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Mathematics of Quantitative Finance

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Abstract. The workshop on Mathematics of Quantitative Finance, organised at the Mathematisches Forschungsinstitut Oberwolfach from 26 February to 4 March 2017, focused on cutting edge areas of mathematical finance, with an emphasis on the applicability of the new techniques and models presented by the participants.

Mathematics Subject Classification (2010): 91G10, 91G20, 91G60, 91G80, 60F10, 60G22, 34E05, 60H07, 60H15, 60H10.

Introduction by the Organisers

The field of Quantitative Finance is not owned by mathematics: statistics, computer science, economics and even physics (econophysics) all have contributed and continue to contribute to this field. That said, there is a rich community within mathematics that is devoted to further applications of mathematics to finance. Early on, many efforts went into a proper understanding of the absence of arbitrage in models. The ultimate result in this direction-the fundamental theorem of asset pricing (FTAP) in its most general form due to Delbaen-Schachermayer [7, 8]-is a deep result rooted both in functional analysis and in stochastic analysis. It has been said ironically that depending where one stands this is either the most or least important theorem in the field, and indeed, there are many problems in practice where general results on the absence of arbitrage yield little insight into a concrete problem. Many of the investigations in recent years have been inspired-in more or less direct ways-from such concrete problems. Some, such as optimal order execution (optimal liquidation of large positions under market impact), are closely related to the financial crisis. Others stem from the desire to extract model-free information from market data only (such as variance swap theory). Again others aim to understand the impact of model-uncertainty in applications. We could not possibly attempt to tackle all directions of ongoing research in this meeting, but we believe it is part of the beauty of this subject that some of the most important recent developments in the field are inspired quite directly by problems from industry. The workshop focused on the following such topics and problems:

- For some time, fractional Brownian motion was considered an object of limited interest in Finance. In essence, this is due to the failure of fBm (with $H \neq 1/2$) to be a semi-martingale, thereby allowing arbitrage and hence making it a poor model for a traded asset. However, recent work by Gatheral, Jaisson and Rosenbaum [14] exhibits strong evidence that volatility is 'rough' (an estimate for SPX volatility actually gives $H \approx 0.14$, which translates to 'quite rough', also note that there are no arbitrage problems for volatility is not a traded asset). Such volatility regimes also turn out to be most relevant for option pricing, as they can resolve the long-standing problem of creating (extreme) volatility smile skews, as seen in markets, impossible to obtain with classical (finite factor Markovian) stochastic volatility models without jumps (in fact, this explosion of the short-time smile was proved earlier in [1]). We like to see fractional Brownian motion as an infinite-dimensional Markovian object. Having said that, it is clear that formulas and theories developed with great finesse for finite factor Markovian models should be carried over to this infinite dimensional setting: geometry, analysis and numerics of infinite dimensional models with components involving, e.g., fractional Brownian motions or, more generally, rough paths.

- In interest rate markets, in early 2000, a stochastic volatility model (SABR, for 'stochastic $\alpha\beta\rho$') was proposed and quickly became industry standard for its seemingly miraculous 'SABR formula', an explicit expression for implied volatility bypassing the need for time-consuming Monte Carlo routines. Behind the miracle are geometric properties of the model and large deviations theory of stochastic analysis (in the spirit of Varadhan, Molchanov and many others). The decisive link to implied volatility asymptotics is due to P. Hagan, A. Lesniewski et al. [17, 18]. Many others have explored this and related topics further, in 2014 *Comm. Pure Appl. Math.* alone published three on that matter [4, 9, 10]. Still many (functional) analytic properties of the SABR model or its relatives like (weak) second order numerical schemes, or FEM formulations, are unclear. Some discussions about these gaps were led in the meeting, by several leading specialists on that topic [2, 3, 11, 16].

- Non-linear PDE theory made its decisive appearance in Finance. The brilliant monograph [15] written by practitioners J. Guyon and P. Henry-Labordère collects a number of ideas useful in practice (typically in form of numerical algorithms), and open many mathematical questions. (A nice example is given by the McKean-Vlasov based calibration algorithm, the practically important propagation of chaos is far from clear and numerical simulations suggest 'bad' regimes in which propagation of chaos actually fails). Here many aspects are wide open: convergence proofs and rates of convergence including complexity estimates.

- The role of information in stochastic models of Finance has been treated traditionally by filtrations of σ-algebras. From a practical point of view it is clear that there is a gap between the instantaneous information arriving through price changes or news and the information entering actual trading or investment strategies, i.e. one should split the market's filtration and the trader's filtration. To consider this delay or, more generally, information gap, as a fundamental property of markets, i.e. to develop a theory where trading decision are made with respect to smaller filtrations whereas models are written on a larger filtration, in other words a theory of Bayesian Finance, has only been considered in special cases. Actual models, where such structures are reflected, could include recently introduced uncertain volatility models [6]. Remarkably such models behave extremely well from the point of view of calibration, but have never been considered systematically from a fundamental point of view. In particular the corresponding FTAP for Bayesian Finance and its connection to robust Finance is unclear.

- In recent years, particular attention has been given to model-free Finance, where, instead of relying on a specific (class of) model(s), data provides the essential structure of the pricing framework, in the form of lower and upper bounds for option prices. Optimal transport tools therefore received a warm welcome in quantitative finance, allowing us to free ourselves from the sometimes too narrow applicability of models.

These fundamental topics were investigated in detail in the 42 talks (of varied lengths) given by some of the participants during the workshop. As the above research topics illustrate, 20 years after settling the fundamental theorems, the field is sparkling with new ideas from all directions of applied mathematics and beyond.

Peter Friz, Antoine Jacquier, Josef Teichmann

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Abstracts

Rough volatility: An overview

Jim Gatheral

The scaling properties of historical volatility time series, which now appear to be universal, motivate the modelling of volatility as the exponential of fractional Brownian motion. This model can be understood as reflecting the high endogeneity of liquid markets and the long memory of order flow. The Rough Bergomi model which is the simplest corresponding model under the pricing measure Q fits the implied volatility surface remarkably well. Recent advances include the computation of an explicit characteristic function for a natural fractional generalization of the Heston model and the construction of an efficient simulation scheme, the so-called Hybrid BSS scheme.

Calibration of rough volatility models is still work in progress. The current focus is on a specific model-free quantity, *stochasticity*, which is straightforward to compute for each expiration both in the model and in principle from the volatility smile. It is an open question whether or not the volatility smile can be extrapolated from visible strikes robustly enough to permit accurate resolution of stochasticity from market data.

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Robust replication of barrier-style claims on price and volatility

Matthew Lorig
(joint work with Peter Carr, Roger Lee)

We consider a frictionless market (i.e., no transaction costs) and fix an arbitrary but finite time horizon $T < \infty$. For simplicity, we assume zero interest rates, no arbitrage, and take as given an equivalent martingale measure (EMM) $\mathbb{P}$ chosen by the market on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. The filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ represents the history of the market. All stochastic processes defined below live on this probability space and all expectations are with respect to $\mathbb{P}$ unless otherwise stated.

Let $B = (B_t)_{0 \leq t \leq T}$ represent the value of a zero-coupon bond maturing at time T. As the risk-free rate of interest is zero by assumption, we have $B_t = 1$ for all $t \in [0, T]$. Let $S = (S_t)_{0 \leq t \leq T}$ represent the value of a risky asset. We assume S is strictly positive and has continuous sample paths. To rule out arbitrage, it is well-known that the asset S must be a martingale under the pricing measure $\mathbb{P}$. As such, there exists a non-negative, $\mathbb{F}$-adapted stochastic process $\sigma = (\sigma_t)_{0 \leq t \leq T}$ such that

$$\mathrm{d}S_t = \sigma_t S_t \mathrm{d}W_t, \qquad S_0 > 0,$$

where W is a Brownian motion with respect to the pricing measure $\mathbb{P}$ and the filtration $\mathbb{F}$. Henceforth, the process σ will be referred to as the *volatility process*. We assume that the volatility process σ is right-continuous and $\mathbb{F}$-adapted, that it evolves independently of W and that it satisfies

$$\int_0^T \sigma_t^2 \mathrm{d}t < c < \infty, \tag{1}$$

for some arbitrarily large but finite constant $c > 0$. Note that σ may experience jumps and is not required to be Markovian. It will be convent to introduce $X = (X_t)_{0 \leq t \leq T}$, the log price process $X_t = \log S_t$. As S is strictly positive by assumption, the process X is well-defined and finite for all $t \in [0, T]$. A simple application of Itô's Lemma yields

$$\mathrm{d}X_t = -\frac{1}{2}\sigma_t^2 \mathrm{d}t + \sigma_t \mathrm{d}W_t, \qquad X_0 = \log S_0. \tag{2}$$

Note that a claim on (the path of) S can always be expressed as a claim on (the path of) $X = \log S$. For any $\mathbb{F}$-stopping time τ, we define its T-bounded counterpart

$$\tau^* := \tau \wedge T.$$

Note that, by construction, τ^* is an $\mathbb{F}$-stopping time. Let $C_{\tau^*}(K)$ denote the time τ^* price of a European call written on S with maturity date T and strike price $K > 0$, and let $P_{\tau^*}(K)$ denote the price of a European put written on S with the same strike and maturity. By no-arbitrage arguments, we have

$$\begin{aligned} C_{\tau^*}(K) &= \mathbb{E}_{\tau^*}(S_T - K)^+ = \mathbb{E}_{\tau^*}(\mathrm{e}^{X_T} - K)^+, \\ P_{\tau^*}(K) &= \mathbb{E}_{\tau^*}(K - S_T)^+ = \mathbb{E}_{\tau^*}(K - \mathrm{e}^{X_T})^+, \end{aligned} \tag{3}$$

where we have introduced the shorthand notation $\mathbb{E}_{\tau^*} \cdot := \mathbb{E}[\,\cdot\,|\mathcal{F}_{\tau^*}]$. For convenience, we will sometimes refer to a European call or put written on X rather than S with the understanding that these are equivalent. We assume that a European call or put with maturity T trades at every strike $K \in (0, \infty)$.

In this work, we consider claims written on the path of X in four varieties

$$\begin{aligned} \text{European-style}: &\qquad \varphi(X_T, \langle X \rangle_T), \\ \text{knock-out}: &\qquad \mathbb{1}_{\{\tau > T\}} \varphi(X_T, \langle X \rangle_T), \\ \text{knock-in}: &\qquad \mathbb{1}_{\{\tau \leq T\}} \varphi(X_T - X_{\tau^*}, \langle X \rangle_T - \langle X \rangle_{\tau^*}), \\ \text{rebate}: &\qquad \mathbb{1}\{\tau \leq T\} \varphi(\langle X \rangle_{\tau^*}), \end{aligned}$$

where τ is the first exit time of some interval I:

$$\tau = \inf \{t \geq 0 : X_t \notin I\}.$$

We use the phrase 'European-style' to indicate that a claim payoff depends only on the terminal values X_T and $\langle X \rangle_T$ and not on any barrier event (e.g., knock-in or knock-out).

The main result of this paper is to show that , under certain growth and regularity conditions on φ, each of path-dependent claims considered above can be replicated by self-financing portfolio consisting of bonds, the underlying stock, and a strip of European puts and calls. In particular, we give an explicit construction of this portfolio.

The main open question is how to relax the assumption that the volatility process σ is independent of the Brownian motion W that drives the stock price process. Relaxing this assumption would enable us to consider models for S that induce an asymmetric implied volatility smile (which is what is observed empirically in equity markets).

On the link between the Malliavin derivative operator and the implied volatility behaviour: Can we expect the Malliavin calculus to be useful in applications?

Elisa Alòs

(joint work with Jorge A. León, Josep Vives)

In this talk, we will take a walk through some recent applications of Malliavin calculus.We will see examples where the Malliavin calculus techniques have been used to solve real problems. We will discuss about how, when and why Malliavin calculus can become a useful tool.

More precisely, we use the Malliavin calculus techniques to obtain an expression for the short-time behaviour of the at-the-money implied volatility skew in term of the Malliavin derivative of the volatility process. Our techniques do not need the volatility to be neither a diffusion, nor a Markov process. The obtained results give us a useful tool in modelling problems. In particular, they prove that stochastic volatility models based on the fractional Brownian motion (fBm) can be an interesting process to describe the short-time behaviour of the implied volatility surface.

It is well-known that classical stochastic volatility diffusion models, where the volatility also follows a diffusion process, capture some important features of the implied volatility. For example, its variation with respect to the strike price, described graphically as a *smile* or *skew* [6]. But the observed implied volatility exhibits dependence not only on the strike price, but also on time to maturity (*term structure*). Unfortunately, the term structure is not easily explained by classical stochastic volatility models. For instance, a popular rule-of-thumb for the short-time behaviour with respect to time to maturity, based on empirical observations, states that the skew slope is approximately $\mathcal{O}((T-t)^{-\frac{1}{2}})$, while the rate for these stochastic volatility models is $\mathcal{O}(1)$ (see [3, 4, 5]). Note that in these models, for reasonable coefficients in their dynamics, volatility behaves almost as a constant, on a very short-time scale. Consequently, returns are roughly normally distributed and the skew becomes quite flat. On the other hand, Fouque, Papanicolaou, Sircar and Solna [2] have introduced continuous diffusion models again to describe the empirical short-time skew. Their idea is to include suitable coefficients that depend on the time till the next maturity date and that guarantee the variability is large enough near the maturity time.

The main goal of this work is to provide a method based on the techniques of the Malliavin calculus to estimate the rate of the short-dated behaviour of the implied volatility stochastic volatility models, where the volatility does not need to be neither a diffusion nor a Markov process. It is well-known that the Malliavin calculus is a powerful tool to deal with anticipating processes. Because of the Clark-Ocone formula, one should expect the properties of the implied volatility to be able to be 'translated' in terms of the Malliavin derivative of the implied volatility. Moreover, as the future volatility is not adapted, this theory becomes a natural tool to analyse this problem. Hence, now it is possible to deal with a volatility in a class that includes either fractional processes with parameter in $(0,1)$, Markov processes, or processes with time-varying coefficients, among others.

In this talk we will consider the following model for the log-price of a stock under a risk-neutral probability measure Q:

$$(1)\quad X_t = x + (r-\lambda k)t - \frac{1}{2}\int_0^t \sigma_s^2 ds + \int_0^t \sigma_s(\rho dW_s + \sqrt{1-\rho^2}dB_s),\ t\in[0,T].$$

Here, x is the current log-price, r is the instantaneous interest rate, W and B are independent standard Brownian motions, $\rho\in(-1,1)$. The volatility process σ is a square-integrable stochastic process with right-continuous trajectories and adapted to the filtration generated by W. Moreover, we denote

- $v_t := \left(\frac{Y_t}{T-t}\right)^{\frac{1}{2}}$, with $Y_t := \int_t^T \sigma_s^2 ds$, denotes the future average volatility.
- For any $\tau > 0$, $p(x,\tau)$ will denote the centered Gaussian kernel with variance τ^2. If $\tau = 1$ we will write $p(x)$.

- $BS(t,x,\sigma)$ will denote the price of an European call option under the classical Black-Scholes model with constant volatility σ, current log stock price x, time to maturity $T-t$, strike price K and interest rate r. Remember that in this case:
$$BS(t,x,\sigma) = e^x N(d_+) - Ke^{-r(T-t)}N(d_-),$$
where N denotes the cumulative probability function of the standard normal law and
$$d_\pm := \frac{x - x_t^*}{\sigma\sqrt{T-t}} \pm \frac{\sigma}{2}\sqrt{T-t},$$
with $x_t^* := \ln K - r(T-t)$.
- $\mathcal{L}_{BS}(\sigma)$ will denote the Black-Scholes differential operator, in the log variable, with volatility σ :
$$\mathcal{L}_{BS}(\sigma) = \partial_t + \frac{1}{2}\sigma^2\partial^2_{xx} + (r - \frac{1}{2}\sigma^2)\partial_x - r\cdot$$
It is well known that $\mathcal{L}_{BS}(\sigma)BS(\cdot,\cdot,\sigma) = 0$.
- $G(t,x,\sigma) := (\partial^2_{xx} - \partial_x)BS(t,x,\sigma)$.

The anticipating Itô's formula allow us to prove the following extension of the Hull and White formula that gives the price of an European call option as a sum of the price when the model has no correlation plus one term that describes the impact of the correlation on option prices.

Theorem 1. *Assume the model (1) holds with $\sigma \in \mathbb{L}^{1,2}$. Then it follows that*
$$V_t = E(BS(t,X_t,v_t)|\mathcal{F}_t) + \frac{\rho}{2}E(\int_t^T e^{-r(s-t)}\partial_x G(s,X_s,v_s)\Lambda_s ds|\mathcal{F}_t),$$
where $\Lambda_s := (\int_s^T D_s\sigma_r^2 dr)\sigma_s$.

The above theorem is the key that, with some limit arguments, allow us to prove the following result.

Theorem 2. *Consider the model (1). Then under some regularity conditions:*

(1) *Assume that there exists a $\mathcal{F}_t$-measurable random variable $D_t^+\sigma_t$ such that, for every $t>0$,*
$$\sup_{s,r\in[t,T]} \left|E\left(\left(D_s\sigma_r - D_t^+\sigma_t\right)\middle|\, \mathcal{F}_t\right)\right| \to 0,$$
a.s. as $T \to t$. Then
$$\lim_{T\to t}\frac{\partial I_t}{\partial X_t}(x_t^*) = -\frac{1}{\sigma_t}\left(\lambda k + \rho\frac{D_t^+\sigma_t}{2}\right).$$

(2) *Assume that there exists a $\mathcal{F}_t$-measurable random variable $L_t^{\delta,+}\sigma_t$ such that, for every $t>0$,*
$$\frac{1}{(T-t)^{2+\delta}}\int_t^T\int_s^T E\left(D_s\sigma_r|\,\mathcal{F}_t\right)drds - L_t^{\delta,+}\sigma_t \to 0,$$

a.s. as $T \to t$. Then

$$\lim_{T\to t}(T-t)^{-\delta}\frac{\partial I_t}{\partial X_t}(x_t^*) = -\frac{\rho}{\sigma_t}L_t^{\delta,+}\sigma_t.$$

Notice that the above result implies that, for models based on the fBm, the short-time limit of the at-the-money skew tends to infinity, as observed in real market data.

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Consistency of option prices under bid-ask spreads

Stefan Gerhold

(joint work with Ismail Cetin Gülüm)

Calibrating martingales to given option prices is a central topic of mathematical finance, and it is thus a natural question which sets of option prices admit such a fit, and which do not. Note that we are not interested in *approximate* model calibration, but in the consistency of option prices, and thus in arbitrage-free models that fit the given prices *exactly*. Put differently, we want to detect arbitrage in given prices. We do not consider continuous call price surfaces, but restrict to the (practically more relevant) case of finitely many strikes and maturities. Therefore, consider a financial asset with finitely many European call options written on it. In a frictionless setting, the consistency problem is well understood: Carr and Madan [1] assume that interest rates, dividends and bid-ask spreads are zero, and derive necessary and sufficient conditions for the existence of arbitrage free models. Essentially, the given call prices must not admit calendar or butterfly arbitrage. Davis and Hobson [3] include interest rates and dividends and give similar results. They also describe explicit arbitrage strategies, whenever arbitrage exists.

As with virtually any result in mathematical finance, robustness with respect to market frictions is an important issue in assessing the practical appeal of these findings. Somewhat surprisingly, not much seems to be known about the consistency problem in this direction, the single exception being a paper by Cousot [2]. He allows positive bid-ask spreads on the options, but not on the underlying, and finds conditions on the prices that determine the existence of an arbitrage-free model explaining them.

The novelty of our work is that we allow a bid-ask spread on the underlying. Without any further assumptions on the size of this spread, it turns out that there is no connection between the quoted price of the underlying and those of the calls: Any strategy trying to exploit unreasonable prices can be made impossible by a sufficiently large bid-ask spread on the underlying. In this respect, the problem is *not* robust w.r.t. the introduction of a spread on the underlying. However, an arbitrarily large spread seems questionable, given that spreads are usually tight for liquid underlyings. We thus enunciate that the appropriate question is not 'when are the given prices consistent', but rather 'how large a bid-ask spread on the underlying is needed to explain them?' We thus put a bound $\epsilon \geq 0$ on the (discounted) spread of the underlying and want to determine the smallest such ϵ that leads to a model explaining the given prices. We then refer to the call prices as ϵ-consistent (with the absence of arbitrage).

We assume discrete trading times and finite probability spaces throughout; no gain in tractability or realism is to be expected by not doing so. In the case of a single maturity, we obtain necessary and sufficient conditions for ϵ-consistency. The multi-period problem, on the other hand, seems to be challenging. We provide two partial results: necessary (but presumably not sufficient) conditions for ϵ-consistency, and necessary and sufficient conditions under simplifying assumptions. The latter, in particular, drop the bid-ask spread on the options, retaining only the spread on the underlying.

Recall that the main technical tool used in the papers [1, 2, 3] mentioned above to construct arbitrage-free models is Strassen's theorem [5], or modifications thereof. In the financial context, this theorem essentially states that option prices have to increase with maturity. This property breaks down if a spread on the underlying is allowed. We will therefore employ a recent generalization of Strassen's theorem, obtained in [4]. It gives necessary and sufficient conditions for the existence of martingales within a prescribed distance, measured in terms of the infinity Wasserstein distance. This generalized Strassen theorem will be the key to obtain the ϵ-consistency conditions under simplified assumptions mentioned above.

Open problem: As mentioned above, we did not succeed in finding necessary and sufficient conditions for ϵ-consistency for option price data with more than one maturity. However, this seems to be a difficult problem, and we are not sure whether simple conditions exist.

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Smoothing the payoff for efficient computation of Basket option prices

Christian Bayer
(joint work with Raul Tempone, Markus Siebenmorgen)

In quantitative finance, the price of an option on an underlying S can typically–disregarding discounting–be expressed as $E[f(S)]$ for some (payoff) function f on S and the expectation operator E induced by the appropriate pricing measure. Hence, option pricing is an integration problem. The integration problem is usually challenging due to a combination of two complications:

- S often takes values in a high-dimensional space. The reason for the high dimensionality may be time discretization of a stochastic differential equation, path dependence of the option (i.e., S is actually a path of an asset price, not the value at a specific time), a large number of underlying assets, or others.
- the payoff function f is typically not smooth.

In this work, we focus on the problem of pricing basket options in models, where the distribution of the underlying is explicitly given to us. Specifically, we consider multivariate Black-Scholes and Variance-Gamma models, i.e., models, for which no time discretization is required. We consider a basket option on a d-dimensional underlying asset $S_T = \left(S_T^1, \dots, S_T^d\right)$ with payoff function

$$f(S_T) = \left(\sum_{i=1}^{d} w_i S_T^i - K \right)^+$$

for some positive weights $w_1, \dots, w_d$, a maturity T and a strike price K. Observe in passing that one could also allow some weights to be negative, an option type known as 'spread option'. Note that in addition, (discrete) Asian options also fall under this framework.

Even in the standard Black-Scholes framework, closed-form expressions for basket option prices are not available, since sums of log-normal random variables are generally not log-normally distributed.

Efficient numerical integration algorithms are even available in high dimensions, but they usually require smoothness of the integrand. Hence, they are a priori not applicable in many option pricing problems. We will specifically focus on (adaptive) sparse-grid methods [1]. Even for quasi Monte Carlo methods, smoothness of the integrand is required in theory, even though the method seems to work well in many practical examples lacking smoothness.

From a numerical analysis point of view, the most obvious solution to the problem is to smoothen the integrand using standard mollifiers, and there is a prominent history of successful application of mollification in quantitative finance. For many financial applications, there seems to be a more attractive approach that avoids the balancing act between providing the smoothness needed for the numerical integration algorithm and introducing bias in the integrand. Indeed, we suggest using the smoothing property of the distribution of the underlying itself for regularizing the integrand. This technique is quite standard in a time-stepping setting, and we indeed plan to explore its applicability in that context in the future.

In this work, however, the regularization will be achieved by integrating against one factor of the multivariate geometric Brownian motion first–conditioning on all the other factors. More specifically, we show that we can always decompose

$$\sum_{i=1}^{d} w_i S_T^i \overset{\mathcal{L}}{=} H e^Y$$

for two independent random variables H and Y. Here, the random variable Y is normally distributed. Therefore, by computing the conditional expectation given H, the basket option valuation problem is reduced to an integration problem in H (corresponding to an integration in $\mathbb{R}^{d-1}$) with a payoff function given in this case by the Black-Scholes formula, a smooth function. The key observations are:

- the mollified integrand is explicitly given and analytic;
- the mollification procedure does not introduce any bias.

At this stage, efficient numerical integration procedures become available. For instance, adaptive sparse grids can be constructed for the mollified integrand and lead to very good performance in low and moderate dimensions (of up to around 35 in our experiments).

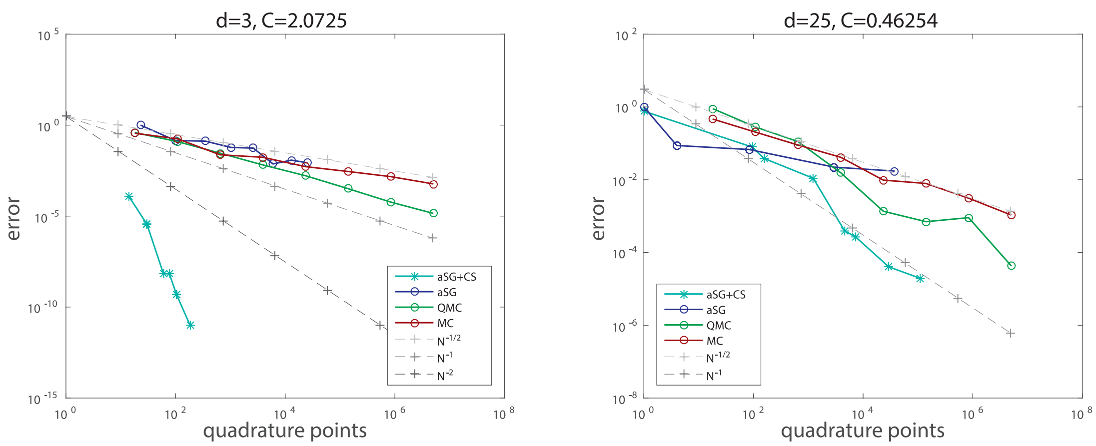

FIGURE 1. Acceleration of the (Q)MC quadrature with a sparse-grid control variate for $d = 3$ and $d = 25$ with volatilities selected randomly from the interval $[0.3, 0.4]$. Depicted are: adaptive sparse grid with (turquoise) and without (blue) mollified integrand, MC (red) and QMC (green) without mollified integrand.

For instance, in Figure 1, we see that the adaptive sparse grid construction without smoothing breaks almost completely even in dimension 3, whereas the adaptive sparse grid with mollified payoff still performs better than MC and QMC in dimension $d = 25$.

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Short dated option pricing under rough volatility

Blanka Horvath

(joint work with Philipp Harms, Antoine Jacquier and also with Christian Bayer, Peter Friz, Archil Gulishashvili, Benjamin Stemper)

Implied volatility, as a unit-less indicator of option prices, is at the very centre of quantitative finance, and understanding its precise behaviour has been the focus of praticitioners' and academics' for several decades. Recently Gatheral, Jaisson and Rosenbaum [11] proposed a new class of models able to remarkably accurately fit and forecast volatility time series. Following this seminal paper, Bayer, Friz and Gatheral [3] studied the pricing problem in this class of models. Specifically, Bayer, Friz and Gatheral [3] report on striking aptitudes of a natural model in this class in reproducing some distinctive features of the implied volatility which traditional volatility models so far were notoriously unable to capture. Asymptotic results in this direction [1, 8, 9, 10] arrive at similar conclusions, reinforcing the potential prowess of this class of models. In this model class, the instantaneous volatility of the price process is stochastic, but driven not by a standard Brownian motion, but by a fractional Brownian motion, hence allowing for memory (aka non Markovianity) of the volatility process. Generalising their model slightly, the stock price process satisfies the following system of stochastic differential equations:

$$\begin{aligned} \mathrm{d}S_t &= \sigma_t S_t \mathrm{d}B_t, & S_0 > 0, \\ \mathrm{d}\sigma_t &= b(\sigma_t)\mathrm{d}t + a(\sigma_t)\mathrm{d}W_t^H, & \sigma_0 > 0, \end{aligned} \tag{1}$$

where the Hurst coefficient $H \in (0,1)$ determines the degree of smoothness (or roughness) of the continuous fractional Brownian motion W^H and where the coefficients $b(\cdot)$ and $a(\cdot)$ are assumed to be regular enough. The two Gaussian drivers B and W^H are correlated via the Volterra representation of the latter.

In this talk I report on two lines of research of this class of models from an asymptotic point of view: One line of results (obtained jointly with Philipp Harms and Antoine Jacquier) focusses on density asymptotics for this class of models, the other line of results (obtained jointly with Christian Bayer, Peter Friz, Archil Gulisashvili and Benjamin Stemper) studies the asymptotics of call prices near the money directly, when the time to maturity becomes small.

- **Density asymptotics for rough stochastic volatility models**: For models in the fractional volatility family, where the existence and smoothness of the density is given, we revisit small-noise expansions in the spirit of Benarous, Baudoin-Ouyang, Deuschel-Friz-Jacquier-Violante for bivariate diffusions driven by fractional Brownian motions with different Hurst exponents. We derive suitable expansions in these fractional stochastic volatility models and infer corresponding expansions for implied volatility. This sheds light (i) on the influence of the Hurst parameter in the time-decay of the smile and (ii) on the asymptotic behaviour of the tail of the smile, including higher orders.
- **Extending density results within the fractional volatility family:** for a fixed time $t \geq 0$, existence and smoothness of the density of S_t or of the couple (S_t, σ_t) is by now classical when $H = 1/2$ (standard Brownian motion), or when the other driver B is also fractional with the same Hurst exponent. These results go back to Malliavin [14] and have been extended by many authors, including Baudoin-Hairer [2], Cass-Friz [5]. However, in this mixed class of models, no precise results exist, and we aim at extending this literature in this direction. We intend to follow two approaches: first following the classical steps of Malliavin's proof, via Hörmander's theorem (combining results by Nualart [16] and Baudoin-Hairer [2]), second via the theory of rough paths-albeit with possibly stronger conditions on the coefficients of the process $(\sigma_t)_{t\geq 0}$. Regarding the latter, in the uncorrelated case, it is possible to build upon results Cass-Friz's results [5]. The correlated hypoelliptic case is less 'obvious' and requires some more work, currently in progress.
- **Call price asymptotics near the money**: With Christian Bayer, Peter Friz, Archil Gulishashvili and Benjamin Stemper, we explore an intriguingly direct novel way of addressing (uniformly with respect to the strike) the asymptotic behaviour of vanilla options as time to maturity becomes small. This general approach applies to a large class of 'classical' (ranging from the Black Scholes to stochastic volatility) models, and carries over to the setting of rough models (as in (1). Both in the standard and in the fractional setting, this approach somehow extends the results by Deuschel-Friz-Jacquier-Violante [6, 7] in the sense that it bypasses the need for the (so far ubiquitously prevalent) derivation of asymptotic expansions of the density of the process. That said, our approach applies in a regime where options are 'moderately out of the money' (with maturity-dependent strike), which interpolates between the 'at-the-money' and the 'out-of-the-money' regimes of option prices.

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On small time asymptotics for rough differential equations driven by fractional Brownian motions

Cheng Ouyang

(joint work with Fabrice Baudoin, Xuejing Zhang)

Stochastic differential equations driven by fractional Brownian motions have been introduced to model random evolution phenomena whose noise has long range dependence, and have found successful applications in biotechnology and biophysics. Fox example, it is used to model the sub diffusion of electrons within a protein molecule.

In this talk, we survey some results on the small time asymptotics of the density function of the solutions to such SDEs, such as Varadhan asymptotics and full expansion of the density function.

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Markovian representation of fractional Brownian motion

Philipp Harms

(joint work with David Stefanovits)

Carmona, Coutin, Montseny and Muravlev [1, 2, 4] observed that fractional Brownian motion (fBm) can be represented as a superposition of infinitely many Ornstein-Uhlenbeck (OU) processes. The key idea is to express the fractional integral in the Mandelbrot-Van Ness representation of fBm by a Laplace transform. Indeed, for each $H < 1/2$, one has by the stochastic Fubini theorem that

$$\int_0^t (t-s)^{H-\frac{1}{2}} dW_s \propto \int_0^t \int_0^\infty e^{-x(t-s)} \frac{dx}{x^{H+\frac{1}{2}}} dW_s = \int_0^\infty \int_0^t e^{-x(t-s)} dW_s \frac{dx}{x^{\frac{1}{2}+H}}.$$

Note that the stochastic integral on the right-hand side is an OU process with speed of mean reversion x. Similar representations exist also in the case $H > 1/2$ [1, 2, 4], but one has to work around the fact that the function $s \mapsto s^{H-1/2}$ is not the Laplace transform of any measure. We found a way of doing this such that the collection of representing processes can be interpreted as a time-homogeneous affine process on a state space of integrable functions (cf. Theorem 2).

Theorem 1 ([3]). *For each $H \in (0,1)$ and $t \in [0,\infty)$ let*

$$W_t^{\mathrm{H}} = \frac{1}{\Gamma(\frac{1}{2}+H)} \int_{-\infty}^t \left((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}} \right) dW_s,$$

where $(W_t)_{t\in\mathbb{R}}$ is a two-sided Brownian motion. Then

$$W_t^H = \begin{cases} \int_0^\infty (Y_t^x - Y_0^x)\mu(dx), & \text{if } H < \frac{1}{2}, \\ \int_0^\infty (Z_t^x - Z_0^x)\nu(dx), & \text{if } H > \frac{1}{2}, \end{cases}$$

where

$$(1) \qquad dY_t^x = -xY_t^x dt + dW_t, \qquad dZ_t^x = -xZ_t^x dt + Y_t^x dt,$$

$$(2) \qquad Y_0^x = \int_{-\infty}^0 e^{sx} dW_s, \qquad Z_0^x = -\int_{-\infty}^0 s e^{sx} dW_s,$$

$$(3) \qquad \mu(dx) = \frac{dx}{x^{\frac{1}{2}+H}\Gamma(\frac{1}{2}+H)\Gamma(\frac{1}{2}-H)}, \qquad \nu(dx) = \frac{dx}{x^{H-\frac{1}{2}}\Gamma(\frac{1}{2}+H)\Gamma(\frac{3}{2}-H)}.$$

Our main contribution is to show that $(Y_t, Z_t)_{t\in[0,\infty)}$ is an affine process on $L^1(\mu) \times L^1(\nu)$, where the spatial variable x is suppressed in our notation.

Theorem 2 ([3]). *The stochastic differential equation* (1) *defines an affine process with continuous sample paths on* $L^1(\mu) \times L^1(\nu)$*, where* μ *and* ν *are given by* (3).

The affine structure exhibited by this result allows one to construct some tractable financial models with fractional features. For example we construct in [3] a fractional extension of the Stein-Stein stochastic volatility model [5]. The extended model can be brought into an affine form by expressing the SDE for the log price X_t in terms of the tensor product $Y_t \otimes Y_t$.

Theorem 3 ([3]). *Let* $(Y_t)_{t\in[0,\infty)}$ *be the affine process of Theorem 2, let* $(\widetilde{W}_t)_{t\in[0,\infty)}$ *be a Brownian motion, possibly correlated with* $(W_t)_{t\in[0,\infty)}$*, and let* $(X_t)_{t\in[0,\infty)}$ *be given by*

$$dX_t = -\frac{1}{2}\left(\int_0^\infty Y_t^x \mu(dx)\right)^2 dt + \int_0^\infty Y_t^x \mu(dx) dW_t, \qquad X_0 = 0.$$

Then $(X_t, Y_t \otimes Y_t)_{t\in[0,\infty)}$ *is an affine process on* $\mathbb{R} \times \{u \otimes u : u \in L^1(\mu)\}$.

These results led to many questions, which were discussed during the workshop and could be investigated in future research. First, it might be of interest to replace the Brownian driver by some Lévy process; conceptually, this should be possible. Second, it is of practical importance to determine the convergence rate when (μ, ν) is replaced by a sum of Dirac measures; Carmona and Coutin [2] obtained some results in this direction. Third, it would be nice to obtain a Markovian representation for the fractional Cox-Ingersoll-Ross process which appears in the works of J. Gatheral, M. Rosenbaum and O. El Euch; it was shown by J. Teichmann and C. Cuchiero during the workshop that this works for a Hawkes process approximations, but it is unclear otherwise. Fourth, it was asked wether the Markovian representation leads to a natural notion of stochastic integral with respect to fBm; this led to extensive and ongoing discussions with F. Biagini and P. Krühner.

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Pathwise large deviations for the rough Bergomi model

Henry Stone
(joint work with Antoine Jacquier, Mikko Pakkanen)

In this presentation we study the small-time behaviour of the correlated rough Bergomi model, introduced by Bayer, Friz and Gatheral [1]. We show that a rescaled version of the log stock price process satisfies a large deviations principle, where the rate function is defined in terms of the reproducing kernel Hilbert space of the Gaussian measure on $C([0,1],\mathbb{R}^2)$ induced by the two dimensional process (Z,B). We then use the large deviations principle to deduce the small-time asymptotic behaviour of the implied volatility in the rough Bergomi model, where log-moneyness is time dependent. Finally, we provide a numerical scheme to compute the rate function as the solution to an infinite dimensional minimisation problem; we then use this scheme to numerically compute the small-time implied volatility.

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Rough paths, Signatures and the modelling of functions on streams

Terry Lyons

Rough path theory allows a new way to describe streamed data; instead of the Kolmogorov perspective of saying what is the state at given times one aims to give consistent if approximate 'descriptions' of the stream over intervals in a partition. This is achieved by looking at the controlling effect of the stream on certain highly canonical nonlinear systems. This description of the stream through the low order terms in the signature over any interval allows effective description of highly complex data streams with a small number of coefficients.

Moreover, these very coordinate signatures are the analogue of monomials and the way that the linear combinations of monomials approximate smooth functions on $\mathbb{R}^d$ has its analogue for functions on stream space. They span an algebra.

As a result they form an ideal local feature set for describing and learning functions on data based around paths and evolving systems.

Recent work of Weixin Yang et al. [1] which achieves state of the art in classifying human behaviour in moving images by reducing people to skeletal structures of strokes reduced by the signature to new strokes capturing the evolution and uses the signature feature set twice as well as a fully connected neural net to achieve these highly competitive results.

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Bounds for VIX Futures given S&P 500 smiles

Julien Guyon

(joint work with Romain Menegaux, Marcel Nutz)

We derive sharp bounds for the prices of VIX futures by using the full information of S&P 500 smiles at two maturities. The VIX (short for volatility index) is published by the Chicago Board Options Exchange (CBOE) and used as an indicator of short-term options-implied volatility. By definition, the VIX is the implied volatility of the 30-day variance swap on the S&P 500; see [4]. Equivalently, using the well-known link between realized variance and log-contracts [10], the VIX at date T_1 is the implied volatility of a log-contract that delivers $\ln(S_{T_2}/S_{T_1})$ at $T_2 = T_1 + \tau$, where $\tau = 30$ days and S_{T_i} is the S&P 500 at date T_i:

$$(\mathrm{VIX}_{T_1})^2 = -\frac{2}{\tau}\mathrm{Price}_{T_1}\left[\ln\left(\frac{S_{T_2}}{S_{T_1}}\right)\right];$$

we are assuming zero interest rates, repos, and dividends for simplicity. The log-contract can itself be replicated at T_1 using call and put options on the S&P 500 with maturity T_2. The VIX index cannot be traded, but VIX futures can: the VIX future expiring at T_1 is an instrument that pays VIX_{T_1} at T_1. While $\mathrm{VIX}^2_{T_1}$ can be replicated, its square root VIX_{T_1} cannot; instead, sub/superreplication in the S&P 500 and its options leads to model-free lower/upper bounds on the price of the VIX future.

The classical sub/superreplication argument is based on the fact that one can replicate any affine function of $\mathrm{VIX}^2_{T_1}$ at T_1 using cash and log-contracts with maturities T_1 and T_2. Thus, one searches for the sub/superreplication of the square root function by an affine function that gives the maximum/minimum portfolio price. Since the square root is a concave function, it is below all its tangent lines, and the classical superreplication boils down to selecting the line that gives the minimum portfolio price. This argument shows that, in the absence of arbitrage, the price of the VIX future at time $T_0 = 0$ cannot exceed the implied volatility σ_{12} of the forward-starting log-contract on the index, starting at the VIX future's expiry T_1 and maturing at T_2,

$$\sigma_{12}^2 \equiv -\frac{2}{\tau}\mathrm{Price}_{T_0}\left[\ln\left(\frac{S_{T_2}}{S_{T_1}}\right)\right].$$

Subreplicating the VIX future using the same instruments corresponds to subreplicating the square root by an affine function. This yields zero as a lower bound for the future's price, which is clearly a poor estimate.

These classical bounds are suboptimal in the sense that they only use the prices of log-contracts. Our aim is, instead, to extract the full information contained in the S&P 500 smiles at T_1 and T_2, by also including all vanilla options at these maturities as (static) hedging instruments, as well as trading (dynamically, i.e., at T_1) in the S&P 500 itself and the log-contract. Moreover, we allow the deltas at T_1 to depend on the information available, that is, the S&P 500 and the VIX index at T_1.

The first part of the paper analyzes this problem for general smiles. We formulate the sub/superreplication as a linear programming (LP) problem and define absence of arbitrage in this setting. The latter leads to the existence of risk-neutral joint distributions μ for $(S_{T_1}, S_{T_2}, \mathrm{VIX}_{T_1})$ which constitute the domain of an optimization problem dual to sub/superreplication. The first two marginals μ_1 and μ_2 are given by the market smiles at T_1 and T_2, whereas the distribution of VIX_{T_1} merely satisfies a certain constraint. The dual problem is thus reminiscent of a (constrained) martingale optimal transport problem, but falls outside the transport framework because the third marginal is not prescribed. This necessitates a novel argument for our duality theorem which establishes the absence of a duality gap, i.e., primal and dual problem have the same value. This theorem holds, more generally, for an option payoff $f(S_{T_1}, S_{T_2}, \mathrm{VIX}_{T_1})$ rather than just the VIX. As a last abstract contribution, we characterize those smiles μ_1, μ_2 for which the classical bounds for the VIX future are optimal. The lower bound is optimal if and only if $\mu_1 = \mu_2$, which never happens in practice. The characterization for the upper bound is more subtle, it states that a convex-order condition in two dimensions holds, or equivalently that a model with constant forward volatility is contained in the dual domain.

While our theoretical bounds are sharper than the classical ones, the corresponding hedging portfolios can only be found numerically, and the numerical problem is far from trivial. Aiming for a balance between flexibility and tractability, we introduce a family of functionally generated portfolios that are determined by a one-dimensional convex/concave function and a constant. The space of one-dimensional convex functions is easy to search numerically, and the generated portfolios are guaranteed to satisfy the sub/superhedging conditions at all values of the underlying, by our construction. We show that the lower price bound obtained by functionally generated portfolios improves the classical one as soon as $\mu_1 \neq \mu_2$ and here the generating function can be chosen explicitly of an inverse 'hockey stick' form.

In the second part of the paper, we study specific families of smiles μ_1, μ_2 and corresponding portfolios. The case where μ_2 is a Bernoulli distribution gives rise to a 'complete market' where the VIX future can be replicated. While the classical upper bound is not sharp unless μ_1 has a very particular form, we show how functionally generated portfolios lead to the sharp bound as given by the unique risk-neutral expectation. When μ_2 is a general distribution with compact support, we present various sufficient conditions for the classical upper bound to be suboptimal. Finally, we discuss a family of examples for which the classical upper bound is already sharp.

The third part of the paper presents numerical experiments using smiles from market data as well as smiles generated by a SABR model. We compare the classical bounds, the bounds obtained from functionally generated portfolios, and the bounds computed by an LP solver that correspond to the theoretical, optimal bounds modulo discretization error. For the generating functions, we use piecewise linear maps and a cut square root; the latter yields the best approximation in our experiments. The results suggest that the classical lower bound can be improved dramatically by functionally generated portfolios; the bound from the LP solver is only slightly better. On the other hand, the classical upper bound is already surprisingly sharp for typical smiles.

Turning to the existing literature on volatility derivatives, the most closely related work is due to De Marco and Henry-Labordère [6] who investigate bounds for VIX options, i.e., calls and puts on the VIX, given the smile of the S&P 500 and the VIX future as liquidly tradable instruments. Thus, compared to [6], we take a step back by investigating bounds for the VIX future itself, given the smile of the S&P 500. The sub/superreplication problem in [6] leads to a linear program with a dual akin to (constrained) martingale optimal transport. The numerical results show that, for typical market smiles, the optimal upper bound on VIX options is equal to an analytical (a priori suboptimal) bound that the authors derive. For a further discussion of numerical solutions to sub/superreplication problems, we also refer to [9], and to [1, 8] for background on martingale optimal transport. While [6] and the present paper consider derivatives on options-implied volatility, previous literature has studied derivatives on realized volatility. Using power payoffs, Carr and Lee [2] show that, if the returns and the volatility of an asset are driven by independent Brownian motions, the asset smile at a given maturity T determines the distribution of the realized variance at T, hence allowing perfect replication of derivatives on realized variance. Using business-time hedging, Dupire [7] derives a lower bound for a call on realized variance at a given maturity T, given the asset smile at T. Carr and Lee [3] extend Dupire's idea to tackle the cases of puts on realized variance as well as forward-starting calls and puts on realized variance. More recently, Cox and Wang [5] have derived the optimal portfolio subreplicating convex functions of realized variance.

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Why rough volatility?

Mathieu Rosenbaum
(joint work with Omar El Euch, Jim Gatheral)

It can be nowadays considered a stylized fact that volatility is rough. Indeed, on all types of assets, one typically obtains that the time series of historical log volatilities is very well approximated by a fractional Brownian motion with Hurst parameter of order 0.1. This finding is confirmed in the derivatives world, where implied volatility surfaces can be remarkably fitted by rough volatility models. Our goal here is to understand the origins of this universal property.

In the first part of this work, we propose some microstructural foundations of rough volatility models based on the structure of modern financial markets. More precisely we consider a market where the four following properties are satisfied:

- High degree of endogeneity, meaning that most of the orders are only sent by reaction to other orders, with somehow no economic value.
- No statistical arbitrage.
- Asymmetry in liquidity on buy and sell sides of the order book.
- Significant presence of metaorders, which are large orders whose execution is split in time.

These four properties are clear features of modern markets in the context of high frequency trading. To combine them in a high frequency price model, we use Hawkes processes. More precisely, our microscopic price process is given by the difference of the two components of a bidimensional Hawkes process. Indeed, each of these properties is very easily translated in term of the parameters of the model. In particular, high degree of endogeneity corresponds to a Hawkes kernel with L^1 norm of its largest eigenvalue close to one and the metorders splitting phenomenon can be obtained with a kernel with fat tails.

We show that after suitable renormalization, our microscopic Hawkes based price model converges in the long run to a rough volatility model, more precisely a rough Heston model. This shows that when the four stylized facts mentioned above are combined (here thanks to the framework of Hawkes processes), rough volatility can naturally arise.

In the second part of this work, we provide even more fundamental explanations for the rough behavior of the volatility. We start from the two following postulates:

- If there is some permanent market impact, it has to be linear (to prevent roundtrip arbitrages).
- The impact of metaorder with volume V right after execution is square root of V (to ensure diffusive prices).

These two postulates are nowadays well admitted and understood. Assuming the order flow is governed by a Hawkes process, we show that under the two preceding postulates, volatility is necessary rough. This result is a first step towards understanding why volatility is systematically rough. Such feature seems in fact related to some type of no-arbitrage principle at high frequency.

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Option market making with competition

Johannes Muhle-Karbe

(joint work with Bruno Bouchard, Martin Herdegen)

Option pricing models typically consider either monopolistic market makers, or settings with infinite competition. Real markets, however, are often dominated by a few large dealers with substantial market power. We study how such imperfect competition is reflected in equilibrium option prices.

To this end, we study a three-stage Stackelberg game of the following form. The dealers move first and quote competitive price schedules. The clients in turn decide how many shares of the option to buy or sell from each dealer. Finally, the dealers hedge their option positions in a market for the underlying of the option, where they interact through their common price impact.

The corresponding hedging strategies can be characterized in terms of a system of coupled but linear forward-backward stochastic differential equations; explicit formulas obtain in the limit for small price impact. These formulas allow to derive the 'effective' risk aversions used by the dealers to determine their optimal price schedules.

A Black-Scholes inequality: applications and generalisations

Michael Tehranchi

1. The semigroup of call prices: binary operation and involution

This note studies the structure of the family of functions

$$\mathcal{C} = \left\{ C : [0,\infty) \to [0,1] : \text{ convex, } C(\kappa) \geq (1-\kappa)^+ \text{ for all } \kappa \geq 0 \right\}$$

Elements of the set $\mathcal{C}$ can be given a probabilistic interpretation:

Proposition 1. *The following are equivalent:*

(1) $C \in \mathcal{C}$.

(2) *There is a non-negative random variable S with $\mathbb{E}(S) \leq 1$ such that*

$$C(\kappa) = \mathbb{E}[(S-\kappa)^+] + 1 - \mathbb{E}(S) = 1 - \mathbb{E}(S \wedge \kappa) \text{ for all } \kappa \geq 0.$$

We say that S is a primal representation of C.

(3) *There is a non-negative random variable S^* with $\mathbb{E}(S^*) \leq 1$ such that*

$$C(\kappa) = \mathbb{E}[(1 - S^*\kappa)^+] = 1 - \mathbb{E}[1 \wedge (S^*\kappa)] \text{ for all } \kappa \geq 0.$$

We say that S^* is a dual representation of C.

The connection between the function $C \in \mathcal{C}$ and its representations is given by the Breeden-Litzenberger formulae

$$\mathbb{P}(S > \kappa) = -C'(\kappa) \text{ and } \mathbb{P}(S^* < 1/\kappa) = C(\kappa) - \kappa C'(\kappa) \text{ for all } \kappa > 0,$$

where C' denotes this right-hand derivative.

For $C \in \mathcal{C}$, let $C^*(0) = 1$ and

$$C^*(\kappa) = 1 - \kappa + \kappa C(1/\kappa) \text{ for all } \kappa > 0.$$

The probabilistic meaning of the operation * is that S is a primal representation of C if and only if S is a dual representation of C^*.

When $C(\infty) = 0$ or equivalently the primal representation has $\mathbb{E}(S) = 1$, the quantity $C(\kappa)$ can interpreted as the price of a call. Consider a market with a stock whose forward price is $(F_{t,T})_{0 \leq t \leq T}$ for a fixed maturity date $T > 0$. We assume that there exists an equivalent measure (a T-forward measure) such that the forward price of every claim is just the expected value of its payout. In particular we have $\mathbb{E}(F_{T,T}) = F_{0,T}$ and hence the normalised initial price of a call with strike K is $\mathbb{E}[(F_{T,T} - K)^+]/F_{0,T} = \mathbb{E}[(S-\kappa)^+] = C(\kappa)$ where $S = F_{T,T}/F_{0,T}$ and $\kappa = K/F_{0,T}$.

The case where $C(\infty) > 0$ is more subtle, but still can be interpreted in terms of call prices in the context of certain continuous-time arbitrage-free markets exhibiting a bubble in the sense of [1] in which the forward price $(F_{t,T})_{0 \leq t \leq T}$ of the underlying asset is a non-negative strictly local martingale.

We now introduce a binary operation $\bullet$ on $\mathcal{C}$ defined by

$$C_1 \bullet C_2(\kappa) = \inf_{\eta > 0} [C_1(\eta) + \eta C_2(\kappa/\eta)] \text{ for } \kappa \geq 0.$$

This operation has a probabilistic interpretation:

Theorem 1. *Let S_1 be a primal representation of $C_1 \in \mathcal{C}$, and S_2^* a dual representation of $C_2 \in \mathcal{C}$, where S_1 and S_2^* are defined on the same space. Then we have*

$$C_1 \bullet C_2(\kappa) \geq 1 - \mathbb{E}[S_1 \wedge (S_2^* \kappa)] \text{ for all } \kappa \geq 0,$$

with equality if S_1 and S_2^* are countermonotonic.

Note that when $\mathbb{E}(S_1) = 1$ we have

$$C_1 \bullet C_2(\kappa) = \max_{S_1, S_2^*} \mathbb{E}[(S_1 - S_2^* \kappa)^+] \text{ for all } \kappa \geq 0,$$

where the maximum is taken over all primal representations S_1 of C_1 and dual representations S_2^* of C_2 defined on the same probability space. In particular, the quantity $C_1 \bullet C_2(\kappa)$ gives the upper bound on the no-arbitrage price of an option to swap κ shares of an asset with price S_2^* for one share of another asset with price S_1, given all of the call prices of both assets. This is interpretation is related to the upper bound on basket options found in [4].

We now come to the key observation of this note. To state it, we distinguish two particular elements $E, Z \in \mathcal{C}$ defined by

$$E(\kappa) = (1 - \kappa)^+ \text{ and } Z(\kappa) = 1 \text{ for all } \kappa \geq 0.$$

Note that the random variables representing E and Z are constant, with $S = 1 = S^*$ representing E, and $S = 0 = S^*$ representing Z.

Theorem 2. *The set $\mathcal{C}$ of call functions is a noncommutative semigroup with respect to the binary operation $\bullet$, with involution *, identity element E and absorbing element Z.*

2. One-parameter semigroups and peacocks

We now study the family of sub-semigroups of $\mathcal{C}$ indexed by a single parameter $y \geq 0$. We will make use of the following notation. For a probability density function f, let

$$C_f(\kappa, y) = \int_{-\infty}^{\infty} (f(z + y) - \kappa f(z))^+ dz = 1 - \int_{-\infty}^{\infty} f(z + y) \wedge [\kappa f(z)] dz$$

for $y \in \mathbb{R}$ and $\kappa \geq 0$. Note that C_φ is the Black-Scholes call pricing function where $\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ is the standard normal density.

The one-parameter sub-semigroups of $\mathcal{C}$ can be characterised completely:

Theorem 3. *Suppose $C(\cdot, y) \in \mathcal{C}$ for all $y \geq 0$, where $C(\cdot, 0) = E$ and $C(\cdot, y) \neq E$ and $C(\cdot, y) \neq Z$ for some $y > 0$. The following are equivalent*
 (1) *$\{C(\cdot, y) : y \geq 0\}$ is a one-parameter semigroup.*
 (2) *$C = C_f$ for a log-concave density f.*

We can introduce a partial order $\leq$ on $\mathcal{C}$ by

$$C_1 \leq C_2 \text{ if and only if } C_1(\kappa) \leq C_2(\kappa) \text{ for all } \kappa \geq 0.$$

The operation $\bullet$ interacts well with this partial ordering:

Proposition 2. *For any $C_1, C_2 \in \mathcal{C}$, we have $C_1 \leq C_1 \bullet C_2$ and $C_2 \leq C_1 \bullet C_2$.*

The partial order can be given a useful probabilistic interpretation when restricted to the family $\mathcal{C}_1$ of call functions C with $C(\infty) = 0$ whose primal representation S satisfies $\mathbb{E}(S) = 1$. The following is well-known; see [3].

Proposition 3. *Given $C_1, C_2 \in C_1$ with primal representations S_1, S_2. Then the following are equivalent*

(1) $C_1 \leq C_2$
(2) *S_1 is dominated by S_2 in the convex order.*

Combining Theorem 3 and Propositions 2 and 3 yields the following tractable family of peacocks.

Theorem 4. *Let f be a log-concave density with $f(z) > 0$ for all $z < 0$, let be a random variable Z have density f and let $Y : [0, \infty) \to [0, \infty)$ be increasing. Set*

$$S_t = \frac{f(Z + Y(t))}{f(Z)} \text{ for } t \geq 0.$$

The family of random variables $(S_t)_{t \geq 0}$ is a peacock.

Define the function Y_φ by

$$y = Y_\varphi(\kappa, c) \Leftrightarrow C_\varphi(\kappa, y) = c.$$

The quantity $Y_\varphi(\kappa, c)$ denotes the implied total standard deviation in the Black-Scholes model of an option of moneyness κ whose normalised price is c. The upshot of Theorem 4 is that we can define a family of arbitrage-free implied volatility surface by

$$(\kappa, t) \mapsto \frac{1}{\sqrt{t}} Y_\varphi(\kappa, C_f(\kappa, Y(t))).$$

Given f and Y, the above formula is reasonably tractable, and could be seen to be in the same spirit as the SVI parametrisation of the SVI implied volatility surface [2]. We can recover the Black-Scholes model by setting $f = \varphi$ and

$$Y(t) = \sigma\sqrt{t},$$

where σ is the volatility of the stock.

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Model-free Pricing of multivariate derivatives

Carole Bernard
(joint work with Oleg Bondarenko, Steven Vanduffel)

Short summary of the talk: There are well developed techniques to infer the risk neutral distribution of an asset return when a wide range of options prices written on this asset is available ([1], [5], [4]). In this paper, we develop a new algorithm to infer the 'implied' dependence among asset returns (i.e. the dependence under the risk neutral probability) by using the information available in the market from index options and options on its components. It is model-free, non parametric, consistent with maximum entropy principle. Our method allows to price any path-independent multivariate derivatives. Our method is inspired by the rearrangement algorithm (RA) of [13] that was originally used to minimize the variance of a portfolio in which the distributions of the components are known but their interdependence is not.

A more detailed presentation is as follows:

Introduction
The goal of this paper is to model dependence among asset prices using information from option prices only. We can find two ways in the literature to answer this question depending on whether we want to make assumptions about a model or we opt for a model-free approach. The first approach consists of building an appropriate model for the multidimensional underlying process. Constructing a multi-asset model consistent with observed index options and individual stock options is a challenging task. For instance, [9] propose a mixture of models that can reproduce some set of multivariate option prices and individual options. They then derive a notion of implied correlation as the correlation matrix such that they can fit their model. They find that there is a set of such matrices and show evidence of model risk and estimation risk for the implied correlation. Other studies are conducted by [2] and [12]. The second approach is model-free and seeks to measure the implied dependence among assets without making model assumptions. This is the direction that we are taking.

An attempt to measure the implied correlation in a model-free way is proposed by the CBOE S&P 500 Implied Correlation Indexes. The CBOE started to disseminate such indexes in July 2009, with historical values back to 2007. They are now well accepted measures based on individual implied volatilities and index implied volatility and thus driven by option prices only. The white paper of [7] describes the CBOE implied correlation indexes in full detail. We show that the CBOE implied correlation has some limitations that we want to overcome. The CBOE implied correlation seeks to be model-free. However, we show using examples that the correlation parameter that is obtained from this procedure is consistent with a very strong assumption that each component of the index as well as the index itself is modeled by a LogNormal distribution.

[11] propose to price correlation risk. [10] is a companion paper in which the authors propose a stochastic correlation model. They then use it to extract an implied correlation within their model assumptions. [10] have a definition of implied correlation inspired by the same expression as in the CBOE implied correlation index formula but the volatilities are not implied volatilities but model free implied variances from index and individual options. Their model is relatively simple as all pairwise correlation are driven by the same correlation $\rho(t)$ at time t that is a mean reverting process (in the same spirit as [8]). Some extensions are proposed by [6].

Model-free Approach to Infer Dependence

In this paper, we propose to use the methodology studied in [3] to infer a dependence (copula) among the assets in the market using the information conveyed by option prices only.

To do so, we need option prices on individual assets, as well as multivariate options such as basket options or spread options, or options on a weighted sum of the individual assets. Using this information, we infer a dependence structure among the assets that is consistent with maximum entropy (in the sense of Shannon entropy of the resulting multivariate distribution) and that is consistent with the option prices.

Consider an index defined as

$$S = \sum_{i=1}^{d} \omega_i X_i \tag{1}$$

where the weights ω_i are constant over some period of time.

Inputs: Marginal distributions of individual components and of index

- $X_1 \sim F_1$, ..., $X_d \sim F_d$
- the cdf of $\omega_1 X_1 + ... + \omega_d X_d \sim G$ is known for some $\omega_i \in \mathbb{R}$

The above input distributions include the case of baskets options and spread options.

Output: A dependence structure (copula) that is consistent with this given set of information.

The methodology can be described concisely as follows

- Fit risk neutral marginal distributions of individual components X_i using 3-month options for example.
- Fit the marginal distribution of the index (with known components X_i, i.e., $S = \sum \omega_i X_i$) using 3-month options on the index.
- Then, run the BRA (Algorithm described in [3]). The algorithm constructs a copula that is consistent with the marginal distributions and the distribution of the sum (or the weighted sum).

Note that the algorithm identifies one particular dependence among all possible candidate solutions that are consistent with prices of individual options and prices of index options. As shown by [3], the resulting dependence has the property of maximum Shannon entropy when marginal distributions are normally distributed

(Gaussian copula and correlation matrix with maximum determinant). When margins are skewed or heavier tailed, only the property on the correlation matrix with maximum determinant is satisfied.

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Perfect Hedging with rough Heston models

Omar El Euch

(joint work with Mathieu Rosenbaum)

It has been recently shown that rough volatility models, where the volatility is driven by a fractional Brownian motion with small Hurst parameter, provide very relevant dynamics in order to reproduce the behavior of both historical and implied volatilities. However, due to the non-Markovian nature of the fractional Brownian motion, they raise new issues when it comes to derivatives pricing and hedging.

We use an original link between nearly unstable Hawkes processes and fractional volatility models to build a microscopic model based on Hawkes processes which behaves on the long run as a rough version of the Heston model. Thanks to this convergence result and by computing the characteristic function of the microscopic price we pass to the limit to derive the characteristic function of rough Heston log price. In the classical Heston model, the characteristic function is expressed in terms of the solution of a Riccati equation. Here we show that rough Heston models exhibit quite a similar structure, the Riccati equation being replaced by a fractional Riccati equation.

Noticing that the conditional law of a rough-Heston model is still of a rough-Heston dynamic with same parameters but with a different forward variance curve, we are also able to compute explicitly the conditional characteristic function of the log price which is a deterministic function of the spot price and the forward variance curve. Hence, we derive the dynamics of the characteristic function process and therefore the dynamics of any Vanilla option price showing that the theoretical hedging instruments are the spot price and the forward variance curve.

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Probability density of lognormal fractional SABR model

Tai-Ho Wang
(joint work with Jiro Akahori, Xiaoming Song)

The celebrated Black and Black-Scholes-Merton models have been the benchmark for European options on currency exchange, interest rates, and equities since the inauguration of the trading on financial derivatives. However, empirical evidences have shown that the main drawback of these models is the assumption of constant volatility; the key parameter required in the calculation of option premia under such models. The volatility parameters induced from market data are in fact non-constant across markets; dubbed as *volatility smile*. The Stochastic $\alpha\beta\rho$ (SABR for short hereafter) model, suggested by Hagan, Lesniewski, Woodward [7], is one of the models, such as local volatility models, stochastic volatility models, and exponential Lévy type of models etc, that attempts to capture the volatility smile effect. Furthermore, as opposed to local volatility models, in SABR model the volatility smile moves in the same direction as the underlying with time [6].

The SABR model is depicted by the following system of stochastic differential equations (SDEs):

$$dF_t = \alpha_t F_t^\beta dW_t, \quad F_0 = F, \tag{1}$$

$$d\alpha_t = \nu\alpha_t dZ_t, \quad \alpha_0 = \alpha, \tag{2}$$

with $\beta \in [0,1]$, where F_t is the forward price and α_t is the instantaneous volatility. W_t and Z_t are correlated Brownian motions with constant correlation coefficient ρ. The SABR model is at times referred to as the lognormal SABR model when $\beta = 1$. The SABR formula is an asymptotic expansion for the implied volatilities of call options with various strikes in small time to expiry. Let $\sigma_{BS}(K,\tau)$ be the implied volatility of a vanilla option struck at K and time to expiry τ. The SABR formula to the lowest order states

$$\sigma_{BS}(K,\tau) = \nu \frac{\log(F/K)}{D(\zeta)} \{1 + O(\tau)\} \tag{3}$$

as time to expiry τ approaches 0. The function D and the parameter ζ involved in (3) are defined respectively as

$$D(\zeta) = \log\left(\frac{\sqrt{1-2\rho\zeta+\zeta^2}+\zeta-\rho}{1-\rho}\right)$$

and

$$\zeta = \begin{cases} \frac{\nu}{\alpha}\frac{F^{1-\beta}-K^{1-\beta}}{1-\beta} & \text{if } \beta \neq 1; \\ \frac{\nu}{\alpha}\log\left(\frac{F}{K}\right) & \text{if } \beta = 1. \end{cases}$$

Generally, the SABR formula is given one order higher, up to order τ.

The geometry of SABR model is isometrically diffeomorphic to the two dimensional hyperbolic space or the Poincaré plane. This isometry leads to a derivation of the SABR formula (3) based on an expression of the heat kernel, known as the McKean kernel, on Poincaré plane. In particular, the lowest order term in (3) has a geometric interpretation. The function D is the shortest geodesic distance from the spot value (F_0, α_0) to the vertical line $F = K$ in the upper half plane $\{(F,\alpha) \in \mathbb{R}^2 : \alpha \geq 0\}$. Hence, the lowest order term in (3) is indeed the ratio between the absolute value of logmoneyness, i.e., $\log(K/F_0)$, and the shortest geodesic distance from (F_0, α_0) to the vertical line $F = K$ in the upper half plane. As expression for heat kernel on hyperbolic space is concerned, Ikeda and Matsumoto in [8] provided a probabilistic approach and obtained, among other interesting results, a representation for the transition density of hyperbolic Brownian motion, i.e., the heat kernel over the Poincaré plane.

The volatility process is generally conceived behaving 'fractionally"in that the driving noise is a fractional process, e.g., a fractional Brownian motion with Hurst exponent other than a half. Models that attempt to incorporate the fractional feature of volatility include: the ARFIMA model in [5] and the FIGARCH model [1] for discrete time models; the long memory stochastic volatlity model in [2] and the affine fractional stochastic volatilility model in [3] for continuous time models. Somewhat on the contrary, in a recent study in [4], the Hurst exponent H is estimated as being less than a half; thereby indicating antipersistency as opposed to persistency of the volatility process.

In order to embed the empirically observed fractional feature of the volatility process into the classical SABR model, we suggest a fractional version of the SABR model as

$$\frac{dS_t}{S_t} = \alpha_t(\rho dB_t + \bar{\rho} dW_t), \tag{4}$$

$$\alpha_t = \alpha_0 e^{\nu B_t^H}, \tag{5}$$

where $\rho \in (-1, 1)$ and $\bar{\rho} = \sqrt{1-\rho^2}$. B_t^H is a fractional Brownian motion with Hurst exponent H driven by B_t. Modulo a mean-reversion component, this model aligns with the model statistically tested in [4]. The main observation in [4] is that, using square root of the realized/integrated variance as a proxy for the instantaneous volatility, the logarithm of the volatility process behaves like a fractional Brownian motion in almost any time scale of frequency. The Hurst exponent H inferred from the time series data is less than a half; indeed, $H \approx 0.1$. This observation of small Hurst exponent in the volatility process makes the analysis of the model more technical and challenging from stochastic analysis point of view. To our knowledge, most of the small time asymptotic expansions for processes driven by fractional Brownian motions have restrictions on the Hurst exponent H of the driving fractional Brownian motion, mostly $H \geq \frac{1}{4}$. One of the advantage of the approach undertaken in the current paper is that it works without restriction on the Hurst exponent H. The key ingredient is a representation in a Fourier space, which we call the bridge representation, for the joint density of log spot and volatility, in the spirit of [8].

A small time asymptotic expansion of the joint density is readily obtained from the bridge representation. The idea is to approximate the conditional expectation in the bridge representation by a judiciously chosen deterministic path since, conditioned on the initial and terminal points, at each point in time a Gaussian process will not wander too far away from its expectation. As long as an asymptotic expansion for the density of the underlying asset is available, obtaining an expansion for implied volatility is almost straightforward by basically comparing the coefficients with a similar expansion obtained by using the lognormal density on the Black or the Black-Scholes-Merton side.

The methodology of deriving the bridge representation can be generalized directly to obtain a bridge representation for the joint density of multiple times; hence inducing a representation for finite dimensional distributions of the fractional SABR model. Based on this bridge representation for finite dimensional distributions, we present a heuristic yet appealing derivation of sample path large deviation principle for the fractional SABR model in small time. This large deviations principle in a sense can be regarded as defining a 'geodesic distance' over the fractional SABR plane since it recovers the energy functional on the Poincaré plane when $H = \frac{1}{2}$. An immediate consequence of this sample path large deviation principle is the fractional SABR formula to the lowest order which recovers the classical SABR formula with $H = \frac{1}{2}$. The fractional SABR formula pertains the guiding principle that the lowest order term in the implied volatility expansion is given by the ratio between the absolute value of the logmoneyness and the geodesic distance to the vertical line $F = K$.

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On VIX Futures in the rough Bergomi model

Aitor Muguruza
(joint work with Antoine Jacquier, Claude Martini)

The rough Bergomi model introduced by Bayer, Friz and Gatheral [1] has been outperforming conventional Markovian stochastic volatility models by reproducing implied volatility smiles in a very realistic manner, in particular for short maturities. We investigate here the dynamics of the VIX and the forward variance curve generated by this model, and develop efficient pricing algorithms for VIX futures and options. We further analyse the validity of the rough Bergomi model to jointly describe the VIX and the SPX, and present a joint calibration algorithm based on the hybrid scheme by Bennedsen, Lunde and Pakkanen [2].

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Volatility and arbitrage

Johannes Ruf
(joint work with Bob Fernholz, Ioannis Karatzas)

The capitalization-weighted cumulative variation $\sum_{i=1}^{d} \int_0^{\cdot} \mu_i(t) \mathrm{d}\langle \log \mu_i \rangle(t)$ in an equity market consisting of a fixed number d of assets with capitalization weights $\mu_i(\cdot)$, is an observable and a nondecreasing function of time. If this observable of the market is not just nondecreasing but actually grows at a rate bounded away from zero, then strong arbitrage can be constructed relative to the market over sufficiently long time horizons. It has been an open issue for more than ten years, whether such strong outperformance of the market is possible also over arbitrary time horizons under the stated condition. We show that this is not possible in general, thus settling this long-open question. We also show that, under appropriate additional conditions, outperformance over any time horizon indeed becomes possible, and exhibit investment strategies that effect it.

In this talk, we highlight an open question concerning the precise bounds on the time horizon for which arbitrage can be guaranteed (see also Section 7 in [1]). Discussions with the workshop participants Paul Gassiat and Mete Soner seem to indicate that this gap can be closed.

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A random surface description of the capital distribution in large markets

Mykhaylo Shkolnikov

(joint work with Praveen Kolli)

We study the capital distribution in the context of the first-order models of Fernholz and Karatzas. We find that when the number of companies becomes large the capital distribution fluctuates around the solution of a porous medium PDE according to a linear parabolic SPDE with additive noise. Such a description opens the path to modeling the capital distribution surfaces in large markets by systems of a PDE and an SPDE and to understanding a variety of market characteristics and portfolio performances therein.

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PV and XVA Greeks for Callable Exotics by Algorithmic Differentiation

Alexandre Antonov

(joint work with Numerix R&D)

Greeks, or sensitivity calculations, are complicated and time-consuming operations performed by financial libraries. Traditionally, Greeks have been used by traders who needed them to hedge their risk. After the financial crisis, an extra layer of complexity was added as banks began systematically hedging different valuation adjustments, starting with the Credit Valuation Adjustment (CVA). These valuation adjustments, often referred to as XVAs (with FVA, KVA, and MVA being other major constituents), depend on the future exposure of a portfolio of instruments (at the netting set level and above in the hierarchy). As a result, we require a distribution of future prices for even a simple swap position; this can be handled using Monte Carlo (MC) methods.

The most common method of computing a sensitivity is the bump-and-reprice method, which is widely used due to the simplicity and universality of its implementation. The main drawback of the bump-and-reprice method is its slow speed: to compute an extra Greek, one needs to rerun the pricing calculation of sensitive instruments. Modern methods for computing Greeks, such as payoff differentiation or likelihood methods, are much faster than the bump-and-reprice method but are more complicated and less universal. They deliver a Greek as a mathematical derivative of a PV or XVA over a given parameter and must be tailored to the specific model or payoff.

A traditional payoff differentiation is applied path-by-path to forward instruments. It can either be direct (i.e., forward in time) or adjoint (i.e., backward in time). Direct differentiation is more efficient when the number of results (outputs) is greater than the number of parameters. Adjoint differentiation (AD) is more efficient in the opposite case and is more natural in finance, where we seek the sensitivity of a single output (the CVA) against the many risk factors influencing the portfolio.

Following the seminal paper by Giles and Glasserman in 2006 [7], AD was introduced to mathematical finance by Capriotti and Giles [5, 6] and Joshi and Yang [8] as a general method for the efficient calculation of sensitivities and has gained wide popularity. Andreasen [1] was the first to apply AD to the complicated world of XVA.

Another important aspect of AD is that it is an adjoint action, meaning that it takes place after the main (pricing) calculation. Therefore, it is necessary to record information that is generated during pricing that will be utilized later on for the AD. All of the pricing operations are recorded in a data structure commonly referred to as the tape. During the adjoint pass, the tape is 'played back' to perform the differentiation using the recorded information. The tape adds a lot of complexity to the AD implementation in the form of potential memory issues due to large storage demands, coding, debugging, and maintenance difficulties, etc.

Callable instrument prices and XVA measures are generally calculated using Monte Carlo regression (also known as least-squares Monte Carlo or American Monte Carlo). The value of a given path at a particular point in time can depend on the values of all paths (including itself) at previous time steps. This property makes a direct AD application difficult. The first detailed description of AD for exotic instruments with full regressions appeared in [3]. In that paper a technical recipe was given for the traditional application of the AD approach to general instruments. As the main result of that paper, a new differentiation technique, named 'backward differentiation" (BD), was proposed. It is applied at the time of pricing and completely avoids the instrument tape and its related complications.

In this presentation we generalize this backward differentiation method to XVA Greeks (details can be found in [4]). We start by treating cases where cashflow derivatives are sufficient for computing PV/XVA Greeks, i.e., where the differentiation of conditional expectations (or regression functions) is not necessary. For example, PV Greeks for Bermudan swaptions can be computed without having to perform the complicated step of regression function differentiation. We modify the BD algorithm to calculate Greeks for such instruments: the method is applied during the backward pricing procedure and has almost no overhead with respect to a pure backward pricing (without the Greeks).

A general XVA calculation cannot be done using only the cashflow derivatives–some exceptions are listed in this article–instead, the differentiation of future instrument values that are results of the regression may be required. We leverage the algorithmic calculation of future values (algorithmic exposures) proposed in [2] and describe adjoint differentiation (AD) and the new BD for XVA Greeks. The latter algorithm is much simpler than the former, in particular, it does not require the use of the instrument tape, i.e., it does not require the storage of certain payoff derivatives during the pricing procedure as is the case for AD. At the same time, both AD and BD enjoy a similar level of performance.

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Maximum Likelihood Estimation for Wishart processes

Aurélien Alfonsi
(joint work with Ahmed Kebaier, Clément Rey)

Wishart processes have been introduced by Bru [4] and take values in the set of positive semidefinite matrices. Let $d \in \mathbb{N}^*$ denote the dimension, $\mathcal{M}_d$ be the set of real d-square matrices, $\mathcal{S}_d^+$ be the subset of positive semidefinite matrices. Wishart processes are defined by the following Stochastic Differential Equation (SDE)

$$(1) \qquad \begin{cases} dX_t = \left[\alpha a^\top a + bX_t + X_t b^\top\right] dt + \sqrt{X}_t dW_t a + a^\top dW_t^\top \sqrt{X}_t, \quad t > 0 \\ X_0 = x \in \mathcal{S}_d^+, \end{cases}$$

where $\alpha \geq d-1$, $a \in \mathcal{M}_d$, $b \in \mathcal{M}_d$ and $(W_t)_{t\geq 0}$ denotes a d-square matrix made of independent Brownian motions. We recall that for $x \in \mathcal{S}_d^+$, $\sqrt{x}$ is the unique matrix in $\mathcal{S}_d^+$ such that $\sqrt{x}^2 = x$. Existence and uniqueness results for the SDE (1) are given by Bru [4] and Cuchiero et al. [5] in a more general affine setting.

In the last decade, there has been a growing interest to use Wishart processes for modelling, especially for financial applications. In equity, Gourieroux and Sufana [10] and Da Fonseca et al. [7] have proposed a stochastic volatility model for a basket of assets that assumes that the instantaneous covariance between the assets follows a Wishart process. This extends the well-known Heston model with many assets. Wishart processes have also been used for interest rates models: affine term structure models involving these processes have been proposed for example by Gnoatto [8] and Ahdida et al. [1]. For all these models, the question of estimating the parameters of the underlying Wishart process is relevant for practical purposes. However, there are still few studies on the estimation of its parameters. This issue has been considered by Da Fonseca et al. [6] by considering the moments, while the Maximum Likelihood Estimator (MLE) of the Cox-Ingersoll-Ross process ($d = 1$ case) has been recently by Ben Alaya and Kebaier [2, 3].

Here, following Lipster and Shiryaev [12] and Kutoyants [11], we study the Maximum Likelihood Estimator (MLE) assuming that we observe the full path $(X_t, t \in [0,T])$ up to time $T > 0$. Thanks to this assumption, $a^\top a$ can be obtained from the quadratic covariations of X and, up to a linear transformation, the estimation problem boils down to estimate the drift parameters (b, α) when $a = I_d$. We then calculate the likelihood of a path $(X_t, t \in [0,T])$ and show that this function has a unique global maximum, which defines the MLE $(\hat{b}_T, \hat{\alpha}_T)$. We study the convergence of this estimator toward (b, α) and obtain precise convergence rates and limits for this estimator in the ergodic case and in some nonergodic cases. We check that the MLE achieves the optimal convergence rate in each case by looking at local asymptotic properties. Motivated by this study, we also present new results on the Laplace transform that extend the recent findings of Gnoatto and Grasselli [9] and are of independent interest.

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Optimal Reward for a Mean Field Game of Racing

Yuchong Zhang
(joint work with Marcel Nutz)

We look at a tractable mean field game of exit time control, where players compete to finish their given projects, and are rewarded based on the ranking of their completion times. Such a model features relative performance evaluation, an important topic in contract theory. They can be applied to various settings such as firms competing to be the first to develop a new product or employees competing to be the first to achieve a certain quantity of sales. Specifically, we model the completion time of each player as the first jump time τ of a doubly stochastic Poisson process with controlled jump intensity λ. The representative player aims at maximizing the expected reward minus the cost of effort:

$$\sup_{\lambda} \mathbb{E}\left[R(1-\rho(\tau)) - \int_0^{\tau} c(1-\rho(t))\lambda_t^2 dt\right],$$

where $\rho(t)$ is the fraction of players that have not finished the game at time t, and R and c are functions of rank, describing the reward scheme and the cost parameter, respectively.

Our goal here is to understand the equilibrium behavior and study the problem of optimal reward design which is of great interest to the central planner. Two specific problems we considered are:

(1) Minimum quantile problem: Given reward budget R_0 and a number $\alpha \in (0,1)$, what rank-based reward scheme minimizes the α-quantile of the equilibrium completion times?
(2) Minimum budget problem: What is the minimum total reward the planner needs to provide in order to secure a completion rate of $\alpha \in (0,1)$ by a given deadline T?

In a one-stage Poisson race without common noise, closed-form solutions to both problems can be found by solving a constrained calculus of variation problem, and the solutions appear to be time inconsistent. When common noise is added to the cost parameter, we obtain a pair of stochastic Hamilton-Jacobi and Kolmogorov equations in equilibrium, which admits a strong mean field game solution.

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On uniqueness and blowup properties for a class of second order SDEs

Eyal Neumann

(joint work with Alejandro Gomez, Jong Jun Lee, Carl Mueller, Michael Salins)

1. Uniqueness

The basic conditions for uniqueness of partial differential equations (PDE) are the same as for ODE: coefficients must be Lipschitz continuous. But the corresponding results for stochastic partial differential equations (SPDE) have only appeared recently. These results are restricted to the stochastic heat equation,

$$\begin{aligned} \partial_t u &= \Delta u + f(u)\dot{W} \\ u(0,x) &= u_0(x). \end{aligned} \tag{1}$$

Here $x \in \mathbf{R}$, $\dot{W} = \dot{W}(t,x)$ is two-parameter white noise, and f is Hölder continuous with index γ. In this case, strong uniqueness holds for $\gamma > 3/4$ [7], but fails for $\gamma < 3/4$ [5]. One can also replace white noise by colored noise, which may allow x to take values in $\mathbf{R}^d$ for $d > 1$, and may change the critical value of γ.

Types of SPDE other than the stochastic heat equations are still unexplored with regard to uniqueness, except for the standard fact that uniqueness holds with Lipschitz coefficients. For example, there is no information about the critical Hölder continuity of $\sigma(u)$ for uniqueness of the stochastic wave equation:

$$\begin{aligned} \partial_t^2 u(t,x) &= \Delta u(t,x) + \sigma(u(t,x))\dot{W}, \\ u(0,x) &= u_0(x), \qquad \partial_t u(0,x) = u_1(x). \end{aligned} \tag{2}$$

Here again $x \in \mathbf{R}$ and $\dot{W} = \dot{W}(t,x)$ is a two-parameter white noise.

In order to shed light on uniqueness for the stochastic wave equation, we proposed in [1] studying the corresponding SDE, $\ddot{X}_t = \sigma(X_t)\dot{B}_t$. By making this equation into a system of first-order equations, we get

$$\begin{aligned} dX_t &= Y_t dt, \\ dY_t &= |X_t|^{\alpha} dB_t, \\ X_0 &= x_0, \qquad Y_0 = y_0. \end{aligned} \tag{3}$$

Here again (B_t) is a standard Brownian motion.

In [1] we proved that if $\alpha > 1/2$ and the initial condition $(x_0, y_0) \neq (0,0)$, then

(3) has a unique solution in the strong sense. Moreover the solution to (3) never reaches the origin with probability 1. On the other hand, when $0 < \alpha < 1$ and $(x_0, y_0) = (0, 0)$, we proved that both strong and weak uniqueness fail for (3).

2. Blowup of the solutions

Another interesting property of SPDE is the blowup of the solution. For example, let $a > 0$ and consider the nonnegative solutions to the one-dimensional stochastic heat equation in compact domain:

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x) + u^\gamma(t,x)\dot{W}, \\ u(t,0) = u(t,a) = 0, \\ u(0,x) = u_0(x). \end{cases} \tag{4}$$

Mueller in [3] showed that if $\gamma < 3/2$, then the solution to (4) exists for all time and is finite. On the other hand, Mueller showed in [4] that if $\gamma > 3/2$, then the solution to (4) blows up in finite time with a positive probability.

The blowup property of the stochastic wave equation appears to be more difficult to analyse. It is still unknown what the conditions are on σ for a finite time blowup of the solution of (2) (see [6]). As in the uniqueness case, we studied the solutions of (3) as the first step of understanding the problem for the stochastic wave equation. The finite time blowup of the solutions of first-order stochastic differential equations can be checked by the Feller test for explosions (for example, see [2]); however, there is no simple way to check in the case of higher-order equations. It is well known that the solution of (3) doesn't blow up if the coefficients have at most linear growth (that is $\alpha \leq 1$). In [1] we proved that when $\alpha > 1$, the solution of (3) blows up in finite time with probability one. It would be interesting to check whether there is a critical exponent for the blowup in finite time of the solutions to the stochastic wave equations.

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Rough volatility from an affine point of view

Josef Teichmann
(joint work with Christa Cuchiero)

Hawkes processes model self-exciting growth phenomena, where the process' past enters its intensity in a linear, rescaled way. These models are important in finance, economics and population dynamics and have recently gained a lot of attention in micro-foundation of rough volatility models, see e.g., [1].

We aim to provide a simple and computationally attractive Markovian framework for Hawkes processes, which allows to understand ubiquitous limit theorems from a different point of view and which allows to easily head for multi-variate extensions.

We consider a process λ, called a Hawkes process lift, taking values in d-dimensional vector measures $\mathcal{M}^d(\mathbb{R}_{\geq 0}, \nu)$ absolutely continuous with respect to a matrix of measures ν, all supported on $\mathbb{R}_{\geq 0}$:

We denote by $\mu : \mathbb{R}_{\geq 0} \to \mathbb{R}^d_{\geq 0}$ and $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}^d_{\geq 0} \otimes \mathbb{R}^d_{\geq 0}$ the finite-valued Laplace transforms of ν and λ_0, i.e.

$$\mu(t) = \int_0^\infty e^{-tx} \lambda_0(dx), \quad \phi(t) = \int_0^\infty e^{-tx} \nu(dx)\,,$$

for $t > 0$. Notice that $\phi(0)$ is not necessarily assumed finite here.

Definition 1. *A Hawkes process lift follows the affine dynamics*

$$d\lambda_t(dx) = -x\lambda_t(dx)dt + \nu(dx)dN_t$$

with initial value λ_0 and a pure jump process N where each component N^i has instantanous predictable jump characteristics $\delta_{\{1\}}\lambda^i_{t-}(\mathbb{R}_{\geq 0})$, for $i = 1, \ldots, d$. In particular λ_{t-} is a finite measure for all $t \geq 0$ if λ_0 is a finite measure componentwise.

Consider a Hawkes process N with instantanous jump characteristic $\delta_{\{1\}}\bar{\lambda}_t$ given by the equation

$$\bar{\lambda}_t = \mu(t) + \int_0^t \phi(t-s)dN_s$$

Then the (predictable) intensity $\bar{\lambda}$ has the representation:

$$\bar{\lambda}_t = \lambda_{t-}(\mathbb{R}_{\geq 0})$$

in terms of the Hawkes process lift.

The above equation defines a unique measure-valued process by explicit construction for finite measures as initial values. The representation property follows from the variation of constants formula. Notice that the caglad version of λ defines a Markov process with vectors of finite measures as state space whereas the cadlag version is at jump times not necessarily finitely valued. However, both agree outside nullsets for each t.

Affine technology easily allows to calculate the Fourier-Laplace transform of the marginal distributions of the Hawkes process lift. Of particular interest for applications, in particular in variance modelling, is the affine transform formulas with respect to term structures of intensity swaps. This turns out to be a generic phenomenon for a large class of affine processes.

More precisely, we aim to express the Fourier-Laplace transform of (λ_t, N_t)

$$\mathbb{E}_{\lambda_s,N_s}\left[e^{\int_0^\infty u(x)\lambda_t(dx)+vN_t}\right] = e^{\int_0^\infty \psi(t-s,u,v)(x)\lambda_s(dx)+vN_s},$$

for $u \in C_b(\mathbb{R}_+;\mathbb{C}^d)$, a lying vector of bounded continuous functions, and a lying vector $v \in \mathbb{C}^d$, in terms of the following term structure of *intensity swap prices*

$$V_s(T) := E_{\lambda_s,N_s}\big[\lambda_{T-}(\mathbb{R}_{\geq 0})\big]$$

for $T \geq s$. This works since by variation of constants

$$V_s(T) = \int_0^\infty \exp(-x(T-s))\lambda_s(dx) + \\ + \int_s^T \int_0^\infty \exp(-x(T-u))\nu(dx)\, V_s(u)du\,,$$

whence λ_s, actually its Laplace transform, can be easily expressed by means of $V_s(T)$ for $T \geq s \geq 0$. So by the classical affine transform formula (and under certain anlyticity assumptions on $T \mapsto V_s(T)$) one obtains the announced affine transform formula.

The previous consideration raises the question whether we can properly use intensity swaps as coordinates of an affine process. The answer is affirmative after a Musiela parametrization, i.e. $V_s(s+y) := W_s(y)$. Actually intensity swaps satisfy after a minor calculation the following equation

$$dW_t(y) = \frac{d}{dy}W_t(y)dt + \widetilde{\phi}(y)\big\{dN_t - W_{t-}(0)dt\big\}\,,$$

where N is a process with instantanous jump characteristic $\delta_{\{1\}}W_{t-}(0)$ and $\widetilde{\phi} := \sum_{k\geq 1}\phi^{*k}$.

The announced representation of Hawkes processes and their limits pave a way towards a unified treatment of micro-foundations and their limits, see [1], from a Markovian and affine perspective, as well as an alternative for the numerical treatment of these roughly driven processes.

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Semimartingale preservation via causal transport

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(joint work with Julio Backhoff Veraguas, Anastasiia Zalashko)

From the seminal works of Monge [9] and Kantorovich [6], the theory of optimal transport has widely developed and established itself as a fervent research area, with growing applications in the most various areas of sciences and engineering. Powerful connections have also been established between the theory of optimal transport and stochastic analysis. In the recent article by Lassalle [8], the author creates another bridge between optimal transport and stochastic analysis, considering the transport problem under the so called *causality* constraint. The origins of this concept can be found in the work of Yamada and Watanabe [10] (see [2, 7] for a generalization of the latter).

The main idea behind my presentation is the exploitation of ideas and techniques from optimal transport under causality, in order to tackle the classical stochastic analysis problem of *enlargement of filtrations.* The central question of enlargements of filtrations is whether the semimartingale property is preserved when passing from a given filtration to a larger one; see [1, 5, 4, 3] for some of the earliest works on the subject. To describe causality, one is first given two Polish filtered probability spaces $(\mathcal{X}, \{\mathcal{F}^{\mathcal{X}}_t\}_{t=0}^T, \mu)$ and $(\mathcal{Y}, \{\mathcal{F}^{\mathcal{Y}}_t\}_{t=0}^T, \nu)$. A transport plan π is a probability measure on $\mathcal{X}\times\mathcal{Y}$ having the prescribed marginals μ,ν; this is denoted by $\pi\in\Pi(\mu,\nu)$. It is further called *causal* if a certain measurability condition holds, roughly: the amount of 'mass' transported by π to a subset of the target space $\mathcal{Y}$ belonging to $\mathcal{F}^{\mathcal{Y}}_t$, is solely determined by the information contained in $\mathcal{F}^{\mathcal{X}}_t$. Thus a causal plan transports μ into ν in an adapted way.

Given a cost function c on $\mathcal{X}\times\mathcal{Y}$, the general *causal transport problem* is defined as

$$\inf\{\mathbb{E}^\pi[\,c\,] : \pi\in\Pi(\mu,\nu),\ \pi \text{ causal}\}. \tag{1}$$

The situation of interest for the purposes of the presentation is when both $\mathcal{X}$ and $\mathcal{Y}$ are the space of continuous functions, possibly endowed with different filtrations. Concretely, let B be a Brownian motion on some probability space $(\Omega,\mathcal{F}^B,\mathbb{P})$, where $\mathcal{F}^B$ is the filtration generated by B, and let $\mathcal{H}\supseteq\mathcal{F}^B$ be a finer filtration (i.e., $\mathcal{H}$ is an *enlargement* of $\mathcal{F}^B$). If B is still a semimartingale with respect to the larger filtration $\mathcal{H}$, then its unique continuous semimartingale decomposition takes the form

$$dB_t = d\tilde{B}_t + dA_t, \tag{2}$$

where $\tilde{B}$ is an $\mathcal{H}$-Brownian motion and A is a continuous $\mathcal{H}$-adapted finite variation process. Then the joint law of $(\tilde{B},B)$ turns out to be a causal transport plan on path space, when considering the canonical and an appropriate enlarged filtration. Since $\tilde{B}$ is an anticipative but deterministic mapping of B, much as a Monge map in classical transport (but mapping a target measure to the source one), one can say that causal transport plans correspond to Kantorovich generalization of such anticipative mappings.

The main result I presented, is a characterization of the preservation of the semimartingale property in an enlarged filtration, for a process which is a Brownian motion in the original filtration. I showed that a necessary and sufficient condition for this preservation property, is that the causal transport problem (1) on continuous path space is finite, where μ is the Wiener measure, ν is some measure equivalent to μ, and the cost function is the total variation of the difference of the coordinate processes on the product space. In addition, when considering transport plans under which this difference is absolutely continuous with respect to Lebesgue measure, I gave a necessary and sufficient condition not only for the semimartingale preservation property to hold, but also to ensure that the finite variation process in (2) is absolutely continuous (which yields the so-called information drift). When the cost function is of Cameron-Martin type, and the filtration enlargement is done entirely at time zero, the causal transport problem can be interpreted in terms of entropy and mutual information. Therefore, irrespective of the cost function and the kind of enlargement, the value of our causal transport problems can be seen as a mutual information in a wider sense. Finally, I mentioned the fact that a generalization of the definition of causality allows to determine necessary and sufficient conditions for a general continuous semimartingale to remain a semimartingale with respect to an enlarged filtration.

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Path-Dependent Volatility

Julien Guyon

Three main volatility models have been used so far in the finance industry: constant volatility, local volatility (LV), and stochastic volatility (SV). The first two models are complete: since the asset price is driven by a single Brownian motion, every payoff admits a unique self-financing replicating portfolio consisting of cash and the underlying asset, therefore its price is uniquely defined as the initial value of the replicating portfolio, independently of utilities or preferences. Unlike the constant volatility models, the LV model is flexible enough to fit any arbitrage-free surface of implied volatilities (henceforth, 'smile')–but then no more flexibility is left.

For their part, SV models are incomplete: the volatility is driven by one or several extra Brownian motions, and as a result perfect replication and price uniqueness are lost. Modifying the drift of the SV leaves the model arbitrage-free, but changes option prices.

Using SV models allows us to gain control on key risk factors like volatility of volatility ('vol of vol'), forward skew, and spot-vol correlation. SV models generate joint dynamics of the asset and its implied volatilities (spot-vol dynamics henceforth) that are much richer than the LV ones. For instance, using a very large mean reversion together with a large vol of vol and a very negative spot-vol correlation, one can generate an almost flat implied volatility surface, together with very negative short term forward skews. If an LV model were used to match this smile, the LV surface would be almost flat as well, producing vanishing forward skew. As a result, cliquets of forward starting call spreads would be much cheaper in the LV model. This is still true even if the smile is not flat: the LV model typically underprices these options. Using SV models prevents possible mispricings.

To allow SV models to perfectly calibrate to the market smile, one can use stochastic local volatility (SLV) models, i.e., multiply the SV by an LV (the so-called 'leverage function') which is fitted to the smile using the particle method; see [6]. This modifies the spot-vol dynamics, but rather slightly: usually the leverage function, seen as a function of the asset price, becomes flatter and flatter as time t grows, so the SLV dynamics become closer and closer to pure SV ones [9].

At this point a natural question arises: can we build *complete* models that have all the nice properties of SLV models, namely, rich spot-vol dynamics, and calibration to the market smile? For instance, can we build a complete model that fits a flat smile, and yet produces very negative short term forward skews? It is tempting but wrong to quickly answer 'no', by arguing that the only complete model calibrated to the smile is the LV model. This is *not* true: in this talk, we show that path-dependent volatility (PDV) models, which are complete, can produce rich spot-vol dynamics and, furthermore, can perfectly fit the market smile. The two main benefits of model completeness are price uniqueness and parsimony: it is remarkable that so many popular properties of SLV models can be captured using a single Brownian motion. Although perfect delta-hedging is unrealistic, incorporating the path-dependency of volatility into the delta is likely to improve the delta-hedge. Not only that: thanks to their huge flexibility, PDV models can generate spot-vol dynamics that are not attainable by SLV models.

PDV models are those models where the instantaneous volatility σ_t depends on the *path* followed by the asset price so far:

$$\frac{dS_t}{S_t} = \sigma(t, (S_u, u \leq t))\, dW_t$$

where, for simplicity, we have taken zero interest rates, repo, and dividends. In practice, the volatility $\sigma_t \equiv \sigma(t, S_t, X_t)$ will often be assumed to depend on the path only through the current value S_t and a finite set X_t of path-dependent variables, which may include for example running or moving averages, maximums/minimums, realized variances, etc.

PDV models have been widely overlooked, compared to LV and SV. The most famous PDV models are probably the ARCH model by Engle [3] and its descendants GARCH [2], NGARCH, IGARCH, etc. But these are discrete-time models which are hardly used in the derivatives industry. The two other main contributions so far are due to Hobson and Rogers [10] and Bergomi [1]. In its discrete setting version, Bergomi's SV model is actually a mixed SV-PDV model in which, given a realization of the variance swap volatility at time $T_i = i\Delta$ for maturity T_{i+1}, $\sqrt{\xi^i_{T_i}}$, the (continuous time) volatility of the underlying on $[T_i, T_{i+1}]$ is path-dependent: it reads $\sigma(S_t/S_{T_i})$, where σ is calibrated to both $\xi^i_{T_i}$ and a desired value of the forward at-the-money (ATM) skew for maturity Δ.

By contrast, the Hobson-Rogers model is a pure PDV model in which the volatility $\sigma_t = \sigma(X_t)$ is a deterministic function of $X_t = (X^1_t, \ldots, X^n_t)$, where

$$X^m_t = \int_{-\infty}^{t} \lambda e^{-\lambda(t-u)} \left(\ln \frac{S_t}{S_u}\right)^m du$$

When $n = 1$, the volatility depends only on the offset

$$X^1_t = \ln S_t - \int_{-\infty}^{t} \lambda e^{-\lambda(t-u)} \ln S_u\, du$$

It is determined by the local trend of the asset price over a period of order $1/\lambda$ years. This assumption is supported by empirical studies. Here the choice of an infinite time window and exponential weights is only guided by computational convenience: it ensures that (S_t, X_t) is a Markovian process, so the time-t price of a European payoff of the type $g(S_T, X_T)$–in particular the price of a vanilla option–reads $u(t, S_t, X_t)$ where u is the solution to a second order parabolic partial differential equation. Note in particular that the implied volatilities at time 0 in the model depend on all the past asset prices through X_0.

At this point four natural questions arise:

(1) Can we specify $\sigma(\cdot)$ and λ so as to exactly fit the market smile? [12, 4] only gave approximate calibration results.

(2) Does the calibrated model have desired dynamics of implied volatility, such as large negative short term forward skews for instance?

(3) In the definition of X_t, can we use general weights and a finite time window $[t-\Delta, t]$ instead of $(-\infty, t]$, so that the volatility truly depends only a limited portion of the past, e.g., the previous month? The generalization in [5] is very partial as it requires positive weights on $[0, t]$.
(4) Much more importantly: how do we generalize to other choices of X_t? The generalization in [11], where the volatility depends on a particular modified version of the offset X_t^1, is also very partial.

In this talk we solve all these questions at a time: first we choose *any* set of path-dependent variables X_t and *any* function $\sigma(t, S, X)$ so that the PDV model with $\sigma_t = \sigma(t, S_t, X_t)$ has desired spot-vol dynamics and/or captures historical features of volatility, and then we define a new PDV model by multiplying $\sigma(t, S, X)$ by a leverage function $l(t, S)$ and we perfectly calibrate l to the market smile of S using the particle method [6]. When we do not calibrate to the smile ($l \equiv 1$), we speak of 'pure' PDV.Usually, multiplying the pure PDV $\sigma(t, S_t, X_t)$ by the calibrated leverage function distorts only slightly the spot-vol dynamics since leverage functions typically flatten over time. This way we mimic SLV models, with the pure PDV $\sigma(t, S_t, X_t)$ playing the role of the SV, but we stay in the world of complete models.

Interesting open questions include the following:

- Existence and uniqueness of a solution to the McKean stochastic differential equation that describes the calibrated model.
- For a given scalar path-dependent variable X_t, calibration of a pure PDV $\sigma(t, X_t)$ to the market smile.
- (Machine learning) Infer from the data a PDV that accurately describes short-term implied volatility.

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Improved Fréchet-Hoeffding bounds and model-free finance

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(joint work with Thibaut Lux)

We consider a random vector $\mathbf{X} = (X_1, \dots, X_d)$ and assume that the marginal distributions of its constituents are known, however the joint distribution is either completely unknown or only partially known. This framework is known in the literature as *model* or, more specifically, *dependence uncertainty*. We are then interested in computing expectations of the form $\mathbb{E}[f(\mathbf{X})]$, for suitable functions f. These quantities correspond to the prices of multiasset derivatives or, subject to an additional transformation, the Value-at-Risk of $\mathbf{X}$, and have several applications in financial and actuarial mathematics.

In the present framework, since the joint distribution is not known, we cannot actually compute $\mathbb{E}[f(\mathbf{X})]$ exactly, thus we will resort to computing bounds on this expectation, and require that these bounds are as tight as possible. One way to compute bounds is via the theory of copulas and the well-known Fréchet-Hoeffding bounds. Indeed, from Sklar's theorem we know that there exists a one-to-one correspondence between the joint law F and the copula C of the random vector $\mathbf{X}$; it holds that

$$F(x_1, \dots, x_d) = C(F_1(x_1), \dots, F_d(x_d))$$

for all $(x_1, \dots, x_d) \in \mathbb{R}^d$, where $F_1, \dots, F_d$ denote the marginal distributions of $X_1, \dots, X_d$. Moreover, the theorem of Fréchet and Hoeffding states that each copula C satisfies the following:

$$W_d(\mathbf{u}) := \max\Big\{0, \sum_{i=1}^{d} u_i - d + 1\Big\} \le C(\mathbf{u}) \le \min\{u_1, \dots, u_d\} =: M_d(\mathbf{u}),$$

for all $\mathbf{u} \in [0,1]^d$. Therefore, using ordering results for copulas, we can translate these bounds into bounds on the expectation $\mathbb{E}[f(\mathbf{X})]$, for suitable functions f. However, these bounds are typically very wide and do not provide meaningful information. In addition, there is typically additional, partial information on the joint law or the copula of $\mathbf{X}$, available that is neglected by these bounds.

The aim of this talk is to present recent results on how to improve and sharpen the Fréchet-Hoeffding bounds by taking advantage of additional information on the copula C. There are various types of additional information than can be used: (i) knowledge of the copula on a subset of $[0,1]^d$, (ii) knowledge of the value of a measure of association, (iii) knowledge of bounds on lower-dimensional copulas, and (iv) knowledge that the copula lies in the vicinity of a reference copula. The improved Fréchet-Hoeffding bounds in case (i) are provided by the following result.

Theorem 1. *Let $S \subset \mathbb{I}^d$ be a compact set and Q^* be a d-quasi-copula. Consider the set*

$$\mathcal{Q}^{S,Q^*} := \big\{Q \colon Q(\mathbf{x}) = Q^*(\mathbf{x}) \text{ for all } \mathbf{x} \in S\big\}.$$

Then, for all $Q \in \mathcal{Q}^{S,Q^*}$*, it holds that*

$$\underline{Q}^{S,Q^*}(\mathbf{u}) \leq Q(\mathbf{u}) \leq \overline{Q}^{S,Q^*}(\mathbf{u}) \quad \textit{for all } \mathbf{u} \in \mathbb{I}^d,$$

where the bounds $\underline{Q}^{S,Q^*}$ *and* $\overline{Q}^{S,Q^*}$ *are provided by*

$$\underline{Q}^{S,Q^*}(\mathbf{u}) = \max\Big(0, \sum_{i=1}^{d} u_i - d + 1, \max_{\mathbf{x}\in S}\Big\{Q^*(\mathbf{x}) - \sum_{i=1}^{d}(x_i - u_i)^+\Big\}\Big),$$

$$\overline{Q}^{S,Q^*}(\mathbf{u}) = \min\Big(u_1, \ldots, u_d, \min_{\mathbf{x}\in S}\Big\{Q^*(\mathbf{x}) + \sum_{i=1}^{d}(u_i - x_i)^+\Big\}\Big).$$

Assuming now that the improved Fréchet-Hoeffding bounds were copulas, we would be able to translate them into bounds on $\mathbb{E}[f(\mathbf{X})]$ using ordering results from Müller and Stoyan [4]. This is the case when $d = 2$, see Tankov [5] and Bernard et al. [1]. However, it turns out that the improved Fréchet-Hoeffding bounds are proper quasi-copulas for $d > 2$, see [2, Section 4], therefore these ordering results do not apply. Even worse, the expectation might not be well-defined, since quasi-copulas do not always define a (signed) measure.

In order to overcome these problems, we have developed in [2] an alternative represenation of the expectation $\mathbb{E}[f(\mathbf{X})]$, that makes also sense when considering quasi-copulas; this is based on an integration-by-parts idea. Moreover, we have also proved ordering results for this new representation, for Δ-monotonic and Δ-antitonic functions. As an application, we have derived in [2] model-free bounds on the prices of multi-asset option prices that significantly improve known results, while [3] contains an application to portfolio Value-at-Risk.

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Robust Markowitz mean-variance portfolio selection under ambiguous covariance matrix

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(joint work with Amine Ismail)

The Markowitz mean-variance portfolio selection problem [20], initially considered in a single period model, is the cornerstone of modern portfolio allocation theory. Investment decisions rules are made according to the objective of maximizing the expected return for a given financial risk quantified by the variance of the portfolio, and lead to the concept of efficient frontier, which proposes a simple illustration of the trade-off between return and risk. The use of Markowitz efficient portfolio strategies in the financial industry has become quite popular mainly due to its natural and intuitive formulation.

In a continuous-time dynamic setting, the mean-variance criterion involves in a nonlinear way the expected terminal wealth due to the variance term, and induces the so-called time inconsistency. This nonstandard feature in stochastic control problem has generated various resolution approaches. A first approach in [25] consists in embedding the mean-variance problem into an auxiliary standard control problem that can be solved by using stochastic linear quadratic theory. A second approach relies on the observation that the dynamic mean-variance problem can be reformulated as a control problem of McKean-Vlasov type, where the cost functional may depend nonlinearly on the law of the wealth state process. It has then been solved in [2] where the authors have derived a version of the Pontryagin maximum principle. More recently, the paper [22] has developed a general dynamic programming approach for the control of McKean-Vlasov dynamics and applied their method for the resolution of the mean-variance portfolio selection problem. We also mention the recent paper [10], where the mean-variance problem is viewed as the McKean-Vlasov limit of a family of controlled many-component weakly interacting systems. These prelimit problems are solved by standard dynamic programming, and the solution to the original problem is obtained by passage to the limit.

In the above cited papers, the continuous-time Markowitz problem was essentially studied in the framework of a Black-Scholes model, and abundant research has been conducted to extend this setup by including models with random parameters. Among this large literature, we cite the recent paper [6] which uses a stochastic correlation model for taking into account the correlation risk between risky assets. In all these works, it is assumed that investors have a perfect knowledge of the stochastic dynamics governing the price process, that is a 'correct' model has to be first specified, and then the parameters have to be accurately estimated or calibrated. However, in finance, a model is clearly an approximation of the reality, and moreover within a model, the estimation problem is a difficult issue. For example, it is known that the estimation of correlation between assets may be extremely inaccurate due to asynchronous data and lead-lag effect, especially when the number of assets is large, and the correlation estimate converges to its true value less rapidly than the estimates of volatilities that are based on the full sets of marginal observations, see e.g. [14], [13] and [1]. On the other hand, optimal portfolios are typically sensitive to the model and the parameters, and may perform badly when the parameters are not sufficiently accurate. Therefore, the impact of model misspecification, due to erroneous models and measurements, is an important issue in the practical implementation of trading strategies, and is usually refereed to as model risk.

In order to address the model risk related to uncertainty or ambiguous model parameters, the robust approach, which consists in taking decisions under the worst-case scenario over all conceivable models, is a notable research direction in mathematical finance. A common robust modeling is to consider a family of probability measures representing all the prior beliefs of the investor on the model parameters. For example, drift uncertainty is modeled via Girsanov's theorem by a set of dominated probability measures, and has been first considered in the context of portfolio selection in [12], and then largely studied in the literature, see the recent paper [15] and the references therein.

We focus here on uncertainty or ambiguity on the covariance matrix of the risky assets, assuming that the instantaneous return (drift) is known (or by considering that we have a strong belief on its value). Uncertain volatility models have been considered in [3], [18], or [7] in the context of option pricing, and in [19], [16] for robust portfolio optimization with expected utility criterion. As in [11], we are also interested in a setting with ambiguous correlation between two risky assets since, as already mentioned above, the correlation parameter is hard in practice to infer with accuracy from market information.

In this paper, we investigate the robust Markowitz mean-variance portfolio selection under uncertainty on the volatilities and correlation of multi risky assets. Robust mean-variance problems have been considered in the economic and engineering literature, mostly on single period or multiperiod models, see e.g. [9], [23] and [17]. Here, in our continuous-time modeling, we adopt the probabilistic framework in [8], related to the theory of G-expectation [21] (see also [24]), in order to capture model uncertainty and ambiguity on the covariance matrix, which leads to a set of non-dominated probability measures for the prior probabilities. We also make some concavity assumption on the set of prior covariance matrix. From a mathematical viewpoint, and compared to robust problem with expected utility, we face two additional difficulties: (i) it cannot be tackled a priori by classical stochastic differential game approach due to the nonlinear variance term, (ii) moreover, since the worst-case scenario is not the same for the mean and the variance, it is not straightforward that it can be put into a min-max problem. We then use the following methodology. We consider a robust mean-variance criterion, which is actually formulated as a min-max problem, and show a posteriori how it is connected to the robust Markowitz problem. We tackle the former problem by a McKean-Vlasov dynamic programming approach: we first reformulate the robust mean-variance problem into a deterministic differential game problem with the law of the wealth process under a prior probability measure as state variable. Then, adapting optimality arguments from dynamic programming principle, and using recent chain rule for flow of probability measures derived in [4] and [5], we state a verification theorem which gives the optimal strategy and performance in terms of a Bellman-Isaacs equation in the Wasserstein space of probability measures. We next apply this analytic partial differential equation characterization of the solution to the robust mean-variance problem, and show that the problem can be reduced into two steps: first, we determine the worst-case scenario, and the remarkable point is that it corresponds to a constant variance/covariance matrix obtained by the minimization of the risk premium, which is a direct input of the model. Secondly, we obtain the optimal mean-variance strategy as in the Black-Scholes model with the known instantaneous return and the worst-case constant covariance matrix. We illustrate our results with closed-form expressions for the optimal portfolio strategies in two examples: uncertain volatilities and ambiguous correlation between two risky assets. Moreover, we are able to derive explicitly the corresponding robust efficient frontier of the robust Markowitz problem. In particular, we obtain a lower bound for the Sharpe ratio of any robust efficient portfolio strategy, which is independent of any modelling on the covariance matrix.

How can robust mean-variance portfolio strategies help to improve performance of investors? We address this question by using simulations to evaluate and compare the Sharpe ratio of a robust investor and a simple investor who implements mean-variance strategies with a misspecified model in two examples: (i) in the first example, the true dynamics of the stock price is assumed to be governed by a Heston type stochastic volatility model that makes the volatility bounded, and the simple investor considers that the risky asset is governed by a Black-Scholes model with constant volatility, (ii) in the second example, the two-assets price is given in reality by a stochastic correlation model, but the simple investor considers a constant correlation between the risky assets. Our results show that the robust Sharpe ratio can perform noticeably better than the misspecified Sharpe ratio for some choice of the parameters describing the true dynamics.

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Viability, Arbitrage and Preferences

Mete Soner
(joint work with Frank Riedel, Matteo Burzoni)

In this paper [2], we consider a financial market with a fixed maturity T. All financial instruments are contracts X of cash flows up to time T. The value of X_t at time t is the cumulative non-discounted cash payments up to time t. Agents are presented a set of contracts that are tradable with no cost and a cloud of possible weak orders among the contracts. A natural notion of viability is then the existence of a preference relation that is consistent with this plausible orders so that all contracts are weakly preferred to any position obtained by adding a replicable contract to itself. Hence in an economy populated with agents with this preference relation every agent is content to remain at her endowment. This is an equivalent statement of viability defined by Harrison & Kreps [1].

We also define the notions of arbitrage and free lunches with vanishing risk. However, these definitions require a careful construction and several definitions which are given in [2]. Two important concepts are the negligible contracts defined through the unanimous partial order and the set of relevant one which we assume to be a part of the common beliefs of the agents.

We prove in this context that a market is viable if and only if there are no free lunches with vanishing risk.

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Continuous time martingale optimal transport and the local vol model

Mathias Beiglböck

(joint work with J. Backhoff, M. Huesmann, S. Källblad, D. Trevisan)

We start by recalling the special role of the *local vol model*: Given a peacock, that is, a family of distributions $(\mu_t)_{t\in[0,1]}$ which increases in convex order and for which $t \mapsto \mu_t$ is weakly continuous, there exists a unique (almost) continuous Markov martingale $(M_t)_{t\in[0,1]}$ such that $M_t \sim \mu_t, t \in [0,1]$.

Arguably, this martingale (i.e. the local vol model in financial terms) is a most natural martingale with the prescribed marginals.

Question 1. Given *two* marginals $\mu \prec_c \nu$, does there exist a 'natural' martingale $(M_t)_{t\in[0,1]}$ such that $M_0 \sim \mu, M_1 \sim \nu$?

Of course, this is not a very precise question. Clearly we will demand that our martingale is a continuous Markov process. To explain somewhat more specifically what we have in mind, we consider the particular case where $\mu = \delta_{\{0\}}$. In this case a particularly simple construction of a continuous martingale terminating at ν is as follows: pick $f : \mathbb{R} \to \mathbb{R}$ increasing such that $f(\gamma) = \nu$, where γ denotes the standard Gaussian measure. Then set for $t \in [0,1]$

$$M_t := E[f(B_1)|F_t] = E[f(B_1)|B_t] = f_t(B_t), \tag{1}$$

where $B = (B_t)_{t\in[0,1]}$ denotes Brownian motion (started in $B_0 \sim \delta_{\{0\}}$), $(F_t)_{t\in[0,1]}$ the Brownian filtration and $f_t(b) := \int f(b+y)\, d\gamma_{1-t}(y)$, $\gamma_s \sim N(0,s)$. Clearly M is a continuous Markov martingale such that $M_0 \sim \delta_0, M_1 \sim \nu$.

Subsequently we will suggest an extension of this construction to the case where we start in a general probability μ rather than a dirac measure.

Before that we pose another question (which is equally imprecise as Question 1 and for which we will suggest basically the same answer as for Question 1).

Question 2. Is there a natural continuous time version of the martingale transport problem?

Before suggesting a martingale version of the classical transport problem, we present the classical Benamou-Brenier transport problem in probabilistic terms: given probabilities μ, ν on the $\mathbb{R}^n, n \geq 1$ consider

$$P_{BB} := \inf_{X_t = X_0 + \int_0^t v_s\, ds, X_0 \sim \mu, X_1 \sim \nu} \mathbb{E}\left[\int_0^1 |v_t|^2\, dt\right]. \tag{BB}$$

Essentially based on work of Brenier we have the following result

Theorem 1. Assume that μ is absolutely continuous with respect to Lebesgue measure and that μ, ν have finite second moment. Then the following hold:

(1) (BB) has a unique optimizer X^*.
(2) A candidate process X is an optimizer if and only if $X_1 = f(X_0)$, where f is the gradient of a convex function $\phi : \mathbb{R}^n \to \mathbb{R}$ and all particles move with constant speed, i.e. $X_t = tX_1 + (1-t)X_0$.

Given the tremendous developments in the theory of optimal transport which followed the publication of Brenier's theorem, it is intriguing to find a martingale counterpart to this result. We suggest the following problem

$$\text{(MBB)} \qquad P_{MBB} := \sup_{M_t = M_0 + \int_0^t \sigma_s \, dB_s, M_0 \sim \mu, M_1 \sim \nu} \mathbb{E}\left[\int_0^1 \sigma_d \, dt\right].$$

Theorem 2. Assume that μ, ν are probabilities on the real line which have finite second moment. Then the following hold:

(1) (BB) has a unique optimizer M^*.
(2) A candidate process M is an optimizer if it is a *stretched Brownian motion.*

We explain what is mean by 'stretched Brownian motion': let α, ν be probabilities on the real line and let $f : \mathbb{R} \to \mathbb{R}$ be the monotone function which pushes $\alpha * \gamma$ to ν. We then take B to be Brownian motion started in $B_0 \sim \alpha$. In analogy to (1) we then define

$$M_t := E[f(B_1)|F_t] = E[f(B_1)|B_t] = f_t(B_t), \tag{2}$$

and call M the stretched Brownian motion between $f_0(\alpha)$ and ν. (Note that $M_0 \sim f_0(\alpha), M_1 \sim \nu$.)

A particular consequence of Theorem 2 is then

Corollary 1. Given measures μ, ν in convex order, there exists a unique stretched Brownian motion from μ to ν.

We note that without Theorem 2 the assertion of Corollary 1 does not seem trivial to us (in particular not in the multidimensional case, see below).

We continue with some remarks on Theorem 2:

(1) The Benamou-Brenier problem gives rise to a natural, time-consistent interpolation between μ and ν: simply set $\mu_t := \text{law} X_t^*$ for $t \in [0,1]$. (This is known as displacement interpolation / McCann interpolation.)

    Analogously, M^* gives rise to a time-consistent interpolation between measures in convex order by setting $\mu_t := \text{law} M_t^*$ for $t \in [0,1]$.
(2) The optimization problems (BB) and (MBB) look rather different. However it is easy to see that both problems are equivalent to optimization problems which look much more similar. We have

$$X^* = \text{argmin}_{X_0 \sim \mu, X_1 \sim \nu} W^2(X, \text{constant speed particle}),$$
$$M^* = \text{argmin}_{M_0 \sim \mu, M_1 \sim \nu} W_c^2(X, \text{constant volatility martingale}),$$

where W^2 denotes Wasserstein distance with respect to squared Cameron-Martin norm, while W_c^2 denotes its *causal* analogue[1] (in the terminology of Lasalle).

(3) The martingale defined in (1) was used by Bass to solve the Skorokhod embedding problem. Hobson asked whether there are natural optimality properties related to this construction and if one could give a version with a non trivial starting law. (MBB) could be seen as an optimality property of the Bass construction and the stretched Brownian motion gives rise to a version of the Bass embedding with non trivial starting law.

(4) Assume that $(\mu_t)_{t\in[0,1]}$ is a peacock. Write M^n for the Markov martingale which is for each $k \in \{1,\dots,n\}$ a time-scaled version of the stretched Brownian motion between $\mu_{(k-1)/n}$ and $\mu_{k/n}$. Then $\lim_{n\to\infty} M^n$ exists and equals the local vol model. Also the optimality properties carry over to the local vol model.

(5) The tremendous importance of Brenier's theorem builds on the fact that it applies to measures on $\mathbb{R}^n, n \geq 1$. Our results extend from the real line to $\mathbb{R}^2$ provided that ν is absolutely continuous with respect to Lebesgue measure and to $\mathbb{R}^n$ under particular assumptions on the marginals. (In the higher dimensional case σ in (MBB) needs to be replaced by $\mathrm{tr}(\sigma)$ and monotonicity of f has to be understood in the Brenier-sense, i.e. f will be given as the gradient of a convex function.) Based on the recent results of DeMarch-Touzi and Obłój-Siorpaes we hope that our results will in fact extend to general 2nd moment measures on $\mathbb{R}^n$.

Model-free portfolio optimization in the long run

Christa Cuchiero

(joint work with Walter Schachermayer, Leonard Wong)

1. Abstract

Cover's celebrated theorem states that the long run yield of a properly chosen 'universal' portfolio is as good as the long run yield of the best retrospectively chosen constant rebalanced portfolio. We formulate an abstract principle behind such a 'universality' phenomenon valid for general optimization problems in the long run. This allows to obtain new results on model-free portfolio optimization, in particular in continuous time, involving larger classes of investment strategies. These results are complemented by a comparison with the log-optimal numéraire portfolio when fixing a stochastic model for the asset prices.

[1] Causal transport plans generalize adapted processes in the same way as classical Kantorovich transport plans extend Monge maps.

2. Cover's universal portfolio

Let us briefly recall the idea of Cover's universal portfolio. Indeed, Cover's insight reveals the striking phenomenon that the 'wisdom of hindsight' does not give any significant advantage as compared to a properly chosen 'universal' portfolio which is constructed in a predictable way. The relevant optimality criterion here is the asymptotic growth rate of the portfolio.

Cover considers a modelfree setting in discrete time, i.e. time t varies in $\mathbb{N}$, where investments into all *constant rebalanced portfolio strategies* are allowed: let $b = (b^1, \ldots, b^d)$ be a fixed element of the d-simplex $\bar{\Delta}^d$, i.e., $b^j \geq 0$ and $\sum_{j=1}^d b^j = 1$. The corresponding constant rebalanced portfolio $(V_t(b))_{t=0}^\infty$ starting at $V_0(b) = 1$ is inductively defined by holding the investment $b^j V_t(b)$ in stock S^j during the period $(t, t+1)$, so that $V_0(b) = 1$ and

$$\frac{V_{t+1}(b)}{V_t(b)}(s) = \sum_{j=1}^{d} b^j \frac{s_{t+1}^j}{s_t^j}, \tag{1}$$

for each scenario $s = ((s_t^j)_{j=1}^d)_{t=0}^\infty$ of strictly positive numbers modeling the evolution of the prices of d stocks $S^1, \ldots, S^d$.

Fix T and define in a pathwise way the quantity V_T^* by

$$V_T^*(s) = \max_{b \in \bar{\Delta}^d} V_T(b)(s), \tag{2}$$

which is a function depending on the scenario $s = (s_t^1, \ldots, s_t^d)_{t=0}^T$.

The idea is that, with hindsight, i.e., knowing the trajectory $(s_t^1, \ldots, s_t^d)_{t=0}^T$, one considers the best weight $b \in \bar{\Delta}^d$ which attains the maximum (2). Cover's goal is to construct a universal portfolio which performs as well as the hypothetical portfolio process $(V_T^*)_{T \geq 0}$, asymptotically for $T \to \infty$. In order to do so, let ν be a probability measure on $\bar{\Delta}^d$. The universal portfolio now consists of investing at time 0 the portion $d\nu(b)$ of one's wealth into the constant rebalanced portfolio $V(b)$ and subsequently following the constant rebalanced portfolio process $(V_t(b))_{t=0}^T$. The explicit formula is

$$V_t^\nu = \int_{\bar{\Delta}^d} V_t(b) d\nu(b), \tag{3}$$

where $V_t(b)$ is defined via (1).

Cover's celebrated result now reads as follows.

Theorem 1. *(Cover [1]): Let ν be a probability measure on $\bar{\Delta}^d$ with full support. Then*

$$\lim_{T \to \infty} \frac{1}{T} \log \frac{V_T^\nu(s)}{V_T^*(s)} = 0, \tag{4}$$

for all trajectories $s = (s_t^1, \dots, s_t^d)_{t=0}^\infty$ *for which there are constants* $0 < c \leq C < \infty$ *such that*

$$c \leq \frac{s_{t+1}^j}{s_t^j} \leq C, \qquad \textit{for all} \quad j = 1, \dots, d \quad \textit{and all} \quad t \in \mathbb{N}. \tag{5}$$

3. Universality principle

The above phenomenon can be formulated abstractly for general long run optimization problems. Consider for $T \in [0, \infty)$ objective functions $V_T : \mathcal{G} \to \mathbb{R}_+, G \mapsto V_T(G)$, where $\mathcal{G}$ corresponds to a set of strategies. Moreover, let ν be a probability measure on $\mathcal{G}$. Define the *best retrospectively chosen strategy* yielding as output

$$V_T^* = \sup_{G \in \mathcal{G}} V_T(G)$$

and a *universal strategy* yielding as output

$$V_T^\nu = \int_{G \in \mathcal{G}} V_T(G) \nu(dG),$$

whenever there exists some G_T^ν such that it is possible to make sense out of $V_T(G_T^\nu)$ and $V_T(G_T^\nu) = V_T^\nu$. The following theorem gives conditions when the universal strategy performs as well as the best retrospectively chosen strategy.

Theorem 2 (C. Cuchiero and J. Teichmann [3]). *Assume that one of the following conditions holds:*

a)

$$\sup \left(\liminf_{T \to \infty} \left(\frac{V_T}{V_T^*} \right)^{\frac{1}{T}} \right) = 1$$

b) *For every* $T > 0$ *and* $\varepsilon > 0$ *there exists some set* $A_{T,\varepsilon} \subset \mathcal{G}$ *such that*

$$\left(\frac{V_T}{V_T^*} \right)^{\frac{1}{T}} \geq 1 - \varepsilon \quad \textit{on } A_{T,\varepsilon}$$

and $\liminf_{T \to \infty} (\nu(A_{T,\varepsilon}))^{\frac{1}{T}} = 1$ *holds true.*

Then

$$\lim_{T \to \infty} \left(\frac{V_T^\nu}{V_T^*} \right)^{\frac{1}{T}} = 1.$$

The result remains true if the sup *and* V_T^* *are replaced by* ess sup *w.r.t* ν.

4. Modelfree portfolio optimization in continuous time

Our main application of the above theorem is modelfree portfolio optimization in continuous time for a larger class of strategies than constantly rebalanced ones, namely so-called *functionally generated portfolios* introduced in [4]. In this case portfolio wealth processes can be defined via H. Föllmer's pathwise approach to stochastic integration in a probability free way (see [5] and also the recent paper [6]). The choice of functionally generated portfolios is in such a continuous time setting best possible for the following two reasons: first it perfectly connects

Cover's theory with stochastic portfolio theory in continuous time, and second functionally generated portfolios are in a Markovian context also the largest class for which wealth processes can be defined in a pathwise way without passing for instance to rough paths theory or Young integration.

When assuming additionally that the log-optimal portfolio is functionally generated, we can prove equality of the asymptotic growth rates of the best retrospectively chosen, the universal and the log-optimal portfolio for certain Markovian Itô-diffusions.

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Moment generating functions and Normalized implied volatilities: Fukasawa's pricing formula

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(joint work with Claude Martini)

We extend the model-free formula of Fukasawa [4] for $\mathbb{E}[\Psi(X_T)]$, where $X_T = \log S_T/F$ is the log-price of an asset, to functions Ψ of exponential growth. The resulting integral representation is written in terms of normalized implied volatilities. Just as Fukasawa's work provides rigourous ground for Chriss and Morokoff's [2] model-free formula for the log-contract (related to the Variance swap implied variance), we prove an expression for the moment generating function $\mathbb{E}[e^{pX_T}]$ on its analyticity domain, that encompasses (and extends) Matytsin's formula [5] for the characteristic function $\mathbb{E}[e^{i\eta X_T}]$ and Bergomi's formula [1] for $\mathbb{E}[e^{pX_T}]$, $p \in [0, 1]$. Besides, we (i) show that put-call duality transforms the first normalized implied volatility into the second, and (ii) analyze the invertibility of the extended transformation $d(p, \cdot) = p\, d_1 + (1 - p)d_2$ when p lies outside $[0, 1]$. As an application of (i), one can generate representations for the MGF (or other payoffs) by switching between one normalized implied volatility and the other.

1. Model-free pricing formulas

Working with dimensionless quantities, we denote $k = \log(K/F)$ the forward log-strike of a Call option, where $F = \mathbb{E}^{\mathbb{P}}[S_T]$ is the forward price for maturity T, and $v(k) = \sqrt{T}\sigma_{\mathrm{BS}}(T, k)$ the dimensionless (or total) implied volatility of the option. It is well-known that any C^2 (or convex) payoff $\varphi(S_T)$ with linear growth can be statically replicated with a strip of call and put options, so that its price can be written via Carr-Madan's formula $\mathbb{E}[\varphi(S_T)] = \varphi(F) + \int_0^F \varphi''(K)P(K)dK + \int_F^\infty \varphi''(K)C(K)dK$, where $P(K)$ and $C(K)$ denote put and call prices for the maturity T. On the other hand, if the law of S_T under $\mathbb{P}$ is absolutely continuous with respect to the Lebesgue measure on $[0,\infty)$, one can write

$$\begin{aligned} (1)\quad \mathbb{E}[\varphi(S_T)] &= \int_0^\infty \varphi(K)\frac{d^2}{dK^2}\mathbb{E}[(S_T-K)^+]dK \\ &= F\int_{\mathbb{R}} \varphi(K)\frac{d^2}{dK^2}\mathrm{Call}_{\mathrm{BS}}(k,v(k))|_{k=\log(K/F)}\, dK. \end{aligned}$$

Applying the chain rule to the rightmost integrand in (1) leads to an integral formula containing Black-Scholes Greeks with respect to strike and volatility, and the derivatives of the implied volatility smile $v(\cdot)$ up to order two. A stream of literature [5, 4, 1] studies the possibility of re-expressing Equation (1) in such a way that the derivatives of the implied volatility do not appear any more on the right hand side. This is a relevant feature in practice, because observed market data is (in any case) discrete. Such investigations required the introduction of the concept of normalizing transformation of the implied volatility smile, introduced by Chriss and Morokoff [2] and Matytsin [5] and formalized in the seminal work of Fukasawa [4], that we recall below.

1.1. **Volatility derivatives.** One of the most important examples in this field is the following formula for the implied variance of the log contract[1], see Chriss and Morokoff [2]:

$$(2)\qquad \mathbb{E}^{\mathbb{P}}\left[-\frac{2}{T}\log\left(\frac{S_T}{F}\right)\right] = \frac{1}{T}\int_{\mathbb{R}} v(g_2(z))^2\phi(z)dz.$$

In (2), ϕ is the standard normal density, and $g_2 : \mathbb{R} \to \mathbb{R}$ is the inverse of the function (called *second normalizing transformation*)

$$f_2(k) := -d_2(k, v(k)) = \frac{k}{v(k)} + \frac{v(k)}{2}.$$

Similarly, the first normalizing transformation (used later on) is given by $f_1(k) := -d_1(k,v(k)) = \frac{k}{v(k)} - \frac{v(k)}{2}$. Apart from its appealing compactness, the formula (2) is amenable for numerical approximations, notably in view of the use of Gauss-Hermite quadrature. Other examples of similar formulas include: other payoffs such as the $S\ln S$ contract (related to the Gamma Swap), and a formula for the characteristic function of $X_T = \log(S_T/F)$ due to Matytsin [5], see below. The important property that the map $f_2 : \mathbb{R} \to \mathbb{R}$ is actually invertible for *any* arbitrage-free implied volatility $v(\cdot)$, implicitly assumed in the aforementioned works, was first proven by Fukasawa [4].

[1] which coincides with the fair strike of the Variance Swap, under the assumption that $(S_t)_{t\leq T}$ follows a diffusion process. When $T = 30$ days and S is the S&P500 stock index, the left hand side of (2) defines the (theoretical value) of VIX^2 at $t = 0$.

2. Moment generating functions

Denote $v_2(z) = v(g_2(z))$. Assuming that v_2 is differentiable, Matytsin [5] gives (only sketching the proof) the following formula for the characteristic function of X_T

$$\mathbb{E}\left[e^{i\eta X_T}\right] = \int_{\mathbb{R}} e^{-i\eta v_2(z)\left(\frac{1}{2}v_2(z)-z\right)} \left(1 - i\eta v_2'(z)\right) \phi(z)dz, \qquad \eta \in \mathbb{R}. \tag{3}$$

Building on this work, Bergomi [1, Section 4.3.1] derives a formula for the moments of S_T/F of order $p \in [0,1]$:

$$\mathbb{E}\left[\left(\frac{S_T}{F}\right)^p\right] = \int_{\mathbb{R}} e^{\frac{1}{2}p(p-1)v^p(z)^2} \phi(z)dz, \qquad p \in [0,1], \tag{4}$$

where the 'p-normalized' implied volatility $v^p(\cdot)$ is defined in the following way: consider the convex interpolation

$$f(p,k) = pf_1(k) + (1-p)f_2(k) = \frac{k}{v(k)} + \left(\frac{1}{2} - p\right)\frac{v(k)}{2}$$

of the two normalizing transformations f_1 and f_2. We know from Fukasawa [4] that the two maps $k \mapsto f_1(k)$ and $k \mapsto f_2(k)$ are strictly increasing from $\mathbb{R}$ to $\mathbb{R}$: therefore, so is $k \mapsto f(p,k)$, for every $p \in [0,1]$. Let now $g(p,\cdot)$ be the inverse of $f(p,\cdot)$ on $\mathbb{R}$: $v^p(\cdot)$ is defined by

$$v^p(z) = v(g(p,z)), \qquad \text{for all } z \in \mathbb{R}. \tag{5}$$

Note we have the following nice interpretation of (4): in the Black-Scholes model, where $S_T = Fe^{\sigma W_T - \frac{1}{2}\sigma^2 T}$ is a geometric Brownian motion with constant volatility parameter $\sigma = \frac{v}{\sqrt{T}}$, one has $\mathbb{E}\left[\left(\frac{S_T}{F}\right)^p\right] = e^{\frac{1}{2}p(p-1)v^2} = \int_{\mathbb{R}} e^{\frac{1}{2}p(p-1)v^2}\phi(z)dz$. Therefore, we can see Equation (4) as an extension of the pricing formula for power payoffs, from the Black-Scholes world to models with non-constant implied volatility.

The formulas (3) and (4) are the starting point of our work. As mentioned above, Bergomi [1] derives (4) from (3). We follow a different route: our starting point is the work of Fukasawa. We first extend the formula for expectations of functions of X_T with polynomial growth given in [4, Theorem 4.6] to exponential functions - carrying out in details the plan addressed in [4, Remark 4.8]. This provides a formula for the generalized characteristic function $p \in \mathbb{C} \mapsto \mathbb{E}[e^{pX_T}]$ on its analyticity domain, written directly in terms of the implied volatility smile. Matytsin's (3) and Bergomi's (4) formulas are embedded in this representation as special cases (along with a dual version of the first, and an extension to the complex plane of the second). By taking real values of p, this formula allows to numerically evaluate the (finite) risk-neutral moments of the underlying asset price from the market smile - therefore identifying model-free quantities that can be used as targets in the calibration of a parametric model.

As addressed in [4, Remark 4.8], it is natural, when evaluating expectations of the form $\mathbb{E}\left[\left(\frac{S_T}{F}\right)^p\right]$, to exploit Lee's moment formulas relating the critical moments of S_T to the asymptotic slopes of the implied volatility for large and small strikes. We stress that our approach here goes the other way round: we prove an integral representation for $\mathbb{E}\big[\big(\frac{S_T}{F}\big)^p\big]$ *without* making use of Lee's result. As a by-product, one can deduce sharp bounds on the exponents that appear in the moment formulas.

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Hedging with price impact

Peter Bank

(joint work with Mete Soner, Moritz Voß)

We consider the problem of hedging a European contingent claim in a financial model with temporary price impact as proposed by [1]. A second order expansion of a utility indifference price asymptotics for small impact suggests to consider the hedging problem as a linear quadratic optimal stochastic control problem with a possibly singular state constraint which amounts to a cost optimal tracking problem of a frictionless hedging strategy. We solve this problem explicitly first for general predictable target hedging strategies and constant coefficients. It turns out that, rather than towards the current target position, the optimal policy trades towards a weighted average of expected future target positions. Our findings complement a number of previous studies in the literature on optimal strategies in illiquid markets where the frictionless hedging strategy is confined to diffusions, see, e.g., [7], [14], [15], [3], [13], [10], [8], [9], where the frictionless hedging strategy is confined to diffusions. We show furthermore that the general structure of the solution actually applies in great generality, i.e. with stochastic volatility and stochastic liquidity along with a possibly singular stochastic terminal state constraint on a set with positive but not necessarily full probability. Here our optimal signal process is built on the solution to a backward stochastic Riccati equation (see [12], [11]. [2], [6]) and it reveals not only necessary and sufficient conditions under which the problem admits a finite value, but also allows us to tackle the delicate random singularity at terminal time via a suitable time consistent approximation of the optimization problem.

We refer to our papers [4] and [5] for further details.

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On the dynamic representation of some time-inconsistent risk measures in a Brownian filtration

Ludovic Tangpi
(joint work with Julio Backhoff Veraguas)

Given a probability space $(\Omega, \mathcal{F}, P)$ equipped with the completed filtration $(\mathcal{F}_t)_{t\in[0,T]}$ of a d-dimensional Brownian motion W with $T \in \mathbb{R}_+$ and $\mathcal{F} = \mathcal{F}_T$, a functional $\rho : L^\infty(\mathcal{F}) \to (-\infty, +\infty]$ that is convex, increasing and cash-invariant[1] is called convex risk measure. For computation purposes as well as the study of optimization problems with objective functions ρ (e.g. risk minimization) it is often desirable to show that $\rho(X)$ admits a dynamic representation. When $\rho = \rho_{0,T}$ and $(\rho_{\nu,\tau})_{0\le\nu\le\tau\le T}$ (with ν, τ being stopping times) is a family of (conditional)

[1] $\rho(X + c) = \rho(X) + c$ for all $X \in L^\infty(\mathcal{F})$ and $c \in \mathbb{R}$. Translation invariance is a synonym for this.

convex risk measures $\rho_{\nu,\tau} : L^\infty(\mathcal{F}_\tau) \to L^\infty(\mathcal{F}_\nu)$, the main condition under which a dynamic representation can be derived is time-consistency, which amounts to

$$\rho_{\sigma,\nu}(X) = \rho_{\sigma,\tau}(\rho_{\tau,\nu}(X))$$

for all stopping times $0 \le \sigma \le \tau \le \nu \le T$ and $X \in L^\infty(\mathcal{F}_\nu)$. In fact, in this case, for every $X \in L^\infty$ and every $t \in [0,T]$ one has $\rho_{t,T}(X) = Y_t$ where (Y, Z) is the unique (minimal super-)solution of the backward stochastic differential equation

$$Y_t = X + \int_t^T g_u(Z_u)\, du - \int_t^T Z_u\, dW_u\,, \tag{1}$$

for a given function $g : [0,T] \times \Omega \times \mathbb{R}^d \to \mathbb{R}$, see [3], [4] and [5].

However, as shown by [6], most commonly used (law invariant) risk measures, such as the conditional value-at-risk (also referred to as tail or average value-at-risk), do not enjoy this property. Notable exceptions are the expected value and the so-called entropic risk measure. Our aim is to show that nevertheless many interesting time-inconsistent risk measures can be computed dynamically.

Our analysis focuses on the so-called optimized certainty equivalents (OCE) risk measures; see [1, 2], given by

$$\rho(X) := \inf_{r \in \mathbb{R}} (E[l(X - r)] + r), \tag{2}$$

where $l : \mathbb{R} \to \mathbb{R}$ is a convex increasing function. Denoting by l^* the convex conjugate of l, we assume

(N) : $l^*(1) = 0 = l(0)$.

(C) : $l(x) > x$ for all x such that $|x|$ is large enough.

This is a class containing time-consistent risk measures such as the entropic one, as well as time-inconsistent ones such as the conditional value-at-risk of [8], the monotone mean-variance of [7], and more generally OCEs linked to infomation-theoretic measures as the ϕ-divergence throught their dual representation. Let $\mathcal{O}$ be the interior of dom(l^*). For every $(s, y) \in [0,T] \times \mathbb{R}^m$, we consider the Itô diffusion $Y^{s,y}$ given by

$$\begin{aligned} dY_t^{s,y} &= b(t, Y_t^{s,y})dt + \sigma(t, Y_t^{s,y})dW_t, \quad t \ge s \\ Y_s^{s,y} &= y \end{aligned}$$

for two given (deterministic) functions b and σ assumed to be Lipschitz with linear growth. In our main result, we prove that for every

$$X^y := f(Y_T^{0,y}) + \int_0^T g(t, Y_t^{0,y})\, dt$$

where f and g are bounded and continuous functions,

$$\rho(X^y) = V(0, y, 1),$$

where $V : [0,T] \times \mathbb{R}^m \times \mathcal{O} \to \mathbb{R}$ is a viscosity solution of the PDE

$$\begin{aligned} \partial_t V + b(s,y)\partial_y V + &\frac{1}{2} tr\left(\sigma(s,y)\sigma(s,y)' \partial_{yy}^2 V\right) \\ &+ \sup_{\beta \in \mathbb{R}^d} \left[\frac{1}{2} z^2 |\beta|^2 \partial_{zz}^2 V + z\, \partial_{yz}^2 V \sigma(s,y)\beta \right] + z g(s,y) = 0, \end{aligned} \tag{3}$$

with

$$V(T,y,z) = f(y)z - l^*(z).$$

Furthermore if either $\mathrm{dom}(l^*)$ is bounded, or l^* is finite and has polynomial growth, then V is the minimal supersolution of (3) in the class of functions with polynomial growth.

The road leading to this result starts with the dual representation of the risk measure

$$\rho(X) \ = \ \sup_\beta E\left[XZ_T^\beta - l^*\left(Z_T^\beta\right)\right], \tag{4}$$

where β runs throught the set of bounded predictable processes and Z_T^β is the stochastic exponential of $\int \beta\, dW$, then a simple 'enlargement of state space' idea which allows to interpret the evaluation of the risk measure as a stochastic optimal control problem of its own: $\rho(X) = V(0,y,1)$ with

$$V(s,y,z) := \sup_\beta E\left[f(Y_T^{s,y})Z_T^{s,z,\beta} - l^*\left(Z_T^{s,z,\beta}\right) + \textstyle\int_s^T g(t,Y_t^{s,y})Z_t^{s,z,\beta}dt\right],$$

Drawing from the primal formulation (2) allows us to prove the dynamic programming principle in the enlarged state space:

$$V(s,y,z) = \sup_\beta E\left[\textstyle\int_s^\theta g(t,Y_t^{s,y})Z_t^{s,z,\beta}dt + V\left(\theta, Y_\theta^{s,y}, Z_\theta^{s,z,\beta}\right)\right],$$

$y \in \mathbb{R}^m, z \in \mathrm{dom}(l^*)$. As usual in stochastic control theory, we leverage on this DPP to obtain the aforementioned HJB equation. Here, the main difficulties lie in that the Hamiltonian in (3) is singular (i.e. not real-valued), the domain of the value function is unbounded and the drift term is not uniformly Lipschitz w.r.t. the control. These make the problem unamenable to existing techniques.

Some open questions which, in our opinion deserve some attention include the uniquess of the viscosity solution, the proof of the dynamic representation for non-Markovian claims or non-Brownian filtrations.

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A fundamental theorem of asset pricing for large financial markets under restricted information (L^p case)

IRENE KLEIN
(joint work with Christa Cuchiero, Josef Teichmann)

We present a version of the fundamental theorem of asset pricing (FTAP) for large financial markets in continuous time under restricted information in an L^p-setting. One often accepted hypothesis in modelling financial markets is the following: *Observations of prices are perfect and can be immediately included in trading decisions.* In the talk we consider a setting where this hypothesis is not true anymore. We consider a financial market, where the prices of the assets are not fully revealed to the trader (like ideas in Platon's cave allegory) or where trading decisions are not immediately executed. We call this a *platonic large financial market*, which in mathematical terms is given by a stochastic basis together with two filtrations $\mathbb{F} \subset \mathbb{G}$ and a (possibly uncountable) family of càdlàg stochastic processes $(S^i)_{i\in I}$ adapted to $\mathbb{G}$. We assume for simplicity that no issues on nullsets occur, i.e. *both* filtrations are completed by the same nullsets. Trading in these assets is possible but only with $\mathbb{F}$-predictable simple strategies including an arbitrary but finite number of assets. This extends the results of Yuri Kabanov and Christophe Stricker [4] to continuous time in a large financial market setting. On the other hand it generalizes Stricker's L^p version of FTAP [5] towards a setting of restricted information. We do neither assume that price processes are semi-martingales, (and it does not follow due to trading with respect to restricted information), nor any property of the two filtrations in question, but rather go for a completely general result, where trading strategies are predictable with respect to a smaller filtration than the one generated by the price processes. So our setting is completely general and reasonable and covers, e.g., the following situations:

- Trading with an execution delay.
- Trading with observational delay.
- For some assets trading is only possible for restricted time sets (for example static trading)
- Prices are uncertain due to liquidity issues, transaction costs.
- Model prices differ from market prices, which means in principle that one believes market prices come with an error.
- Lack of information (e.g. discrete information versus continuous time modelling).
- Model uncertainty.

To be able to prove a FTAP we use an L^p-setting including some integrability properties of the assets, for $1 \leq p < \infty$, as in [5]. In this way we can avoid the use of general admissible integrands and the stochastic integral in full generality, which does not yet exist for the $\mathbb{F} - \mathbb{G}$ situation. In contrast to the case $p = \infty$ where general integrands would be needed (and which we therefore exclude from our setting), for $p < \infty$ we only need simple integrands which pose no technical difficulties. Our Theorem then reads as follows (leaving away exact definitions of NAFLp(P') and exact choice of $\mathbb{F}$ which can be found in the slides of the talk):

Theorem 1. *The following statements are equivalent:*

(1) *There exists $P' \sim P$ and $1 \leq p < \infty$ such that* no asymptotic $L^p(P')$ free lunch *(NAFLp(P')) holds.*
(2) *There exists a probability measure $Q \sim P$ such that the $\mathbb{F}$ optional projections of all assets S^i are Q-martingales.*

Note that we do not have any dependence on the physical measure P of our L^p-no asymptotic free lunch condition (i), which would have been expected for an L^p-condition. However, we do not use the L^p condition for the fixed measure P and fixed $p < \infty$ but only for *some* $P' \sim P$ and *some* $1 \leq p < \infty$. And the equivalence with the condition (ii) above shows that there is no dependence on P of condition (i). In particular, this implies that, for example for bounded assets S^i, condition (i) is in fact equivalent with no asymptotic free lunch with vanishing risk as defined in Cuchiero, Klein, Teichmann [3]. For the measure P' we also have a superreplication theorem in the $L^p(P')$-$L^q(P')$-duality setting which can be found on the slides.

Finally, let us remark on model uncertainty. In this connection we refer in particular to the talk of Mete Soner which was given at the present workshop in Oberwolfach and which encouraged that this example is particularly stressed in the present talk. We define a new variant of a robust ("model free") setting, see also [1] and [2].

- Given a pathspace D with canonical filtration $\tilde{\mathbb{F}}$ and a family $\mathcal{P} = \{P^\theta : \theta \in \Theta\}$.
- Assume an a priori given probability measure ν on $(\Theta, \mathcal{B}(\Theta))$ and assume that $\theta \to P^\theta(A)$ is measurable for each $A \in \tilde{\mathcal{F}}$.

Define

$$P(A) = \int_\Theta P^\theta(A)\nu(d\theta)$$

Here we assume that we do not know what our correct model is, but it depends on the choice of a parameter θ which is chosen according to an a priori fixed measure ν on Θ. This can be understood as a kind of robust model, where the set of possible measures is $\mathcal{P}$. By the definition of the measure P we reduce our robust setting to the classical setting with respect to P. In order to characterize arbitrage in this setting we have to specify $X \geq 0$ P-a.s. and $P(X > 0) > 0$. It is clear that $X \geq 0$ P a.s. means $X \geq 0$ outside a set A for which ν-almost all θ we have that $P^\theta(A) = 0$. Compare this notion with quasi-sure nullsets as in [2], which means $P^\theta(A) = 0$ for *all* $P \in \mathcal{P}$ (without the additional measure ν). This setting gives rise to two filtrations in a natural way. On one hand we have the full information on $D \times \Theta$: $\mathcal{G}_t = \tilde{\mathcal{F}}_t \bigotimes \mathcal{B}(\Theta)$. On the other hand, as we cannot observe the correct parameter θ directly, trading will be based on a smaller filtration $\mathbb{F}$. Moreover we could include even more restrictions, for example if observations come on a discrete time grid, this would then lead to three filtrations.

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A regularity structure for rough volatility

Paul Gassiat

(joint work with Christian Bayer, Peter Friz, Jörg Martin, Benjamin Stemper)

We are interested in rough stochastic volatility models of the form

$$dS_t/S_t = \sigma dB_t \equiv \sigma \left(\rho dW_t + \bar{\rho} d\bar{W}_t\right) \tag{1}$$

where $W, \bar{W}$ are two independent Brownians, $\rho^2 + \bar{\rho}^2 = 1$, more specifically, for suitable (nice) f, we postulate

$$\begin{aligned} \sigma(t,\omega) &= f(\hat{W}_t(\omega)), \\ \hat{W}_t &= \int_0^t K(t,s)\, dW_s(\omega). \end{aligned} \tag{2}$$

The example to have in mind is $K(t,s) = |t-s|^{H-1/2}$ in which case $\hat{W}$ is a (Riemann-Liouville) fractional Brownian motion, with "rough" (i.e. rougher than Brownian) samples paths whenever $H \in (0, 1/2)$. The interest in such models is now widely acknowledged, e.g. [3, 1]. In particular, evidence for $H < 1/2$ (in fact, as low as $H \approx 0.05$) is overwhelming both under the physical and the pricing measure.

In contrast to classical stochastic volatility models in which (S, σ) is jointly Markov, this is not so here, for $\hat{W}$ already is a non-Markovian process. As a consequence, PDE based option pricing is no more available. The only option then is Monte Carlo simulation. It is interesting to note that a type of quasi-Monte Carlo on Wiener space, Kusuoka-Lyons-Victoir cubature, often applied in the form of the popular Ninomia-Victoir scheme, relies on writing the model in

Stratonovich form. But this does not work in the *correlated* ($\rho \neq 0$) rough case! Indeed,one does not have existence, in general, of $\int f\left(\hat{W}\right) \circ dW_t$, as already for $f(x) = x$, we have the (infinite) Ito-Stratonovich correction $[\hat{W}, W]$, due to $H < 1/2$.

Another - practical - problem with the above model, (1, 2), is that it lies outside the bulk of results known for Markovian diffusion processes. For instance, many stochastic volatility models have been analyzed using short-time / small-noise expansions, typically building on classical Freidlin-Wentzell estimates (That said, the model is formulated within Ito-calculus and can be analyzed by its methods. Indeed one can proceed by careful application of the approximate contraction principle, mimicking the (pre-rough paths) proofs of Freidlin-Wentzell theory, as was understood by Forde-Zhang [2].)

In this work we aim at showing how Hairer's theory of *regularity structures,* [4], a major generalization of rough path theory, is a very appropriate tool to analyze rough volatility. Recall that this theory identifies certain polynomial functions of the noise, which then are used as "extended monomials" in local Taylor-like expansions for the object of interest. In our setting, these monomials are given by the cross-integrals

$$\int \hat{W}^k dW, \quad \int \hat{W}^k d\bar{W} \quad k = 0, \dots, M$$

In the framework of regularity structures one is then reduced to studying these integrals under the so-called "model topology" and the continuity results provided by the pathwise approach then allow to find pleasing answers to both points raised above :

- We obtain an approximation theory taking into account "infinite" Itô-Stratonovich correction (in the form or diverging renormalization constants). More precisely, if W^ε are regular approximations to W (e.g. piecewise linear interpolation), there exist constants $C_\varepsilon > 0$ such that
$$\lim_{\varepsilon \to 0} \int_0^t f(\hat{W}_s^\varepsilon) dW_s^\varepsilon - C_\varepsilon \int_0^t f'(\hat{W}_s^\varepsilon) ds = \int_0^t f(\hat{W}_s) dW_s.$$
- We also obtain natural proofs of small noise large deviation estimates, recovering results of [2] (see also forthcoming work of Jacquier-Pakkanen-Stone).

To conclude, we note that one of the advantages of the pathwise approach lies in its robustness to small changes in the considered model. For instance, let us now consider a stochastic volatility model as in (1), but now with $\sigma(t, \omega) = f(Z_t)$, where Z_t is the solution to the stochastic Itô-Volterra equation

$$Z_t = z + \int_0^t K(s,t) \left(\psi(Z_s) dW_s + \phi(Z_s) ds\right).$$

Then (at least in the case $H \geq \frac{1}{4}$) one can directly use our "regularity structure" results to obtain again (with essentially no further work) small-noise large deviation results, and approximation results, again with an (explicit) infinite Itô-Stratonovich correction.

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Density Estimates and Applications

Paul Krühner
(joint work with Tilmann Blümmel, David Baños, Julia Eisenberg)

In this talk we try to find 'good' upper and lower estimates for the density ρ_t of X_t $(t > 0)$ where X is an Itô-process, i.e. $X_t := x + \int_0^t \beta_s ds + \int_0^t \sigma_s dW_s$ where $x \in \mathbb{R}^d$, β a suitable $\mathbb{R}^d$-valued process, σ a suitable $\mathbb{R}^{d\times d}$-valued process and W is a standard d-dimensional Brownian motion. These estimates for the density can be used to obtain estimates for expectations and to obtain error estimates for suboptimal controls in stochastic control problems. The most relevant case for density bounds is the so-called Markovian case, i.e.

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t c(X_s)dW_s, \quad t \geq 0$$

where the density is the weak fundamental solution of the corresponding Focker-Plank equation. The corresponding PDE does not yield good estimates for the density of the coefficient if those coefficients are merely assumed to be measurable. We find optimal density bounds for the cases where β is bounded by a constant $C \geq 0$ and σ is constant itself, namely $\rho_t(x) \leq \left(\frac{1}{\sqrt{t}}\varphi(C\sqrt{t}) + C\Phi(C\sqrt{t})\right)^d$ for any $t > 0$, $x \in \mathbb{R}^d$. Also we find optimal density bounds for the integrated density if β, σ are bounded $\mathbb{R}$-valued and σ is elliptic, i.e. there are $a, b > 0$ such that $a \leq \sigma_t \leq b$. This bound is given by

$$\int_0^t \rho_s(x)ds \leq \frac{b}{a^2}\int_0^t \left[\frac{1}{\sqrt{s}}\varphi\left(C\sqrt{s}\frac{b}{a^2}\right) + C\frac{b}{a^2}\Phi\left(C\frac{b}{a^2}\sqrt{s}\right)\right] ds, \quad t > 0, x \in \mathbb{R}.$$

Finally, we discuss the improvement in terms of travel distance, i.e. if the process is started in a position x, then we find sharper bounds for the density in $y \neq x$ and show some applications.

Semi-Static and Sparse Variance-Optimal Hedging

Martin Keller-Ressel
(joint work with Martin Haubold, Paolo Di Tella)

We consider a financial market consisting of a liquid traded asset S, a contingent claim H^0 on S which shall be hedged, and d other supplementary assets $H = (H^1, \ldots, H^d)$. Typically, the supplementary assets are European options on S or other assets which are strongly correlated with S. As hedging strategies for H^0 we consider so-called *semi-static* strategies, which consist of a dynamic (i.e. continuously rebalanced) position in S and a static (i.e. buy-and-hold) position in the assets $(H^1, \ldots, H^d)$.

There are multiple reasons to consider such semi-static strategies: First, the supplementary assets may be substantially less liquid than S or incur large transaction costs, such that it is not feasible to continuously rebalance positions in them. Second, it is well-known that in certain settings - most notably when H^0 is a variance swap and the supplementary claims are European options - semi-static strategies with an infinite number of supplementary assets allow for perfect hedging even in incomplete markets, cf. [5, 1].

Under a minimal-variance criterion, as introduced by [2, 3], the problem of finding the optimal semi-static hedging strategy (ϑ, v) and the optimal initial endowment c can be formulated as the minimization problem

$$(1)\qquad \epsilon^2 = \min_{(\vartheta,v)\in \mathrm{L}^2(S)\times\mathbb{R}^d, c\in\mathbb{R}} \mathbb{E}\left[\left(c - v^\top \mathbb{E}[H] + \int_0^T \vartheta_t dS_t - (H^0 - v^\top H)\right)^2\right].$$

Here, $\mathrm{L}^2(S)$ denotes the space of admissible dynamic strategies and is given by

$$\mathrm{L}^2(S) := \left\{\vartheta \text{ predictable and } \mathbb{R}\text{-valued: } \mathbb{E}\left[\int_0^T \vartheta_t^2 d\langle S, S\rangle_t\right] < +\infty\right\}.$$

To solve the variance-optimal semi-static hedging problem, we decompose it into an inner and an outer minimization problem and rewrite (1) as

$$\begin{cases} \epsilon^2(v) & = \min_{\vartheta\in \mathrm{L}^2(S), c\in\mathbb{R}} \mathbb{E}\left[\left(c - v^\top \mathbb{E}[H] + \int_0^T \vartheta_t dS_t - (H^0 - v^\top H)\right)^2\right], \\ \epsilon^2 & = \min_{v\in\mathbb{R}^d} \epsilon(v)^2. \end{cases}$$

The inner problem is of the same form as the classic variance-optimal hedging problem considered in [2, 3] and is solved by the so-called Galtchouk-Kunita-Watanabe-decomposition of the claims $(H^0, \ldots, H^d)$ with respect to S. The outer problem, on the other hand, turns out to be a finite dimensional quadratic optimization problem and hence is easy to solve.

Furthermore, we consider the problem of reducing the number of supplementary assets used in the static part of the hedge to a smaller number $k < d$, without substantially increasing the hedging error of the strategy. This problem can be seen as an optimal hedging problem under a sparsity constraint on the cardinality of non-zero positions in the supplementary assets. Again, the motivation to introduce such constraints is to reduce transaction and liquidity costs that are incurred from setting up the static part of the position. We propose two mathematical formulations of this problem, one where the cardinality constraint $\|v\|_0 \leq k$ is added to (1), and a second one where a penalty term $\lambda\|v\|_1$ with $\lambda > 0$ is added to (1). The second formulation can be seen as a convex relaxation of the (non-convex) cardinality constraint and is similar to the LASSO-approach to the variable selection problem in linear regression, cf. [4].

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Pathwise large deviations and variance reduction for affine stochastic volatility models

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(joint work with Zorana Grbac, David Krief)

1. Large deviations for affine stochastic volatility models

Definition 1. *An affine stochastic volatility model* [4] $(X_s, V_s)_{s\leq t}$, *is a stochastically continuous, time-homogeneous Markov process such that* $\left(e^{X_s}\right)_{s\leq t}$ *is a martingale and there exist deterministic functions* ϕ *and* ψ *such that*

$$\mathbb{E}\left(e^{uX_s+wV_s}\middle|X_0=x, V_0=v\right) = e^{\phi(s,u,w)+\psi(s,u,w)\,v+u\,x}\ ,$$

for all $(s,u,w)\in\mathbb{R}_+\times\mathbb{C}^2$.

The functions ϕ and ψ satisfy generalised Riccati equations

$$\partial_t\phi(t,u,w) = F(u,\psi(t,u,w))\ ,\quad \phi(0,u,w)=0$$
$$\partial_t\psi(t,u,w) = R(u,\psi(t,u,w))\ ,\quad \psi(0,u,w)=w\ ,$$

Large deviations for X_t/t as $t\to\infty$ have been studied in [3] using Gärtner-Ellis theorem and limiting results from [4]. To apply Gärtner-Ellis theorem to $\varepsilon X_{t/\varepsilon}$ as $\varepsilon\to 0$, as was done in [3] one needs to compute the limit

$$\lim_{\varepsilon\to 0}\varepsilon\log\mathbb{E}[e^{uX_{t/\varepsilon}}] = \lim_{\varepsilon\to 0}\varepsilon\{\phi(t/\varepsilon,u,0)+\psi(t/\varepsilon,u,0)V_0\}.$$

The study of these limits reduces to the study of equilibrium points of the Riccati equations. In [4] it has been shown that there exists an interval $I\supseteq[0,1]$ such that for each $u \in I$, the equation $\partial_t \psi = R(u, \psi)$ admits a stable equilibrium $w(u)$ with basin of attraction of $\mathcal{B}(u)$ and therefore, for $w \in \mathcal{B}(u)$ and all $t > 0$, $\psi(t/\varepsilon, u, w) \underset{\varepsilon \to 0}{\longrightarrow} w(u)$. Further, let $J = \{u \in I : F(u, w(u)) < \infty\}$. Then $[0,1] \subseteq J \subseteq I$ and there exists a function h such that for all $u \in J$ and $w \in \mathcal{B}(u), t > 0$,

$$\varepsilon \phi(t/\varepsilon, u, w) \underset{\varepsilon \to 0}{\longrightarrow} t\, h(u) \quad \text{so that} \quad \lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E}\left[e^{uX_{t/\varepsilon}}\right] = h(u).$$

The functions w and h can be extended uniquely to cumulant generating functions of infinitely divisible random variables.

Our first contribution is to extend the one-dimensional LDP of [3] to the finite-dimensional setting.

Theorem 1. *Let (X, V) be an affine stochastic volatility model such that h is essentially smooth, and assume in addition that $w(u_+) = w(u_-)$, where $J = [u_-, u_+]$. Then $(\varepsilon X_{t_1/\varepsilon}, ..., \varepsilon X_{t_n/\varepsilon})$ satisfies a LDP on $\mathbb{R}^n$ with good rate function*

$$\Lambda^*_{t_1,\dots,t_n}(x) = \sup_{\lambda \in J^n} \left\{ \sum_{j=1}^n \lambda_j (x_j - x_{j-1}) - \sum_{j=1}^n (t_j - t_{j-1})\, h(\lambda_j) \right\}.$$

This result is not a direct extension of the one-dimensional LDP based on convergence of $\varepsilon \log \mathbb{E}[e^{uX_{t/\varepsilon}}]$: here one needs convergence of $\varepsilon \log \mathbb{E}[e^{uX_{t/\varepsilon} + vV_{t/\varepsilon}}]$ which may not hold even if the former holds due to lack of integrability. This difference explains the presence of the condition $w(u_+) = w(u_-)$ which is easy to check but rather restrictive: for Heston model it is equivalent to having zero correlation between stock price and volatility. It is an open problem to understand whether and how this condition may be relaxed.

A direct application of the Dawson-Gärtner theorem allows to extend this result to the pathwise setting. Let $\mathbb{F}([0,T], \mathbb{R})$ be the set of all functions from $[0,T]$ to $\mathbb{R}$, equipped with the topology of pointwise convergence, and let $\tau = \{t_1, \dots, t_n\}$ with $0 < t_1 < \dots < t_n = T$.

Theorem 2. *Let (X, V) be an affine stochastic volatility model such that h is essentially smooth, and assume in addition that $w(u_+) = w(u_-)$, where $J = [u_-, u_+]$. Then $(\varepsilon X_{s/\varepsilon})_{0 \le s \le T}$ satisfies a LDP on $\mathcal{F}([0,T], \mathbb{R})$ with good rate function*

$$\Lambda^*(x) = \sup_\tau \Lambda^*_\tau(x).$$

The rate function $\Lambda^*(x)$ is finite only if $x : [0,T] \to \mathbb{R}$ is of bounded variation, with $x(0) = 0$. In this case it is given by

$$\Lambda^*(x) = \int_0^T h^*(\dot{x}^{ac}_s)\, ds + u^+ \nu^+([0,T]) + u^- \nu^-([0,T]),$$
$$\text{where} \quad h^*(y) = \sup_{\lambda \in J} \{\lambda y - h(\lambda)\}.$$

Here, x^{ac} is the absolutely continuous component of x and ν^+ and ν^- are the positive and negative parts of the singular component of dx.

2. Application to variance reduction

Consider a financial market which consists of a risk-free asset and a risky asset with price $S_t = S_0 e^{X_t}$, where (X) is an affine stochastic volatility process under the risk-neutral probability $\mathbb{P}$. We are interested in a derivative product whose value (pay-off) at time T is given by a functional $P(S)$ which depends of the entire trajectory of the stocks. To evaluate its price, we consider the importance sampling estimator

$$\widehat{P}_N^{\mathbb{Q}} := \frac{1}{N}\sum_{j=1}^{N}\left[\frac{d\mathbb{P}}{d\mathbb{Q}}\right]^{(j)} P(S_{\mathbb{Q}}^{(j)}),$$

where $S_{\mathbb{Q}}^{(j)}$ are sample trajectories of S under the measure $\mathbb{Q}$. For our models a natural choice of the importance sampling measure is provided by is the time-dependent Essher transform

$$\frac{d\mathbb{P}^\theta}{d\mathbb{P}} = \frac{e^{\int_{[0,T]} X_t\cdot\theta(dt)}}{\mathbb{E}\left[e^{\int_{[0,T]} X_t\cdot\theta(dt)}\right]}$$

where θ is a (deterministic) bounded $\mathbb{R}^n$-valued measure on $[0,T]$.

The optimal θ should minimize the variance of the estimator under $\mathbb{P}^\theta$,

$$\mathrm{Var}_{\mathbb{P}^\theta}\left(P\frac{d\mathbb{P}}{d\mathbb{P}^\theta}\right) = \mathbb{E}_{\mathbb{P}}\left[P^2\frac{d\mathbb{P}}{d\mathbb{P}^\theta}\right] - \mathbb{E}\left[P\right]^2$$

Denoting $G(X) = \log P(S)$, the minimization problem writes

$$\inf_{\theta\in M}\mathbb{E}_{\mathbb{P}}\left[\exp\left\{2G(X) - \int_{[0,T]} X_t\cdot\theta(dt) + \log\mathbb{E}\exp\Big(\int_{[0,T]} X_t\cdot\theta(dt)\Big)\right\}\right].$$

Nevertheless, this problem is as diffucult to solve numerically as the original problem of computing the option's price. Inspired by the works of [1, 2, 5], we choose instead to minimize a proxy for the variance computed using Varadhan's lemma. Under appropriate conditions, letting $X_t^\varepsilon = \varepsilon X_{t/\varepsilon}$,

$$\text{(1)}\quad \lim_{\varepsilon\to 0}\varepsilon\log\mathbb{E}\left[\exp\left(\frac{2G(X^\varepsilon) - \int_0^T X_s^\varepsilon\, d\theta_s + \log\mathbb{E}\exp\left(\int_0^T X_s^\varepsilon d\theta_s\right)}{\varepsilon}\right)\right]$$
$$= \sup_{x\in V_r}\left\{2G(x) - \int_0^T x_s d\theta_s - \Lambda^*(x)\right\} + \int_0^T h(\theta([s,T]))\, ds$$

where V_r denotes the space of functions with bounded variation. Therefore, we say that the variance reduction measure θ^* is asymptotically optimal if it minimizes the right-hand side of (1).

In some cases, the asymptotically optimal measure may be computed more easily. Let $G : D \to \mathbb{R}_+$ be concave. Then, under appropriate conditions, asymptotically optimal measure is the solution to

$$\inf_{\theta \in M} \{\widehat{G}(\theta) + \int_{[0,T]} h(\theta([t,T]))dt\}, \quad \text{where} \quad \widehat{G}(\theta) = \sup_{x \in V_r} \{G(x) - \int_{[0,T]} x_t \theta(dt)\}.$$

We have tested the method on the Heston model with and without jumps, both for European and Asian options and obtained numerical results confirming the efficiency of the algorithm.

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How Leverage Transforms a Volatility Skew

Roger Lee

(joint work with Ruming Wang)

To model leveraged investments such as leveraged ETFs, define the *β-leveraged product* on a positive semimartingale S to be the stochastic exponential of β times the stochastic logarithm of S.

In various asymptotic regimes, we relate rigorously the implied volatility surfaces of the β-leveraged product and the underlying S, via explicit shifting/scaling transformations. In particular, a family of regimes with *jump* risk admit a shift coefficient of $-3/2$, unlike the previously conjectured $+1/2$ shift. The $+1/2$, we prove, holds in a family of continuous (including fBm-driven) stochastic volatility regimes at short expiry and at small volatility-of-volatility.

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Efficient estimation of distributions of present values for long-dated contracts

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(joint work with Scott Robertson, Alexandra Tsimbalyuk)

In many applications–especially in the areas of risk management, insurance and actuarial mathematics–of interest is the distribution of present values for long-dated contracts, modelled by random variables of the form

$$X_0 = \int_0^{\tau_0} D_u a_u du. \tag{PV}$$

Above, $(D_t;\, t \in \mathbb{R}_+)$ denotes a (possibly, stochastic) discount factor, $(a_t;\, t \in \mathbb{R}_+)$ the cash-flow rate, and τ_0 is the stopping time modelling the contract termination.

Under the understanding that the underlying drivers for the contract are typically stationary in time, we assume the existence of a potentially multidimensional Markovian ergodic diffusion factor process[1] $Z = (Z_t;\, t \in \mathbb{R})$ with dynamics of the form

$$dZ_t = \mu(Z_t) + \sigma(Z_t) dW_t, \quad t \in \mathbb{R},$$

where W is a standard (possibly, multi-dimensional) Brownian motion, for appropriate functions μ and σ that ensure existence, uniqueness and ergodicity of the solution of the above differential equation. Given this driving process Z, we further assume that:

- $-dD_t/D_t = r(Z_t)dt + \eta(Z_t)dZ_t + \theta(Z_t)dB_t$ holds for $t \in \mathbb{R}$, with $D_0 = 1$, where B is a standard Brownian motion, independent of W, and r, η and θ are appropriate smooth functions;
- $(a_t;\, t \in \mathbb{R}) = (\alpha(Z_t);\, t \in \mathbb{R})$ for appropriate measurable function α;
- $\tau_0 = \inf\{u \geq 0 : \Delta N_u = 1\}$, where $N = (N(t);\, t \in \mathbb{R})$ is a Cox process with rate $(\lambda(Z_t);\, t \in \mathbb{R})$ for given non-negative measurable function λ.

Estimation of the conditional distribution of random variables as X_0 in (PV) above given Z_0 is typically difficult. Methods based on PDEs are not really applicable, since they require specification of boundary conditions that are generically unknown. When Monte-Carlo methods are utilised, simulation for each path realisation can take an prohibitive amount of time, leading to extremely slow and poor results. We propose an alternative simulation method, using ergodicity and time-reversal, that leads to significantly better results. To wit, we first extend the definition of X_0 from (PV) to a whole process $X = (X_t;\, t \in \mathbb{R})$ via

$$X_t = \int_t^{\tau_t} \frac{D_u}{D_t} \alpha(Z_u) du, \quad t \in \mathbb{R},$$

where $\tau_t = \inf\{u \geq t : \Delta N_u = 1\}$, and we note that the joint process $(Z, X) = ((Z_t, X_t);\, t \in \mathbb{R})$ is stationary. As such, one could simulate its paths and use the ergodic theorem for estimation of the joint law of (Z_0, X_0). The issue with simulating

[1]Note that the time-index of Z is the whole real line, instead of only the positive half-line, which is possible since it is ergodic.

(Z, X) is that X is anticipative, i.e, "forward-looking'; the remedy we propose is to consider the time-reversed process $(\zeta, \chi) = ((\zeta_t, \chi_t);\, t \in \mathbb{R}) = ((Z_{-t}, X_{-t});\, t \in \mathbb{R})$. Of course, the invariant measures of the forward and backward processes coincide. The dynamics for ζ are available, in terms of the coefficients μ and σ, as well as the invariant density of Z (or, equivalently, of ζ). Simple integral manipulations also provide dynamics for the process χ, given ζ, and then the ergodic theorem enables to obtain an estimator of the joint law of $(Z_0, X_0) = (\zeta_0, \chi_0)$ by using the occupation measure of (ζ, χ). Under additional assumptions, versions of the central limit theorem may also be obtained.

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