathrm{Diff}(M))$ in several ways (ball, solid torus). Consider the long exact sequence in homotopy groups $\mathrm{Diff}_0(M) \to \mathrm{Met}^{<0} \to \mathcal{T}(M)$. Where does the above free part of homotopy of $\mathrm{Diff}(M)$ live?

**DW:** Assume that $(M, g)$ is closed and Ricci flat. Is it true that $(M, g)$ admits a globally parallel harmonic spinor?

**TS, BB:** Denote by $\mathcal{R}^{>0}(M)$ ( $\mathcal{R}^{\geq 0}(M)$) the space of positive Ricci (non-negative Ricci) curvature metrics on $M$. When is $\mathcal{R}^{>0}(M) \subset \mathcal{R}^{\geq 0}(M)$ a (weak) homotopy equivalence? Comments: If $M$ does not admit Ricci flat metrics then the answer is positive, $\mathcal{R}^{>0}(M) = \mathcal{R}^{\geq 0}(M)$. Example: $M = N \# S^3 \times T^{d-3}$ where $N$ admits a positive Ricci metric.
For Calabi-Yau manifolds there exist isolated (finite dimensional?) islands of Ricci flat metrics and, hence, the answer is negative for such manifolds.

**WT:** Do Botvinnik-Gilkey results about about moduli spaces of positive scalar curvature metrics with non-trivial fundamental group (using $\eta$-invariant technique) generalize to the non-negative scalar curvature setting?

**TF, TS:** Recall that $D = \mathrm{Diff}_0(M)$ acts freely on $\mathrm{Met}^{<0}(M)$ which gives a principal locally trivial fiber bundle whose base is the Teichmuller/Moduli space $\mathcal{T}(M) = \mathrm{Met}^{<0}(M)/D$. Without the curvature restriction, $D$ does not act freely and non-trivial isotropy groups yield orbifold singularities in the $\mathcal{T}(M)$. What can be done in this case?

- Life is hard. Analyze singularities.
- Replace with "observer space" $\mathcal{M}_x(M) = \mathrm{Met}^*(M)/D_x$, where $D_x = \{\varphi \in D : \varphi(x) = x, D\varphi_x = Id\}$.
- Equip elements of $\mathrm{Met}^*(M)$ with orthonormal frames, then $D$-action becomes free. Issue: frame bundles of non-diffeomorphic bundles can become diffeomorphic?
- Study algebraic geometry. Use "stacky" quotient.
- Stabilize by multiplying by a flat torus or Bott manifold (or anything Ricci positive?). The index theory will be the same.

**TF:** Can one replace diffeomorphisms by homeomorphisms and Riemannian metrics by singular metrics and discuss moduli spaces in such a setting?

**MW:** Can one extend the result on non-triviality of rational homotopy in the stable range of the observer moduli space $\mathcal{M}_x^{Ric>0}(S^d)$ to other manifolds? What can be said about higher homotopy of the usual moduli space $\mathcal{M}^{Ric>0}(S^d)$. Comment: Probably extends to products of spheres and their connected sums, and also sphere bundles over bases with positive Ricci curvature.

**TS:** Look at the long exact sequence in homotopy

$$\pi_{i+1}(\mathcal{M}_x^{\geq 0}(M)) \otimes \mathbb{Q} \to \pi_i(D_x(M)) \otimes \mathbb{Q} \to \pi_i(\mathcal{R}^{>0}(M)) \otimes \mathbb{Q}$$

When $M = S^d$, Hsiang-Farrell stable rational homotopy lives in the moduli space. Can some of it, for some $M$, live in the space of metrics of positive Ricci curvature?

**TF:** Work of Farrell-Knopf-Ontaneda-Zhou implies that stable homotopy of $\mathrm{Diff}(S^d)$ lives in the space of $\frac{1}{4}$-pinched metrics on $S^d$ — $\mathrm{Met}^{(1/4,1)}(S^d)$. What can be said about the moduli space of positively curved metrics?

**TF, WT:** What can be said if one replaces $S^d$ by a complex projective space?

## Day 2

**AG:** (a well known open problem) Let $(S, g)$ be a non-positively curved surface of genus $\geq 2$. Equip $T^1S$ with the Liouville measure and let $K \subset T^1S$ be the closed set of rank 2 vector $v$; that, is the curvature along the geodesic through $v$ equals to 0. Prove that measure of $K$ is zero. Is it true that $K$ consists of finitely many flat bands of closed geodesics? Positive solution to this conjecture implies ergodicity of the geodesic flow by work of Pesin.

**TF, AG:** Assume that $M$ and $N$ are closed negatively curved manifolds with isomorphic fundamental groups. Is it true that their $k$-frame bundles are isomorphic? (Yes, when $k = 1, 2$ by Cheeger-Gromov remarks.)

**TF, AG:** Is the space of smooth conjugacy classes of Anosov diffeomorphisms a Hausdorff space (dim $> 2$)? Can Ebin's infinite dimensional analysis techniques be adapted to the Anosov setting?

**AG:** (in the spirit of M. Herman) Find a closed manifold $M$ which does not admit a metric of negative curvature, but which admits a metric whose geodesic flow is partially hyperbolic? Of course, a difficult open problem is to decide if there exist a closed manifold $M$ which does not admit a metric of negative curvature, but admits a metric whose geodesic flow is Anosov.

**MB:** Is $\mathrm{Met}^{<0}(M) \subset \mathrm{Met}^{Anosov}(M)$ a homotopy equivalence?

**MB, WT:** Let $M$ be genus $\geq 2$ surface. Is $\mathrm{Met}^{Anosov}(M)$ contractible? For example, consider an Anosov metric $g$ on a surface, is it true that the Ricci flow flows $g$ to negative curvature?

Day 3

**AD:** When $M$ is closed, spin of dimension $4k-1 \geq 7$ then (if not empty) the positive scalar curvature moduli space $\mathcal{M}^{>0}(M)$ has infinitely many connected components (Gromov-Lawson). Can the same be said about the moduli space of positive Ricci curvature?

**AD:** Are there connected components of $\mathcal{M}^{>0}(M)$ as above, which are different from the ones obtained by Gromov-Lawson?

**AD:** Let $\Sigma$ be a homotopy sphere. Is it true that the moduli space of metrics of non-negative sectional curvature $\mathcal{M}^{\mathrm{sec}\geq 0}(\Sigma)$ has infinitely many connected components? Comment: The answer is yes for 7-dimensional homotopy spheres which are $S^3$-bundles over $S^4$ via the Kreck-Stolz invariant.

**AD:** What can be said about connected components of moduli spaces of positively curved metric in dimensions $\neq 4n-1$?

**IB:** Let $V$ be a non-compact manifold. Consider the space $\mathcal{R}^{\geq 0}(V)$ of $C^\infty$ metrics on $V$ with non-negative sectional curvature and let $\mathcal{M}^{\geq 0}(V)/\operatorname{Diff}$. What is the right topology on $\mathcal{R}^{\geq 0}(V)$ and $\mathcal{M}^{\geq 0}(V)$? One would like to have the following properties:

- $\mathcal{R}^{\geq 0}(V)$ is metrizable and $\mathcal{M}^{\geq 0}(V)$ is metrizable at most points,
- continuous paths in $\mathcal{M}^{\geq 0}(V)$ can be lifted to $\mathcal{R}^{\geq 0}(V)$,
- "obviously continuous" paths (e.g. the path from the flat plane to a positively curved surface of revolution) are indeed continuous,
- convergence properties in $\mathcal{M}^{\geq 0}(V)$.

**IB:** As an application of work of Volkov one obtains an explicit description of the moduli space $\mathcal{M}^{>0}(S^2)$ of $\mathcal{R}^{>0}(S^2)$ and it has the homotopy type of a point. Can one determine homotopy types of $\mathcal{M}^{\geq 0}(S^2)$, $\mathcal{M}^{>0}(\mathbb{R}^2)$ and $\mathcal{M}^{\geq 0}(\mathbb{R}^2)$?
Comment: The explicit description of $\mathcal{M}^{\geq 0}(S^2)$ as quotient of centered convex bodies by $O(3)$ still works, the issue is that the straight line homotopy to the round metric may escape the space smooth metrics. This happens because in non-negative curvature the boundary of convex body realizing a smooth Riemannian metric can have non-smooth (just $C^1$?) points. Convex bodies realizing positively curved metrics on $\mathbb{R}^2$ are highly non-unique, hence, $\mathcal{M}^{\geq 0}(\mathbb{R}^2)$ doesn't have an analogous description.

Day 5

**SY:** By work of Kreck and Su Yang $S^1 \times \mathbb{C}P^3$ is determined by algebraic topology up to connected sum with certain exotic spheres (0,1,2,3 only). Can one distinguish the $S^1 \times \mathbb{C}P^3$ by geometry? For example if $M$ is homeomorphic to $S^1 \times \mathbb{C}P^3$ and it admits non-negative sectional curvature, then $M$ is diffeomorphic to $S^1 \times \mathbb{C}P^3$? Or, assume that $M$ viewed as the total space of the bundle $\mathbb{C}P^3 \to M \to S^1$ admits a fiberwise metric of positive sectional curvature varying smoothly, then is $M$ diffeomorphic to $S^1 \times \mathbb{C}P^3$?

**TF:** Equip $M$ above with a Riemannian metric and consider the harmonic map to the circle. Can this map be a fibration in the exotic cases?

**TS:** Distinguish homotopy types of

$$\mathcal{R}^?(M) \subset \mathcal{R}^{scal>0}(M) \subset \mathcal{R}^{\text{inv}}(M),$$

where $\mathcal{R}^{\text{inv}}(M)$ are metrics whose Dirac operators are invertable; especially in the case when $M$ is simply connected. Note that the Dirac operator methods do not seem to work. Also what about non-spin $M$?

**TF:** (work in progress) Is it possible to have the following composition rationally non-trivial

$$\pi_i \operatorname{Diff}(\mathbb{D}^n, \partial) \otimes \mathbb{Q} \simeq \pi_{i-1}\Omega \operatorname{Diff}(\mathbb{D}^n, \partial) \otimes \mathbb{Q} \to \pi_{i-1} \operatorname{Diff}(\mathbb{D}^{n+1}, \partial) \otimes \mathbb{Q}$$

A positive answer implies $\pi_i \mathcal{T}^{<0}(M) \otimes \mathbb{Q} \neq 0$.

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