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Set Theory

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Abstract. This workshop included selected talks on pure set theory and its applications, simultaneously showing diversity and coherence of the subject.

Mathematics Subject Classification (2010): 03E35, 03E55, 03E45, 03E17, 03E05.

Introduction by the Organisers

Forcing and inner model theory were introduced by Gödel and Cohen in their spectacular resolution of Hilbert's First Problem, Cantor's Continuum Hypothesis. Sophisticated refinements of these two methods, as well as applications of set theory, were the major themes of this successful and diverse workshop.

1. Inner Model Theory and Large Cardinals

In 1939 Gödel constructed the minimal inner model of set theory, L. Fine analysis of L and other inner models of set theory was initiated and lead by Jensen. Large cardinals form a transfinite hierarchy of axioms transcending the consistency strength of ZFC. These axioms (sometimes combined with forcing) provide the only means of resolving higher mathematical problems, such as Lebesgue measurability of projective sets of real numbers. Fine structure of inner models of large cardinals provides best known justification for consistency of these axioms. It depends on iterability of small transitive models of fragments of ZFC (the so-called 'mice'). While iteration strategies provide definable well-orderings of the reals, large cardinals provide counterbalance and imply Lebesgue measurability of definable sets of reals. The recent introduction of hybrid structures (known as 'HOD mice') by Woodin and Sargsyan is a technical breakthrough comparable to the 1980's Martin–Steel proof of Projective Determinacy and introduction of iteration trees for Woodin cardinals. In his talk Steel reported on the intriguing correlation between inner models of determinacy and HOD (the inner model of hereditarily ordinal definable sets). Sarsgyan talked about inner models close to the set-theoretic universe and the associated covering theorems. In his talk Cramer explored Woodin's AD-conjecture and I_0, possibly the strongest large cardinal principle known to man. In his talk Wilson proved that the assertions that every universally Baire set of reals has the perfect set property, and that every set of reals in $L(\mathbb{R}, \mathrm{uB})$ has the perfect set property have the same consistency strength.

Zeman talked about full iterability of the core model, and related topics were covered in talks by Chan and Zhu.

2. Applications

Some of the most successful talks demonstrated recent uses of set-theoretic methods in previously unexpected settings. A striking example of this was Viale's talk on the 1960s Schnauel's conjecture concerning the transcendence degree of field extensions of the rational numbers. Viale combined forcing with an absoluteness argument, and C*-algebras as spaces of names, to prove the existence of a countable field that satisfies Schnauel's conjecture. Foreman reported on joint work with B. Weiss in which they resolved von Neumann's isomorphism problem in ergodic theory in its original (and arguably the most important) formulation, for the Lebesgue measure-preserving diffeomorphisms of manifolds. Foreman and Weiss proved that a satisfactory classification in this context is *impossible*. This example of a necessary use of set theory will also be covered in 'Snapshots of modern mathematics from Oberwolfach.' Marks gave a completely constructive solution to Tarski's circle squaring problem, improving on an earlier equidecomposition theorem due to Laczkovich. The proof—as interesting as the result—used ideas from the study of flows in graphs, and a recent result of Gao, Jackson, Krohne, and Seward on the hyperfiniteness of free Borel actions of $\mathbb{Z}^d$. The concept of amenability for locally compact groups was introduced in late 1920s by von Neumann and has since permeated much of modern mathematics. Question of the amenability and structure of Thompson's groups has attracted attention of group theorists, operator algebraists, and set theorists alike. In his talk Moore provided a high lower bound for the complexity of the structure of finitely-generated subgroups of Thompson's group F.

3. Forcing

A special treat in this meeting was an opportunity to see two novel iterated forcing constructions. Our meeting started with an inspiring talk by Jensen, who isolated assumptions under which Namba forcing is subcomplete and thus obtained a new model of the Continuum Hypothesis. In his talk Raghavan reported on a new method for iterating forcing via boolean ultrapowers using a supercompact cardinal, and its application to cardinal characteristics. The latter topic was also explored in talks by Brendle and Horowicz. In his talk Krueger used forcing to prove that Shelah's approachability ideal $I[\kappa^+]$ consistently does not contain a maximal set modulo the club filter. Hrušák talked about forcing, filters, and a deep combinatorial property of trees, the Halpern-Läuchli theorem.

4. Combinatorics

The tree property of cardinals holds at $\aleph_0$ (known as König's lemma) and fails at $\aleph_1$ (witnessed by Aronszajn trees). While true for weakly compact cardinals, it fails for all other regular cardinals in Gödel's inner model L. In his talk Unger blended large cardinals and forcing to construct a model in which many regular cardinals have the tree property. Combinatorics of singular cardinals remains one of the most intriguing and deepest areas of set theory. It involves intricate blend of forcing and large cardinals. The main tools are the higher relative of Namba forcing, Prikry forcing, and its refinement, Magidor forcing. In her talk Sinapova presented a sophisticated diagonal extender-based supercompact Prikry forcing and used it to show that the power-set of a singular strong limit cardinal need not be included in HOD relativized to one of its subsets, providing a limiting example to a result of Shelah. In his talk Hayut used a variation on Prikry forcing and a supercompact cardinal to prove the relative consistency of new instances of the two-cardinal model-theoretic transfer principle, Chang's conjecture. New uses and properties of Prikry forcing were given in talks by Gitik and Koepke. The intricate connection between coloring and chromatic numbers of finite graphs is naturally even more intricate in infinite graphs. In his talk Rinot demonstrated how combinatorial principles such as $\square$ isolated in inner models affect these problems. Zapletal considered anticliques in closed graphs on $\mathbb{R}$. Soukup talked about surprising properties of uncountable linear orderings.

5. Topological Dynamics

This recurring theme in modern descriptive set theory was well-represented. Fraïssé limits are highly homogeneous structures generically constructed from finite substructures. In the recent years Fraïssé constructions were generalized to previously unexpected settings and used to analyze categories from Ramseyan categories to strongly self-absorbing C*-algebras. In his talk Solecki studied projective Fraïssé constructions and the associated homology theory, opening new alleys of research, and Nguyen Van Thé talked about the relation between Fraïssé constructions and Ramsey theory. Melleray and Tsankov talked about Polish group actions.

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Workshop: Set Theory

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Abstracts

On the subcompleteness of some Namba-type forcings

Ronald Jensen

We show that - under the assumption of CH and $2^{\omega_1} = \omega_2$ - two Nambda-like forcings $\mathbb{N}'$ and $\mathbb{N}^*$ are subcomplete.

$\mathbb{N}'$ is the set of Nambda trees with a finite stem $s = \text{stem}(T)$ such that for all $t \in T$ either $t = s \upharpoonright n$ for some $n < |s|$ or $t \supset s$ has ω_2 many immediate successors. The salient feature of $\mathbb{N}'$-generic sequences $\langle \gamma_i : i < \omega \rangle$ is that whenever $F : \omega_2 \to \omega_2$ is a function in the ground model then there is an $n \in \omega$ such that

$$F(\gamma_m) < \gamma_{m+1}$$

for all $m \geq n$.

$\mathbb{N}^*$ is defined like $\mathbb{N}'$ except that we impose the stronger requirement that if $t \in T$ and $t \supset \text{stem}(T)$ then

$$\{\alpha < \omega_2 : t^\frown \langle \alpha \rangle \in T\}$$

is stationary in ω_2. The salient feature of $\mathbb{N}^*$-generic sequences $\langle \gamma_i : i < \omega \rangle$ is that whenever $A \subseteq \omega_2$ is a club in the groundmodel then there is an $n \in \omega$ so that $\gamma_m \in A$ for all $m \geq n$. Both forcings add no reals assuming CH in the groundmodel.

$\mathbb{N}'$ has been treated extensively in the literature, especially [1]. In [2], we generalized Shelah's notion of "dee-complete" and "ω_1-proper" forcing to "dee-subcomplete" and "ω_1-subproper". We showed that under CH, $\mathbb{N}'$ had both properties and, therefore, can be iterated without adding reals. Quite recently we introduced the notion of "almost subcomplete forcing" and proved an iteration theorem. Assuming CH and $2^{\omega_1} = \omega_2$, we showed that $\mathbb{N}'$ and $\mathbb{N}^*$ are both almost subcomplete. We then belatedly realized that our proof showed $\mathbb{N}'$ and $\mathbb{N}^*$ to be, in fact, fully subcomplete.

In our talk, we will deal mainly with $\mathbb{N}^*$ though we shall briefly indicate the changes to be made in proving the same result for $\mathbb{N}'$. For details, we refer the reader to
`https://www.mathematik.hu-berlin.de/~raesch/org/jensen.html`.

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`https://www.mathematik.hu-berlin.de/~raesch/org/jensen.html`

Incompactness for chromatic number is compatible with many reflection principles

Assaf Rinot
(joint work with Chris Lambie-Hanson)

Introduction. A *graph* is a pair $G = (V, E)$, where $E \subseteq [V]^2$. The *neighborhood* of a vertex $x \in V$ is denote by $N_G(x) := \{y \in V \mid \{x, y\} \in E\}$. For any ordering $\lhd$ of V and $x \in V$, denote $N_G^{\lhd}(x) := \{y \in N_G(x) \mid y \lhd x\}$.

A *chromatic coloring* of G is a function $c : V \to \kappa$ satisfying $c(x) \neq c(y)$ for all $\{x, y\} \in E$. The *chromatic number* of G, denoted $\mathrm{Chr}(G)$, is the least cardinal κ for which such a chromatic coloring exists. The *coloring number* of G, denoted $\mathrm{Col}(G)$, is the least cardinal κ for which there exists a well-ordering $\lhd$ of V satisfying $|N_G^{\lhd}(x)| < \kappa$ for all $x \in V(G)$. Note that if $\lhd$ witnesses the value of $\mathrm{Col}(G)$, then one can obtain a chromatic coloring $c : V \to \mathrm{Col}(G)$ by straight-forward recursion over $(V, \lhd)$. That is, $\mathrm{Chr}(G) \leq \mathrm{Col}(G)$ for every graph G.

By a classic result of de Bruijn and Erdős, if G is a graph, k is a positive integer, and all finite subgraphs of G have chromatic number $\leq k$, then $\mathrm{Chr}(G) \leq k$. Questions involving generalizations of this theorem (to infinite cardinal numbers, as well as to other cardinal functions) have attracted a lot of attention. Let us list some key results:

Fact 1. *Compactness for the chromatic number:*

- *(essentially de Bruijn-Erdős, 1951) If χ is strongly compact, $\theta < \chi$, and G is a graph such that every subgraph of size $< \chi$ has chromatic number at most θ, then* $\mathrm{Chr}(G) \leq \theta$.
- *(Foreman-Laver, 1988) Relative to a large cardinal hypothesis, it is consistent with* GCH *that any graph of size and chromatic number $\aleph_2$ contains a subgraph of size and chromatic number $\aleph_1$.*
- *(Shelah, 1990) Relative to a large cardinal hypothesis, it is consistent with* GCH *that, whenever $1 \leq n < \omega$ and G is a graph such that every subgraph of size $< \aleph_\omega$ has chromatic number at most $\aleph_n$, it follows that G has chromatic number at most $\aleph_n$.*[1]
- *(Unger, 2015) Relative to a large cardinal hypothesis, it is consistent with* GCH *that, whenever $1 \leq \alpha < \omega_1$ and G is a graph such that every subgraph of size $< \aleph_{\omega_1}$ has chromatic number at most $\aleph_{\alpha+1}$, it follows that G has chromatic number at most $\aleph_{\alpha+1}$.*

Compactness for the coloring number:

- *(Shelah, 1975) If G is a graph on a singular cardinal λ, $\aleph_0 \leq \theta \leq \lambda$, and every subgraph of G of size $< \lambda$ has coloring number at most θ, then* $\mathrm{Col}(G) \leq \theta$.

[1] The case $n = 0$ remains open to this date.

- *(Magidor-Shelah, 1994) If χ is strongly compact, $\aleph_0 \le \theta < \chi$, and G is a graph such that every subgraph of size $< \chi$ has coloring number at most θ, then $\mathrm{Col}(G) \le \theta$.*

  Furthermore, relative to a large cardinal hypothesis, it is consistent that the above statement holds after replacing the strongly compact χ by the first cardinal fixed-point.
- *(Shelah, 1990's) Relative to a large cardinal hypothesis, it consistent that for every infinite θ, any graph of coloring number $> \theta$ has a θ^+-sized subgraph of coloring number θ^+.*
- *(Fuchino et al., 2012)* FRP *is equivalent to the assertion that any graph of uncountable coloring number has an $\aleph_1$-sized subgraph of uncountable coloring number.*

We say that a graph G is (θ, κ)*-chromatic* (resp. (θ, κ)*-coloring*) if $\mathrm{Chr}(G) = \kappa$ (resp. $\mathrm{Col}(G) = \kappa$) and $\mathrm{Chr}(G') \le \theta$ (resp. $\mathrm{Col}(G') \le \theta$) for every strictly smaller subgraph G' of G.

Fact 2. *Incompactness for the chromatic number:*

- *(Erdős-Hajnal, 1968) If $2^{\aleph_0} = \aleph_1$, then there is an $(\aleph_0, \aleph_1)$-chromatic graph of size $\aleph_2$.*
- *(Galvin, 1973) If $2^{\aleph_0} = 2^{\aleph_1} < 2^{\aleph_2}$, then there is an $(\aleph_0, \aleph_2)$-chromatic graph of size $(2^{\aleph_1})^+$.*
- *(Todorcevic, 1983) If κ is a regular uncountable cardinal and there exists a nonreflecting stationary subset of E^κ_ω, then there is an $(\aleph_0, \ge \aleph_1)$-chromatic graph of size κ.*
- *(Baumgartner, 1984)* GCH *is consistent with an $(\aleph_0, \aleph_2)$-chromatic graph of size $\aleph_2$.*
- *(Komjáth, 1988) $2^{\aleph_0} = \aleph_3$ is consistent with an $(\aleph_0, \aleph_2)$-chromatic graph of size $\aleph_2$.*
- *(Todorcevic, 1986) Martin's Axiom entails the existence of an $(\aleph_0, 2^{\aleph_0})$ graph of size $2^{\aleph_0}$.*
- *(Komjáth, 1988) $2^{\aleph_0} = \aleph_{\omega_1+1}$ is consistent with an $(\aleph_0, \aleph_1)$-chromatic graph of size $\aleph_{\omega_1}$.*
- *(Shelah, 1990)* GCH *is consistent with an $(\aleph_0, \aleph_1)$-chromatic graph of size $\aleph_{\omega_1}$.*
- *(Soukup, 1990) The existence of an $(\aleph_0, (2^{\aleph_0})^+)$-chromatic graph of size $(2^{\aleph_0})^+$ is consistent with arbitrarily large value for $2^{\aleph_0}$.*
- *(Shelah, 1990) If $V = L$, then (*GCH *holds, and) for every regular non-weakly compact cardinal κ, there is an $(\aleph_0, \kappa)$-chromatic graph of size κ.*
- *(Shelah, 2013) If $\theta < \kappa$ are regular cardinals, $\kappa^\theta = \kappa$, and there exists a non-reflectioning stationary subset of E^κ_θ, then there is a $(\theta, \ge \theta^+)$-chromatic graph of size κ.*
- *(Rinot, 2015) If λ is an infinite cardinal, $2^\lambda = \lambda^+$, and $\Box_\lambda$ holds, then there is an $(\aleph_0, \theta)$-chromatic graph of size λ^+ for all infinite $\theta \le \lambda$. If,*

additionally, λ is singular, then there is an $(\aleph_0, \lambda^+)$-chromatic graph of size λ^+.

Incompactness for the coloring number:

- *(Shelah, 1975) If $\theta < \kappa$ are infinite regular cardinals and there exists a non-reflecting stationary subset of E^κ_θ, then there is a (θ, θ^+)-coloring graph of size κ.*

The above list makes it clear that unlike the chromatic number (for which *large gaps* are known to be consistent), the only known consistency result for the coloring number is that of *gap 1*.

We do not know whether *gap 2* is consistent, but can prove it is an upper bound:

Question 1. *Is it consistent that for some infinite θ, there exists a (θ, θ^{+2})-coloring graph?*

Theorem 1 ([LHR16]). *For any infinite θ and $\alpha \geq 3$, there exists no $(\theta, \theta^{+\alpha})$-coloring graph.*

We can also show that certain instances of Chang's Conjecture give us situations in which the preceding is not sharp. For instance, if $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ holds, then for all infinite θ and $\alpha \geq 2$, there exists no $(\theta, \theta^{+\alpha})$-coloring graph of size $\leq \aleph_{\omega+1}$. Nevertheless, $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ is compatible with the existence of an $(\aleph_0, \aleph_1)$-coloring graph of size $\aleph_{\omega+1}$.

Main results. The main question that motivates our research reads as follows.

Question 2. *What is the relationship between compactness for the chromatic number and compactness for the coloring number?*

Our main result shows that incompactness for the chromatic number — even with very large gaps — is compatible with compactness for the coloring number (and hence with stationary set reflection). For this, we introduce the following variation of the square principle: $\square(\kappa, \sqsubseteq_\chi)$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha < \kappa \rangle$ satisfying the following:

- for every limit ordinal $\alpha < \kappa$, C_α is a club in α;
- for every $\alpha < \kappa$ and $\bar{\alpha} \in \mathrm{acc}(C_\alpha)$, if $\mathrm{otp}(C_\alpha) \geq \chi$, then $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$;
- for every club D in κ, there exists some $\alpha \in \mathrm{acc}(D)$ such that $D \cap \alpha \neq C_\alpha$.

Definition 1 ([Rin15]). *To any sequence $\vec{C} = \langle C_\alpha \mid \alpha < \kappa \rangle$, we attach a graph $G(\vec{C}) := (\mathrm{acc}(\kappa), E)$, by letting $\alpha E \beta$ iff $\min(C_\alpha) > \sup(C_\beta \cap \alpha) \geq \min(C_\beta)$ or $\min(C_\beta) > \sup(C_\alpha \cap \beta) \geq \min(C_\alpha)$.*

Theorem 2 ([LHR16]). *If $\vec{C}$ is a $\square(\kappa, \sqsubseteq_\chi)$-sequence, then $\mathrm{Chr}(G(\vec{C}) \restriction \alpha) \leq \chi$ for all $\alpha < \kappa$.*

Now, let $\mathcal{E}(\kappa, \chi)$ stand for the assertion that there exists a $\square(\kappa, \sqsubseteq_\chi)$-sequence $\vec{C}$ for which $\mathrm{Chr}(G(\vec{C})) = \kappa$. In [LHR16], it is proved that $\square(\lambda^+, \sqsubseteq_\chi) + \mathsf{GCH}$ entails $\mathcal{E}(\lambda^+, \chi)$ for every singular cardinal λ. In addition, we have the following consistency results.

Theorem 3 ([LHR16]). *Relative to various large cardinal hypotheses, the following are consistent:*

(1) $\mathsf{FRP}(\aleph_2)$ *together with* $\mathcal{E}(\aleph_2, \aleph_0)$*;*
(2) $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ *together with* $\mathcal{E}(\aleph_{\omega+1}, \aleph_0)$*;*
(3) RC / FRP / MM *together with* $\mathcal{E}(\kappa, \aleph_2)$ *holding for all regular* $\kappa > \aleph_2$*;*
(4) χ *is a supercompact cardinal together with* $\mathcal{E}(\kappa, \chi)$ *holding for all regular* $\kappa > \chi$*;*
(5) *Reflection of stationary subsets of* κ *together with* $\mathcal{E}(\kappa, \aleph_0)$*, where* κ *is the least inaccessible cardinal;*
(6) (a) $\Delta_{\aleph_{\omega^2}, \aleph_{\omega^2+1}}$ *together with* $\mathcal{E}(\aleph_{\omega^2+1}, \aleph_0)$*;*
(b) Δ_κ *together with* $\mathcal{E}(\kappa, \aleph_0)$*, where* κ *is inaccessible.*

To motivate Clause (6) of the preceding, we mention:

Fact 3 (Magidor-Shelah, 1994). *Suppose* $\theta < \chi \leq \kappa$ *are infinite cardinals such that* κ *is singular or* $\Delta_{\chi,\kappa}$ *holds. Then every graph of size* κ *and coloring number* $> \theta$ *has a strictly smaller subgraph of coloring number* $> \theta$*.*

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Projective Fraïssé limits and homology

Slawek Solecki

(joint work with Aristotelis Panagiotopoulos)

The point of view described below allows one to approach certain problems in topological dynamics and topology using purely combinatorial/model theoretic methods. Certain important compact topological spaces E are obtained as **canonical quotients** of a "generic" inverse limits of a family of finite structures. This procedure was first described in [5]. One defines a family of finite structures and a family of epimorphisms among them. We denote by $\mathcal{E}$ the family of epimorphsims assuming that the class of structures is remembered by $\mathcal{E}$. The language of the structures contains a distinguished binary relation symbol R. The family $\mathcal{E}$, in the situations of interest, fulfills the projective amalgamation property, that is, for $f_1 \colon B_1 \to A$ and $f_2 \colon B_2 \to A$ coming from $\mathcal{E}$, there exist $g_1 \colon C \to B_1$ and $g_2 \colon C \to B_2$ such that $f_1 \circ g_1 = f_2 \circ g_2$ in $\mathcal{E}$. (Note that the arrows are reversed when compared with the standard Fraïssé set-up.) One then produces the **projective Fraïssé limit**

$$\mathbb{E} = \varprojlim \mathcal{E}$$

by an infinite, generic amalgamation procedure analogous to the classical Fraïssé construction. The underlying set of the structure $\mathbb{E}$ carries a compact zero-dimensional topology. In the situations of interest, the interpretation of R in $\mathbb{E}$, $R^{\mathbb{E}}$, turns out to be a compact equivalence relation. Then the topological space E is the canonical quotient $\mathbb{E}/R^{\mathbb{E}}$. We often refer to $\mathbb{E}$ the pre-E. Several important compact, second countable spaces have been produced this way for natural combinatorially defined classes $\mathcal{E}$ [1], [3], [4], [5], [7].

Two aspects of this situation are significant to us:

— properties of the space $E = \mathbb{E}/R^{\mathbb{E}}$ and its homeomorphism group can be often established by transferring properties of $\mathbb{E}$ and its automorphism group; the latter situation is easier to handle because it is essentially combinatorial with the complications of the topology on E being translated into combinatorics of the zero-dimensional space $\mathbb{E}$ and the equivalence relation $R^{\mathbb{E}}$;

— the structure $\mathbb{E}$ is of independent interest, much like the structure of certain objects produced by applications of the standard Farïssé method, for example, the random graph or the Urysohn universal metric space..

The aim of the work described below is to develop the right notion of the simplex and the boundary operation for homology theory of projective Fraïssé limits and their quotients. A homology theory that is appropriate for projective Fraïssé limits needs to fulfill certain conditions: the simplexes should themselves be projective Farïssé limits; the epimorphisms in the projective Fraïssé class that define simplexes should be combinatorial; on the other hand, they should be coinitial in a larger class of epimorphisms that is flexible enough to make applications possible.

Recall that a **simplicial complex** C is a family of non-empty subsets of a set with the family assumed to be closed under taking non-empty subsets. The elements of the underlying set $V(C)$ are called **vertices** and the elements of the family are called **faces**. For two simplicial complexes C and D, a **simplicial map** $f\colon C \to D$ is a function from the vertices of C to the vertices of D that maps faces of C to faces of D. Recall that the **barycentric subdivision** of a simpicial complex C is a simplicial complex βC whose vertices are faces of C and whose faces are sets of faces of C that are linearly ordered by inclusion. For each simplicial map $f\colon C \to D$ there exists a simplicial map $\beta f\colon \beta C \to \beta D$ defined by $(\beta f)(s) = f[s]$. Note that s being a vertex of βC is a face of C, so taking its pointwise image $f[s]$ makes sense and gives a face of D, so a vertex of βD, as its result.

We fix a natural number n. We define a class $\mathcal{D}_n$ of structures and epimorphisms among them. Let Δ_n be the n-dimensional simplex, that is, Δ_n is a simplicial complex whose set of vertices has $n+1$ elements and whose family of faces consist of all non-empty subsets of the vertex set. Now, our class of finite structures consists of all barycentric subdivisions $\beta^k \Delta_n$ of Δ_n, $k \in \mathbb{N}$. The distinguished binary relation R holds on a pair of vertices if they belong to a face. A function $\delta\colon \beta^{k+1}\Delta_n \to \beta^k \Delta_n$, $k \in \mathbb{N}$, is called a **selection** if, for each face s of $\beta^{k+1}\Delta_n$, we have $\delta(s) \in s$. The class of epimorphisms in $\mathcal{D}_n$ is the smallest class of simplicial maps from some $\beta^l \Delta_n$ to $\beta^k \Delta_n$, for some $k, l \in \mathbb{N}$, that contains the identity map from Δ_n to Δ_n, contains all selections, and is closed under composition and operation β.

We have the following theorem.

Theorem 1 (Panagiotopoulos–S.). (i) *The class $\mathcal{D}_n$ is a projective Fraïssé class, that is, the projective amalgamation holds for $\mathcal{D}_n$.*
(ii) *The projective Fraïssé limit $\mathbb{D}_n$ of $\mathcal{D}_n$ is such that $\mathbb{D}_n/R^{\mathbb{D}_n}$ is homeomorphic to the geometric n-dimensional simplex.*

The next theorem identifies the broadest (in a precise sense not specified here) class of maps $\mathcal{H}_n$ in which $\mathcal{D}_n$ is coinitial. Recall that a function is called a **near-homeomorphism** if it is a uniform limit of homeomorphisms. A function f from a geometric n-dimensional simplex to itself is called a **restricted near-homeomorphism** if, for each face S (of each dimension $k \leq n$), f maps S to S and $f \restriction S\colon S \to S$ is a near-homeomorphism. We put a simplicial function from $\beta^l\Delta_n$ to $\beta^k\Delta_n$, for some $k, l \in \mathbb{N}$, in $\mathcal{H}_n$ if its geometric realization is a restricted near-homeomorphism.

Theorem 2 (Panagiotopoulos–S.). *The class $\mathcal{D}_n$ is coinitial in $\mathcal{H}_n$, that is, for each morphism f in $\mathcal{H}_n$ there exists a morphism g in $\mathcal{H}_n$ such that $f \circ g$ is in $\mathcal{D}_n$.*

The proof of the above theorem crucially uses the combinatorial/topological tools from stellar subdivision theory as exposed in [6].

The theorem above implies that $\mathcal{H}_n$ is a projective Fraïssé class whose limit is equal to $\mathbb{D}_n$. Class $\mathcal{H}_n$ appears to be flexible enough for applications we have in mind to Menger universal compacta [2].

Open problems. The immediate open problem is to investigate the interaction of homology coming from the notion of simplex as defined above with the operation of projective Fraïssé limit. There is a test case here: the development of the universal Menger compacta through projective Fraïssé limits. Jointly with Panagiotopoulos, we found projective Fraïssé classes that lead to projective Fraïssé limits whose canonical quotients appear to be the Menger compacta. A verification of this fact seems to require appropriate homology theory for projective Fraïssé limits. If successful this development would lead to a fully combinatorial treatment of Menger compacta and their homogeneity; see Bestvina's work [2].

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Halpern-Läuchli property of filters

Michael Hrušák
(joint work with Osvaldo Guzmán and David Chodounský)

We study the existence of *Sacks-indestructible* ultrafilters. To that end we consider the following related notions - idealized versions of the Halpern-Läuchli property according to Laver [2]: An ideal $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is

- *Halpern-Läuchli (HL)* if for every $c : 2^{<\omega} \to 2$ there is an $X \in \mathcal{I}^+$ and a perfect tree $T \subseteq 2^{<\omega}$ such that $c \restriction T \cap \bigcup_{n\in X} 2^n$ is constant, and
- *ω-Halpern-Läuchli (HL$^\omega$)* if for every $c : \oplus_{n\in\omega} 2^{<\omega} \to 2$ there is an $X \in \mathcal{I}^+$ and perfect trees $T_n \subseteq 2^{<\omega}$ such that $c \restriction \oplus_{n\in\omega} T_n \cap \bigcup_{n\in X} 2^n$ is constant, where
$$\oplus_{n\in\omega} T_n = \bigcup_{n\in\omega} \Pi_{i\leq n} T_i \cap 2^n.$$

We note that Sacks-indestructible ultrafilters *exist generically*, i.e. any filter of size $< \mathfrak{c}$ can be extended to a Sacks-indestructible ultrafilter if and only if the cardinal invariant $\mathfrak{hl} = \min\{\mathsf{cof}(\mathcal{I}) : \mathcal{I} \text{ is not HL}\}$ is equal to $\mathfrak{c}$. By analyzing the Halpern-Läuchli property in Borel ideals we prove $\mathfrak{d} \leq \mathfrak{hl} \leq \mathsf{cof}(\mathcal{N})$. In the process we note that

- Every P^+-ideal is HL.
- Every ideal extendible to an F_σ-ideal is HL
- nwd, Fin^α are HL
- (essentially due to Steprāns) $\mathcal{Z}$ is not HL.
- $tr(\mathcal{N}) = \{A \subseteq 2^{<\omega} : \{x \in 2^\omega : \exists^\infty n\ (x \restriction n \in A)\} \in \mathcal{N}\}$ is HL.
- If $\mathcal{U}$ is Sacks-idestructible ultrafilter then $\mathcal{Z} \not\leq_K \mathcal{U}^*$, i.e. Sacks-indestructible ultrafilter is a $\mathcal{Z}$-ultrafilter (in the sense of Baugartner [1]).

There is no difference between HL vs. HL$^\omega$ in all of these results, in particular:

- If a definable $\mathcal{I}$ is HL$^\omega$ then $\mathrm{Con}(\mathfrak{r}(\mathcal{P}(\omega)/\mathcal{I}) < \mathfrak{c})$.

We conclude with a list of related open problems:

- (A. Miller) Is there a Sacks-indestructible ultrafilter in ZFC?
- Is there (consistently) an ultrafilter which is Sacks-indestructible but not $\mathbb{S}^\omega$-indestructible?
- Is there a $\mathcal{Z}$-ultrafilter in ZFC?
- Is there a (Borel) ideal which is HL but not HL$^\omega$?
- Is $\mathfrak{d} < \mathfrak{hl}$ consistent?
- Is $\mathfrak{hl} < \mathsf{cof}(\mathcal{N})$ consistent?

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Uncountable strongly surjective linear orders

Daniel T. Soukup

The starting point of this project is the simple observation that if L, K are linear orders and $f : L \to K$ is order preserving and surjective then K embeds into L. In [7], the authors initiated the study of linear orders for which this implication is reversible:

Definition 1. *A linear order L is* strongly surjective *iff $L \twoheadrightarrow K$ for all $K \subseteq L$.*

That is, L maps onto any of its suborders in an order preserving way. It is easy to see that $\omega, -\omega$ or $\mathbb{Q}$ is strongly surjective but $\omega + 1$ is not. Indeed, suppose that $K \subseteq \mathbb{Q}$. Form the ordered sum $K^* = \sum_{k \in K} \mathbb{Q}$, that is blow up each point in K to a copy of the rationals. This linear order is countable, dense and has no endpoints (no matter what K is). So K^* is isomorphic to $\mathbb{Q}$; furthermore, K^* can be clearly mapped onto K by collapsing back the copies of $\mathbb{Q}$.

The main question we tackle is the following: are there any strongly surjective linear orders which are not countable? If yes, what are the possible order types of these linear orders?

It was proved in [7] already that, in the class of separable linear orders, this question is independent: if the Continuum Hypothesis holds then every separable, strongly surjective linear order is actually countable; if the Proper Forcing Axiom holds then any $\aleph_1$-dense set of real numbers is strongly surjective. In particular, uncountable strongly surjective linear orders may exist.

Our first result generalizes the above significantly; recall that a linear order L is Aronszajn iff L is short (contains no copies of ω_1 or $-\omega_1$) and contains no uncountable separable suborders. It is easy to see that any strongly surjective linear order is short.

Theorem 1. *Any strongly surjective linear order is Aronszajn if $2^{\aleph_0} < 2^{\aleph_1}$ or in the Cohen-model.*

Now, is it possible that there are strongly surjective Aronszajn orders? Baumgartner [2] showed that any Aronszajn order can be realized as a lexicographically ordered Aronszajn tree.

Theorem 2. *Under $\diamondsuit^+$, there is a strongly surjective, lexicographically ordered Suslin tree.*

In [2], Baumgartner constructs a minimal Suslin tree which gives such an example (this was already noted in [7]), in fact a lexicographically ordered Suslin tree which is doubly isomorphic to all large subtrees. By doubly isomorphic we mean that both the lexicographic and tree order are preserved by the maps. However, recently Hossein Lamei Ramandi pointed out that Baumgartner's crucial [2, Lemma 4.14] is false. Our proof fills this gap using a similar construction from [3].

Finally, it is possible that there are no uncountable strongly surjective linear orders at all.

Theorem 3. *Suppose that $2^{\aleph_0} = \aleph_1$ and axiom (A) holds. Then every strongly surjective linear order is countable.*

Axiom (A) was introduced by J. Moore [5] and says that any ladder system colouring on ω_1 can be uniformized on any Aronszajn tree T. CH together with (A) was used in [5] to show that consistently, the only minimal uncountable linear orders are ω_1 and $-\omega_1$. One can force axiom (A) from a model of CH by a countable support iteration of proper posets which do no add new reals (see [5] for details).

We believe that there are still plenty of interesting open problems in this topic. Some are listed below:

Problem 1. *Is it consistent that there is a strongly surjective linear order of size $> \aleph_2$?*

Such an order necessary has large real suborders so this problem could be as hard as proving that all κ-dense sets of reals are isomorphic for some $\kappa > \aleph_2$.

Problem 2. *Is it consistent that there is a strongly surjective linear order which contains no minimal (strongly surjective) linear orders?*

Let us mention that not every stronly surjective linear order is minimal:

Theorem 4. *Consistently, there is a strongly surjective suborder of $\mathbb{R}$ which is not minimal and not homogeneous.*

Indeed, one can make use of a model from [1] where $\mathrm{MA}_{\aleph_1}$ and OCA holds and there is an *increasing set of reals A* i.e. for any uncountable set of pairwise disjoint n-tuples from A (with $n \in \omega$), one can select $\langle a_i : i < n \rangle$ and $\langle b_i : i < n \rangle$ so that $a_i < b_i$ for all $i < n$.

Problem 3. *Does $MA_{\aleph_1}$ imply that there is an uncountable strongly surjective $L \subseteq \mathbb{R}$?*

We conjecture that the answer is no. To support this recall that $\mathrm{MA}_{\aleph_1}$ is consistent with the statement that every uncountable set of reals contains a 2-entangled set [1] while 2-entangled sets cannot be strongly surjective.

Problem 4. *Suppose that L is strongly surjective and $x \in L$. Is $L \setminus \{x\}$ strongly surjective?*

A particularly fascinating class of Aronszajn orders are the Countryman lines. L is Countryman if the poset L^2 can be covered by countably many chains. It is well known that Countryman lines are Aronszajn, exist in ZFC but cannot have Suslin suborders.

Problem 5. *Are there strongly surjective Countryman lines? What about under $MA_{\aleph_1}$ or PFA?*

Recall that under $\mathrm{MA}_{\aleph_1}$ every Countryman line is minimal and under PFA, each Aronszajn order contains a Countryman line [4]. Furthermore, Moore [6] contructs a linear order η_C from a Countryman line which is universal for Aronszajn orders under PFA.

Problem 6. *Is η_C strongly surjective under PFA?*

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Borel circle squaring

Andrew Marks
(joint work with Spencer Unger)

We give a completely constructive solution to Tarski's circle squaring problem. More generally, we prove a Borel version of an equidecomposition theorem due to Laczkovich. If $k \geq 1$ and $A, B \subseteq \mathbb{R}^k$ are bounded Borel sets with the same positive Lebesgue measure whose boundaries have upper Minkowski dimension less than k, then A and B are equidecomposable by translations using Borel pieces. This answers a question of Wagon. Our proof uses ideas from the study of flows in graphs, and a recent result of Gao, Jackson, Krohne, and Seward on special types of witnesses to the hyperfiniteness of free Borel actions of $\mathbb{Z}^d$.

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Woodin's AD-Conjecture for I_0

Scott Cramer

In this talk we discuss the definition, motivation, aspect of the proof and consequences of Woodin's AD-Conjecture for I_0. This conjecture comes out of the relationship between models of the Axiom of Determinacy (AD) and models of large cardinal axioms which exists, for instance, at the level of Woodin cardinals. In this case, there is a method for obtaining from a model of large cardinals, a model for AD, by considering the derived model, and there is a method for obtaining a model of large cardinals from a model of determinacy, by considering HOD of the determinacy model. One interpretation of Woodin's AD-Conjecture is that it gives evidence that one can continue to obtain strong determinacy models even for the strongest large cardinals known.

I_0 is the statement that there exists a nontrivial elementary embedding

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$$

with critical point below λ. This large cardinal is stronger than strong, supercompact, huge, I_1 and I_2, and is one of the strongest large cardinals not known to be inconsistent. The AD-Conjecture deals with the existence of a certain representation for subsets of $V_{\lambda+1}$ called a $U(j)$-representation, due to Woodin[1]. These representations are very similar to weakly homogeneously Suslin representations, which in the presence of large cardinals, allow one to obtain determined sets of reals. Hence it is natural to think that the existence of $U(j)$-representations is connected with the existence of models of determinacy. Our main theorem is the following.

Theorem 1 (C.)**.** *Assume I_0 holds at λ. Then every subset of $V_{\lambda+1}$ has a $U(j)$-representation in $L(V_{\lambda+1})$ and hence the AD-Conjecture holds at I_0.*

There are a number of consequences of this result which we highlight. First of all, in terms of models of determinacy we have the following.

Theorem 2 (C., Woodin)**.** *Suppose that λ is a limit of supercompact cardinals and there is a proper class of Woodin cardinals. Also assume that I_0 holds at λ. Let $G \subseteq Coll(\omega, \lambda)$ be V-generic. In $V[G]$, let Γ^∞_G be the set of universally Baire sets which are in $L(V_{\lambda+1})[G]$. Then*

(1) *$L(\Gamma^\infty_G) \models LSA$ (a strong determinacy axiom, which includes AD),*
(2) *$\Theta^{L(\Gamma^\infty_G)} = \Theta^{L(V_{\lambda+1})}$.*

The key point for showing (2) is that $\Theta^{L(\Gamma^\infty_G)} \not< \Theta^{L(V_{\lambda+1})}$.

One also obtains the consistency of I_0 with I_0 and the failure of SCH at λ (due independently to Woodin and Dimonte-Friedman).

The proof of the AD-Conjecture for I_0 yielded some interesting additional structure, and we obtained a new type of representation for subsets of $V_{\lambda+1}$ called a j-Suslin representation. These representations have a simple definition, which we can give.

Definition 1. *Fix $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$. $A \subseteq V_{\lambda+1}$ has a (j,κ)-Suslin representation T if the following hold for some $\vec{\lambda}$ increasing cofinal in λ.*

(1) *T is a tree on $V_\lambda \times L_\kappa(V_{\lambda+1})$ such that for all $(s,a) \in T$*

$$s = (s_0, \ldots, s_n), \qquad s_i \subseteq V_{\lambda_i}, \qquad s_i = s_{i+1} \cap V_{\lambda_i}.$$

(2) *'$A = p[T]$', that is, T is (basically) a tree representation for A.*
(3) *for all $(s,a) \in T$ there is an n such that for j_n the nth iterate of j,*

$$j_n(s,a) = (s,a).$$

(4) *for all s there exists an n such that $j_n(T_s) = T_s$.*

These representations do not follow directly from the existence of $U(j)$-representations, and they seem more analogous to Suslin representations in the context of I_0, as they do not involve the concept of a tree augmented with measures. Their existence in fact implies a generic absoluteness result which is slightly stronger than the generic absoluteness implied by $U(j)$-representations. We have the corresponding existence theorem for j-Suslin representations as well.

Theorem 3 (C.). *Assume I_0 holds at λ. Then every subset of $V_{\lambda+1}$ has a j-Suslin representation in $L(V_{\lambda+1})$.*

Whether these results can be extended beyond throughout the E^0_α-hierarchy, is still open.

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Least branch hod pairs

John R. Steel

We describe some results on the structure of HOD in models of the axiom of determinacy. Our results apply to models which have not reached iteration strategies for mice with long extenders.

Definition 1. *"No long extenders" (*NLE*) is the assertion: there is no ω_1 iteration strategy for a countable pure extender mouse with a long extender on its sequence.*

We show that below long extenders, there is a simple general notion of *least branch hod pair*, and a general comparison theorem for them. They have a fine structure. *Modulo the existence of iteration strategies*, they can be used to analyze HOD, and they can have subcompact cardinals:

Theorem 1. [1] *Suppose that κ is supercompact, and there are arbitrarily large Woodin cardinals. Suppose that V is uniquely iterable above κ; then*

(1) *for any $\Gamma \subseteq \mathrm{Hom}_\infty$ such that $L(\Gamma,\mathbb{R}) \models \mathsf{NLE}$, $\mathrm{HOD}^{L(\Gamma,\mathbb{R})} \models \mathsf{GCH}$, and*
(2) *there is a $\Gamma \subseteq \mathrm{Hom}_\infty$ such that $\mathrm{HOD}^{L(\Gamma,\mathbb{R})} \models$ "there is a subcompact cardinal".*

The large cardinal and iterability hypotheses on V are used to show that there are enough hod pairs that HOD is their direct limit. "Enough" is made precise by:

Definition 2. *(AD^+)* HOD pair capturing *(*HPC*) is the statement: for every Suslin, co-Suslin set of reals A, there is an lbr hod pair (P,Σ) such that A is Wadge reducible to Code(Σ).*

HPC is a cousin of Sargsyan's "Generation of full pointclasses". It holds in the minimal model of $\mathsf{AD}_\mathbb{R}$ + "θ is regular", and beyond, by Sargsyan's work.

Theorem 2. [2, 4] *Assume there is a supercompact cardinal, and arbitrarily large Woodin cardinals. Suppose V is uniquely iterable. Let $\Gamma \subseteq \mathrm{Hom}_\infty$ be such that $L(\Gamma, \mathbb{R}) \models \mathsf{NLE}$; then $L(\Gamma, \mathbb{R}) \models \mathsf{HPC}$.*

Theorem 3. *Assume $\mathsf{AD}^+ + V = L(P(\mathbb{R})) + \mathsf{HPC}$; then $\mathrm{HOD}|\theta$ is an least branch premouse. Thus $\mathrm{HOD} \models \mathsf{GCH}$.*

The main open problem is, as it has been for a long time, how to eliminate the iterability hypothesis. This would be done by a proof of the following conjecture:

Conjecture. $(\mathsf{AD}^+ + \mathsf{NLE}) \Rightarrow \mathsf{HPC}$.

We have two results of a general nature on this conjecture. First, HPC localizes:

Theorem 4. [2] *Assume $\mathsf{AD}^+ + \mathsf{HPC}$, and let $\Gamma \subseteq P(\mathbb{R})$; then $L(\Gamma, \mathbb{R}) \models \mathsf{HPC}$.*

The key to localization of HPC is to compute optimal Suslin representations for the iteration strategies in lbr hod pairs. This suggests a proof of the conjecture that goes by induction on scaled pointclasses. It turns out that crossing gaps in scales is not the problem:

Theorem 5. [4] *Assume AD^+, and let Γ be an inductive-like pointclass with the scale property. Suppose that the iteration strategies of lbr hod pairs are Wadge cofinal in $\Gamma \cap \check{\Gamma}$, and that all sets in $\check{\Gamma}$ are Suslin; then there is an lbr hod pair (P, Σ) such that Code(Σ) is not in Γ.*

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The higher sharp revisited

Yizheng Zhu

The collection of projective sets of reals is the smallest one containing all the Borel sets and closed under complements and continuous images. The Axiom of Projective Determinacy (PD) is the correct axiom that settles the regularity properties of projective sets. For instance, PD implies that every projective set is Lebesgue measurable, has the Baire property and is either at most countable or equinumerous with the continuum. Further exploration of PD relies on tools from both inner model theory and higher-level descriptive set theory.

Assuming $\boldsymbol{\Delta}^1_{2n+1}$-determinacy, Moschovakis shows that there is a Π^1_{2n+1}-norm φ on a good universal Π^1_{2n+1} set G onto $\boldsymbol{\delta}^1_{2n+1}$. We define the universal Σ^1_{2n+2} subset of $\boldsymbol{\delta}^1_{2n+1}$ by

$$\mathcal{O}_{\Sigma^1_{2n+2}} = \{\langle \theta, \alpha \rangle : \theta \text{ is a } \Sigma^1_{2n+2} \text{ formula}, \exists w (\varphi(w) = \alpha \wedge \theta(w))\}.$$

The canonical structure associated to this level is $L_{\boldsymbol{\delta}^1_{2n+1}}[\mathcal{O}_{\Sigma^1_{2n+2}}]$. It does not depend on the choice of φ and G by Moschovakis. Steel gives an inner-model-theoretic meaning of this structure: Let $M_{k,\infty}$ be the direct limit of all the iterates of M_k via countable iteration trees and let $\delta_{k,\infty}$ be the least Woodin of $M_{k,\infty}$. Then $\boldsymbol{\delta}^1_{2n+1}$ is the least $< \delta_{2n,\infty}$-strong cardinal in $M_{2n,\infty}$ and $L_{\boldsymbol{\delta}^1_{2n+1}}[\mathcal{O}_{\Sigma^1_{2n+2}}] = M_{2n,\infty}|\boldsymbol{\delta}^1_{2n+1}$.

In this talk, we show that a similar pattern holds at odd levels. Let WO_ω be standard coding set of ordinals in u_ω, i.e., the set of $\langle \tau, x^\# \rangle$ for which τ is a Skolem term for a ordinal. The ordinal coded by $\langle \tau, x^\# \rangle$ is $|\langle \tau, x^\# \rangle| = \tau^{L[x]}(x, u_1, \ldots, u_k)$, where τ is $k+1$-ary. We define the universal Σ^1_3 subset of u_ω by

$$\mathcal{O}_{\Sigma^1_3} = \{\langle \theta, \alpha \rangle : \theta \text{ is a } \Sigma^1_3 \text{ formula}, \exists w \in WO_\omega(|w| = \alpha \wedge \theta(w))\}.$$

The canonical structure associated to this level is $L_{u_\omega}[\mathcal{O}_{\Sigma^1_3}]$. It is equal to $L_{\kappa_3}[T_2] \cap V_{u_\omega}$ by Q-theory, where $L_{\kappa_3}[T_2]$ is the admissible closure of the Martin-Solovay tree projecting to Π^1_2. We show that this structure is also a canonical model in inner model theory:

Theorem 1. *Assume $\boldsymbol{\Delta}^1_3$-determinacy. Then $M_1|\delta_{1,\infty} = L_{u_\omega}[\mathcal{O}_{\Sigma^1_3}]$.*

Theorem 1 generalizes to arbitrary odd levels with $M_{2n-1,\infty}$ and $\mathcal{O}_{\Sigma^1_{2n+1}}$ by Zhu [1]. A natural question on the uniqueness of $L_{u_\omega}[\mathcal{O}_{\Sigma^1_3}]$ arises:

Question 1. *Let $\vec{\varphi}$ be a Δ^1_3-scale on a good universal Π^1_2 set such that each φ_i is $\Game(<\omega^2\text{-}\Pi^1_1)$. Let $\mathcal{O}_{\Sigma^1_3,\vec{\varphi}}$ be the universal Σ^1_3 subset of u_ω relative to $\vec{\varphi}$. Does $L_{u_\omega}[\mathcal{O}_{\Sigma^1_3}] = L_{u_\omega}[\mathcal{O}_{\Sigma^1_3,\vec{\varphi}}]$?*

Based on Theorem 1 and its generalization, we are able to prove the existence of the mouse $M^\#_{2n}$ from determinacy principles:

Theorem 2. *Assume that both boldface $\boldsymbol{\Delta}^1_{2n+1}$ and lightface Σ^1_{2n+1} games are determined. Then there is a countable iterable $M^\#_{2n}$.*

We do not know if $\boldsymbol{\Delta}^1_{2n+1}$ can be replaced by $\boldsymbol{\Sigma}^1_{2n}$ in Theorem 2. On the other hand, Zhu [2] proves the existence of a countably iterable $M^\#_{2n+1}$ from the determinacy of boldface $\boldsymbol{\Sigma}^1_{2n+1}$ and lightface Σ^1_{2n+2} games.

Assuming $\boldsymbol{\Delta}^1_2$-determinacy, Woodin shows that for a Turing cone of x, $\omega_2^{L[x]}$ is a Woodin cardinal in $HOD^{L[x]}$. The internal structure of $HOD^{L[x]}$ has been a mystery. In particular, we don't know whether $HOD^{L[x]}$ is a model of GCH. The proof of Theorem 1 includes the following result which suggests a candidate for $HOD^{L[x]}$.

Theorem 3. *Assume $\boldsymbol{\Delta}^1_3$-determinacy. There is an iterate N of M_1 via a countable iteration tree with the following properties: Let $\pi_{N,\infty} : N \to M_{1,\infty}$ be the direct limit of iteration maps and let δ^N be the Woodin of N. Then there is $(\gamma_n : n < \omega)$, cofinal in δ^N, such that each $\pi_{N,\infty} \restriction \gamma_n \in N$ and is definable over N from parameters in $\gamma_n \cup \{u_1, \ldots, u_n\}$.*

Let N be as in Theorem 2. Let π^*_n be defined over N by the same formula as $\pi_{N,\infty} \restriction \gamma_n$ but replacing the parameters $(u_1, \dots, u_n)$ by the first n indiscernibles above δ^N. Let $\pi^* = \bigcup_{n<\omega} \pi^*_n$. Then π^* is a cofinal map from δ^N to the ω-th indiscernible of N above δ^N. $L[N|\delta^N, \pi^*]$ is our candidate model.

Question 2. *Is $L[N|\delta^N, \pi^*]$ elementarily equivalent to $HOD^{L[x]}$ for a cone of x?*

A positive answer to Question 2 would suggest a form of the least non-trivial hybrid mouse: It could be the structure obtained from adding into M_1 a certain cofinal map from the Woodin of M_1 to the ω-th indiscernible of M_1.

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On metrizable universal minimal flows

Todor Tsankov

Let G be a topological group. A *G-flow* is a continuous action of G on a compact Hausdorff space. A flow is *minimal* if it has no proper subflows, or equivalently, if every orbit is dense. It is an old result of Ellis that there exists a minimal G-flow that maps onto every other minimal flow of the group and that this universal property characterizes the flow up to isomorphism. This flow is called the *universal minimal flow (UMF)* of G and will be denoted by $M(G)$. For some groups G (for example discrete or locally compact), $M(G)$ is not metrizable and does not admit a concrete description. However, for many "large" Polish groups, the UMF is metrizable, can be computed, and carries interesting combinatorial information.

The group G is called *extremely amenable* if $M(G)$ is reduced to a point, or equivalently, if every G-flow has a fixed point. It turns out that many large Polish groups are extremely amenable: for example, the unitary group of an infinite-dimensional Hilbert space (Gromov–Milman), the automorphism group of the order $(\mathbf{Q}, <)$ of the rationals, or the orientation-preserving homeomorphisms of the reals (Pestov [6]). The proof for the latter two groups uses the classical Ramsey theorem; this approach was generalized by Kechris, Pestov, and Todorcevic [4], who found a precise correspondence between *structural Ramsey theory* on the one hand and extreme amenability of automorphism groups on the other.

One can use large extremely amenable subgroups to construct more interesting metrizable UMFs as follows. Let G be a Polish group and $H \leq G$ be a closed subgroup. We equip the homogeneous space G/H with the quotient of the right uniformity of G whose entourages are given by

$$\mathcal{U}_V = \{(vgH, gH) : g \in G, v \in V\},$$

where V varies over symmetric neighborhoods of 1_G. We say that H is *co-precompact* in G if the uniform space G/H is precompact; in that case, we denote by $\widehat{G/H}$ its compact completion. Then $G \curvearrowright \widehat{G/H}$ is a G-flow that has the following universal property: whenever $G \curvearrowright X$ is a G-flow and $x_0 \in X$ is such that $H \cdot x_0 = x_0$, there is a natural G-map $\widehat{G/H} \to X$ that sends the coset H to x_0. In particular, if H is extremely amenable, any minimal subflow of $\widehat{G/H}$ is isomorphic to $M(G)$. This construction is due Pestov.

Two easy examples where this applies are given by

$$M(S_\infty) = S_\infty / \widehat{\mathrm{Aut}(\mathbf{Q}, <)} = \mathrm{LO},$$

the compact space of linear orderings of a countable set, and

$$M(\mathrm{Homeo}^+(S^1)) = \mathrm{Homeo}^+(S^1)/\mathrm{Homeo}^+(\mathbf{R}) = S^1.$$

The first of these is due Glasner and Weiss [3] (by a different method), and the second to Pestov.

Our main theorem states that this construction is the only way to produce metrizable universal minimal flows. It is joint work with Itaï Ben Yaacov, Julien Melleray, and Lionel Nguyen Van Thé and is obtained as the union of the results of the two papers [5] and [1].

Theorem 1. *Let G be a Polish group. Then the following are equivalent:*

(1) *$M(G)$ is metrizable;*
(2) *There exists a closed, co-precompact, extremely amenable subgroup $H \leq G$ such that $M(G) = \widehat{G/H}$.*

A version of Theorem 1 where G is assumed to be a closed subgroup of S_∞ was proved by Zucker [7].

The main open problem that remains is to find a general sufficient condition on the group G, that can be verified more easily than item 2 above, which implies that $M(G)$ is metrizable. A natural candidate for such a condition was Roelcke precompactness (cf. [2, Problem 30] and [5, Question 1.4]), however this was refuted by Evans. It is open whether if G is the automorphism group of a countable structure homogeneous in a finite relational language, $M(G)$ must necessarily be metrizable [2, Problem 29].

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Anticliques in closed graphs

Jindrich Zapletal
(joint work with Francis Adams)

Let G be a graph on a Polish space X. Write $\mathtt{non}(G)$ for the smallest possible size of a subset of X which cannot be covered by countably many compact G-anticliques. We propose to characterize the graphs for which $\mathtt{non}(G)$ is consistently smaller than the bounding number $\mathfrak{b}$.

In order to limit the problem to manageable size, we consider only closed graphs on compact spaces. For those, there is a canonical poset that adds a large compact G-anticlique:

Definition 1. *The poset P_G consists of conditions $p = \langle a_p, o_p\rangle$ where $a_p \subset X$ is a finite G-anticlique and o_p is a function assigning to each $x \in a_p$ an open set $o_p(x)$ containing x such that if $x \neq y \in a_p$ implies $o_p(x) \times o_p(y) \cap G = 0$. The ordering is defined by $q \leq p$ if $a_p \subset a_q$ and for each $x \in a_q$ there is $y \in a_p$ such that $o_q(x) \subset o_p(y)$.*

It is not difficult to see that if $H \subset P_G$ is a generic filter, then the closure of $\bigcup_{p\in G} a_p$ is a compact G-anticlique. The question now becomes, how do combinatorial properties of the graph G influence the forcing properties of the poset P_G?

In order to answer this question, we introduce several properties of graphs, some old, some new:

Definition 2. *Let G be a graph on a Polish space X.*

(1) *The chromatic number $\chi(G)$ is the smallest number of G-anticliques covering X;*
(2) *The loose number $\lambda(G)$ is the smallest number of G-loose sets covering X. Here, a set $A \subset X$ is G-loose if every point $x \in X$ has a neighborhood containing no elements of A connected with x.*
(3) *The coloring number $\theta(G)$ is the smallest cardinal such that G has an orientation in which the outflow of all points has size less than θ.*

It is not diificult to show that $\chi(G) \leq \lambda(G) \leq \theta(G)$ holds, and there are elementary examples of closed graph on compact spaces that show that the inequalities cannot be reversed. It is not easy to evaluate the cardinals $\chi(G)$ and $\lambda(G)$ even for closed graphs G, and it is not clear if the status of $\chi(G) \leq \aleph_0$ or $\lambda(G) \leq \aleph_0$ is absolute among models of ZFC containing the code for the graph G. On the other hand, the status of $\theta(G) \leq \aleph_0$ is fairly easy to check, it is a coanalytic question about the graph G. Some examples are described in the following:

Example 1. (1) *Every locally countable graph has $\theta(G) \leq \aleph_0$.*

(2) *Every acyclic graph has* $\theta(G) \leq \aleph_0$.
(3) *Let* $n < 4$, *let* d *be the usual metric on* $[0,1]^n$, *let* $a \subset \mathbb{R}^+$ *be a set with* 0 *as its only accumulation point, and let* G *connect elements* $x, y \in [0,1]^n$ *if* $d(x,y) \in a$. *Then* $\theta(G) \leq \aleph_0$.
(4) *In the previous example, if* $n \geq 4$ *is a natural number, then* $\lambda(G) \leq \aleph_0$ *but* $\theta(G) > \aleph_0$.
(5) *In the previous example, if one deals with the Hilbert cube and arbitrary compatible metric* d *on it, then* $\chi(G) > 0$.

We also introduce a useful property of c.c.c. forcing notions.

Definition 3. *Let* P *be a poset.*

(1) *A set* $A \subset P$ *is* liminf-centered *if for every infinite subset* $B \subset A$ *there is a condition* $p \in P$ *forcing that infinitely many elements of* B *belong to the generic filter.*
(2) *The poset* P *is* σ-LIP *if it is a union of countably many liminf-centered sets.*

Zapletal and Chodounsky proved that posets with this property do not add dominating reals, and their finite support iterations maintain this feature as well. σ-LIP is similar to σ-centeredness, but there are no implications between the two notions: the random forcing is σ-LIP but not σ-centered, and the Hechler forcing is σ-centered but not σ-LIP.

Theorem 1. *Let* G *be a closed graph on a compact metrizable space* X.

(1) *(Todorcevic)* P_G *is c.c.c. iff* G *contains no perfect cliques;*
(2) *(Todorcevic)* P_G *is* σ*-centered iff* $\chi(G) \leq \aleph_0$;
(3) P_G *is* σ*-LIP iff* $\lambda(G) \leq \aleph_0$.

Corollary 1. *If* G *is an locally countable or acyclic closed graph on a compact space, then in some c.c.c. extension* $\mathfrak{b} < \texttt{non}(G)$ *holds.*

The AIM forcing

Dima Sinapova

(joint work with James Cummings, Sy Friedman, Menachem Magidor and Assaf Rinot)

The results were obtained during a SQuaRE program at the American Institute of Mathematics (AIM) in San Jose, CA.

It is a familiar phenomenon in the study of singular cardinal combinatorics that singular cardinals of countable cofinality can behave very differently from those of uncountable cofinality. For example, we may contrast Silver's theorem [3] with the many consistency results producing models where GCH first fails at a singular cardinal of countable cofinality [2]. We show that there is a similar sharp dichotomy involving questions about the definability of subsets of a cardinal rather than the size of its powerset.

A remarkable result by Shelah [4] states that if κ is a singular strong limit cardinal of uncountable cofinality then there is a subset $x \subseteq \kappa$ such that $P(\kappa) \subseteq \mathrm{HOD}_x$. We develop a version of diagonal extender-based supercompact Prikry forcing, and use it to show that singular cardinals of countable cofinality do not in general have this property [1]:

Theorem 1. *Suppose that $\kappa < \lambda$ where $\mathrm{cf}(\kappa) = \omega$, λ is inaccessible and κ is a limit of λ-supercompact cardinals. There is a forcing poset $\mathbb{P}$ such that if G is $\mathbb{P}$-generic then:*

- *The models V and $V[G]$ have the same bounded subsets of κ.*
- *Every infinite cardinal μ with $\mu \leq \kappa$ or $\mu \geq \lambda$ is preserved in $V[G]$.*
- $\lambda = (\kappa^+)^{V[G]}$.
- *For every $x \subseteq \kappa$ with $x \in V[G]$, $(\kappa^+)^{\mathrm{HOD}_x} < \lambda$.*

From stronger assumptions we can use $\mathbb{P}$ to obtain a model in which κ is a singular strong limit cardinal of cofinality ω, and κ^+ is supercompact in HOD_x for all $x \subseteq \kappa$.

The idea is to use many supercompact measures to add ω sequences through $\mathcal{P}_{\kappa_n}(\alpha)$ for unboundedly many $\alpha < \lambda$. To do this we define an *extender based diagonal supercompact Prikry forcing.*

Let $\kappa < \lambda$ with λ strongly inaccessible and $\kappa = \sup_n \kappa_n$, where $(\kappa_n)_{n<\omega}$ is a strictly increasing sequence of λ-supercompact cardinals. Fix U_n a κ_n-complete fine normal ultrafilter on $P_{\kappa_n}\lambda$, and for $\kappa \leq \alpha < \lambda$ let $U_{n,\alpha}$ be the projection of U_n to $P_{\kappa_n}\alpha$ via the map $x \mapsto x \cap \alpha$. We will use the sequence of measures $\langle U_{n,\alpha} \mid n < \omega\rangle$ to add sequences $\langle x_n^\alpha \mid n < \omega\rangle$ for many α's below λ. More precisely, we define a forcing $\mathbb{P}$, such that if G is $\mathbb{P}$-generic, then G adds:

- An unbounded $F \subset \lambda$, and
- For all $\alpha \in F$, a $\subset$-increasing sequence $\langle x_n^\alpha \mid n < \omega\rangle$ with union α, witnessing that α is collapsed to κ.
- For $\alpha < \beta$ both in F, for all large n, $x_n^\alpha = x_n^\beta \cap \alpha$.

Also, for each $\alpha \in F$, there is forcing poset $\mathbb{Q}_\alpha$, which adds precisely the sequence $\langle x_n^\alpha \mid n < \omega\rangle$. The main forcing $\mathbb{P}$ projects to $\mathbb{Q}_\alpha$ and $\mathbb{P}/\mathbb{Q}_\alpha$ is homogeneous.

Let us give the actual definition of the main forcing. Conditions in $\mathbb{P}$ are of the form:

$$p = \langle f_0, ..., f_{n-1}, \langle a_n, A_n, f_n\rangle, \langle a_{n+1}, A_{n+1}, f_{n+1}\rangle, ...\rangle,$$

where

- each f_k is a function with
  - $\mathrm{dom}(f_k) \subset [\kappa, \lambda)$ of size less than λ, and
  - each $f_k(\eta) \in \mathcal{P}_{\kappa_k}(\eta)$,
- each $a_k \subset [\kappa, \lambda)$ of size less than λ and is disjoint from $\mathrm{dom}(f_k)$,
- A_k is a measure one set in $U_{k,\max(a_k)}$,
- $a_n \subset a_{n+1} \subset ...$

The forcing has the λ^+ c.c. and the Prikry property, i.e. for any p, and ϕ, there is $q \leq p$ with the same length, such that q decides ϕ. It follows that no new bounded subsets of κ are added and that cardinals greater than or equal to λ are preserved. Then in the forcing extension, λ becomes the successor of κ. Moreover, for any subset $x \subset \kappa$ in the generic extension, there some $\alpha \in F$, such that x is in the extension by $\mathbb{Q}_\alpha$. This together with homogeneity of $\mathbb{P}/\mathbb{Q}_\alpha$ gives that $(\kappa^+)^{\mathrm{HOD}_x} < \lambda$.

We conclude with the following open problems:

- Can we obtain the above for every singular cardinal of countable cofinality on an interval?
- Can we interleave collapses to make κ be $\aleph_\omega$ or $\aleph_{\omega^2}$? The main obstacle is the big gap between κ and λ.
- Can we combine this forcing with simultaneously adding subsets of κ?

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Towards a model where every regular cardinal greater than $\aleph_1$ has the tree property

Spencer Unger

A tree is a wellfounded partial order where the predecessors of every point are linearly ordered. The height of a node in a tree is just the order type of the set of its predecessors. For an ordinal α, the α^{th} level of a tree is the set of nodes with height α. The height of a tree is the least α for which the α^{th} level is empty.

An old theorem of König states that every tree of height $\aleph_0$ with finite levels has a *cofinal branch*, that is, a linearly ordered subset whose order type is the height of the tree. The analogous theorem for $\aleph_1$ is false. A construction due to Aronszajn shows that there is a tree of height $\aleph_1$ with countable levels and no cofinal branch.

König's theorem is an instance of the compactness of $\aleph_0$ and Aronszajn's construction shows that $\aleph_1$ is not compact in the analogous sense. These notions generalize in the obvious way to larger cardinals. For a cardinal μ we call a tree of height μ with levels of size less than μ and no cofinal branch, a μ-Aronszajn tree. The natural question is "Which cardinals carry Aronszajn trees?".

Some early theorems provide connections with the generalized continuum hypothesis and large cardinals. Specker showed that if $\kappa^{<\kappa} = \kappa$, then there is a special κ^+-Aronszajn tree. Here special means that there is a function $f : T \to \kappa$ such that $s < t$ implies $f(s) \neq f(t)$. On the other hand, the nonexistence of μ-Aronszajn trees is connected with large cardinals. Theorems of Erdös and Tarski, and Monk and Scott show that μ is inaccessible and has no μ-Aronszajn trees if and only if μ is weakly compact.

The modern tool of forcing provides a method for resolving the above question. The first theorem in this direction is due to Mitchell who proved that the theory ZFC + "there is a weakly compact cardinal" is consistent if and only if the theory ZFC + "there are no $\aleph_2$-Aronszajn trees" is consistent. In particular, modulo the consistency of a weakly compact cardinal one cannot prove in ZFC that there is an $\aleph_2$-Aronszajn tree.

To give a complete answer to the above question, we seek to produce a model where for every regular cardinal μ greater than $\aleph_1$ there are no μ-Aronszajn trees. Of course the existence of such a model will require the consistency of large cardinals. Significant partial progress has been made towards such a model. We refer the reader to [4] and [5] for a summary of known results.

We present the following partial result:

Theorem 1 (U. [6]). *Modulo the consistency of large cardinals, there is a model of ZFC where there are no special Aronszajn trees for every regular cardinal in the interval* $[\aleph_2, \aleph_{\aleph^2+3}]$ *and* $\aleph_{\aleph^2}$ *is strong limit.*

The following question looks quite difficult.

Question 1. *Can the above result be improved to "no Aronszajn trees"?*

Motivated by [1], we ask:

Question 2. *Is there a Radin forcing which adds a club* $C \subseteq \kappa$ *such that for all cardinals* $\mu \in C$*, there are no special* μ^+*-Aronszajn trees?*

Motivated by [2], we ask:

Question 3. *Is it consistent that for all limit* $\alpha < \omega_1$*, there are no* $\aleph_{\alpha+1}$*-Aronszajn trees?*

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New results and open problems on the non-existence of mad families

Haim Horowitz
(joint work with Saharon Shelah)

As the axiom of choice is needed for the construction of mad families, we may regard them as pathological sets of reals and investigate models of ZF where no mad families exist. The first result along this line was obtained by Mathias, who proved the following:

Theorem 1. [5] *There are no mad families in the Solovay model constructed from a Mahlo cardinal.*

Two years ago, the above result was improved by Toernquist, who proved the following:

Theorem 2. [7] *There are no mad families in Solovay's model.*

Finally, we were recently able to find the exact consistency strength of the non-existence of mad families:

Theorem 3. [1] $ZF + DC +$ *"there are no mad families" is equiconsistent with* ZFC*.*

As for the non-existence of mad families under determinacy hypotheses, the following results are known:

Theorem 4. [5] *There are no mad families under* $AD_{\mathbb{R}}$*.*

Theorem 5. [6] *There are no mad families under* AD^+*.*

Our motivating problem is the following:

Question 1. *How does the non-existence of mad families interact with other regularity properties?*

Of particular interest are the following more specific questions:

Question 2. [7] *Does AD imply that there are no mad families?*

Question 3. [5] *Does* $ZF + DC +$ *"every set of reals has the Ramsey property" imply that there are no mad families?*

We may ask more generally:

Question 4. *What are the connections/similarities/differences between the Ramsey property and the non-existence of mad families?*

By an old result of Mathias, if every set of reals has the Ramsey property, then every filter on ω is meager. The following question naturally arises:

Question 5. [6] *If there are no infinite mad families, does it follow that every filter is meager?*

We answer the above question negatively:

Theorem 6. [2] *$ZF+DC+$"there are no mad families" $+$"there is a non-meager filter on ω" is consistent relative to ZFC.*

Remark: A similar result was obtained independently by Larson and Zapletal in [4] under large cardinal assumptions.

Returning to the AD problem, we might consider the following approach to attacking the problem: Find a regularity property Γ such that AD implies that all sets of reals have the property Γ, and such that "all sets of reals have the property Γ" implies that there are no mad families. The next result shows that this is impossible for a natural family of regularity properties:

Theorem 7. [3] *Let $\mathbb{P}$ be an ω^ω-bounding arboreal forcing notion, the following is consistent relative to an inaccessible cardinal: $ZF+DC+$"all sets of reals are $\mathbb{P}-$measurable" $+$"there exists a mad family".*

Question 6. *Can we extend the above result to include other regularity properties (such as the Ramsey property)?*

We expect the answer to be positive, which should provide a solution to the question of Mathias mentioned above.

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Covering with derived models

Grigor Sargsyan

A fundamental belief among inner model theorists is that no matter how *complex* the universe is, there is a *canonical inner model* that is *close* to the universe. As is traditionally the case, we measure complexity by the kind of large cardinals that could exist in inner models of V. The large cardinals naturally split into two classes that can be characterized as *short* and *long*. The short ones correspond to the so-called short extenders and the long ones correspond to the long extenders. The talk was about the short region, and so we assume there are no inner models with long large cardinals.

Taking a rather liberal view, we say that a transitive set M is *close* to V at an uncountable cardinal κ if

(1) $M \models ZFC - Powerset$

(2) $H_\kappa \subseteq M$,
(3) $M \models$ "there is a largest cardinal η",
(4) $M \models \square_\eta$,
(5) The $\square_\eta$-sequence of M is not threadable in V.

In many situations, inner model theorists isolated an extender model M, i.e. a model build from a sequence of extenders, such that for every κ, $M|(\kappa^+)^M$ is close to V at κ. One such celebrated result is due to Mitchell-Shimmerling-Steel-Jensen who showed that the core model is close to V at every κ. However an insight from Woodin showed that the core model theory is not the route towards the identification of the canonical model that is close to V. The talk started by asking what can be done and presented some new structures that can be close to V.

In recent years, inner model theorists, out of necessity, started considering structures that are not only build from extender sequences but also are closed under their own iteration strategies. Such structures naturally occur in models of determinacy. The covering properties that were explored in this talk had to do with this sort of hybrid structures. First the speaker introduced the *Lp* covering with derived models.

Suppose κ is a measurable cardinal that is a limit of Woodin cardinals and strong cardinals. Let $\mathcal{H}^-$ be $V_\Theta^{\mathcal{K}}$ where $\mathcal{K}$ is HOD of the derived model at κ and Θ is Θ of the derived model. Let Σ be the strategy of $\mathcal{H}^-$ and let $\mathcal{H}$ be the union of all Σ-extender models that extend $\mathcal{H}$, are sound and project to Θ. It is customary to write $\mathcal{H} = Lp^\Sigma(\mathcal{H}^-)$.

Lp-covering with der models: $\mathcal{H}$ is close to V at κ.

Unfortunately, the speaker has shown that Lp-covering with der models fails in many "small" extender models, like the minimal one that contains a Woodin cardinal that is itself a limit of Woodin cardinals. The talk proceeded to describe a generalized version of the covering that the speaker believes must hold at least below Woodin limit of Woodins. Here is the conjecture presented during the talk.

Conjecture 1. *Suppose there is no inner model with a Woodin cardinal that is a limit of Woodin cardinals. Suppose κ is a measurable cardinal that is a limit of Woodin cardinals and strong cardinals. There are then two hybrid structures $\mathcal{M}_0 \triangleleft \mathcal{M}_1 \subseteq H_{\kappa^+}$ such that whenever $g \subseteq Coll(\omega, <\kappa)$ is generic and $\Gamma = \{A \subseteq \mathbb{R}^{V[g]} : L(A, \mathbb{R}^{V[g]}) \models AD^+\}$ then*

(1) $\mathcal{H}^- \triangleleft \mathcal{M}_0$,
(2) $L(\mathcal{M}_0^\omega, \Gamma, \mathbb{R}) \models AD^+$,
(3) *letting* $\eta = Ord \cap \mathcal{M}_0$, η *is the largest cardinal of* $\mathcal{M}_1$,
(4) $\mathcal{M}_1$ *is close to* V *at* κ.

The author then consider a similar conjecture, the Generation of Pointclasses and outline the proof of the following recent theorem. Here "Generation of Pointclasses" refers to the statement that under AD^+ every set of reals is Wadge reducible to one coding an iteration strategy of a hybrid extender model.

Theorem 1. *Assume there is no hybrid extender model with a non-domestic cardinal. Then Generation of Pointclasses holds.*

Here non-domestic cardinal is one that has a measure concentrating on strong cardinals that are limit of Woodin cardinals.

Improved upper bounds for the consistency strength of Chang's Conjecture with gaps

Yair Hayut
(joint work with Monroe Eskew)

The talk is based on the paper [1].

Definition 1. *Let $\kappa < \lambda$, $\mu < \nu$ be cardinals. We say that Chang's Conjecture holds between the pair (ν, μ) and the pair (λ, κ) if for every countable language with distinguished unary predicate R, $\mathcal{L}$, and a model $\mathcal{A}$ over $\mathcal{L}$, $|\mathcal{A}| = \nu$, $|R^{\mathcal{A}}| = \mu$, there is an elementary submodel $\mathcal{B} \prec \mathcal{A}$, with $|\mathcal{B}| = \lambda$, $|R^{\mathcal{B}}| = \kappa$.*

Instances of Chang's Conjecture for pairs of the form $(\kappa^+, \kappa) \twoheadrightarrow (\mu^+, \mu)$ are equivalent to variants of Löwenheim-Skolem Theorem for the extension of the first order logic by Chang's quantifier. Thus, they belong naturally to the zoo of reflection principles.

Theorem 1. *Assume* GCH*. Let κ be κ^{++}-supercompact cardinal and let $\mu < \kappa$ be a regular cardinal. There is a generic extension in which $(\mu^{+3}, \mu^{++}) \twoheadrightarrow (\mu^+, \mu)$.*

The proof is similar to Neeman's proof of the consistency of the tree property at $\aleph_{\omega+1}$.

Proof. Let us sketch the proof. We show that there is a cardinal $\rho < \kappa$ such that in the generic extension by the forcing $\mathrm{Col}(\mu, \rho^+) \times \mathrm{Col}(\rho^{++}, \kappa)$, the desired result holds.

Assume that this is not the case. Then, for every $\rho < \kappa$, there is a name for a counterexample $\dot{f}_\rho$. Namely, $\dot{f}_\rho \colon (\kappa^{++})^{<\omega} \to \kappa^+$ such that whenever $I \subseteq \kappa^{++}$, $\operatorname{otp} I = \rho^{++}$, $\Vdash |\dot{f}_\rho``(I^{<\omega})| = \rho^{++}$.

Let $j \colon V \to M$ be a κ^{++}-supercompact embedding. In M, we know that if we apply the function $j(\dot{f})_\kappa$ on all finite subsets of $j``\kappa^{++}$, we should get κ^{++} different outputs. Using the properties of the forcing, one can construct a sequence of length κ^{++} of conditions $(p_\star, q_\zeta) \in \mathrm{Col}(\mu, \kappa^+) \times \mathrm{Col}(\kappa^{++}, j(\kappa))$ and finite sequences in κ^{++}, a_i such that for every $\xi < \zeta$, $(p_\star, q_\zeta) \Vdash j(\dot{f})_\kappa(j(a_\xi)) < j(\dot{f})_\kappa(j(a_\zeta))$. Reflecting this back to V, we obtain for every pair of ordinals $\xi < \zeta < \kappa^{++}$:

$$\exists \rho < \kappa,\ \exists r \in \mathrm{Col}(\mu, \rho^+) \times \mathrm{Col}(\rho^{++}, \kappa),\ r \Vdash \dot{f}_\rho(a_\xi) < \dot{f}_\rho(a_\zeta) < \kappa^+$$

This is a coloring from pairs of ordinals below κ^{++} to V_κ. By Erdős-Rado Theorem, there is a homogeneous set of order type $\kappa^+ + 1$. Using this homogeneous set, one can find in the generic extension an increasing sequence of length $\kappa^+ + 1$ below κ^+ - a contradiction. $\square$

Using similar methods, one can prove also:

Theorem 2. *Let κ be κ^{++}-supercompact cardinal and let $\mu < \kappa$ be regular. There is a generic extension in which* $(\mu^{+\omega+2}, \mu^{+\omega+1}) \twoheadrightarrow (\mu^+, \mu)$.

Theorem 3. *Let κ be $\kappa^{+\omega+1}$-supercompact cardinal. There is a generic extension in which* $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$.

The following questions remain open:

Question 1. *Assume that κ is supercompact. Is there a generic extension in which* $(\aleph_3, \aleph_2) \twoheadrightarrow (\aleph_2, \aleph_1)$*?*

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The structure of finitely generated subgroups of Thompson's Group *F*

Justin Moore

(joint work with Collin Bleak, Matthew Brin, Martin Kassabov and Matthew Zaremsky)

Richard Thompson's group F and, more generally, the group of piecewise linear homeomorphisms of the interval $\mathrm{PL}_+(I)$ have played an important role in group theory. Thompson himself, for instance, has shown that F is not elementarily amenable and yet does not contain a nonabelian free group. The question of whether F or $\mathrm{PL}_+(I)$ are amenable has been well known and well studied since it was first popularized by Geoghegan in the 1980s. Brin and Sapir have conjectured that every non elementarily amenable subgroup of F (or of $\mathrm{PL}_+(I)$) contains an isomorphic copy of F.

Recently we have initiated a program of parametrizing the biembeddability classes of finitely generated subgroups of F and of $\mathrm{PL}_+(I)$. This can be regarded as an extension of Bleak's work in [1]. In order to be satisfactory, such a parameterization should be sufficiently effective and illuminating so as to make the embeddability relation between finitely generated subgroups of F more transparent and tractable. A confirmation of Brin and Sapir's conjecture stated above, for instance, should be an immediate corollary of a successful completion of this program. In fact, we make the following conjecture:

Conjecture 2. *The finitely generated subgroups of F are well quasi-ordered by the embeddability relations: if $(G_i \mid i < \infty)$ is a sequence of finitely generated subgroups of F, then there is an $i < j$ such that G_i embeds into G_j.*

While this does not have the Brin-Sapir conjecture as a corollary, it seems highly likely that a proof of this conjecture would allow for an inductive verification of the Brin-Sapir conjecture. Currently we seem a long way from the verification of Conjecture 2. We have, however, developed some of the internal structure within the finitely generated subgroups of F which seems to suggest that the conjecture should be true. It also provides tools and heuristics for how to proceed with the program in general.

The first stage of our project has been to develop a method for describing subgroups of F. We say that a finite subset X of $\mathrm{PL}_+(I)$ is *fast* if the boundaries of the supports of its elements are disjoint and for any $n > 1$ the map $f \mapsto f^n$ defined on X extends to a monomorphism of groups. The qualitative dynamics of the elements of a fast generating set are encoded in its *dynamical diagram* — specifically this diagram records the orientation and relationship of the components of the supports of the elements of X. We have proved in [2] (the first result depends on the second):

Theorem 1 (Bleak, Brin, Kassabov, M., Zaremsky). *If X and Y are finite fast subsets of $\mathrm{PL}_+(I)$ which have isomorphic dynamical diagrams, then the isomorphism induces a bijection from X to Y which extends to an isomorphism $\langle X\rangle \cong \langle Y\rangle$.*

Theorem 2 (Bleak, Brin, Kassabov, M., Zaremsky). *If $X \subseteq \mathrm{PL}_+(I)$ is finite and fast and $M \subseteq I$ intersects every support component of an element of X, then $f \mapsto f \upharpoonright M\langle X\rangle$ defines a monomorphism of groups on $\langle X\rangle$ where $M\langle X\rangle$ is the orbit of M under $\langle X\rangle$.*

These results show that every finitely generated subgroup of $\mathrm{PL}_+(I)$ with a fast generating set embeds into F. It is currently unclear whether every subgroup of F is biembeddable with a group admitting a fast generating set, but we conjecture this is true.

In the course of studying the fast generated subgroups of F, we have isolated a certain subclass and showed that it is highly structured.

Theorem 3 (Bleak, Brin, M.). *There is a sequence $(G_\xi \mid \xi \in \epsilon_0)$ of elementary amenable groups, each with finite fast generating sets, such that:*

- *if $\xi \in \eta$, then G_ξ embeds into G_η but G_η does not embed into G_ξ;*
- *if $\xi \in \epsilon_0$, then there is an $\eta \in \epsilon_0$ such that G_η is 2-generated and has EA-class greater than ξ (and less than ϵ_0);*
- *if $\xi \in \epsilon_0$, then $G_{\xi+1}$ is isomorphic to $G_\xi \oplus \mathbb{Z}$.*

(Recall that the ordinal ϵ_0 is the supremum of the sequence of ordinals ω, ω^ω, $\omega^{\omega^\omega}, \ldots$) Previously Brin had proved that for each $\alpha \in \omega^2$, there is an elementarily amenable subgroup of F of EA-class $\alpha + 1$ [3]; until the proof of Theorem 3 it was unknown if there was an EA-subgroup of F of class at least $\omega^2 + 2$.

In fact the class of groups in Theorem 3 seems likely to play a central role in developing an understanding of all finitely generated subgroups of F. For instance, we conjecture that every finitely generated elementarily amenable subgroup of F (or even $\mathrm{PL}_+(I)$) embeds into G_ξ for some $\xi \in \epsilon_0$. This would show, in particular, that $\epsilon_0 + 1$ is the maximum EA-class of a subgroup of F. In fact we conjecture that if G is a finitely generated subgroup of $\mathrm{PL}_+(I)$ which does not contain a copy of F, then G embeds into G_ξ for some $\xi \in \epsilon_0$.

The proof of Theorem 3 hinged on developing a new way of representing the ordinal ϵ_0 in terms of an ordering on certain symmetric matrices with nonnegative integer entries. In addition to the immediate use of this representation in the proof of Theorem 3, it seems likely that it will provide a useful tool for future analysis of the subgroups of F with fast generating sets.

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Generalized cardinal invariants

Jörg Brendle

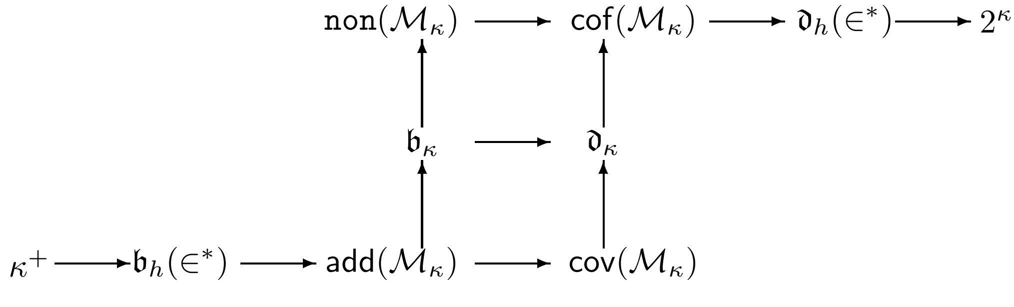
Cichoń's diagram describes the order relationship between cardinal invariants characterizing the meager and null ideals, $\mathcal{M}$ and $\mathcal{N}$, on the real numbers 2^ω, as well as the unbounding and dominating numbers $\mathfrak{b}$ and $\mathfrak{d}$.

In joint work with Andrew Brooke-Taylor, Sy Friedman, and Diana Montoya [BBFM] we develop a version of Cichoń's diagram for cardinal invariants on the generalized Cantor space 2^κ or the generalized Baire space κ^κ where κ is an uncountable regular cardinal. The global behavior of the generalizations of $\mathfrak{b}$ and $\mathfrak{d}$, $\mathfrak{b}_\kappa$ and $\mathfrak{d}_\kappa$, was investigated by Cummings and Shelah [CS]. The meager ideal has a natural analogue $\mathcal{M}_\kappa$ on 2^κ if we equip 2^κ with the topology generated by basic open sets of the form $[s] = \{f \in 2^\kappa : s \subseteq f\}$ where $s \in 2^{<\kappa}$, and call a subset of 2^κ *κ-meager* if it is a κ-union of nowhere dense sets in this topology. With these notions we obtain analogues of classical ZFC-results like the easy inequalities $\mathfrak{b}_\kappa \leq \mathtt{non}(\mathcal{M}_\kappa)$ and $\mathsf{cov}(\mathcal{M}_\kappa) \leq \mathfrak{d}_\kappa$ and the more tricky $\mathsf{add}(\mathcal{M}_\kappa) \leq \mathfrak{b}_\kappa$ and $\mathfrak{d}_\kappa \leq \mathsf{cof}(\mathcal{M}_\kappa)$. In fact, the proof for $\kappa = \omega$ for the latter generalizes to the case when κ is strongly inaccessible, while for general uncountable regular κ a new argument is necessary. These ZFC-results, as well as further results mentioned below, are obtained by Galois-Tukey reductions.

An important dividing line is whether or not $2^{<\kappa} = \kappa$ holds. Landver [La] remarked that $2^{<\kappa} > \kappa$ implies $\mathsf{add}(\mathcal{M}_\kappa) = \mathsf{cov}(\mathcal{M}_\kappa) = \kappa^+$ while the same argument shows that $\mathtt{non}(\mathcal{M}_\kappa) \geq 2^{<\kappa}$, as observed by Blass, Hyttinen, and Zhang [BHZ]. Furthermore $\mathsf{cof}(\mathcal{M}_\kappa) > 2^{<\kappa}$ holds. Still generalizing results for ω to arbitrary uncountable regular κ, we obtain $\mathsf{add}(\mathcal{M}_\kappa) = \min\{\mathfrak{b}_\kappa, \mathsf{cov}(\mathcal{M}_\kappa)\}$ and, if in addition $2^{<\kappa} = \kappa$ holds, $\mathsf{cof}(\mathcal{M}_\kappa) = \max\{\mathfrak{d}_\kappa, \mathtt{non}(\mathcal{M}_\kappa)\}$. The latter assumption is necessary for, assuming GCH and adding κ^+ Cohen reals, one obtains a model where $\mathsf{cof}(\mathcal{M}_\kappa) = \kappa^{++} = 2^{\kappa^+}$ (because of the inequality $\mathsf{cof}(\mathcal{M}_\kappa) > 2^{<\kappa}$) while all other cardinal invariants are equal to $\kappa^+ = 2^\omega = 2^{<\kappa} = 2^\kappa$. In fact, in the – somewhat degenerate – case that $2^{<\kappa} > \kappa$ holds, one can quite freely monkey around with these cardinal invariants in forcing extensions, e.g., $\mathsf{add}(\mathcal{M}_\kappa) = \mathsf{cov}(\mathcal{M}_\kappa) = \kappa^+$, $\mathfrak{b}_\kappa = \kappa^{++}$, $\mathfrak{d}_\kappa = \kappa^{+++}$, $\mathtt{non}(\mathcal{M}_\kappa) = 2^\omega = 2^\kappa = \kappa^{+4}$, and $\mathsf{cof}(\mathcal{M}_\kappa) = \kappa^{+5}$ is consistent: start with a model of GCH, first force the statement about $\mathfrak{b}_\kappa$ and $\mathfrak{d}_\kappa$, and then add κ^{+4} Cohen reals.

In the interesting $2^{<\kappa} = \kappa$ context, however, generalizing independence results about the order relationship of the cardinal invariants is a much harder problem and only two models separating the cardinal invariants are known so far: the *generalized Cohen model*, obtained by adding at least κ^{++} κ-Cohen subsets of κ to a model of GCH, which yields the consistency of $\kappa^+ = \mathsf{add}(\mathcal{M}_\kappa) = \mathtt{non}(\mathcal{M}_\kappa)$ and $\mathsf{cov}(\mathcal{M}_\kappa) = \mathsf{cof}(\mathcal{M}_\kappa) = 2^\kappa \geq \kappa^{++}$; and a very complicated *model of Shelah* [Sh] with κ being supercompact, $\mathsf{cov}(\mathcal{M}_\kappa) = \kappa^+$ and $\mathfrak{d}_\kappa = \kappa^{++}$. Dualizing his construction one should obtain the consistency of $\mathfrak{b}_\kappa = \kappa^+$ and $\mathtt{non}(\mathcal{M}_\kappa) = \kappa^{++}$ for supercompact κ. We conjecture that, unlike for ω, for uncountable regular κ with $2^{<\kappa} = \kappa$, both $\mathsf{add}(\mathcal{M}_\kappa) = \mathfrak{b}_\kappa$ and $\mathfrak{d}_\kappa = \mathsf{cof}(\mathcal{M}_\kappa)$ hold. Also, for successor κ with $2^{<\kappa} = \kappa$, $\mathfrak{b}_\kappa = \mathtt{non}(\mathcal{M}_\kappa)$ and $\mathsf{cov}(\mathcal{M}_\kappa) = \mathfrak{d}_\kappa$ may well be a theorem of ZFC. The main problem with consistency results in this context is that attempts to generalize preservation theorems break down in limit steps of cofinality less than κ.

While it is unclear how to generalize the null ideal, many of the cardinal invariants in Cichoń's diagram have combinatorial characterizations which can be easily generalized to the uncountable context. One such generalization goes as follows: let κ be inaccessible and let $h \in \kappa^\kappa$ be a function converging to κ. A function φ with domain κ and $\varphi(\alpha) \in [\kappa]^{|h(\alpha)|}$ for $\alpha < \kappa$ is called an *h-slalom*. Let $\mathfrak{b}_h(\in^*)$ be the least size of a family F of functions in κ^κ such that for all h-slaloms φ, there is $f \in F$ such that $f(\alpha) \notin \varphi(\alpha)$ for cofinally many α. Dually, $\mathfrak{d}_h(\in^*)$ is the smallest cardinality of a set Φ of h-slaloms such that for all $f \in \kappa^\kappa$ there is $\varphi \in \Phi$ such that $f(\alpha) \in \varphi(\alpha)$ for a final segment of α. Note that if κ is successor, $\mathfrak{b}_h(\in^*) = \mathfrak{b}_\kappa$ and $\mathfrak{d}_h(\in^*) = \mathfrak{d}_\kappa$. For $\kappa = \omega$, a classical result of Bartoszyński says $\mathfrak{b}_h(\in^*) = \mathsf{add}(\mathcal{N})$ and $\mathfrak{d}_h(\in^*) = \mathsf{cof}(\mathcal{N})$. If κ is strongly inaccessible, we obtain a natural generalization of the *Bartoszyński-Raisonnier-Stern Theorem* ($\mathsf{add}(\mathcal{N}) \leq \mathsf{add}(\mathcal{M})$ and $\mathsf{cof}(\mathcal{M}) \leq \mathsf{cof}(\mathcal{N})$) saying that $\mathfrak{b}_h(\in^*) \leq \mathsf{add}(\mathcal{M}_\kappa)$ and $\mathsf{cof}(\mathcal{M}_\kappa) \leq \mathfrak{d}_h(\in^*)$ and, thus, the following version of Cichoń's diagram.



Furthermore $\mathfrak{b}_h(\in^*) < \mathsf{add}(\mathcal{M}_\kappa)$ and $\mathsf{cof}(\mathcal{M}_\kappa) < \mathfrak{d}_h(\in^*)$ are both consistent for strongly inaccessible κ. Finally, in the *generalized Sacks model*, obtained by adding

κ^{++} κ-Sacks subsets of κ to a model of GCH, either with an iteration or a product with supports of size κ, $\mathfrak{d}_h(\in^*) = \kappa^+$ while $\mathfrak{d}_{id}(\in^*) = \kappa^{++} = 2^\kappa$ where id is the identity and h is the power set function $h(\alpha) = 2^\alpha$. This is different from the case $\kappa = \omega$ where all $\mathfrak{d}_h(\in^*)$ are the same (and equal to $\mathsf{cof}(\mathcal{N})$).

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An anti-classification result for measure preserving diffeomorphisms

Matthew Foreman

(joint work with Benjamin Weiss)

The isomorphism problem in ergodic theory was formulated by von Neumann in 1932 in his pioneering paper [1]. It has been solved for various classes of transformations that have concrete representations. In 1942, Halmos and von Neumann used the unitary operators defined by Koopman to completely characterize ergodic measure preserving transformations with pure point spectrum, the transformations that can be concretely realized (in a Borel way) as translations on compact groups. Another notable success is the use of Kolmogorov-Sinai entropy to distinguish between measure preserving systems. Ornstein's work showed that entropy classifies a large class of highly random systems, such as geodesic flows on surfaces of negative curvature and i.i.d. stochastic processes.

However the problem remained open (as originally formulated) for Lebesgue measure preserving diffeomorphisms of manifolds. This talk presented joint work with B. Weiss showing that such a classification is *impossible*, at least using inherently countable methods.

The precise result is as follows.

Theorem 1. *The collection G of diffeomorphisms $T \in Diff^\infty(\mathbb{T}^2, \lambda)$ such that:*

(1) *T is ergodic,*
(2) *T is isomorphic to T^{-1}*

is not Borel in the C^∞-topology.

This has as an immediate corollary:

Corollary 1. *The collection of pairs $(S, T) \in Diff^\infty(\mathbb{T}^2, \lambda) \times Diff^\infty(\mathbb{T}^2, \lambda)$ such that S and T are ergodic and isomorphic is not a Borel set in the product topology on $Diff^\infty(\mathbb{T}^2, \lambda) \times Diff^\infty(\mathbb{T}^2, \lambda)$*

The theorem is proved using the Anosov-Katok method of *Approximation by Conjugacy.* As a byproduct of the proof we get new examples of diffeomorphisms:

Theorem 2. *For all countable ordinals α, there is a measure-distal ergodic diffeomorphism of the 2-torus of distal height α.*

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An isometric action of a Polish group which induces a universal equivalence relation

Julien Melleray

Recall that, when E, F are equivalence relations on standard Borel spaces X, Y respectively, one says that E is *Borel reducible to* Y if there exists a Borel map $f\colon X \to Y$ such that

$$\forall x, x' \in X \quad (xEx') \Leftrightarrow (f(x)Ff(x')) \ .$$

Intuitively, the classification problem which induced the equivalence relation E has been reduced, by applying f, to the classification problem which gave birth to F; the fact that f is computable is important, otherwise one would just be comparing the cardinalities of the quotient spaces X/E and Y/F. We say that E and F are *Borel bireducible* if E Borel reduces to F and F Borel reduces to E.

This notion was introduced by Friedman and Stanley [2] in the late eighties and has been much studied since; the structure of definable equivalence relations, considered up to Borel bireducibility, is now much better understood even though many questions remain. The talk was concerned with a particular level of this complexity hierarchy: equivalence relations which are universal among those which are induced by a Borel action of a Polish group. It is well known [1] that there exists one such relation (up to Borel bireducibility); in [3] Gao and Kechris proved that the classification problem of all Polish metric spaces up to isometry sits exactly at this level of complexity. In order to prove this result (which was also established independently by Clemens), they used properties of the Urysohn space $\mathbb{U}$; it is the unique Polish metric space which is :

- *universal*, that is, $\mathbb{U}$ contains an isometric copy of every separable metric space.
- *ultrahomogeneous*, that is, any isometry between finite subsets of $\mathbb{U}$ extends to an isometry of $\mathbb{U}$.

The proof of Gao and Kechris goes through using universality of $\mathbb{U}$ to encode the isometry relation of Polish metric spaces as a relation on the standard Borel space $\mathcal{F}(\mathbb{U})$ of all closed subsets of $\mathbb{U}$; and proving both that this relation is bireducible to the relation induced by the natural action of the isometry group $\mathrm{Iso}(\mathbb{U})$ on $\mathcal{F}(\mathbb{U})$, and that this latter relation is universal for relations induced by Borel actions of a Polish group.

Gao and Kechris then raised the following problem.

Question 1 (Gao–Kechris [3]). *Does there exist a Polish metric space X, and a closed subgroup G of the isometry group of X, such that the relation induced by the action of G on X is universal for equivalence relations induced by a Borel action of a Polish group?*

The talk presented a positive answer to that question, which we proceed to describe now: let $\mathbb{U}_1$ denote the Urysohn space for metric spaces of diameter 1 (it has the same properties as $\mathbb{U}$ above, except that it is of diameter 1 and is only universal for separable metric spaces of diameter at most 1) and G be its isometry group. Then, given any left-invariant metric d on G, the completion $\hat{G}$ of G may be identified with the space of all isometric embeddings of $\mathbb{U}_1$ into itself; using this fact, and the fact that the relation of isometry of Polish metric spaces of diameter at most 1 is Borel bireducible with the universal equivalence relation for Polish group actions, we prove the following result.

Theorem 1. *The action of G on its left-completion $\hat{G}$ induces a universal equivalence relation for actions induced by a Borel action of a Polish group.*

Note that of course this action is by isometries, since d was assumed to be left-invariant. So the above result answer the question of Gao and Kechris mentioned above. One can even improve it somewhat, using a trick from [4] to prove the following.

Theorem 2. *There exists a Polish metric space X such that the natural action of its isometry group on X induces a universal equivalence relation for Polish group actions.*

After my talk, T. Tsankov asked whether G is exactly the isometry group of $\hat{G}$ (thus the first theorem above would immediately imply the second). I do not know the answer to that question, though I suspect it is negative.

The results presented in this talk will be published in a paper accepted by Fundamenta Mathematicae and a preprint version is available on the author's webpage at
`http://math.univ-lyon1.fr/~melleray/universal_isometric.pdf`

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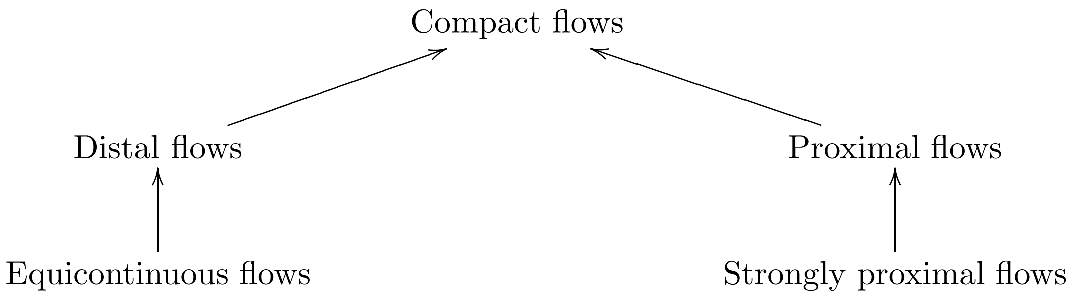
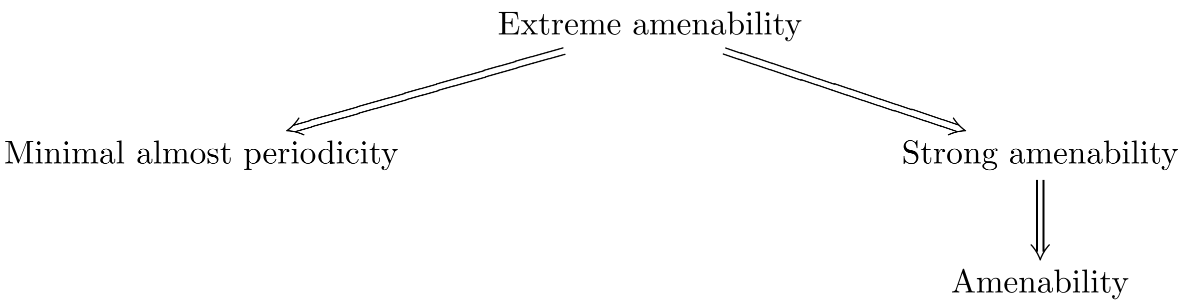
Fixed points in compactifications and combinatorial counterparts

Lionel Nguyen Van Thé

In [1], Kechris, Pestov and Todorcevic established a striking correspondence between topological dynamics of non-Archimedean Polish groups and structural Ramsey theory (for a precise statement, see Theorem 1 below). This turned out to be an invaluable tool to produce extremely amenable groups and to reach a better understanding of the dynamics of non-Archimedean Polish groups. The purpose of the present work is to recast the Kechris-Pestov-Todorcevic correspondence as an instance of a more general construction, allowing to show that Ramsey-type statements actually appear naturally when expressing combinatorially the existence of fixed points in certain compactifications of groups. As a consequence, similar correspondences in fact exist in various dynamical contexts, whose landmarks appear in the following diagram:



To each of the aforementioned classes of flows, one can associate a natural fixed-point property: a topological group G is *extremely amenable* when every G-flow has a fixed point, *strongly amenable* when every proximal G-flow has a fixed point, *amenable* when every strongly proximal G-flow has a fixed point (equivalently, every G-flow admits a G-invariant Borel probability measure), and *minimally almost periodic* when every equicontinuous G-flow has a fixed point (which is known to be equivalent to having a fixed point on any *distal* G-flow, having no non-trivial finite-dimensional unitary representation, and admitting no non-trivial continuous morphism to a compact group). This leads to the following "dual" form of the previous diagram:



On the combinatorial side, the general setting is that of first-order structures, which we restrict here to the relational case. Given a first-order relational language, recall that a structure in that language is *ultrahomogeneous* when any isomorphism between any two of its finite substructures extends to an automorphism. Countable ultrahomogeneous structures are called *Fraïssé* structures. In the recent developments of Fraïssé theory, a main concern is the study of the interaction between the combinatorics of the set $\mathrm{Age}(\mathbf{F})$ of all finite substructures of $\mathbf{F}$ and the dynamics of the automorphism group $\mathrm{Aut}(\mathbf{F})$. The main theorem of [1] is a striking illustration of this:

Theorem 1 (Kechris-Pestov-Todorcevic [1])**.** *Let $\boldsymbol{F}$ be a Fraïssé structure. TFAE:*

i) $\mathrm{Aut}(\boldsymbol{F})$ *is extremely amenable.*
ii) $\mathrm{Age}(\boldsymbol{F})$ *has the Ramsey property.*

The Ramsey property (for embeddings) referred to in the previous result means that for every $\mathbf{A} \in \mathrm{Age}(\mathbf{F})$, every function χ taking finitely many values on $\binom{\mathbf{F}}{\mathbf{A}}$ (such a χ is usually referred to as a coloring) is necessarily constant on arbitrarily large finite set. Precisely: given any $\mathbf{B} \in \mathrm{Age}(\mathbf{F})$, in which $\mathbf{A}$ typically embeds in many ways, χ is constant of some set of the form $\binom{\mathbf{b}(\mathbf{B})}{\mathbf{A}}$, for some $b \in \binom{\mathbf{F}}{\mathbf{B}}$. The present work establishes results of the same flavor, in the context that is described by the aforementioned diagrams.

For an example belonging to the left side of the diagram: A *joint embedding* $\langle a, z\rangle$ of two structures $\mathbf{A}$ and $\mathbf{Z}$ is a pair (a, z) of embeddings of $\mathbf{A}$ and $\mathbf{Z}$ into some common structure $\mathbf{C}$. To such objects is attached a natural notion of isomorphism. These notions can be defined in the same way in the case of finitely many structures $\mathbf{A}$, $\mathbf{Z}^1$,...,$\mathbf{Z}^k$.

Definition 1. *Let $\mathcal{K}$ be a class of finite structures in some first order language, and $\boldsymbol{A}, \boldsymbol{Z} \in \mathcal{K}$. An* unstable $(\mathbf{A}, \mathbf{Z})$-sequence *is a pair of sequences $(a_m)_{m\in\mathcal{N}}$ and $(z_n)_{n\in\mathcal{N}}$ such that there exist two different joint embeddings $\tau_<$ and $\tau_>$ satisfying:*

$$\forall m, n \in \mathcal{N} \quad (m < n \Rightarrow \langle a_m, z_n\rangle = \tau_<) \wedge (m > n \Rightarrow \langle a_m, z_n\rangle = \tau_>)$$

When there is no unstable $(\boldsymbol{A}, \boldsymbol{Z})$-sequence, the pair $(\boldsymbol{A}, \boldsymbol{Z})$ is stable.

Definition 2. *Let $\mathcal{K}$ be a class of finite structures in some first order language. It has the* stable Ramsey property *when the following holds: for every $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{K}$, every $\boldsymbol{Z}^1, ..., \boldsymbol{Z}^k \in \mathcal{K}$ so that every pair $(\boldsymbol{A}, \boldsymbol{Z}^i)$ is stable, there exists $\boldsymbol{C} \in \mathcal{K}$ such that for every joint embedding $\langle c, z^1, ..., z^k\rangle$, there is $b \in \binom{\boldsymbol{C}}{\boldsymbol{B}}$ so that for every $i \leq k$, the isomorphism type of the joint embedding $\langle a, z^i\rangle$ does not depend on $a \in \binom{\boldsymbol{b}(B)}{\boldsymbol{A}}$.*

With these notions in mind, here is the characterization of minimal almost periodicity in the spirit of the Kechris-Pestov-Todorcevic correspondence:

Theorem 2. *Let $\boldsymbol{F}$ be a Fraïssé structure with Roelcke-precompact automorphism group. TFAE:*

i) $\mathrm{Aut}(\boldsymbol{F})$ *is minimally almost periodic.*
ii) *For every* $\boldsymbol{A} \in \mathrm{Age}(\boldsymbol{F})$*, every* $\mathrm{Aut}(\boldsymbol{F})$*-invariant equivalence relation on* $\binom{\boldsymbol{F}}{\boldsymbol{A}}$ *with finitely many classes is trivial.*
iii) $\mathrm{Age}(\boldsymbol{F})$ *has the stable Ramsey property.*

Note that the equivalence between the first two items already follows from the work of Tsankov [2] where unitary representations of oligomorphic groups are classified, or of Ben Yaacov [4] where the relationship between the Bohr compactification and the algebraic closure of the empty set is identified.

Here is now an example of a result belonging to the right side of the diagram:

Definition 3. *Let* $\boldsymbol{F}$ *be a Fraïssé structure and* χ *be a coloring of* $\binom{\boldsymbol{F}}{\boldsymbol{A}}$*. Say that* χ *is* proximal *when for every* $\boldsymbol{D} \in \mathrm{Age}(\boldsymbol{F})$*, there exists* $\boldsymbol{E} \in \mathrm{Age}(\boldsymbol{F})$ *such that for every* $e_1, e_2 \in \binom{\boldsymbol{F}}{\boldsymbol{E}}$*, there exists* $d \in \binom{\boldsymbol{E}}{\boldsymbol{D}}$ *such that* $\chi \circ e_1$ *and* $\chi \circ e_2$ *agree on* $\binom{d(\boldsymbol{D})}{\boldsymbol{A}}$*.*

Definition 4. *Let* $\boldsymbol{F}$ *be a Fraïssé structure. Say that* $\boldsymbol{F}$ *has the* proximal Ramsey property *when for every* $\boldsymbol{A}, \boldsymbol{B} \in \mathrm{Age}(\boldsymbol{F})$ *and every finite proximal coloring* χ *of* $\binom{\boldsymbol{B}}{\boldsymbol{A}}$*, there is* $b \in \binom{\boldsymbol{F}}{\boldsymbol{B}}$ *such that* χ *is constant on* $\binom{b(\boldsymbol{B})}{\boldsymbol{A}}$*.*

Theorem 3. *Let* $\boldsymbol{F}$ *be a Fraïssé structure. TFAE:*

i) *Every zero-dimensional proximal* $\mathrm{Aut}(\boldsymbol{F})$*-flow has a fixed point.*
ii) $\boldsymbol{F}$ *has the proximal Ramsey property.*

Under additional technical assumptions, these statements are equivalent to $\mathrm{Aut}(\boldsymbol{F})$ *being strongly amenable.*

Those results are obtained as corollaries of a single master theorem. The first step towards it is to realize that under a suitable interpretation, every finite coloring χ of $\binom{\mathbf{F}}{\mathbf{A}}$ may be seen as an element of the algebra $\mathrm{RUC}_b(\mathrm{Aut}(\mathbf{F}))$ of complex valued right-uniformly continuous functions. The second step is to realize that the group $\mathrm{Aut}(\mathbf{F})$ acts continuously in two different ways on $\mathrm{RUC}_b(\mathrm{Aut}(\mathbf{F}))$: by left shift $(g \cdot f(x) = f(g^{-1}x))$ in the norm topology, and by right shift $(g \bullet f(x) = f(xg))$ with the pointwise convergence topology (more details can be found in [3, Chapter IV, Sections 4 and 5]). The third step is to define a notion of the Ramsey property that is localized to a particular kind of colorings.

Definition 5. *Let* $\mathcal{F} \subset \mathrm{RUC}_b(\mathrm{Aut}(\boldsymbol{F}))$*. Say that* $\boldsymbol{F}$ *has the* Ramsey property for colorings in $\mathcal{F}$ *when for every* $\boldsymbol{A}, \boldsymbol{B}$ *in* $\mathrm{Age}(\boldsymbol{F})$*, every finite set* $\mathcal{C} \subset \mathcal{F}$ *of finite colorings of* $\binom{\boldsymbol{F}}{\boldsymbol{A}}$*, there exists* $b \in \binom{\boldsymbol{F}}{\boldsymbol{B}}$ *such that every* $\chi \in \mathcal{C}$ *is constant on* $\binom{b(\boldsymbol{B})}{\boldsymbol{A}}$*.*

With all this in mind:

Theorem 4. *Let* $\boldsymbol{F}$ *be a Fraïssé structure,* $\mathcal{X}$ *be a class of* $\mathrm{Aut}(\boldsymbol{F})$*-flows such that the class of* $\mathcal{X}$*-*$\mathrm{Aut}(\boldsymbol{F})$*-ambits is closed under suprema and factors, and that every* $\mathrm{Aut}(\boldsymbol{F}) \curvearrowright X \in \mathcal{X}$ *admits some* $x \in X$ *such that* $\mathrm{Aut}(\boldsymbol{F}) \curvearrowright \overline{\mathrm{Aut}(\boldsymbol{F}) \cdot x} \in \mathcal{X}$*. Let* $\mathcal{A}$ *denote the unital, left shift invariant, closed* C^**-subalgebra of* $\mathrm{RUC}_b(\mathrm{Aut}(\boldsymbol{F}))$ *defined by* $\mathcal{A} = \{f \in \mathrm{RUC}_b(G) : G \curvearrowright \overline{G \bullet f} \in \mathcal{X}\}$*. TFAE:*

i) *Every zero-dimensional* $\mathrm{Aut}(\boldsymbol{F})$*-flow in* $\mathcal{X}$ *has a fixed point.*
ii) $\boldsymbol{F}$ *has the Ramsey property for the finite colorings in* $\mathcal{A}$*.*

When the finite colorings are dense in $\mathcal{A}$*, those statements are equivalent to:*

i') *Every* $\mathrm{Aut}(\boldsymbol{F})$*-flow in* $\mathcal{X}$ *has a fixed point.*

While such a general theorem is satisfactory in view of a global understanding of the interaction between the combinatorics of $\mathbf{F}$ and the dynamics of $\mathrm{Aut}(\mathbf{F})$, it remains unclear whether it will be as useful as the original Kechris-Pestov-Todorcevic correspondence in practice.

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Counterexamples to Vaught's Conjecture and the Admissibility Equivalence Relation in L

William Chan

Definition 1. *An equivalence relation* E *on a Polish space* X *is thin if and only if for all perfect sets* $P \subseteq X$*, there exists* $x, y \in P$ *with* $x \neq y$ *and* $x\ E\ y$*.*

Silver showed there are no thin $\mathbf{\Pi}^1_1$ equivalence relation with uncountably many classes:

Fact 1. *(Silver's Dichotomy,* [4]*) If* E *is a* $\mathbf{\Pi}^1_1$ *equivalence relation on* X*, then exactly one of the following holds:*
(1) E *has countably many classes.*
(2) There is a perfect set $P \subseteq X$ *so that* $\neg(x\ E\ y)$ *for all* $x, y \in P$ *with* $x \neq y$*.*

$\mathbf{\Sigma}^1_1$ thin equivalence relations with uncountably many classes do exist. The following are some examples:

Definition 2. *Let* E_{ω_1} *be the equivalence relation on* ${}^\omega 2$ *defined by*

$$x\ E_{\omega_1}\ y \Leftrightarrow (x \notin \mathrm{WO} \wedge y \notin \mathrm{WO}) \vee (x \cong y)$$

where WO *denotes the collection of reals coding well orderings and* $\cong$ *is isomorphism in the langauge of linear orderings.*

E_{ω_1} *is a* Σ^1_1 *thin equivalence relation with uncountably many classes and has* $\mathbf{\Delta}^1_1$ *classes except for the single* Σ^1_1 *non-*$\mathbf{\Delta}^1_1$ *class consisting of the non-well-orderings.*

Definition 3. *The countable admissible ordinal equivalence relation is F_{ω_1} defined on ${}^\omega 2$ by*

$$x \ F_{\omega_1} \ y \Leftrightarrow \omega_1^x = \omega_1^y$$

where ω_1^z is the least ordinal not recursive in z.

F_{ω_1} is a Σ_1^1 thin equivalence relation with uncountable many $\mathbf{\Delta}_1^1$ classes.

Definition 4. *Let $\mathscr{L}$ be a recursive language. Let T be a countable $\mathscr{L}$ theory in the infinitary logic $\mathscr{L}_{\omega_1,\omega}$. Define the equivalence relation E_T on ${}^\omega 2$ (considered as coding $\mathscr{L}$-structures) by*

$$x \ E_T \ y \Leftarrow (x \not\models T \wedge y \not\models T) \vee (x \cong_{\mathscr{L}} y)$$

where $\cong_{\mathscr{L}}$ is isomorphism as $\mathscr{L}$-structures. E_T is generally a $\mathbf{\Sigma}_1^1$ equivalence relation with all $\mathbf{\Delta}_1^1$ classes.

T is a counterexample to Vaught's conjecture if and only if E_T is a thin equivalence relation with uncountably many classes.

The existence of counterexamples to Vaught's conjecture is the content of the long standing eponymous open question known as the Vaught's conjecture. Counterexamples to Vaught's conjecture also have characterizations using scatterness and Morley trees.

A natural question is to compare these three types of thin $\mathbf{\Sigma}_1^1$ equivalence relations. A common form of comparison for equivalence relations is the $\mathbf{\Delta}_1^1$ reduction:

Definition 5. *Let E and F be equivalence relations on Polish spaces X and Y, respectively. E is $\mathbf{\Delta}_1^1$ reducible to F, denoted $E \leq_{\mathbf{\Delta}_1^1} F$, if and only there is a $\mathbf{\Delta}_1^1$ function $\Phi : X \to Y$ so that $x \ E \ y$ if and only if $\Phi(x) \ F \ \Phi(y)$.*

Note that $E_{\omega_1} \leq_{\mathbf{\Delta}_1^1} F_{\omega_1}$ and $E_{\omega_1} \leq_{\mathbf{\Delta}_1^1} E_T$, when T is a theory in $\mathscr{L}_{\omega_1,\omega}$, is trivially impossible since E_{ω_1} has a single class in $\Sigma_1^1 \setminus \mathbf{\Delta}_1^1$ yet F_{ω_1} and E_T have all $\mathbf{\Delta}_1^1$ classes. Also $F_{\omega_1} \leq_{\mathbf{\Delta}_1^1} E_{\omega_1}$ and $E_T \leq_{\mathbf{\Delta}_1^1} E_{\omega_1}$, when T is a counterexample to Vaught's conjecture, are also impossible: any such reduction would yield a $\mathbf{\Sigma}_1^1$ subset of WO such that the ordinal ranks of its elements are unbounded below ω_1 which violates the boundedness principle.

The latter failure is global. The former is local since it is due to E_{ω_1} having a single Σ_1^1 but not $\mathbf{\Delta}_1^1$ class. To accounts for local problems, Zapletal defined the almost $\mathbf{\Delta}_1^1$ reduction, denoted $E \leq_{a\mathbf{\Delta}_1^1} F$, which is a $\mathbf{\Delta}_1^1$ function which fails to be a reduction of E to F on at most countably many E-classes.

Using results of Zapletal about equivalence relations with infinite pinned cardinals which hold assuming the existence of a measurable cardinal,

Fact 2. *([5]) Assume there is a measurable cardinal. $E_{\omega_1} \leq_{a\mathbf{\Delta}_1^1} F_{\omega_1}$. $E_{\omega_1} \leq_{a\mathbf{\Delta}_1^1} E_T$, whenever T is a counterexample to Vaught's conjecture. (The first is provable with just $0^\sharp$.)*

Therefore a natural question is whether the above result is provable in ZFC. The answer is no:

Theorem 1. *([2]) In L and set generic extension of L, $\neg(E_{\omega_1} \leq_{a\mathbf{\Delta}_1^1} F_{\omega_1})$.*

Sy-David Friedman asked about comparisons of E_T and F_{ω_1}, when T is a counterexample to Vaught's conjecture: Can $E_T \leq_{\mathbf{\Delta}^1_1} F_{\omega_1}$?

This talk will show using infinitary logic in countable admissible fragments that the answer can consistently be no. The material of this talk appears in [2] and [1].

Theorem 2. *([2] and [1]) In L and set-generic extension of L, $\neg(E_T \leq_{\mathbf{\Delta}^1_1} F_{\omega_1})$, whenever T is a counterexample to Vaught's conjecture.*

Both Theorem 1 and Theorem 2 are shown by establishing that the existence of such reductions imply the existence of a real z so that there is no ordinal β for which all $\alpha > \beta$, α is z-admissible if and only if α is admissible. Note that $0^\sharp$ is a real with such properties. However, Sy-David Friedman ([3]) has shown that it is consistent with ZFC that such a real exists.

The answer for whether $E_{\omega_1} \leq_{a\mathbf{\Delta}^1_1} F_{\omega_1}$ has been shown to be consistently no and yes, relative to large cardinals. A natural question would then be whether the question of Sy-David Friedman can consistently be yes:

Question 1. *Assume counterexamples to Vaught's conjecture exist, is it consistent (possibly relative $0^\sharp$ or a measurable cardinal) that there is some counterexample to Vaught's conjecture, T, so that $E_T \leq_{\mathbf{\Delta}^1_1} F_{\omega_1}$?*

However, the following is provable in ZFC:

Fact 3. *([1]) (ZFC) If T is a non-minimal counterexample to Vaught's conjecture, then $\neg(E_T \leq_{\mathbf{\Delta}^1_1} F_{\omega_1})$.*

Question 2. *What is the consistency strength of $E_{\omega_1} \leq_{a\mathbf{\Delta}^1_1} F_{\omega_1}$?*

Since this reduction does not exist in set-generic extensions of L, one may need to consider class forcing extensions if one seeks to show it is consistent with ZFC. The almost $\mathbf{\Delta}^1_1$ reduction does not exists in set-generic extensions of L since it can be shown that such reductions imply the existence of some real z so that the next admissible ordinal after any z-admissible ordinal is not z-admissible. Does the existence of such an almost $\mathbf{\Delta}^1_1$ reduction imply that there is a real z whose admissibility spectrum is so spread out that z-admissible ordinals become L-cardinals in which case Harrington's principle would allow $0^\sharp$ to be recovered?

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Boolean ultrapowers and iterated forcing

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(joint work with Saharon Shelah)

In [1], Shelah invented two different techniques for proving the consistency of $\mathfrak{d} < \mathfrak{a}$. The first method is known as *iteration along a template*. It may be thought of as a generalized version of finite support iteration. Shelah used this method to produce a model of $\aleph_2 = \mathfrak{d} < \mathfrak{a} = \aleph_3$ without any large cardinal assumptions. The second method involves starting with a κ-complete ultrafilter $\mathcal{U}$ on a measurable cardinal κ and iterating the process of taking ultrapowers of some poset by $\mathcal{U}$. Shelah used this method to produce models of both $\mathfrak{d} < \mathfrak{a}$ and $\mathfrak{u} < \mathfrak{a}$. These two methods have their own advantages and disadvantages. The first method works in ZFC and it yields models where the invariants have accessible values; the second one requires the existence of a measurable cardinal κ in the ground model and can only produce models where the invariants lie above κ. However the second method is applicable to a larger class of forcing notions. Both methods have the disadvantage that the posets that are used in the iterations must be either Suslin c.c.c. or σ-centered. So one test question for refining these existing techniques was whether one could produce a model of $\mathfrak{d} < \mathfrak{a}$ while having Martin's axiom for all c.c.c. posets of size $< \mathfrak{d}$.

In [2], we develop a new method for proving consistency results on cardinal invariants, particularly results involving the invariant $\mathfrak{a}$. This method can be used with a wide range of forcings notions, including arbitrary c.c.c. posets. However the new method always requires a supercompact cardinal θ in the ground model and produces forcing extensions in which the desired invariants sit above θ. Another feature of our method is that it generalizes to cardinal invariants above ω, and can be used to give uniform consistency proofs that work at any regular cardinal. It can also be used to treat situations where three cardinal invariants must be separated. In particular, our technique solves various long standing open problems about cardinal invariants at uncountable regular cardinals. The following theorems summarize the results we are able to obtain using our technique.

Theorem 1. *Suppose θ is a supercompact cardinal. Suppose $\theta \leq \mu = \mu^{<\theta} < \chi = \chi^{\mu^+}$. Then there is a c.c.c. forcing extension in which $\mathfrak{d} = \mu^+$, $\mathrm{MA}_{<\mu^+}$ holds and $\mathfrak{a} = \mathrm{cf}(\chi)$.*

Theorem 2. *Suppose θ is a supercompact cardinal. Suppose $\theta \leq \mu = \mu^{<\theta} < \chi = \chi^{\mu^{++}}$. Then there is a c.c.c. forcing extension in which $\mathfrak{b} = \mu^+ < \mu^{++} = \mathfrak{s} < \mathfrak{a} = \mathrm{cf}(\chi)$*

Theorem 3. *Suppose that $\aleph_0 < \kappa = \kappa^{<\kappa} < \theta < \mathrm{cf}(\mu_1) = \mu_1 < \mathrm{cf}(\mu_2) = \mu_2 < \mathrm{cf}(\mu_3) = \mu_3 < \chi = \chi^{\mu_3}$ and suppose that θ is supercompact. Then there is a forcing extension in which $\mu_1 = \mathfrak{p}(\kappa) < \mathfrak{b}(\kappa) = \mu_2 < \mu_3 = \mathfrak{d}(\kappa) < \mathrm{cf}(\chi) = \mathfrak{a}(\kappa)$.*

Theorem 4. *Suppose that $\kappa < \theta \leq \mu = \mu^{<\theta} < \chi = \chi^{\mu^+}$. Assume also that θ is supercompact and that κ is Laver indestructible supercompact. Then there is a forcing extension in which $\mu^+ = \mathfrak{u}(\kappa) < \mathfrak{a}(\kappa) = \mathrm{cf}(\chi)$.*

All of these results rely on Boolean ultrapowers, studied by Keisler and other model theorists in the 1960s. In the case of cardinal invariants on ω a certain finite support iteration of c.c.c. forcings in built. Then we take the Boolean ultrapower of this iteration by a carefully constructed θ-complete ultrafilter on a complete Boolean algebra $\mathcal{B}$. Then we force with this ultrapower to obtain the desired model. In the case when κ is uncountable, we begin with a $< \kappa$-support iteration of $< \kappa$-strategically closed posets that satisfy an iterable form of the κ^+-c.c. Then, just as in the case of cardinal invariants on ω, we take the Boolean ultrapower of this iteration by a suitable ultrafilter constructed on $\mathcal{B}$. In all of the above stated theorems, $\mathcal{B}$ is the complete Boolean algebra for adding χ many subsets of μ with conditions of size less than θ.

The ultrafilter D on $\mathcal{B}$ needs to be sufficiently rich to ensure that a large number of types of size μ and a few types of size $\leq \chi$ are realized in any Boolean ultrapower by D. To this end, we introduce the notion of an *optimal* ultrafilter on a complete Boolean algebra $\mathcal{B}$. This notion is related to but is distinct from the notion of an optimal ultrafilter studied by Malliaris and Shelah in [3]. One major difference is that in [3] it was easy to realize all types of size $\leq \mu$, meaning that all of the relevant ultrapowers there were μ^+-saturated. However in our situation it is impossible to make the ultrapowers μ^+-saturated because we are in the context of $\mathcal{L}_{\theta,\theta}$ logic. Hence there will always be types of size $\leq \theta$ that cannot be realized in the ultrapower. We get around this by introducing the notion of a *μ^+-plentiful* model, which is a model in which all types of size $\leq \mu$ that are suitably "continuous" with respect to a normal fine measure on $[H(\lambda)]^{<\theta}$ are realized, where λ is a sufficiently large regular cardinal. The definition of an optimal ultrafilter is then appropriately modified to reflect this, and we prove that the Boolean ultrapower by such an optimal ultrafilter is always μ^+-plentiful. It is further proved that an optimal ultrafilter always exists on the complete Boolean algebra for adding χ many subsets of μ with conditions of size $< \theta$. Another difficulty is that it becomes necessary to translate the problem of realizing types of size $\leq \mu$ in the Boolean ultrapower into the problem of finding multiplicative refinements of certain monotonic functions defined on the Boolean algebra. We accomplish this by introducing the notion of a *(μ, θ)-regular* ultrafilter on $\mathcal{B}$. Finally we ensure that certain "homogeneous" types of cardinality $> \mu$ are also realized in the Boolean ultrapower. This is needed for showing that $\mathfrak{a}(\kappa) = \mathrm{cf}(\chi)$ in the Theorems 1–4.

Many problems remain open. The most obvious one is whether the use of a supercompact cardinal is needed for any of these consistency results. Another open problem is the question of whether it is possible to arrange the invariants occurring in Theorems 1–4 to have smaller more "accessible" values. Theorems 1–4 produce models where the invariants sit above the first supercompact cardinal greater than κ from the point of view of the ground model. Regarding Theorem 3, we are able to show in this connection that for any uncountable regular cardinal κ, if $\mathfrak{b}(\kappa) = \kappa^+$, then $\mathfrak{a}(\kappa) = \kappa^+$. Thus if one wants to have $\mathfrak{b}(\kappa) < \mathfrak{a}(\kappa)$, then the smallest possible value for $\mathfrak{b}(\kappa)$ is κ^{++}. We end with two open problems that do not seem to be amenable to the technique of Boolean ultrapowers.

Question 1. *Is $\mathfrak{b} < \mathfrak{s} < \mathfrak{d}$ consistent? What about $\mathfrak{b} < \mathfrak{a} < \mathfrak{s}$?*

Question 2 (Kunen)**.** *Is $\mathfrak{u}(\omega_1) < 2^{\aleph_1}$ consistent?*

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The approachability ideal without a maximal set

John Krueger

Let κ be an uncountable cardinal. For a given sequence $\vec{a} = \langle a_i : i < \kappa^+ \rangle$ of subsets of κ^+ with size less than κ, define $S_{\vec{a}}$ to be the set of limit ordinals $\beta < \kappa^+$ for which there exists a set $c \subseteq \beta$, which is cofinal in β, and has order type equal to $\mathrm{cf}(\beta)$, which is approximated by the sequence $\vec{a} \restriction \beta$ in the sense that for all $\gamma < \beta$, $c \cap \gamma \in \{a_i : i < \beta\}$.

Define the approachability ideal $I[\kappa^+]$ as the collection of sets $S \subseteq \kappa^+$ such that for some sequence $\vec{a}$ as above, and for some club $C \subseteq \kappa^+$, $S \cap C \subseteq S_{\vec{a}}$. In other words, $I[\kappa^+]$ is the ideal on κ^+ which is generated over the nonstationary ideal on κ^+ by sets of the form $S_{\vec{a}}$. The collection $I[\kappa^+]$ is a normal ideal on κ^+.

The approachability ideal was introduced by Shelah in the 1970's ([6]), and since then it has played a role as an important tool in combinatorial set theory and forcing. A major result on the approachability ideal is that if κ is a regular uncountable cardinal, then the set $\kappa^+ \cap \mathrm{cof}(<\kappa)$ is a member of $I[\kappa^+]$ ([7, Section 4]). Hence, when κ is a regular uncountable cardinal, the structure of $I[\kappa^+]$ is completely determined by which stationary subsets of $\kappa^+ \cap \mathrm{cof}(\kappa)$ are in $I[\kappa^+]$.

Shelah [7] raised the question whether it is consistent that there are no stationary subsets of $\kappa^+ \cap \mathrm{cof}(\kappa)$ in $I[\kappa^+]$. This problem was solved by Mitchell [4], who proved that it is consistent, relative to the consistency of a greatly Mahlo cardinal, that there is no stationary subset of $\omega_2 \cap \mathrm{cof}(\omega_1)$ in $I[\omega_2]$. In the process of solving this problem, Mitchell introduced a number of powerful new ideas in forcing, including strongly generic conditions, strong properness, and a method for using side conditions to add by forcing a club subset of ω_2 with finite conditions (see Friedman [2] for a similar method which was introduced independently).

Assuming that $(\kappa^+)^{<\kappa} = \kappa^+$, we can enumerate all subsets of κ^+ of size less than κ in a single sequence $\vec{b} = \langle b_i : i < \kappa^+ \rangle$. It is not hard to show that if $\vec{a} = \langle a_i : i < \kappa^+ \rangle$ is any sequence of subsets of κ^+ of size less than κ, then there exists a club $C \subseteq \kappa^+$ such that $S_{\vec{a}} \cap C \subseteq S_{\vec{b}}$. It follows that $I[\kappa^+]$ is generated over the nonstationary ideal on κ^+ by the single set $S_{\vec{b}}$. Another way of describing this conclusion is that $I[\kappa^+]$ has a maximal set modulo clubs.

A natural question is whether the approachability ideal $I[\kappa^+]$ must *always* have a maximal set modulo clubs, regardless of any cardinal arithmetic assumptions.

That is, is it consistent that $I[\kappa^+]$ does not have a single generator over the nonstationary ideal. By the normality of $I[\kappa^+]$, this possibility is equivalent to having not fewer than κ^{++} many generators. This question was first raised by Shelah in [7], in the same place where he mentions the possibility of $I[\kappa^+]$ not containing any stationary subset of $\kappa^+ \cap \text{cof}(\kappa)$. The problem also appears at the end of [4], where Mitchell suggests that the methods introduced in his paper are likely to be useful for answering the question.

Theorem 1 (K. 2016)**.** *Assuming the consistency of a greatly Mahlo cardinal, it is consistent that $I[\omega_2]$ does not have a maximal set modulo clubs.*

Part of the proof involves developing a forcing poset with finite conditions for adding a partial square sequence to a given stationary set $S \subseteq \omega_2 \cap \text{cof}(\omega_1)$. Recall that a *partial square sequence on* S is a sequence $\langle c_\alpha : \alpha \in S \rangle$ satisfying that for each $\alpha \in S$, c_α is a club subset of α with order type equal to ω_1, and whenever γ is a common limit point of c_α and c_β, then $c_\alpha \cap \gamma = c_\beta \cap \gamma$.

If there exists a partial square sequence on S, then S is in $I[\omega_2]$. Namely, define a sequence $\vec{a} = \langle a_\gamma : \gamma < \omega_2 \rangle$ as follows. For a given ordinal γ, if there exists some $\alpha \in S$ strictly greater than γ such that γ is a limit point of c_α, then let $a_\gamma := c_\alpha \cap \gamma$. Define a_ξ for all other ordinals ξ in such a way as to include any initial segment of any set of the form a_γ, where γ is an ordinal of the first type. One can easily check that for some club $C \subseteq \omega_2$, $S \cap C \subseteq S_{\vec{a}}$. Therefore, $S \in I[\omega_2]$.

Forcing a square sequence with finite conditions was first achieved by Dolinar and Dzamonja [1], using Mitchell's style of models as side conditions [4]. Later, Krueger [3] developed a forcing poset for adding a square sequence with finite conditions using the framework of coherent adequate sets. And Neeman [5] defined a forcing poset for adding a square sequence using his framework of two-type side conditions.

The forcing poset we develop for adding a partial square sequence is similar to the forcings of [3] and [5] for adding a square sequence. However, we need to develop the properties of our forcing poset in much greater detail than was done in those papers, so that we can use it to prove the consistency result.

We then develop a forcing poset $\mathbb{Q}$ which simultaneously adds a partial square sequence on multiple sets. This forcing poset is similar to a product forcing, since the different posets which are incorporated in the forcing are independent of each other, except for the presence of a shared side condition. We believe that it is likely that this kind of side condition product will have other applications in the future.

A crucial property of the product forcing $\mathbb{Q}$ is that certain quotients of it satisfy the ω_1-approximation property. More specifically, for certain uncountable models P, $P \cap \mathbb{Q}$ is a regular suborder of $\mathbb{Q}$, and the quotient forcing $\mathbb{Q}/\dot{G}_{P \cap \mathbb{Q}}$ has the ω_1-approximation property in $V^{P \cap \mathbb{Q}}$.

A similar result about certain quotients having the ω_1-approximation property was used by Mitchell [4] in his proof of the consistency that $I[\omega_2]$ does not contain a stationary subset of $\omega_2 \cap \mathrm{cof}(\omega_1)$. This result followed from the equation

$$(p \wedge q) \restriction P = (p \restriction P) \wedge (q \restriction P),$$

where $\wedge$ denotes greatest lower bound, which holds below a strongly P-generic condition which is tidy (see [4, Definition 2.20, Lemma 2.22]). Unfortunately, our forcing poset $\mathbb{Q}$ does not satisfy this equation. First, our forcing poset $\mathbb{Q}$ does not even have greatest lower bounds. Secondly, even if the definition of $\mathbb{Q}$ is adjusted so that $\mathbb{Q}$ has greatest lower bounds, which is possible, the above equation still fails, even on any dense set.

Nonetheless, we are able to make use of some of the ideas in Mitchell's original argument for the ω_1-approximation property [4, Lemma 2.22], by replacing the above equation with something weaker, and more complicated, namely,

$$(q \oplus^N p) \restriction P = (q \restriction P) \oplus^{N \cap P} (p \restriction P).$$

In this equation, $a \oplus^M b$ denotes the amalgam of a condition b with a condition a which is in the model M and is below the projection $b \restriction M$. We believe that this equation will be useful in future applications for verifying the approximation property, in cases where Mitchell's original tidy property fails.

Finally, we complete the proof of the consistency that $I[\omega_2]$ does not have a maximal set modulo clubs. Assuming that κ is a greatly Mahlo cardinal, we get a sequence $\langle B_i : i < \kappa^+ \rangle$ of Mahlo sets. We use the forcing poset $\mathbb{Q}$ to simultaneously add partial square sequences on $B_i \setminus B_{i+1}$, for each $i < \kappa^+$, while collapsing κ to become ω_2. This will place each such set in the approachability ideal $I[\omega_2]$. We make use of the approximation property of certain quotients of $\mathbb{Q}$ to show that $I[\omega_2]$ does not have a maximal set.

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Intermediate models of Prikry extensions

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(joint work with Moti Gitik, Vladimir Kanovei)

Theorem 1. *Let M be a transitive model in which U is a normal measure on the measurable cardinal κ. Let $M \subseteq M[C]$ be a generic extension by Prikry forcing with U. Then every intermediate transitive model N, $M \subseteq N \subseteq M[C]$ of* ZFC *is of the form $N = M[D]$ for some $D \subseteq C$. If D is finite then $N = M$, and if D is infinite then $N = M[D]$ is again a Prikry generic extension of M. Hence Prikry forcing is a "minimal" forcing.*

By Theorem 1, the intermediate models of the Prikry extension $M \subseteq M[C]$ are parametrized by $\mathcal{P}(\omega) \cap M$ modulo finite; we have the following $\subseteq$-$\subseteq$ /fin-isomorphisms:

$$(\{N|M \subseteq N \subseteq M[C]\}, \subseteq) \cong (\mathcal{P}(C) \cap M[C], \subseteq /\text{fin}) \cong (\mathcal{P}(\omega) \cap M, \subseteq /\text{fin})$$

Since every intermediate model of $M \subseteq M[C]$ is generated by a set of ordinals it suffices to show

Lemma 1. $\forall X \in M[C], X \subseteq \text{Ord}\, \exists D \subseteq C, D \in M[C] M[X] = M[D]$.

One can prove the Lemma using forcing arguments or, as we shall do below, within the framework of iterated ultrapowers. Let

$$(M_m, U_m, \kappa_m, \pi_{mn})_{m \leqslant n \in \text{Ord}}$$

be the iteration of $M_0 = M$ by $U_0 = U$. Note that

$$M_m = \{\pi_{0m}(f)(\kappa_0, \ldots, \kappa_{m-1}) | f \in M_0, f : \kappa_0^m \to M_0\}$$

for $m < \omega$. The sequence $\{\kappa_m | m < \omega\}$ of critical points is a Prikry sequence for the measure U_ω:

$$\forall A \in \mathcal{P}(\kappa_\omega) \cap M_\omega (A \in U_\omega \leftrightarrow \{\kappa_m | m < \omega\} \setminus A \text{ is finite}).$$

By the elementarity of $\pi_{0\omega}$ and the homogeneity properties of Prikry forcing it suffices to prove the theorem for M_ω and $\{\kappa_m | m < \omega\}$ instead of M and C. For simplicity we write M for M_ω and C for $\{\kappa_m | m < \omega\}$. The generic extension $M[C]$ was identified by Patrick Dehornoy [1] as the intersection model

$$M[C] = \bigcap_{m<\omega} M_m.$$

Proof of Lemma 1 in the special Case: $X \subseteq \kappa_\omega$, $X \in M[C] \setminus M$.

Lemma 2. *κ_ω is singular in $M[X]$.*

Proof. For $m < \omega$ let

$$X = \pi_{0m}(f_m)(\kappa_0, \ldots, \kappa_{m-1}) \in M_m.$$

Then

$$X \cap \kappa_m = \pi_{0\omega}(f_m)(\kappa_0, \ldots, \kappa_{m-1}) \cap \kappa_m.$$

In $M[X]$,

$$\forall \zeta < \kappa_\omega \exists m < \omega \exists \xi_0, \ldots, \xi_{m-1} < \zeta : X \cap \zeta = \pi_{0\omega}(f_m)(\xi_0, \ldots, \xi_{m-1}) \cap \zeta.$$

This defines *regressive* functions. If κ_ω were regular, Fodor's theorem would yield values m_0 and $\eta_0, \ldots, \eta_{m_0}$ such that for some stationary set $S \subseteq \kappa_\omega$

$$\forall \zeta \in S X \cap \zeta = \pi_{0\omega}(f_{m_0})(\eta_0, \ldots, \eta_{m_0-1}) \cap \zeta.$$

But then

$$X = \pi_{0\omega}(f_{m_0})(\eta_0, \ldots, \eta_{m_0-1}) \cap \kappa_\omega \in M,$$

contradiction. So κ_ω must be singular in $M[X]$. □

Wellorder sequences $\alpha_0 < \ldots < \alpha_{m-1}$ and $\beta_0 < \ldots < \beta_{n-1}$ lexicographically from the top:

$(\alpha_0, \ldots, \alpha_{m-1}) \prec (\beta_0, \ldots, \beta_{n-1})$ iff there is some i such that

$$\alpha_{m-1} = \beta_{n-1}, \ldots, \alpha_{m-i} = \beta_{n-i}, \beta_{n-i-1} \text{ exists,}$$
$$\text{and if } \alpha_{n-i-1} \text{ exists, then } \alpha_{m-i-1} < \beta_{n-i-1}.$$

Lemma 3. *Let $x \in M_n$. Let $\alpha_0 < \ldots < \alpha_{m-1}$ be $\prec$-minimal such that there is $f \in M_0, f : \kappa_0^m \to M_0$ such that*

$$x = \pi_{0n}(f)(\alpha_0, \ldots, \alpha_{m-1}).$$

Then $\{\alpha_0, \ldots, \alpha_{m-1}\} \subseteq \{\kappa_0, \ldots, \kappa_{n-1}\}$. In case that $x \subseteq \kappa_n$ then $\alpha_0 < \ldots < \alpha_{m-1}$ is also $\prec$-minimal such that

$$x = \pi_{0\omega}(f)(\alpha_0, \ldots, \alpha_{m-1}) \cap \kappa_n.$$

The proof rests on the fact that any ordinal α_i with $\kappa_{l-1} < \alpha_i < \kappa_l \leqslant \kappa_n$ is of the form

$$\alpha_i = \pi_{0n}(g)(\kappa_0, \ldots, \kappa_{l-1})$$

so that α_i can be replaced by $\kappa_0, \ldots, \kappa_{l-1}$ whilst descending in the $\prec$-ordering.

Lemma 4. *In $M[X]$, there is an infinite subset $D_0 \subseteq C$ (which is cofinal in κ_ω).*

Proof. By Lemma 2 let $\{\alpha_\nu | \nu < \gamma\} \in M[X]$ be cofinal in κ_ω where $\gamma < \kappa_\omega$. Without loss of generality, $\gamma < \kappa_0$.

Work in M_0. For $\nu < \gamma$ consider the minimal κ_m such that $\alpha_\nu < \kappa_m$. By Lemma 3 let $\vec{\kappa}_\nu \subseteq C$ be a $\prec$-minimal sequence such that for some $f_\nu \in M_0$

$$\alpha_\nu = \pi_{0\omega}(f_\nu)(\vec{\kappa}_\nu).$$

Since $\gamma < \kappa_0$

$$(\pi_{0\omega}(f_\nu) | \nu < \gamma) = \pi_{0\omega}((f_\nu | \nu < \gamma)) \in M.$$

In $M[X]$, $\vec{\kappa}_\nu$ is also defined as the $\prec$-minimal sequence such that

$$\alpha_\nu = \pi_{0\omega}(f_\nu)(\vec{\kappa}_\nu).$$

Let $D_0 = \bigcup_{\nu<\gamma} \vec{\kappa}_\nu \in M[X]$, $D_0 \subseteq C$. If D_0 were finite then

$$\{\alpha_\nu | \nu < \gamma\} \subseteq \{\pi_{0\omega}(f_\nu)(\vec{\kappa}) | \nu < \gamma, \vec{\kappa} \subseteq D_0\} \in M$$

would make κ_ω singular in M, contradiction. □

Work in M_0. Let $\lambda_0 < \lambda_1 < \dots$ enumerate D_0. For $m < \omega$ let $\vec{\kappa}_m \subseteq C$ be $\prec$-minimal such that there is a function $f_m \in M_0$ such that

$$X \cap \lambda_m = \pi_{0\omega}(f_m)(\vec{\kappa}_m) \cap \lambda_m. \tag{1}$$

Let $D = D_0 \cup \bigcup_{m<\omega} \vec{\kappa}_m \subseteq C$. Observe that

$$(\pi_{0\omega}(f_m)|m < \omega) = \pi_{0\omega}((f_m|m < \omega)) \in M. \tag{2}$$

By (1) and (2), $X \in M[D]$.

Conversely, $D_0 \in M[X]$, and $(\vec{\kappa}_m|m < \omega)$ can be defined in $M[X]$ by: $\vec{\kappa}_m$ is $\prec$-minimal such that

$$X \cap \lambda_m = \pi_{0\omega}(f_m)(\vec{\kappa}_m) \cap \lambda_m.$$

Hence $D \in M[X]$. Thus $M[X] = M[D]$. **QED**

The **Proof of Lemma 1 in the general Case**: $X \subseteq \lambda$, $X \in M[C] \setminus M$ for $\lambda > \kappa_\omega$ by *forcing methods* is based on ideas by Moti Gitik, analysing $M[D] \subseteq M[C]$ as a forcing extension by a quotient of Prikry forcing which satisfies the κ_ω-chain condition in $M[D]$. On the day after my talk at Oberwolfach I found an alternative argument working with the above *intersection model*.

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Forcing the truth of a weak form of Schanuel's conjecture

Matteo Viale

Schanuel's conjecture states that the transcendence degree over $\mathbb{Q}$ of the $2n$-tuple $(\lambda_1, \dots, \lambda_n, e^{\lambda_1}, \dots e^{\lambda_n})$ is at least n for all $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ which are linearly independent over $\mathbb{Q}$; if true it would settle a great number of elementary open problems in number theory, among which the transcendence of e over π.

Wilkie [7], and Kirby [3, Theorem 1.2] have proved that there exists a smallest countable algebraically and exponentially closed subfield K of $\mathbb{C}$ such that Schanuel's conjecture holds relative to K (i.e. modulo the trivial counterexamples, $\mathbb{Q}$ can be replaced by K in the statement of Schanuel's conjecture). We prove a slightly weaker result (i.e. that there exists such a countable field K without specifying that there is a smallest such) using the forcing method and Shoenfield's absoluteness theorem.

Specifically:

- We first use a canonical identification of the B-names for complex numbers for a complete boolean algebra B with the space $C^+(St(\mathsf{B}))$ of continuous functions from the Stone space of B into the one point compactification of the complex numbers with preimage of the point at infinity nowhere dense. This characterization is due indipendently to myself and Vaccaro [5], Jech [2], Ozawa [4].

- Next we show, by means of Ax's theorem on the Schanuel property for differential fields with exponentiation [1], that a strong form of Schanuel's conjecture holds between $C^+(St(\mathsf{B}))$ and the complex numbers for any complete boolean algebra B.
- Finally we use the forcing notion that collapses the ground model complex numbers to become countable to argue that the above Schanuel property holds in this generic extension between the complex numbers and a countable subfield. We conclude using Shoenfield's absoluteness, since the latter statement is Σ^1_2.

This result suggests that forcing can be a useful tool to prove theorems (rather than independence results) and to tackle problems in domains which are apparently quite far apart from set theory.

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Full Iterability of the $\mathbf{K}^c$ Construction in the Presence of Woodin Cardinals

Martin Zeman
(joint work with Grigor Sargsyan)

In inner model theory, a $\mathbf{K}^c$ construction serves, among many other things, as an important step in the construction of the true core model $\mathbf{K}$. In order to construct $\mathbf{K}$, it is necessary to prove the full iterability of $\mathbf{K}^c$, that is, to find an iteration strategy for $\mathbf{K}^c$ that acts on iteration trees of arbitrary size. Typical proofs of full iterability of $\mathbf{K}^c$ are based on a reflection argument [2], which, in turn, uses a smallness condition for the levels of $\mathbf{K}^c$ in a substantial way.

If the universe does not have a proper class inner model with a Woodin cardinal, the reflection argument can be carried out, and produces an iteration strategy for levels of $\mathbf{K}^c$ which is guided by Q-structures, that is, structures which definably destroy the Woodinness of a given ordinal. More precisely, for each iteration tree $\mathcal{T}$ such a strategy assigns the unique cofinal well-founded branch b such that the branch model $\mathcal{M}^{\mathcal{T}}_b$ has the Q-structure for $\delta(\mathcal{T})$ as an initial segment. As already mentioned above, this argument breaks down if the universe has a proper class model with a Woodin cardinal. Moreover, if the universe has a Woodin cardinal then, by an argument of Woodin, $\mathbf{K}$ does not exist, so $\mathbf{K}^c$ is not fully iterable.

In certain type of universes, however, it is possible to prove a full iterability of $\mathbf{K}^c$ and therefore construct $\mathbf{K}$. We give one such example. This universe has a highly specific feature, namely it has a pair (P, Σ) where P premouse with a Woodin cardinal and Σ is a iteration strategy for P which is fullness preserving and has branch condensation [1]. The existence of such a pair makes it possible to run a reflection argument for iterability of levels of the $\mathbf{K}^c$-construction, but this time the iteration strategy for these levels is not guided only by Q-structures, but also by certain iteration maps which arise from iterating P according to Σ. The latter applies in situations where the Woodinness of $\delta(\mathcal{T})$ cannot be destroyed. We carry out the proof of iterability under the smallness assumption that all levels of the $\mathbf{K}^c$-construction are tame [3], but conjecture that this hypothesis can be significantly weakened.

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Ultrafilters over measurable cardinals

Moti Gitik

Aki Kanamori asked in [1] the following three questions:

(1) Is there a $\kappa-$complete ultrafilter over a measurable κ with an infinite number of Rudin-Frolik predecessors?

(2) If $\{U_\tau \mid \tau < \kappa\}$ is a family of distinct $\kappa-$complete ultrafilters over a measurable κ and E is any $\kappa-$complete ultrafilter over κ, is there an $X \in E$ so that $\{U_\tau \mid \tau \in X\}$ is a discrete family?

(3) If $\mathcal{U}$ and $\mathcal{V}$ are $\kappa-$complete ultrafilters over a measurable κ such that $\mathcal{U} \times \mathcal{V} \leq_{R-K} \mathcal{V} \times \mathcal{U}$, is there a $\mathcal{W}$ and integers n and m so that $\mathcal{U} \simeq \mathcal{W}^n$ and $\mathcal{V} \simeq \mathcal{W}^m$?

We give an affirmative answer to the first question and negative answers to the second and the third.

Namely, the following was shown:

Theorem 1. *(a) If $o(\kappa) \geq 2$ in the core model, then there is a cardinal preserving generic extension with a $\kappa-$complete ultrafilter over κ with infinitely many predecessors in the Rudin-Frolik ordering.*
(b) The existence of a $\kappa-$complete ultrafilter over κ with infinitely many predecessors in the Rudin-Frolik ordering implies that $o(\kappa) \geq 2$ in the core model.

Kanamori showed in [1], that at least $0^\dagger$ is needed.

Theorem 2. *(a) If $o(\kappa) \geq \kappa + 1$ in the core model, then there is a cardinal preserving generic extension with a family $\{U_\tau \mid \tau < \kappa\}$ of distinct $\kappa-$complete ultrafilters over κ and a $\kappa-$complete ultrafilter E over κ, so that $\{U_\tau \mid \tau \in X\}$ is a not discrete family for any $X \in E$, is at most $o(\kappa) = \kappa + 1$.*
(b) Suppose that there are a family $\{U_\tau \mid \tau < \kappa\}$ of distinct $\kappa-$complete ultrafilters over κ and a $\kappa-$complete ultrafilter E over κ, so that $\{U_\tau \mid \tau \in X\}$ is not a discrete family for any $X \in E$. Then $\{o(\alpha) \mid \alpha < \kappa\}$ is unbounded in κ in the Mitchell core model.

Concerning the third question, Solovay gave an affirmative answer once "$\mathcal{U} \times \mathcal{V} \leq_{R-K} \mathcal{V} \times \mathcal{U}$" is replaced by "$\mathcal{U} \times \mathcal{V} \simeq \mathcal{V} \times \mathcal{U}$", and Kanamori once $\mathcal{U}$ is a $p-$point, see [1] 5.7, 5.9. We show the following:

Theorem 3. *Assume $o(\kappa) = \kappa$. Then in a cardinal preserving generic extension there are two $\kappa-$complete ultrafilters $\mathcal{U}$ and $\mathcal{V}$ over κ such that*

(1) $\mathcal{V} >_{R-K} \mathcal{U}$,
(2) $\mathcal{V} \times \mathcal{U} >_{R-K} \mathcal{U} \times \mathcal{V}$.

Theorem 4. *Assume $o(\kappa) = \kappa$. Then in a cardinal preserving generic extension there are two $\kappa-$complete ultrafilters $\mathcal{U}$ and $\mathcal{V}$ over κ such that*

(1) *$\mathcal{V}$ is a normal measure,*
(2) *$\mathcal{V}$ is the projection of $\mathcal{U}$ to its least normal measure,*
(3) $\mathcal{V} \times \mathcal{U} >_{R-K} \mathcal{U} \times \mathcal{V}$.

Theorem 5. *Suppose that there is no inner model in which κ is a measurable with $\{o(\alpha) \mid \alpha < \kappa\}$ unbounded in it. Then for any two $\kappa-$complete ultrafilters $\mathcal{U}$ and $\mathcal{V}$ over κ, if $\mathcal{V} \times \mathcal{U} \geq_{R-K} \mathcal{U} \times \mathcal{V}$, then there is an integer n such that $\mathcal{V} =_{R-K} \mathcal{U}^n$.*

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The consistency strength of the theory ZFC + "every universally Baire set has the perfect set property"

Trevor Wilson
(joint work with Ralf Schindler)

In joint work with Ralf Schindler, we show that the perfect set property for universally Baire sets of reals is equiconsistent, modulo ZFC, to the existence of a virtual large cardinal that we call "Shelah for remarkability."

The perfect set property was formulated by Cantor in an attempt to prove the continuum hypothesis. It says that a given set of reals either is countable or has a perfect subset. Clearly every open set of reals has this property. If every set of reals could be shown to have this property, the continuum hypothesis would follow. As a first step, Cantor and Bendixson showed that every closed set of reals has the perfect set property. However, this approach to the continuum hypothesis failed as Bernstein proved the existence of a set of reals without the perfect set property.

A question remained: how far beyond the closed sets of reals does the perfect set property hold? (From now on, we will follow the custom of using "reals" to denote ω^ω with the product topology; the results of Cantor, Bendixson, and Bernstein still hold in this setting.) We can then define the *analytic* sets of reals as the projections of the closed subsets of $\omega^\omega \times \omega^\omega$, and the *coanalytic* sets of reals as the complements of the analytic sets of reals. Suslin adapted the arguments of Cantor and Bendixson to show that every analytic set of reals has the perfect set property. On the other hand, Gödel showed that if $V = L$, then there is a coanalytic set of reals without the perfect set property.

Regarding the extent of the perfect set property in the projective hierarchy, these results of Gödel and Suslin established the boundary of what can be proved in ZFC. To get more, one has to assume more, namely the existence of an inaccessible cardinal. Solovay showed that the following statements are equiconsistent modulo ZFC: (1) there is an inaccessible cardinal, (2) every coanalytic set of reals has the perfect set property, and (3) every set of reals in $L(\mathbb{R})$ has the perfect set property. Solovay showed that after forcing with the Levy collapse of an inaccessible cardinal, every set of reals in $L(\mathbb{R})$—and in particular every coanalytic set—has the perfect set property. On the other hand, if every coanalytic set has the perfect set property, then $\aleph_1$ is inaccessible in L. For a further exposition of the results mentioned above, see Kanamori [2].

In order to further investigate the perfect set property along these lines, one needs a higher complexity class of sets of reals for which the perfect set property is not already decided. One such class is that of the *universally Baire* sets of reals, defined by Feng, Magidor, and Woodin [1]. Such sets of reals have all the symmetric regularity properties such as Lebesgue measurability and the property of Baire, but do not necessarily have the perfect set property.

Because every coanalytic set of reals is universally Baire, the consistency strength of "every universally Baire set of reals has the perfect set property" is at least that of an inaccessible cardinal. On the other hand, Woodin showed that if there is a Woodin cardinal then every universally Baire set of reals is weakly homogeneously Suslin (see Larson [3].) Consequently, if there is a Woodin cardinal then every universally Baire set of reals has the perfect set property, and moreover, after forcing with the the Levy collapse of the least inaccessible cardinal, every set of reals in $L(\mathbb{R}, \mathrm{uB})$ has the perfect set property where uB denotes the class of all universally Baire sets of reals.

To precisely determine the consistency strength of the perfect set property for universally Baire sets of reals and the sets of reals constructible from them, we introduce a new large cardinal property. We call a cardinal κ *Shelah for remarkability* if for every function $f : \kappa \to \kappa$ there are ordinals $\lambda > \kappa$ and $\bar{\lambda} < \kappa$ such that in some generic extension there is an elementary embedding $j : V_{\bar{\lambda}} \to V_\lambda$ with $j(\mathrm{crit}(j)) = \kappa$, $\bar{\lambda} \geq f(\mathrm{crit}(j))$ and $f \in \mathrm{ran}(j)$.

Like the property of remarkability introduced by Schindler [4], this is a "virtual" large cardinal property, meaning that it asserts that certain elementary embeddings exist in a generic extension. It follows from κ being Shelah, similarly to how the remarkability of κ follows from κ being strong. Like remarkability, it is very weak: if $0^\sharp$ exists then every Silver indiscernible is Shelah for remarkability in L. It has slightly higher consistency strength than remarkability: if κ is Shelah for remarkability, then V_κ has a proper class of remarkable cardinals.

We show that the following statements are equiconsistent modulo ZFC: (1) there is a cardinal that is Shelah for remarkability, (2) every universally Baire set of reals has the perfect set property, and (3) every set of reals in $L(\mathbb{R}, \mathrm{uB})$ has the perfect set property. A brief outline of the argument follows.

After forcing with the Levy collapse of a cardinal that is Shelah for remarkability, every set of reals in $L(\mathbb{R}, \mathrm{uB})$—and in particular every universally Baire set—has the perfect set property. The key step here is to show that every universally Baire set in such a generic extension is definable from trees added by a proper initial segment of the generic filter. On the other hand, if every universally Baire set has the perfect set property, then $\aleph_1$ is Shelah for remarkability in L. To prove this, we assume that $\aleph_1$ is not Shelah for remarkability in L and use a function $f : \aleph_1 \to \aleph_1$ witnessing this, along with the axiom of choice in V, to construct a universally Baire set of cardinality $\aleph_1$ with no perfect subset.

Our argument also shows that after forcing with the Levy collapse of a cardinal that is Shelah for remarkability, every set of reals in $L(\mathbb{R}, \mathrm{uB})$ is Lebesgue measurable and has the property of Baire. For Lebesgue measurability, we can reverse this using a result of Shelah [5] to obtain equiconsistency. The consistency strength of the property of Baire for all sets of reals in $L(\mathbb{R}, \mathrm{uB})$ remains open.

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