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Real Algebraic Geometry With a View Toward Moment Problems and Optimization

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Abstract. Continuing the tradition initiated in MFO workshop held in 2014, the aim of this workshop was to foster the interaction between real algebraic geometry, operator theory, optimization, and algorithms for systems control. A particular emphasis was given to moment problems through an interesting dialogue between researchers working on these problems in finite and infinite dimensional settings, from which emerged new challenges and interdisciplinary applications.

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Introduction by the Organisers

In this workshop we brought together experts, as well as young researchers, working on the following four themes:

(A) Positivity, Sums of Squares and the Moment Problem.
(B) Polynomial Optimization.
(C) Infinite Dimensional Moment Problem and Applications.
(D) Relations to Operator Theory and Free Positivity.

Topic **(A)** is at the very heart of real algebraic geometry. Positivity of polynomials and rational functions and their representations as sums of squares go back to the foundational papers of Hilbert and of Artin–Schreier. The relation between moment problems and positivity goes back to the work of F. Riesz, M. Riesz, and

Haviland, but it is especially from the beginning of the 1990s that real algebraic geometry and the study of (multidimensional) moment problems started deeply influencing each other. For topic **(B)**, polynomial optimization has been successfully using the tools of real algebraic geometry during the last two decades in order to apply semidefinite programming techniques, leading to a host of new problems and results in real algebraic geometry itself. With regard to topic **(D)**, there has been a steady increase in the interaction between real algebraic geometry and the study of positivity and factorizations in multivariable operator theory. This is especially true of the "hot" new areas of free noncommutative real algebraic geometry and free noncommutative analysis. Topic **(C)** is a relative newcomer, as it became evident during the MFO workshop in 2014 that there was a great need of a dialogue between the community of researchers working in real algebraic geometry on the one hand and those working on applications of infinite dimensional moment problems on the other.

To stimulate discussions the organizers asked 9 senior participants to give an introductory lecture, keeping their presentations to a level accessible to a mixed audience of non-specialists and specialists. These survey-expository talks can be roughly divided according to the four main themes mentioned above (speakers in each area are listed in order of appearance in the schedule):

(A) Marie-Françoise Roy, Christian Berg, Raúl Curto,
(B) Levent Tunçel, Monique Laurent,
(C) Sergio Albeverio, Tobias Kuna,
(D) Michael Dritschel, Igor Klep,

though some of the talks were clearly touching more than one area in perfect accordance with the interdisciplinary spirit of this meeting. The survey-expository talks were scheduled at the beginning of each session on the first four days of the workshop, while regular research talks of 40 minutes were scheduled in all the remaining 19 slots (5 of these were delivered by graduate students and postdocs). To encourage the dialogue between the various areas we decided to keep a mixed thematic structure in the schedule.

Let us give now a summary of the main topics discussed at the workshop.

Positive polynomials and Sums of Squares

Marie-Françoise Roy opened the workshop with a survey talk on representations of positive polynomials as sums of squares of rational functions (and the more general Positivstellensatz), starting with the classical non-constructive proof of Artin and leading all the way to the latest constructive approaches. Antonio Lerario described novel, for most participants, methods of random geometry. The talk of Claus Scheiderer presented (for the first time) a solution to a major open problem in real algebraic geometry that was originally motivated by semidefinite programming (the Helton–Nie conjecture). It lead to numerous repercussions during the workshop in both later talks (such as the one of Levent Tunçel, see below) and private discussions. The talk of Rainer Sinn dealt with the large ongoing research effort of placing Hilbert's original theorem on when (in terms of the degree and the number of variables) a positive polynomial is always a sum of squares of polynomials (without denominators) in the framework of varieties of minimal degree. Charu Goel's talk revisited Hilbert's theorem for the much subtler case of symmetric and even symmetric forms. These appeared also in the talk of Bruce Reznick that discussed various methods of establishing positivity.

Finite dimensional Moment Problems

Christian Berg offered a survey-expository talk about the finite dimensional moment problem, tracing the historical development since its origins with Stieltjes and giving special attention to the indeterminate Hamburger moment problem. The problem of uniqueness of probability distributions in terms of their moments was further developed in the talk of Jordan Stoyanov through a review of the most known determinacy conditions and new challenges related to them. The survey talk of Raúl Curto gave an exhaustive account of the state of the art for the truncated multidimensional moment problem. This subject of fundamental importance, both pure and applied, was treated at further three closely related talks of Greg Blekherman, Lawrence Fialkow, and Konrad Schmüdgen, with numerous discussions between these speakers (as well as other workshop participants). Their talks dealt with the support of the representing measure, especially for discrete measures, including the notion of the core variety that was recently introduced by Fialkow and that seems destined to play a key role. The talk of Fialkow pointed also at a large number of open problems for further investigation. In a somewhat opposite direction and using very different techniques (that originated in multivariable operator theory) Gregory Knese discussed in his talk some absolutely continuous representing measures for the two-dimensional trigonometric moment problem.

Polynomial Optimization and Hyperbolic polynomials

The survey talk of Levent Tunçel focused on hyperbolic (polynomial) programming, which is still conjectured to be a significant extension of semidefinite programming — itself a major extension a linear programming — that gained momentum in the 1990s. While semidefinite programming is linear optimization in slices of the semidefinite cone (i.e. the cone of non-negative quadratic forms), hyperbolic programming is linear optimization in slices of the hyperbolicity cone (i.e. the cone of directions along which a polynomial has only real roots). Whereas the existence of efficient interior-point algorithms for hyperbolic programming is still elusive, Tunçel's talk reported on recent progress in the field. The real algebraic geometry of the hyperbolicity cone is also subject of active research, as evidenced by the talks of Mario Kummer and Cynthia Vinzant on hyperbolic subvarieties, a natural extension of hyperbolic polynomials, as well as the talk of Petter Brändén on mixed characteristic polynomials. Relations with semidefinite representability of convex semialgebraic sets were mentioned. This allowed to make a connection with the result by Claus Scheiderer already mentioned above stating that there are many convex semialgebraic sets which are not semidefinite representable, i.e. that cannot be expressed as projections of slices of the semidefinite cone. Semidefinite representability was also covered by the survey-tutorial talk of Monique Laurent, who dealt with tracial polynomial optimization, a class of non-commutative polynomial optimization problems relevant for bounding the rank of matrix factorization. Such optimization problems appeared also in the talk of Igor Klep on free positivity, see below. It turns out that this rank is direct related to the size of the semidefinite representation of polytopes. Finally, the talk of David de Laat explored the Thomson problem, a classical optimization problem related to optimal distribution of points on the sphere. Applying an infinite dimensional generalization of the Lasserre hierarchy and exploiting the specific problem structure and symmetry gives new bounds for this problem.

Infinite dimensional Moment Problem and Applications

Sergio Albeverio presented an overview of infinite dimensional moment problems appearing in quantum field theory and stochastic analysis, illustrating through concrete examples how such problems are naturally constructed in these fields and which challenging questions are related to them. He also gave a glimpse of the various structural approaches to these classes of infinite dimensional moment problems known in literature, highlighting their interplay with real algebraic geometry and finite dimensional moment problems. This same feature was also pointed out by Tobias Kuna in his survey aimed to explain the relevance of the moment problem in the treatment of dynamics in statistical physics. This lead to a review of the main results known about the moment problem for point processes and so to a general formulation of moment problem for measures supported on infinite-dimensional function spaces. Its particular instance for random measures on locally compact Polish spaces was presented by Eugene Lytvynov, who gave a step-by-step introduction to this type of infinite dimensional moment problem and showed a characterization in terms of moments of all the random measures which are also random discrete measures. Maria João Oliveira described an extension of the classical umbral calculus on $\mathbb{R}$ to the space of distributions on $\mathbb{R}^d$ outlining several applications in infinite dimensional analysis.

As a sequel to Albeverio's talk (and other talks and discussions) an informal group gathered on one of the evenings in the main lecture hall in order to go slowly and in complete details through one concrete recent result, breaking the rather common language barrier that happens often between mathematical physicists and other analysts, let alone algebraists.

Relations to Operator Theory and Free Positivity

The survey talk of Michael Dritschel described a variety of results on sums of squares representations, both commutative and noncommutative, that can be obtained by operator theoretic methods. Alongside the fairly standard separation argument and GNS-type construction, he described other tools such as Schur complements and operator spaces that were novel to most participants. The survey talk of Igor Klep started with some general noncommutative Positivstellensätze and then proceeded to some key results, techniques, and applications in the free noncommutative setting. This set the stage for subsequent research talks. Joe Ball discussed robust control and linear matrix inequalities in both commutative and free setting; the free noncommutative setting appeared both as a relaxation of the commutative setting (as it did also in Klep's talk) and as presenting a structured uncertainty. Jurij Volčič presented new results on "zero loci" of (noncommutative) linear pencils that are likely to play a foundational role in future developments of free noncommutative algebraic geometry (both complex and real). Scott McCullough described state of the art results and techniques on the classification of free spectrahedra up to free noncommutative isomorphism; while the case of free balls is by now quite well understood, the case of general spectrahedra presents new challenges with unexpected phenomena occuring.

Other Events

To celebrate International Women's Day, a "fishbowl discussion" took place on the evening of March 8, with a title "Real nonnegative representations for Women in Mathematics". The discussion generated considerable interest and a lively exchange of views on the situation and the policies regarding gender equality in various countries, that lasted well beyond the workshop. A report of the discussion is available at the websites `http://homepages.laas.fr/henrion/mfo17` and `http://www.math.uni-konstanz.de/~infusino/KWIM-16-17/KWIM`.

As a follow up to the workshop, 4 participants agreed to contribute "snapshots" on various related topics for the MFO project "Snapshots of Modern Mathematics".

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Workshop: Real Algebraic Geometry With a View Toward Moment Problems and Optimization

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Abstracts

Algebraic representations of positive polynomials

Marie-Françoise Roy
(joint work with Henri Lombardi, Daniel Perrucci)

Hilbert '1900 17th problem:

Is a non-negative polynomial a sum of squares of rational functions ?

All bibliographical references in the talk can be found in [1].

1. Artin's proof

Artin '27 gives an affirmative answer:

A non-negative polynomial is a sum of squares of rational functions.

We give a sketch of this beautiful proof. Suppose P is not a sum of squares of rational functions. Sums of squares form a proper cone of the field of rational functions, and do not contain P (a cone contains squares and is closed under addition and multiplication, a proper cone do not contain -1). Using Zorn's lemma, get a maximal proper cone of the field of rational functions which does not contain P. Such a maximal cone defines a total order on the field of rational functions. Every totally ordered field has a real closure (A real closed field is a totally ordered field where positive elements are squares and a polynomial of odd degree has a root). Taking the real closure of the field of rational functions for this order, get a field in which P takes negative values (when evaluated at the "generic point" = the point $(X_1, \ldots, X_k)$). Then P takes negative values over the reals. This is the first instance of a transfer principle in real algebraic geometry : a first order statement involving elements of $\mathbb{R}$ which is true in a real closed field containing $\mathbb{R}$ is true in $\mathbb{R}$. Example of such statement: if $\exists x_1 \ \ldots \ \exists x_k \ P(x_1, \ldots, x_k) < 0$ is true in a real closed field containing $\mathbb{R}$ it is also true in $\mathbb{R}$

The transfer principle can be proved through Quantifier Elimination (QE). Example from high school mathematics

$$\exists \ x \quad ax^2 + bx + c = 0, a \neq 0 \iff b^2 - 4ac \geq 0, a \neq 0.$$

QE is valid for any first order formula, due to Tarski, based on generalizations of Sturm's theorem, or Hermite's quadratic form.

Artin's proof is very indirect (by contraposition, uses Zorn). No hint on denominators: what are the degree bounds? Artin himself notes effectivity is desirable but difficult. QE decides whether the polynomial is everywhere non negative, but how to construct the representation as a sum of squares? What are the best possible bounds on the degrees of the polynomials in this representation ?

2. Constructive proofs

Kreisel '57 - Daykin '61 - Lombardi '90 - Schmid '00: Constructive proofs with primitive recursive degree bounds on k and $d = \deg P$.

Our work '14 another constructive proof with elementary recursive degree bound:

$$2^{2^{2^{d^{4^k}}}} \tag{1}$$

More general problem: find algebraic identities certifying that a system of sign condition is inconsistant. $\mathbf{K}$ an ordered field, $\mathbf{R}$ a real closed extension of $\mathbf{K}$, Given

$$\mathcal{H} : \begin{cases} P_i(x) & \neq & 0 & \text{for} & i \in I_{\neq} \\ P_i(x) & \geq & 0 & \text{for} & i \in I_{\geq} \\ P_i(x) & = & 0 & \text{for} & i \in I_{=} \end{cases} \quad \downarrow \mathcal{H} \downarrow : \quad \underbrace{S}_{>0} + \underbrace{N}_{\geq 0} + \underbrace{Z}_{=0} = 0$$

with

$$S \in \left\{ \prod_{i \in I_{\neq}} P_i^{2e_i} \right\} \qquad \leftarrow \quad \text{monoid associated to } \mathcal{H}$$

$$N \in \left\{ \sum_{I \subset I_{\geq}} \left(\sum_j k_{I,j} Q_{I,j}^2 \right) \prod_{i \in I} P_i \right\} \quad \leftarrow \quad \text{cone associated to } \mathcal{H}$$

$$Z \in \langle P_i \mid i \in I_{=} \rangle \qquad \leftarrow \quad \text{ideal associated to } \mathcal{H}$$

is an **incompatibility** of $\mathcal{H}$, certifying that the set defined by the sign conditions in $\mathcal{H}$ is inconsistant. The Positivstellensatz asserts that a system of sign condition is inconsistant if and only if it is incompatible.

Classical proofs of Positivstellensatz are based on Zorn's lemma and Tranfer principle. Constructive proofs use QE. Method: transform a proof that a system of sign conditions is empty, based on QE, into an incompatibility. Most QE methods eliminate variables one after the other, using the projection method: the realizable sign conditions for $\mathcal{P} \subset \mathbf{K}[x_1, \ldots, x_k]$ are determined by the list of non empty sign conditions for $\text{Proj}(\mathcal{P}) \subset \mathbf{K}[x_1, \ldots, x_{k-1}]$ Cohen-Hormander QE method is very simple conceptually but primitive recursive. Projection method can be made efficient = elementary recursive; Classical cylindrical decomposition uses the geometric notion of connected component. We design a new projection method based only on algebra (using Thom's encoding of real roots by sign of derivatives and sign determination)

Lombardi '90: Primitive recursive degree bounds on k, $d = \max \deg P_i$ and $s = \#P_i$ for Positivstellensatz, based on Cohen-Hörmander algorithm for quantifier elimination :exponential tower of height $k+4$, $d\log(d) + \log\log(s) + c$ on the top.

Our work: Based on our new projection method based only on algebra (using Thom's encoding of real roots by sign of derivatives and sign determination) . Elementary recursive degree bound in k, d and s:

$$2^{2^{2^{\max\{2,d\}^{4^k}} + s^{2^k} \max\{2,d\}^{16^k \text{bit}(d)}}}.$$

It is classical to deduce the representation of a non-negative polynomial as a sum of squares of polynomials from a Positivstellensatz identity.

$$\left\{\begin{array}{rcl} P(x) & \neq & 0 \\ -P(x) & \geq & 0 \end{array}\right. \quad \text{no solution}$$

$$\iff \quad \underbrace{P^{2e}}_{>0} + \underbrace{\textstyle\sum_i Q_i^2 - (\sum_j R_j^2)P}_{\geq 0} = 0 \qquad (\star)$$

$$\implies \quad P = \frac{P^{2e} + \sum_i Q_i^2}{\sum_j R_j^2} = \frac{(P^{2e} + \sum_i Q_i^2)(\sum_j R_j^2)}{(\sum_j R_j^2)^2} (\star\star).$$

Strategy of the proof: for every system of sign conditions with no solution, construct an algebraic incompatibility and control the degrees for the Positivstellensatz. Recover Hilbert's 17 th problem as a special case Uses the key concept of weak inference introduced in Lombardi '90. For example the weak inference $\vdash\ P \geq 0$ means that for any $\mathcal{H}$, if we are given an initial incompatibility of $\mathcal{H}$ with $P \geq 0$, we can construct a final incompability of $\mathcal{H}$ itself.

The proof transforms into a construction of incompatibilities several tools, from classical algebra to modern computer algebra: a real polynomial of odd degree has a real root, a real polynomial has a complex root (using an algebraic proof due to Laplace) , the signature of Hermite's quadratic form is determined by the number of real roots of a polynomial and also by the sign conditions on principal minors, Sylvester's inertia law: the signature of a quadratic form is well defined, realizable sign conditions for a family of univariate polynomials at the roots of a polynomial fixed by sign of minors of several Hermite quadratic form (using Thom's encoding of real roots and sign determination), finally : the realizable sign conditions for $\mathcal{P} \subset \mathbf{K}[x_1, \ldots, x_k]$ are determined by the list of non empty sign conditions for $\mathrm{Proj}(\mathcal{P}) \subset \mathbf{K}[x_1, \ldots, x_{k-1}]$: new efficient projection method using only algebra.

We sketch our effective method for Hilbert 17 th problem. Suppose that P takes always non negative values. The proof that $P \geq 0$ is transformed, step by step, in a proof of the weak inference $\vdash\ P \geq 0$. Now, $P \neq 0, -P \geq 0$, is incompatible with $P \geq 0$, since

$$\underbrace{P^2}_{>0} + \underbrace{P \times (-P)}_{\geq 0} = 0 \quad .$$

Hence, taking $\mathcal{H} = [P \neq 0, -P \geq 0]$ and using the weak inference $\vdash\ P \geq 0$ we construct an incompatibility of $\mathcal{H}$ itself, which is the final incompatibility we are looking for !! We expressed P as a sum of squares of rational functions (using $(\star)$ and $(\star\star)$!!! Keeping track of degre estimates we obtain (1).

Why a tower of five exponentials ? outcome of our method ... no other reason ... Very far from existing lower bounds. There are only single exponential lower bounds (Grigorev Vorobjov) for Positivstellensatz, while the best lower bound for Hilbert 17th problem has a degree linear in k (recent result by Bleckherman and co).

References

[1] H. Lombardi, D. Perrucci, M. -F. Roy, *An elementary recursive bound for effective Positivstellensatz and Hilbert 17-th problem* (preliminary version, arXiv:1404.2338).

Infinite dimensional moment problems in quantum field theory and stochastic analysis

Sergio Albeverio

1. Moment problems in quantum physics: the constructive approach

Moment problems (MP), with their associated linear functionals and measures, arise in practically all areas of classical and quantum physics, both for particles and fields. In our lecture we illustrated particularly the cases of quantum field theory and quantum mechanics (Example 1) and their Euclidean versions (Example 2). In this section we present "constructive aspects", in the next one "structural approaches". In section 3 we discuss the K-MP for the case of Brownian motion.

1.1. **Example 1.** A relativistic (scalar) quantum field (QF) can be looked upon as the quantization of a classical real-valued field $\varphi_{cl}(t,x)$, where $t \in \mathbb{R}$ is time, $x \in \mathbb{R}^s$, $s \in \mathbb{N}_0$, is the space variable, evolving according to the Newton-type equation of motion $\frac{\partial^2}{\partial t^2}\varphi_{cl}(t,x) = \Delta\varphi_{cl}(t,x) - V'(\varphi_{cl}(t,x))$. Δ is the Laplacian on $\mathbb{R}^s$, V' the derivative of the real-valued C^1-function V on $\mathbb{R}$. This is the Klein-Gordon equation with nonlinearity V' (in the case $s = 0$ we have $\frac{\partial^2}{\partial t^2}\varphi_{cl}(t) = -V'(\varphi_{cl}(t))$ describing the position at time t of a classical particle moving under the potential V). The quantization φ of φ_{cl} consists in looking at $\varphi(t,x)$ as self-adjoint operator-valued distributions, acting in a certain (complex) Hilbert space $(\mathcal{H}, (\, , \,))$. The "mean values" of their products in a certain "vacuum state Ω" belonging to $\mathcal{H}$

$$(1) \qquad (\Omega, \varphi(t_1,x_1)\ldots\varphi(t_n,x_n)\Omega), \quad t_1 \leq \ldots \leq t_n,$$

called (time-ordered) Wightman functions, have the physical meaning of "correlation functions". According to Feynman's heuristic approach, (1) can be expressed in the form

$$(2) \qquad \text{``}Z^{-1}\int_\Gamma \exp(iS(\gamma))f(\gamma)\,\mathrm{d}\gamma\text{''},$$

where $f(\gamma) := \prod_{i=1}^n \gamma(t_i,x_i)$. The integration is over all elements γ in a "path space" Γ of mappings from $\mathbb{R}\times\mathbb{R}^s$ into $\mathbb{R}$. S is the classical action functional

$$(3) \qquad S(\gamma) = \frac{1}{2}\int_{\mathbb{R}^d}\left[\left|\frac{\partial\gamma}{\partial t}(\underline{x})\right|^2 - |\nabla_x\gamma(\underline{x})|^2\right]\mathrm{d}\underline{x} - \int_{\mathbb{R}^d} V(\gamma(\underline{x}))\,\mathrm{d}\underline{x},$$

$\underline{x} = (t,x) \in \mathbb{R}^d$, $d := s+1$, ∇_x being the gradient with respect to the space variable x (in the case $s = 0$ we simply do not have the term involving ∇_x), $\mathrm{d}\gamma$ is a flat measure on Γ, Z is a normalization constant. Note that (3) is formally relativistic invariant. In this way the solution of the MP given by the ordered

n-tuples, "moments", (1) is presented as the complex linear functional $I(f)$ given by (2). A rigorous meaning to $I(f)$ has been given when V is of the form $V(y) = \frac{|y|^2}{2} + \lambda W(y)$, with $\lambda \geq 0$ a "coupling constant", $y \in \mathbb{R}$, W a continuous real function on $\mathbb{R}$, and the term $-\lambda \int_{\mathbb{R}^d} W(\gamma(\underline{x}))\, \mathrm{d}\underline{x}$ in (3) is regularized, both by replacing $\gamma(\underline{x})$ by its convolution with a $C_0^\infty(\mathbb{R}^d)$-function ("ultraviolet cut-off") and by multiplying the integrand by the characteristic function of a bounded set in $\mathbb{R}^d$ ("infrared cut-off"), see [3], [15]. The regularization destroys the relativistic invariance, its removal is an open problem for $s = 3$ (for $s = 1, 2$ it has been achieved for special cases of W via an "analytic continuation in time", "Wick rotation", accompanied by a renormalization, see Example 2).

1.2. **Example 2.** Since the late 40's resp. 60's an alternative approach to the QM resp. QF theoretical (complex) MP as formulated in Example 1 has been obtained by formally replacing the real time t by a purely imaginary time $\tau = it$. By such a replacement the unitary Schrödinger evolution group e^{itH}, for $t \geq 0$, goes over to the heat semigroup $e^{-\tau H}$, $\tau \geq 0$ (hence Schrödinger equation goes over to the heat equation, the nonlinear wave operator to a Laplace operator in the new time variable τ and in the space variable x). Moreover, the term iS in the path integral (2) goes over to the term $-S_E$, where S_E is the "Euclidean action", obtained by replacing in S as given by (3) all but the $|\dot{\gamma}(\underline{x})|^2$-term by their opposite terms, getting $S_E \geq 0$ when $\lambda W \geq 0$. The $Z^{-1}e^{iS(\gamma)}\, \mathrm{d}\gamma$ term in (2) is then replaced by $Z^{-1}\exp(-S_E(\gamma))\, \mathrm{d}\gamma$ and can then be interpreted as a heuristic positive measure $\mu(\mathrm{d}\gamma)$ on Γ. The correlation functions (1) and their analogoues for the Euclidean fields $\varphi_E(\tau, x)$, $\tau \in \mathbb{R}$, $x \in \mathbb{R}^s$ are then expressible as moments $\int_\Gamma \gamma(\tau_1, x_1) \dots \gamma(\tau_n, x_n)\mu(\mathrm{d}\gamma)$ with respect to μ, called "Schwinger functions".

Remark. *(1)* In Example 1 for $s = 0$, $V = 0$, μ (restricted to paths from the time interval $[0, t]$, $t > 0$) can be interpreted as Wiener measure, looked upon as a path space measure given the distribution of a Wiener (Brownian motion) process in time $[0, t]$ with values in $\mathbb{R}$ (realized rigorously on Wiener space $C_{(0)}([0, t]; \mathbb{R})$, the zero standing for functions vanishing at the origin). *(2)* The advantages of having introduced the imaginary time is to have at least heuristically a positive measure μ instead of a complex functional, as associated to the moments (correlation functions). Technically this transformation from hyperbolic to elliptic problems permit to better handle, at least for $s \leq 2$, the construction of the functional I, via the associated positive measure μ. One can then go back to the original problem via analytic continuation. This is the essence of the "Euclidean constructive approach to QF" developed in the late sixties, see e.g. [10], [17], [7], [12]. For $W = 0$, μ is realized as the Nelson (global Markov, Euclidean invariant) free field measure μ_0 with mean zero and covariance operator given by the fundamental solution of $-\Delta + 1$. For $d \geq 2$, μ_0 is supported only on spaces of negative Sobolev index, so that for μ_0, and consequently for μ, the moments can only be expected to exist in a generalized sense; moreover the W-term in S_E is ill defined. However for $2 \leq d \leq 3$ and one can "regularize and renormalize special W" in such a way that the Schwinger functions to μ can be obtained as limits in the sense of moments convergence of corresponding moment functions with "regularized and renormalized W". For a subclass of such W the expansion in powers of λ of the term $\exp\left(-\lambda \int W(\gamma(\underline{x}))\, \mathrm{d}\underline{x}\right)$ in μ can even be shown to be asymptotic (even Borel summable for W a 4th order power) to the moments of μ, around those of μ_0. It is still a major challenge to provide an extension of such constructions for the physical case $d = 4$. *(3)* The relations with MP proper are numerous but somewhat scattered in the literature, they would deserve a more systematic investigation; those between moment convergence and weak convergence have been studied by several authors, e.g. Yu. Kondratiev, T. Kuna, E. Lytvynov, H. Zessin. Uniqueness/non uniqueness results have been obtained in connection with self-adjoint extensions of associated relativistic field operators and their Wick powers, see e.g. [1]. Such questions for corresponding Hamiltonians have induced a number af analytic results (involving analytic, -quasi-analytic, -Stieltjes vectors, see, e.g. [16], [13]). Among the open problems is a systematic study of K-MP in these settings (see Section 2 for some remarks on this).

2. Structural approach to MP in QF and infinite dimension

In the structural approach the correlation functions (Wightman resp. Schwinger functions) are looked upon as coming from linear functionals defined on certain topological algebras, with suitable continuity properties and satisfying certain properties (inspired by physical requirements and formulated axiomatically). This approach has been initiated by H.T. Borchers and T. Yngvason, and continued by many authors, see, e.g., [5], [6], [8], [9], [13], [14], [18], [19]. In this approach one considers a $*$-algebra $\underline{\mathcal{S}} := \bigoplus_{n=0}^{\infty} \mathcal{S}_n$ of test functions, $\mathcal{S}_n := \mathcal{S}(\mathbb{R}^{dn})$ (inductive limit), with a suitable product and involution $*$, so that $\underline{\mathcal{S}}$ becomes a $*$-algebra with unit, called Borchers-Uhlmann algebra. The axioms for Wightman resp. Schwinger functions are then translated into requirements for continuous linear functionals W on $\underline{\mathcal{S}}$, which roughly express invariance under the relevant group (Poincaré resp. Euclidean group), locality of associated field operators and above all positive definiteness, permitting a Hilbert space realization (and thus a probabilistic interpretation as usual in quantum mechanics). It is possible to make $\underline{\mathcal{S}}$ into a $*$-Banach algebra. Let us also remark in passing that non commutative extensions of such settings have also been developed, see, e.g. [6], [8]. The complex interrelations between relativistic and Euclidean QF have also been studied in these frameworks, by various authors. An interesting problem which arose for both types of QF concerns the corresponding truncated MP, trying thus to determine the family of possible functionals W's on the basis of their values limited to the $\mathcal{S}_k$, for all $k \leq N$, and some fixed finite N. Due to the constraints posed on W, specific as well as general results have been obtained. E.g. by a result of R. Jost and B. Schroer, essentially if W coincides with W_0 on $\mathcal{S}_2$, where W_0 is the linear functional (to the measure μ_0 introduced above) associated with the relativistic resp. Euclidean free field , then $W = W_0$ on the whole algebra. This implies that to obtain a physically non trivial W (describing interactions) it is necessary (but by no means sufficient!) that $W \neq W_0$ on $\mathcal{S}_2$.
The work on functionals on $\underline{\mathcal{S}}$ can also be put in relation with very recent work [9]

on general infinite dimensional MP and associated linear continuous functionals. Here $\underline{\mathcal{S}}$ is replaced by a general symmetric algebra on a locally convex space. Full MP and K-MP have been throroughly discussed in this setting, in particular general conditions for existence and uniqueness of solutions have been found. Let me give a few details for a related nuclear space setting as presented in recent work [13], [14]. The authors consider the distributional $\mathcal{D}'(\mathbb{R}^d)$-space with the projective topology. For a probability measure μ on $\mathcal{D}'(\mathbb{R}^d)$, if the moment $\int \langle f, \eta\rangle^n \mu(\mathrm{d}\eta)$ exists absolutely and is continuous in f, for any $f \in C_c^\infty(\mathbb{R}^d)$, then there exists a symmetric functional (SF) $m_\mu^{(n)} \in \mathcal{D}'(\mathbb{R}^{dn})$ such that for any $f^{(n)} \in C_c^\infty(\mathbb{R}^{d_n})$, $\langle f^{(n)}, m_\mu^{(n)}\rangle = \int \langle f^{(n)}, \eta^{\otimes n}\rangle \mu(\mathrm{d}\eta)$ (μ is then the probability distribution of a generalized process indexed by $C_c^\infty(\mathbb{R}^d)$). The K-moment problem (MP) ("realization problem") asks the converse question: given a measurable subset K in $\mathcal{D}'(\mathbb{R}^d)$ and an N-tuple of SF $m^{(n)}$, $n = 0, \ldots, N$, on $\mathcal{D}'(\mathbb{R}^{dn})$, for any $N \in \mathbb{N}_0 \cup \{+\infty\}$, can we find a positive bounded measure μ with $\operatorname{supp} K$ s.t. the n-th moment $m_\mu^{(n)}$ coincides with $m^{(n)}$? For $N = +\infty$ we have a "full MP", for $N < \infty$ a truncated MP. In an important case investigated in [14] K is of the semialgebraic form $K = \bigcap_{I \in Y} \{\eta \in \mathcal{D}'(\mathbb{R}^d) | P_i(\eta) \geq 0\}$, where Y is an arbitrary index set, P_i belonging to the set $\mathcal{P}_{C_c^\infty}$ of polynomials on $\mathcal{D}'(\mathbb{R}^d)$ with coefficients in $C_c^\infty(\mathbb{R}^d)$. Let us consider the family $\mathcal{F}(\mathcal{D}'(\mathbb{R}^d)) := \{m = (m^{(n)})_{n \in \mathbb{N}_0}, m^{(n)} \text{ SF on } \mathcal{D}'(\mathbb{R}^{dn})\}$. To $m \in \mathcal{F}(\mathcal{D}'(\mathbb{R}^d))$ there is associated the linear real-valued Riesz functional L_m, given for $P \in \mathcal{P}_{C_c^\infty}$, with coefficients $p^{(n)}$, by $L_m(P) := \sum_{n=0}^N \langle p^{(n)}, m^{(n)}\rangle$. The following theorem is proven in [14]:

Theorem. Suppose there exists a total set E in $C_c^\infty(\mathbb{R}^d)$ s.t. the following conditions hold: $C\{m_n\}$ is quasi-analytic and $\left[\sup \left|\langle f_1 \otimes \ldots \otimes f_{2n}, m^{(2n)}\rangle\right|\right] < \infty$, the sup being taken over all $f_i \in E$. Then m is realized uniquely by a positive measure μ with $\operatorname{supp} \mu = K$ iff $L_m(h^2) \geq 0$ and $L_m(P_i h^2) \geq 0$, $\forall i \in Y$, $\forall h \in P_{C_c^\infty}$ (the P_i and Y being as in the definition of K).

Remark. *(1)* The inequalities involving L_m are equivalent with saying that L_m is positive on the quadratic module associated with the positive polynomials P_i entering the definition of K. *(2)* This result (related to the approach in [9]) extends in various ways previous results by, a.a., Berezansky and Kondratiev [5], Kuna, Lasserre, Oliveira, Šifrin, as well as results obtained in the framework of the Borchers-Uhlmann algebra. It has also applications in statistical mechanics (cfr. contribution by T. Kuna). *(3)* Inasmuch as K is a proper subspace of $\mathcal{D}'(\mathbb{R}^d)$, applications to certain quantum field theoretical models with "functional boundaries conditions" (like those discussed e.g. in [11]) could be expected. In the case where the free field measure is replaced by the Wiener measure (mentioned in Example 2 for $s = 1$) a K-moment problem has been worked out in the work [4], shortly presented in the next section.

3. A K-Moment Problem on Wiener space

Let b_t, $t \in [0, 1]$ be a standard real-valued Brownian motion run in time $[0, 1]$, $\Omega = C_{(0)}([0, 1]; \mathbb{R})$ the corresponding Wiener space, with Wiener measure $\mathbb{P}$. Define

$\langle \pi_F \rangle$, for $F = \mathbb{Q}$ resp. $\mathbb{R}$, as the vector space generated by the set π_F of all polynomials with real coefficients built with the b's, and its multinomials, the time index t running in $F \cap (0,1]$. Consider a real-valued functional L on $\langle \pi_{\mathbb{Q}} \rangle$, normalized so that $L(1) = 1$. Assume L is $L^1(\mathbb{P})$-continuous. Consider the semi-algebraic set $K = \{P_i \geq 0, i = 1, \ldots, m\}$ in Ω, where $P_i \in \langle \pi_{\mathbb{Q}} \rangle$ are given, $i = 1, \ldots, m$, for some $m \in \mathbb{N}_0$. Assume that L is positive on K in the sense that $L\left(g^2 \prod_{i=1}^m P_i^{k_i}\right) \geq 0$ for all $k_i = 0, 1$, and all $g \in \langle \pi_{\mathbb{Q}} \rangle$. Assume also that h has a suitable additional continuity property (of commuting with the quadratic variation of b scaled by a constant $c > 0$). Then L extends uniquely and continuously to $\langle \pi_{\mathbb{R}} \rangle$ and there is a positive measure $\tilde{\mathbb{P}}$ on a certain space $\tilde{\Omega}$ of continuous paths such that $\tilde{\mathbb{P}}$ realizes L and the conditions expressed by K are satisfied almost surely with respect to $\tilde{\mathbb{P}}$. In this sense the K-MP is solved. The proof in [4] is obtained by using essentially Haviland's theorem and Schmüdgen Positivstellensatz (as applied originally to finite dimensional K-MP, with K compact), the lifting to the infinite dimensional setting being achieved by hyperfinite methods of nonstandard analysis. It would be interesting to extend this kind of result to the case of other continuous time processes, e.g. those arising from S(P)DEs. This would have applications to K-MP in areas like quantum fields, statistical mechanics and hydrodynamics ([2]).

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Convex bodies in real enumerative geometry

Antonio Lerario

How many lines intersect four *generic* lines in three-space? A *line* in three-dimensional projective space consists of the set of common solutions of two independent equations:

$$\begin{aligned} a_{00}x_0 + a_{01}x_1 + a_{02}x_2 + a_{03}x_3 = 0 \\ a_{10}x_0 + a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0 \end{aligned} \tag{1}$$

We denote by $\mathbb{G}(1,3)$ the Grassmannian of all lines in $\mathbb{RP}^3$. Topologically it is a smooth, orientable, 4-dimensional real algebraic manifold. Given a line $\ell \subset \mathbb{RP}^3$ we consider the set $\Omega(\ell)$ (a *Schubert variety*) of all lines intersecting ℓ: it is a 3-dimensional real algebraic set with one singular point, the line ℓ itself.

If we pick lines $\ell_1, \ldots, \ell_4 \in \mathbb{RP}^3$ we can ask for the number of lines intersecting all of them. This is a classical question in *enumerative geometry.* It turns out that if the four lines are generic enough there are always two *complex* lines intersecting all of them. The number of real lines instead depends on the arrangement of the four given lines: there are configurations for which the two complex solutions are actually real, but also examples with no real solutions at all (see [7]).

In the geometry of the Grassmannian we are looking for the cardinality

$$\#\Omega(\ell_1) \cap \cdots \cap \Omega(\ell_4), \tag{2}$$

and the number "2" is computed using the duality between intersection of cycles and cup product in the cohomology ring of the Grassmannian, using a technique called *Schubert Calculus* [3, 5]. The result of the calculation is defined over $\mathbb{Z}$ in the complex case because cycles are naturally oriented, but the orientability is lost in the real case and the answer is only defined over $\mathbb{Z}_2$.

How many lines intersect four *random* lines in three-space? There is no *generic* number of real solutions to our enumerative problem, and a natural approach is to look at the same question from a probabilistic point of view. We can generate a random line by taking independent standard Gaussian variables for the coefficients of equation (1): the resulting probability distribution on $\mathbb{G}(1,3)$ is called the *uniform distribution.* It is characterized by the fact that it is invariant under the action of the group $O(4)$ on projective space (which in turn induces an action on the Grassmannian): for this distribution there are no preferred points or directions in $\mathbb{RP}^3$.

Together with P. Bürgisser [2] we have called *expected degree* of $\mathbb{G}(1,3)$ the average number $\delta_{1,3}$ of real lines intersecting four random independent lines $\ell_1, \ldots, \ell_4$:

$$\delta_{1,3} = \mathbb{E}\#\Omega(\ell_1) \cap \cdots \cap \Omega(\ell_4). \tag{3}$$

This number $\delta_{1,3}$ turns out to be the key quantity governing questions of random enumerative geometry of lines.

Example 1 (Lines intersecting four random curves). The average number of lines intersecting curves $C_1, \ldots, C_4$ in random position[1] in $\mathbb{RP}^3$ equals:

$$\delta_{1.3} \cdot \frac{|C_1|}{|\mathbb{RP}^1|} \cdots \frac{|C_4|}{|\mathbb{RP}^1|}, \tag{4}$$

where the $|C_i|$ denote the length of these curves. Thus the answer "decouples" into the product of some normalized volumes (the lengths of the curves) and $\delta_{1,3}$. Note that a similar decoupling phenomenon over the complex numbers is a consequence of the ring structure of the cohomology of the Grassmannian, where the expected degree is replaced by the degree (which is "2") and the normalized length of the curves by their degree (which in fact is again a normalized volume).

Example 2 (Lines tangent to four random hypersurfaces). The average number of lines tangent to four copies of a sphere of radius r in random position in $\mathbb{RP}^3$ equals [4, Example 1.6]:

$$\delta_{1,3} \cdot \left(\frac{8}{\pi} \cos r \sin r \right)^4. \tag{5}$$

More generally with K. Kozhasov [4] we have studied the problem of the number of lines *tangent* to surfaces in random position in $\mathbb{RP}^n$, decoupling the answer into the product of $\delta_{1,3}$ and some curvature integrals of the surfaces, see [4].

A convex body associated to the problem. The exact value of $\delta_{1,3} = 1.7262...$ is not known, but it admits an interesting interpretation as the volume of a special convex body in the tangent space to the Grassmannian. Given a matrix $X \in \mathbb{R}^{2\times 2}$, we denote by σ_1, σ_2 its singular values (i.e. the square roots of the eigenvalues of XX^T) and we define $C(2,2) \subset \mathbb{R}^{2\times 2}$ to be the convex body with support function

$$h(X) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^2} \left(\sigma_1^2 x^2 + \sigma_2^2 y^2\right)^{\frac{1}{2}} e^{-\frac{x^2+y^2}{2}} dxdy, \quad X \in \mathbb{R}^{2\times 2}. \tag{6}$$

In [2] we have called this convex body the *Segre zonoid.* It turns out [2, Corollary 5.2] that, up to a constant, the expected degree of $\mathbb{G}(1,3)$ equals the volume of the Segre zonoid:

$$\delta_{1,3} = 3\pi^2 |C(2,2)|. \tag{7}$$

The explanation for this surprising connection is as follows. Using a variation of integral geometry arguments, the expectation in (3) can be written as the product

[1] We say that smooth sets in $\mathbb{RP}^3$ are in *random position* if each one of them is randomly translated by independent elements sampled independently from the uniform distribution on the Orthogonal group $O(4)$.

of the volume of the Schubert varieties $\Omega(\ell_1), \dots, \Omega(\ell_4)$ times the "average" of the angle that they make at every point $\ell \in \mathbb{G}(1,3)$ where they intersect. This average angle is the expectation of the *absolute value* of the determinant of the matrix whose columns are the normal vectors of the Schubert varieties in $T_\ell\mathbb{G}(1,3) \simeq \mathbb{R}^{2\times 2}$ (in the last identification these normal vectors turn out to be rank-one matrices). Finally there is a tool [8] that relates the expected value of the modulus of a random determinant (i.e. the expected volume of a random parallelepiped) to the volume of a convex body, the Segre zonoid in our case.

Lines on a cubic. A *random cubic* in $\mathbb{R}\mathrm{P}^3$ can be defined as the zero set of a random polynomial

$$f(x) = \sum_{|\alpha|=3} \xi_\alpha x_0^{\alpha_0} \cdots x_3^{\alpha_3}, \tag{8}$$

where the ξ_α are independent centered gaussian variables with variance $\frac{3!}{\alpha_0! \cdots \alpha_3!}$ (with this choice the model is $O(4)$-invariant). With S. Basu, E. Lundberg and C. Peterson we have proved [1] that there are on average $6\sqrt{2} - 3$ real lines on a random cubic. This number equals $\frac{3}{2}\mathbb{E}|\det J_3|$ where:

$$J_3 = \begin{bmatrix} \xi_1 & 0 & \xi_4 & 0 \\ \sqrt{2}\xi_2 & \xi_1 & \sqrt{2}\xi_5 & \xi_4 \\ \xi_3 & \sqrt{2}\xi_2 & \xi_6 & \sqrt{2}\xi_5 \\ 0 & \xi_3 & 0 & \xi_6 \end{bmatrix} \quad (\xi_1, \dots, \xi_6 \text{ are i.i.d. Gaussian}).$$

Some open questions.

(1) Is there an explicit formula for $\delta_{1,3}$ in terms of special functions? (Replying to a question posed by M. Firsching [6], Adam P. Goucher has found an expression for $\delta_{1,3}$ in terms of an integral with no absolute values.)

(2) More generally one can define [2] the expected degree $\delta_{k,n}$ of the Grassmannian of k-flats in $\mathbb{R}\mathrm{P}^n$. When $k = 1$ (i.e. for the Grassmannian of lines) we have [2, Theorem 6.8]:

$$\delta_{1,n} = \frac{8}{3\pi^{5/2}} \cdot \frac{1}{\sqrt{n}} \cdot \left(\frac{\pi^2}{4}\right)^n \cdot \left(1 + \mathcal{O}(n^{-1})\right). \tag{9}$$

For a fixed k, what is the asymptotic of $\delta_{k,n}$ when $n \to \infty$? (For the asymptotic in the logarithmic scale see [2, Theorem 6.5].)

(3) Is there a convex body associated to the problem of lines on random cubics?

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Relations to operator theory and free positivity

Michael Dritschel

We survey some results from operator theory, primarily as applied to sums of squares problems. To begin with, we look at Schur complements and their use in proving the Fejér-Riesz theorem with operator coefficients, and some multivariable generalizations. We also mention Cimprič's matrix version of Krivine's Striktpositivstellensatz, where Schur complements are used to reduce the problem back to the scalar case. We then turn to operator space techniques used for factorization of operator valued hereditary strictly positive polynomials over discrete groups. This is done by identifying the hereditary polynomials with an abstract operator space and employing the Effros-Ruan theorem along with a hyperplane separation argument. This leads into a discussion of Scott McCullough's free version of the Fejér-Riesz theorem, using a clever argument involving Caratheodory interpolation to prove a version of the Naĭmark dilation theorem, which in turn is used to show that a certain map is completely positive. Application of the standard machinery gives the non-commutative Fejér-Riesz factorization. At the end we turn to a few real versions of these theorems (so involving factorizations of Hankel matrices), including Helton's Positivstellensatz.

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High precision computations for energy minimization

David de Laat

In the *Thomson problem* we minimize

$$\sum_{1\le i<j\le N} \frac{1}{\|x_i - x_j\|_2}$$

over all sets $\{x_1,\dots,x_N\}$ of N distinct points in the unit sphere $S^2 \subseteq \mathbb{R}^3$. Variations are obtained by replacing the *Coulomb potential* $\|x-y\|_2^{-1}$ by, for example, the *Riesz s-energy* $\|x-y\|_2^{-s}$.

How can we prove that a configuration attains the global minimum? One approach is to discretize the configuration space and search through all configurations by using derivative bounds. This has has been done successfully by Schwartz for the $N = 5$ case [11, 12]. In [7] we take the approach of deriving a hierarchy of relaxations and computing high precision dual solutions of these relaxations (which in principle can be used to construct optimality certificates). The first step in this hierarchy specializes to Yudin's bound [13], which has been used to obtain optimality certificates for $N = 2, 3, 4, 6, 12$ particles.

We show how the second step of this hierarchy can be computed for energy minimization on the sphere. This shows the computational applicability of the infinite dimensional moment techniques from [8] that we use to derive these relaxations. Interestingly, the computational results suggest that for $N = 5$ the second level in the hierarchy is not just sharp for the Thomson problem, but for a large class of pair potentials.

Relaxations. To construct these relaxations we use the infinite dimensional moment techniques from [8], in which the Lasserre hierarchy [9] for the independent set problem is generalized to infinite graphs.

Let I_t be the compact metric space consisting of subsets of S^2 of cardinality at most t, where the distance between any two points in a subset is strictly larger than some small $\epsilon > 0$. Define the continuous function $f \in \mathcal{C}(I_N)$ by

$$f(S) = \begin{cases} \|x-y\|_2^{-1} & \text{if } S = \{x,y\} \text{ with } x \neq y, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{M}(I_{2t})_{\geq 0}$ be the space of Radon measures and define the relaxation

$$E_t = \min\Big\{\lambda(f) : \lambda \in \mathcal{M}(I_{2t})_{\geq 0} \text{ of positive type, } \lambda(I_{=i}) = \tbinom{N}{i} \text{ for } 0 \le i \le 2t\Big\},$$

where $I_{=t}$ is the subset of I_t containing the sets of cardinality t, and where $\binom{N}{i} = 0$ for $i > N$. Here we say that a measure $\lambda \in \mathcal{M}(I_{2t})$ is of *positive type* if

$$\int_{I_{2t}} A_t K(S)\, d\lambda(S) \geq 0 \quad \text{for all} \quad K \in \mathcal{C}(I_t \times I_t)_{\succeq 0},$$

where $\mathcal{C}(S^2 \times S^2)_{\succeq 0}$ is the cone of continuous positive definite kernels, and where

$$A_t \colon \mathcal{C}(I_t \times I_t)_{\mathrm{sym}} \to \mathcal{C}(I_{2t}),\; A_t K(S) = \sum_{J,J' \in I_t : J \cup J' = S} K(J, J')$$

is an infinite dimensional generalization of the dual operation of mapping a moment vector y to its moment matrix $M(y)$. Given a configuration $S \in I_N$, the measure

$$\lambda_S = \sum_{Q \subseteq S : |Q| \leq 2t} \delta_Q,$$

where δ_Q is the Dirac measure at Q, is feasible for E_t, which shows E_t lower bounds the ground state energy E. We have $E_t \leq E_{t+1}$ and we prove $E_N = E$.

Computations. In the dual optimization problem E_t^* we maximize over continuous kernels $K \in \mathcal{C}(I_t \times I_t)_{\succeq 0}$, and because of the symmetry of the problem we may restrict to $O(3)$-invariant kernels. To find good energy lower bounds we need to find good feasible solutions of E_t^*. For this we optimize over the first few Fourier coefficients of K, which in this case are positive semidefinite matrices. That is, we truncate the inverse Fourier transform

$$K(J, J') = \sum_{\pi} \sum_{i,j} \widehat{K}(\pi)_{i,j} Z_\pi(J, J')_{i,j}.$$

To do this explicitly we show how the matrix valued kernels $Z_\pi(\cdot, \cdot)$ can be computed explicitly.

After this reduction to a finite dimensional variable space, and a variable transformation, we are left with a maximization problem where the variables are positive semidefinite matrices, and where we have constraints of the form

$$p(x_1, \ldots, x_i) \geq 0 \quad \text{for} \quad \{x_1, \ldots, x_i\} \in I_{=i}, \quad (i = 0, \ldots, 2t). \tag{1}$$

Here p is a polynomial in $3i$ variables whose coefficients depend linearly on the entries of the matrix variables. Since K can be assumed to be $O(3)$-invariant, we have that $p(\gamma x_1, \ldots, \gamma x_i)$ does not depend on $\gamma \in O(3)$, so from invariant theory [6] we know there exists a polynomial q such that

$$p(x_1, \ldots, x_i) = q(x_1 \cdot x_1, x_1 \cdot x_2, \ldots, x_i \cdot x_i)$$

To find the q polynomials explicitly up to high precision we solve a number of large linear systems by using high precision sparse Cholesky factorizations. We can then rephrase (1) as

$$q(u_1, \ldots, u_l) \geq 0 \quad \text{for} \quad (u_1, \ldots, u_l) \in \text{some semialgebraic set}, \tag{2}$$

where again the coefficients of q depend linearly on the entries of the matrix variables. Using Putinar's sum-of-squares representation we can then model these constraints as semidefinite constraints.

Since the particles are interchangeable, the expression $p(x_{\sigma(1)}, \ldots, x_{\sigma(i)})$ does not depend on $\sigma \in S_i$. This translates into extra symmetries for the polynomial and the semialgebraic set in (2). We use this to block diagonalize the sum of squares representations, which leads to significant computational savings. For this we extend techniques from [5] to a constrained setting (similar formulations for the constrained setting can be found in [3] and [10]).

Using the above techniques we obtain SDPs that are given as high precision floating point numbers with small enough blocks to be solved by computer. Here we insure the SDPs are strictly feasible and do not have linearly dependent constraints, which is important for current available high precision solvers. The optimal value of this SDP converges from below to E_2 as we increase the SOS degrees and the number of terms in the Fourier truncation. We compute this bound for $N = 5$ with the Coulomb potential and other potentials. In all cases the 28 decimal digits given by the high precision solver SDPA-QD [4] agree with the first 28 decimals of the energy of the corresponding optimal configuration. This is a strong indication that the relaxation E_2 is sharp for these problems. Here we use high precision arithmetic, because machine precision arithmetic would only give 6 or so decimals, which would make it difficult to distinguish between the bound being sharp or just very close. We conjecture E_2 is *universally sharp* for 5 particles on S^2, which would mean E_2 is sharp for a large and important class of potential functions (see [2] for a definition of universal optimality).

The hierarchy E_t is an adaptation to energy minimization of the hierarchy from [8] for packing problems in discrete geometry. It would be very interesting to use the computational techniques developed here to compute 4-point bounds for packing problems. Of particular interest would be the spherical code problem $A(4, \arccos(1/3))$, where a construction of 14 points exists, and where the 2 and 3-point bounds give the upper bounds 16 and 15 [1].

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Existence of semidefinite representations

Claus Scheiderer

A subset $K \subseteq \mathbb{R}^n$ has a semidefinite representation if it can be written

$$K \;=\; \Big\{x \in \mathbb{R}^n \colon \exists\, y \in \mathbb{R}^m \;\; M_0 + \sum_{i=1}^{n} x_i M_i + \sum_{j=1}^{m} y_j N_j \succeq 0\Big\}$$

with $m \geq 0$ and suitable real symmetric matrices M_i, N_j of some size. Semidefinitely representable sets are convex and semi-algebraic, but no other restriction was known so far. Helton and Nie [1] conjectured that conversely every convex semi-algebraic set has a semidefinite representation. We disprove this conjecture as follows [3]. Let $S \subseteq \mathbb{R}^n$ be an arbitrary semi-algebraic set, let $K = \mathrm{conv}(S)$ be its convex hull in $\mathbb{R}^n$, and let $\phi\colon X \to \mathbb{A}^n$ be a morphism of affine algebraic $\mathbb{R}$-varieties with $S \subseteq \phi(X(\mathbb{R}))$. Assume there is a finite-dimensional linear subspace $U \subseteq \mathbb{R}[X]$ such that, whenever $f \in \mathbb{R}[x_1, \ldots, x_n]$ is linear with $f|_S \geq 0$, the pullback $\phi^*(f) \in \mathbb{R}[X]$ is a sum of squares of elements of U. Then we obtain a semidefinite representation for the closed convex hull $\overline{K}$. This representation is completely explicit if ϕ and U are given concretely. Conversely we show that if $\overline{K}$ has a semidefinite representation then there exist ϕ and U with the above properties. The necessary and sufficient condition for semidefinite representability obtained in this way is non-trivial, as we demonstrate by employing smoothness properties of morphisms of algebraic varieties. Some concrete applications: For any semi-algebraic set $S \subseteq \mathbb{R}^n$ with $\dim(S) \geq 2$ there exists a polynomial map $\varphi\colon \mathbb{R}^n \to \mathbb{R}^m$ (for some m) such that the closed convex hull $\overline{\mathrm{conv}(\varphi(S))}$ in $\mathbb{R}^m$ has no semidefinite representation. This is in sharp contrast with the case $\dim(S) \leq 1$, where we had previously shown [2] that $\overline{\mathrm{conv}(S)}$ always has a semidefinite representation. As for concrete examples, we consider the sets $\Sigma_{n,2d} \subseteq P_{n,2d} \subseteq \mathbb{R}[x_1, \ldots, x_n]_{2d}$ of n-ary forms of degree $2d$ that are sums of squares resp. that are nonnegative. These are closed convex cones. We show that $P_{n,2d}$ has a semidefinite representation only in the obvious cases where $\Sigma_{n,2d} = P_{n,2d}$ (i.e. when $n \leq 2$ or $2d = 2$ or $(n, 2d) = (3, 4)$, according to Hilbert).

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Moment problems in one and several dimensions with special emphasis on the indeterminate case

Christian Berg

I gave a historical introduction to the fundamental papers of Stieltjes (1894), Hamburger (1920-21) and Marcel Riesz (1922-23). Hamburger's result can be formulated that a sequence (s_n) of real numbers is of the form $s_n = \int x^n \, d\mu(x),\ n \geq 0$ for a positive measure μ on $\mathbb{R}$, if and only if the Hankel matrices

$$\mathcal{H}_n = (s_{j+k})_{j,k=0}^{n}, \quad n = 0, 1, \ldots$$

are all positive semi-definite. Riesz proved this in a talk in Stockholm already in 1918 by using a "Hahn-Banach-argument" like in later proofs of Haviland (1935).

Stieltjes had to "invent" general distributions of mass via increasing functions and their associated Stieltjes integrals before he could formulate his main result: A sequence (s_n) of real numbers is the sequence of moments of a measure on $[0, \infty[$ if and only if the Hankel matrices $\mathcal{H}_n$ as well as the Hankel matrices of the shifted sequence s_{n+1} are all positive semi-definite. These sequences are now called Stieltjes moment sequences.

In k dimensions, a multi-sequence $s : \mathbb{N}_0^k \to \mathbb{R}$ is positive definite as a function on the abelian semigroup $(\mathbb{N}_0^k, +)$, defined e.g. in [2] , if and only if the associated linear functional $L : \mathbb{R}[\underline{X}] \to \mathbb{R}$ defined by $L(x^\alpha) = s(\alpha), \alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}_0^k$ is non-negative on the smallest preordering Σ_k of sums of squares of polynomials from $\mathbb{R}[\underline{X}]$ (where this symbol denotes the set of real polynomials in k variables with $x^\alpha = x_1^{\alpha_1} \cdots x_k^{\alpha_k}$ and $|\alpha| = \alpha_1 + \ldots + \alpha_k$). When $k \geq 2$ it was known already to Hilbert that Σ_k is a proper subset of the convex cone $\mathbb{R}[\underline{X}]_+$ of polynomials non-negative on $\mathbb{R}^k$, while it is elementary that $\Sigma_1 = \mathbb{R}[X]_+$.

The role of sums of squares is a unifying theme in the workshop.

My recent research has been centered around the indeterminate Hamburger moment problem, i.e., the case where there is more than one and hence an infinite convex set V of measures with the given moments. Already in Stieltjes memoir from 1894 there is an example of an indeterminate moment problem, namely the log-normal distribution with moments $s_n = q^{-n(n+2)/2}$, where $0 < q < 1$ is a parameter. (Stieltjes considered only a special value of q.) Much later Chihara and Leipnik found discrete solutions with the same moments. It is in fact a general result, cf. [1]that any indeterminate Hamburger moment problem has many solutions of each of the following types: (a) with a C^∞ density, (b) countable infinite discrete support, (c) continuous singular. Here "many" means dense subsets of V in the weak topology.

Let us look at the behaviour of the smallest eigenvalue λ_n of the n'th Hankel matrix $\mathcal{H}_n$ as n tends to infinity and assume that all matrices $\mathcal{H}_n$ are not only positive semi-definite but positive definite. We have

$$\lambda_n = \min \left\{ \sum_{j,k=0}^{n} s_{j+k} c_j c_k \mid \sum_{j=0}^{n} c_j^2 = 1 \right\},$$

and since (λ_n) is a decreasing sequence of positive numbers, we can define $\lambda_\infty = \lim_{n\to\infty} \lambda_n$. A theorem of Berg-Chen-Ismail [3] states that the determinate (resp. indeterminate) case occurs if $\lambda_\infty = 0$ (resp. $\lambda_\infty > 0$). In work with Szwarc [4] we have shown that arbitrary slow and fast decrease to 0 can occur in the determinate case.

An analogue of λ_n for multidimensional moment sequences is the quantity

$$\lambda_n := \min\left\{ \int_{\mathbb{R}^k} p^2(x)\, d\mu(x) \mid p(x) = \sum c_\alpha x^\alpha, |\alpha| \le n, \sum c_\alpha^2 = 1 \right\}.$$

We can again define $\lambda_\infty = \lim_{n\to\infty} \lambda_n \ge 0$.

It seems to be an open question if $\lambda_\infty = 0$ characterizes determinacy of the multi-dimensional moment problem. One should compare this with another quantity studied by Putinar and Vasilescu in [8].

A Hamburger moment problem is characterized either by the moment sequence (s_n) or by two real sequences $(a_n), (b_n)$ with $b_n > 0$. These sequences determine the orthonormal polynomials $P_n, n \ge 0$ via the three-term recurrence relation

$$xP_n(x) = b_n P_{n+1}(x) + a_n P_n(x) + b_{n-1} P_{n-1}(x), \quad n \ge 0$$

with the initial conditions $P_{-1}(x) = 0, P_0(x) = 1$.

The indeterminate case is characterized by $\sum_{n=0}^\infty |P_n(z)|^2 < \infty$ for all $z \in \mathbb{C}$ (in fact it is enough that the series converges for one $z \in \mathbb{C} \setminus \mathbb{R}$). Unfortunately nothing similar is known in the multidimensional case, where all solutions can be described via the spectral measures of commuting self-adjoint extensions of the operators of multiplication T_j in $\mathbb{R}[\underline{X}]$ defined by $T_j p(\underline{X}) = x_j p(\underline{X})$, see [7].

In the one-dimensional indeterminate case the complete description of all solutions to the moment problem can be made much more explicit using four entire functions A, B, C, D of common order and type called the order and type of the moment problem. The function D is given as

$$D(z) = z \sum_{n=0}^\infty P_n(z) P_n(0), \quad z \in \mathbb{C}.$$

(It is entire because $(P_n(z)) \in \ell^2$ for all $z \in \mathbb{C}$). It is in general difficult to calculate D or any of the other functions A, B, C and only few concrete examples are known. It is therefore of some interest to be able to calculate the order and type of the moment problem directly from the coefficients $(a_n), (b_n)$ without calculating first (P_n) and D. This has been achieved in [5],[6] for certain classes of sequences $(a_n), (b_n)$.

Let us just mention the symmetric moment problems with $a_n = 0$ for all n and (b_n) is supposed to be *regular* in the following sense: Either $b_n^2 \le b_{n-1}b_{n+1}$ for n sufficiently large, or $b_n^2 \ge b_{n-1}b_{n+1}$ for n sufficiently large. In other words (b_n) is either eventually log-convex or eventually log-concave.

Examples are $b_n = (n+1)^\alpha, \alpha > 1$ and $b_n = \exp(cn^\alpha), c > 0, \alpha > 0$.

Theorem *Let $a_n = 0$ and let (b_n) be regular. Then the symmetric moment problem is indeterminate if and only if $\sum(1/b_n) < \infty$, and in the affirmative case* *the order of the moment problem is the exponent of convergence of* (b_n) *defined as*

$$\mathcal{E}(b_n) = \inf \left\{ c > 0 \mid \sum_{n=0}^{\infty} \frac{1}{b_n^c} < \infty \right\},$$

hence ≤ 1.

In the examples above the exponent of convergence is equal to $1/\alpha$ and 0.

In case of order 0 the so-called logarithmic order of the moment problem equals $\mathcal{E}(\log b_n)$, cf. [5]. In the second example the logarithmic order is $1/\alpha$.

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Polynomial optimization with a focus on hyperbolic polynomials

Levent Tunçel

A problem of minimizing (or maximizing) a multivariate polynomial over a subset of an Euclidean Space defined by the solution set of finitely many polynomial equations and inequalities is called a *Polynomial Optimization Problem* (PoP). PoPs describe a class of optimization problems with a nontrivial amount of geometric, algebraic and analytic properties. At the same time, POPs are general enough to capture as a special case, a very wide swath of finite dimensional optimization problems and even some semi-infinite optimization problems. We can reformulate PoPs in many equivalent forms. For example, by introducing a new variable (increasing the dimension of the space by one), we can push the objective function's multivariate polynomial into the constraints (hence, without loss of generality, we may assume that the objective function is always linear).

In this setting, an interesting, nicely structured, and powerful class of convex PoPs is Semidefinite Programming (SDP) problems. Objective functions of SDPs are linear, and their feasible solution sets are defined as the intersection of the convex cone of n-by-n symmetric positive semidefinite matrices (all real entries), denoted $\mathbb{S}^n_+$ with an affine subspace. Such convex sets are called *spectrahedra.*

In SDP problems we may also use additional auxiliary variables that effectively get projected away due to our carefully picked choices for the objective function of the SDP. This observation shows that SDPs can also deal with convex sets that are orthogonal projections of spectrahedra. These latter convex sets are called *spectrahedral shadows*. Indeed, spectrahedral shadows yield a strict superset of spectrahedra. However, except for utilization of facial exposedness property (of spectrahedra), we do not have many elegant, useful certificates helping us distinguish these two families of convex sets precisely (see [25, 8, 23, 20, 3, 16]).

We consider PoPs from a convex optimization viewpoint (see for instance [9, 11, 19]). Then a central question is "when is the feasible region of a PoP convex?" This leads us to *hyperbolic polynomials* (a.k.a. *stable polynomials*, under a suitable transformation) which naturally define convex domains. For the sake of convenience, we work with homogeneous hyperbolic polynomials so that the underlying convex domains become convex cones called *hyperbolicity cones*. Let p be a homogeneous polynomial (this is without loss of generality in our current context) of degree d in n variables, and let $e \in \mathbb{R}^n$. p is said to be *hyperbolic in direction* e if $p(e) > 0$ and, for all $x \in \mathbb{R}^n$, the scalar polynomial $\lambda \mapsto p(x - \lambda e)$ has only real roots. Studies of hyperbolic polynomials go back at least to the work of Petrovsky (from the 1930s). Considerable amount of work has been done by Gårding, Atiyah, Bott and Gårding as well as Hörmander. Since the early 1990's there has been an amazing amount of activity allowing the subject to branch into systems and control theory, operator theory (see Marcus-Spielman-Srivastava [13] solution of Kadison-Singer problem) interior-point methods (see, for instance, [5, 20, 15] and the references therein), discrete optimization and combinatorics (see, for instance, Gurvits [4], Wagner [27] and references therein) semidefinite programming and semidefinite representations, matrix theory as well as theoretical computer science.

Fix a direction e and a polynomial p hyperbolic in direction e. We call the roots of $\lambda \mapsto p(x - \lambda e)$ the *eigenvalues of* x. Let Λ_{++} denote the set of points that have only positive eigenvalues and let Λ_+ denote its closure. Λ_+ is called the *hyperbolicity cone of* p *in direction* e. It is a convex cone. A very nice example is $\mathbb{S}^n_+$ associated with the hyperbolic polynomial $p(x) := \det(x)$ and the direction e given by the n-by-n identity matrix. Helton-Vinnikov Theorem [26, 7, 12] implies that all three dimensional hyperbolicity cones are spectrahedra and every hyperbolic polynomial giving rise to a 3-dimensional hyperbolicity cone admits a very strong determinantal representation. Using Helton-Vinnikov Theorem, one can prove: some general facts about all hyperbolicity cones, some general facts about all hyperbolic polynomials, and generalizations of many theorems from matrix analysis to "hyperbolicity cone optimization" setting. There are many generalizations of Helton-Vinnikov theorem (see [21] and the references therein), counter examples to certain proposed generalizations of Helton-Vinnikov Theorem (see Brändén [2] and the references therein), various spectrahedral and spectrahedral-shadow representations for interesting hyperbolicity cones (see Netzer and Sanyal [17] and the references therein, in the light of [18]).

If $K = \Lambda_+(p)$, then $F : \mathbb{R}^n \to \mathbb{R}$, $\quad F(x) := \begin{cases} -\ln(p(x)), & \text{if } x \in \Lambda_{++}(p); \\ +\infty, & \text{otherwise.} \end{cases}$

has very useful properties for modern interior-point methods (see [5, 15]). Let F be a normal barrier (see [15] for a definition) for the regular cone K. We say that F has *negative curvature* if for every $x \in \text{int}(K)$ and $h \in K$ we have $\nabla^3 F(x)[h]$ negative semidefinite. Negation of logarithms of hyperbolic polynomials have negative curvature [10, 5]. While the dual cone of a hyperbolicity cone is not necessarily hyperbolic [3], the *dual barrier function* $F_*(s) := \max_{x \in \text{int}(K)} \{-\langle s, x\rangle - F(x)\}$ is always a normal barrier for the dual cone K^*. F_* does not necessarily have negative curvature.

Open Problems: 1. Does there exist an algebraic convex cone (defined as the solution set of homogeneous multivariate polynomial inequalities) which admits a normal barrier with negative curvature but it is not a hyperbolicity cone? **2.** [15] Characterize the set of convex cones which admit normal barriers with negative curvature. **3.** (*Generalized Lax Conjecture*) Every hyperbolicity cone is a spectrahedron.

Conjecture 1: [24] Every hyperbolicity cone is a spectrahedral shadow.

A few months before this writing, Scheiderer [22] answered a related question of Nemirovski [14] by disproving the Helton-Nie conjecture [6] (Helton-Nie conjecture is a stronger version of our Conjecture 1 above).

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An infinite dimensional umbral calculus

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(joint work with Dmitri Finkelshtein, Yuri Kondratiev and Eugene Lytvynov)

Umbral calculus has applications in combinatorics, theory of special functions, approximation theory, probability and statistics, topology, and physics, see e.g. the survey paper [9] for a long list of references.

In its modern form, umbral calculus is a study of shift-invariant linear operators acting on polynomials, their associated polynomial sequences of binomial type, and Sheffer sequences (including Appell sequences). We refer to the seminal papers [24, 29, 30], see also the monographs [20, 28].

Many extensions of umbral calculus to the case of polynomials of several, or even infinitely many variables were discussed e.g. in [5, 8, 11, 23, 25, 26, 27, 31, 32], for a longer list of such papers see the introduction to [10]. Appell and Sheffer sequences of polynomials of several noncommutative variables arising in the context of free probability, Bollean probability, and conditionally free probability were discussed in [2, 3, 4], see also the references therein.

The paper [10] was a pioneering (and seemingly unique) work in which elements of basis-free umbral calculus were developed on an infinite dimensional space, more precisely, on a real separable Hilbert space $\mathcal{H}$. This paper discussed, in particular, shift-invariant linear operators acting on the space of polynomials on $\mathcal{H}$, Appell sequences, and examples of polynomial sequences of binomial type.

In fact, examples of Sheffer sequences, i.e., polynomial sequences with generating function of a certain exponential type, have appeared in infinite dimensional analysis on numerous occasions. Some of these polynomial sequences are orthogonal with respect to a given probability measure on an infinite dimensional space, while others are related to analytical structures on such spaces. Typically, these polynomials are either defined on a co-nuclear space Φ' (i.e, the dual of a nuclear space Φ), or on an appropriate subset of Φ'. Furthermore, in majority of examples, the nuclear space Φ consists of (smooth) functions on an underlying space X. For simplicity, we choose to discuss the Gel'fand triple

$$\Phi = \mathcal{D} \subset L^2(\mathbb{R}^d, dx) \subset \mathcal{D}' = \Phi'.$$

Here $\mathcal{D} := \mathcal{D}(\mathbb{R}^d)$ is the space of smooth compactly supported functions on $\mathbb{R}^d$, $d \in \mathbb{N}$, and $\mathcal{D}' := \mathcal{D}'(\mathbb{R}^d)$ is the dual space of $\mathcal{D}$, that is, the space of distributions on $\mathbb{R}^d$. The dual pairing between $\mathcal{D}'$ and $\mathcal{D}$ is obtained by continuously extending the inner product in $L^2(\mathbb{R}^d, dx)$.

Let us mention several known examples of Sheffer sequences on $\mathcal{D}'$ or its subsets:

(i) In infinite dimensional Gaussian analysis, also called white noise analysis, Hermite polynomial sequences on $\mathcal{D}'$ (or rather on $S' \subset \mathcal{D}'$, the Schwartz space of tempered distributions) appear as polynomials orthogonal with respect to the Gaussian white noise measure see e.g. [6, 12].

(ii) Charlier polynomial sequences on the configuration space $\Gamma \subset \mathcal{D}'$ of counting Radon measures on $\mathbb{R}^d$ appear as polynomials orthogonal with respect to Poisson point process on $\mathbb{R}^d$ [13, 15, 18].

(iii) Laguerre polynomial sequences on the cone of discrete Radon measures on $\mathbb{R}^d$ appear as polynomials orthogonal with respect to the gamma random measure [17, 18].

(iv) Meixner polynomial sequences on $\mathcal{D}'$ appear as polynomials orthogonal with respect to the Meixner white noise measure [21, 22].

(v) Special polynomials on the configuration space Γ are used to construct the K-transform, see e.g. [7, 14, 16]. Recall that the K-transform determines the duality between point processes on $\mathbb{R}^d$ and their correlation measures. These polynomials will be identified in our discussion as the infinite dimensional analog of the falling factorials.

(vi) Polynomial sequences on $\mathcal{D}'$ with generating function of a certain exponential type are used in biorthogonal analysis related to general measures on $\mathcal{D}'$ [1, 19].

Note, however, that even the very notion of a general polynomial sequence on an infinite dimensional space has never been discussed.

We discuss an extension of the classical umbral calculus on $\mathbb{R}$ to the infinite dimensional space $\mathcal{D}'$. We define monic polynomial sequences on $\mathcal{D}'$, polynomial sequences of binomial type and Sheffer sequences. Equivalent conditions are given for a sequence of monic polynomials on $\mathcal{D}'$ to be, respectively, of binomial type or a Sheffer sequence. Our theory has remarkable similarities to the classical setting of polynomials on $\mathbb{R}$. For example, the form of the generating function of a Sheffer sequence on $\mathcal{D}'$ is similar to the generating function of a Sheffer sequence on $\mathbb{R}$, albeit the constants appearing in the latter function are replaced in the former function by appropriate linear continuous operators.

A procedure for lifting a polynomial sequence of binomial type (resp., a Sheffer sequence) on $\mathbb{R}$ to a polynomial sequence of binomial type (resp., a Sheffer sequence) on $\mathcal{D}'$ is discussed as well. Using this procedure, in particular, we recover, on $\mathcal{D}'$, the Hermite polynomials, the Charlier polynomials, the orthogonal Laguerre polynomials, mentioned above.

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Free Real Algebraic Geometry

Igor Klep
(joint work with Bill Helton, Scott McCullough, Markus Schweighofer)

Free real algebraic geometry (RAG) is the study of positivity (e.g. positive elements, sums of squares, quadratic modules, semialgebraic sets, etc.) in free algebras, i.e., algebras of (free) noncommutative polynomials.

Let $x = (x_1, \dots, x_g)$ be freely noncommuting indeterminates, and consider the free monoid $\langle x\rangle$ generated by x. It consists of words in x, including the empty word denoted by 1. The free algebra on x, denoted $\mathbb{R}\langle x\rangle$, consists of noncommutative polynomials. It comes equipped with the involution $*$ fixing x_j. The positivity under consideration comes from evaluating polynomials in $\mathbb{R}\langle x\rangle$ at g-tuples of $n\times n$ real symmetric matrices $A \in \mathbb{S}_n^g$.

The starting point of free RAG is considered to be Helton's sum of squares theorem, describing positive noncommutative polynomials as sums of squares:

Theorem 1 ([5]). *For $f \in \mathbb{R}\langle x\rangle$ of degree d, the following are equivalent:*
(i) for all $n \in \mathbb{N}$ and all $A \in \mathbb{S}_n^g$, we have $f(A) \succeq 0$;

(ii) for all $A \in \mathbb{S}_N^g$*, we have* $f(A) \succeq 0$*, where* $N = \dim \mathbb{R}\langle x \rangle_d$*;*
(iii) there exist $g_i \in \mathbb{R}\langle x \rangle$ *with* $f = \sum g_i^* g_i$*.*

Later several Positivstellensätze were proved, often inspired by results in classical RAG. This talk surveyed some of the recent progress in free RAG with emphasis on convexity. Convexity in free RAG is governed by linear matrix inequalities (LMIs), so we next introduce those.

Let $A_1, \dots, A_g \in \mathbb{S}_d$. The formal affine linear combination $L_A(x) = I_d + \sum_j A_j x_j$ is a (monic) linear pencil, and the expression $L_A(x) \succeq 0$ is a linear matrix inequality (LMI). Its solution is the spectrahedron $\mathcal{S}_A = \{x \in \mathbb{R}^g \mid L(x) \succeq 0\}$. Using Kronecker's tensor product, we can evaluate L at tuples of symmetric matrices $X \in \mathbb{S}_n^g$, giving rise to $L_A(X) = I_d \otimes I_n + \sum_j A_j \otimes X_j \in \mathbb{S}_{dn}$. The free semialgebraic set

$$\mathcal{D}_A := \bigcup_{n \in \mathbb{N}} \{X \in \mathbb{S}_n^g \mid L_A(X) \succeq 0\}$$

is called a free spectrahedron. These admit a "perfect" Positivstellensatz.

Theorem 2 (Convex Positivstellensatz [6, 7])**.** *A matrix-valued polynomial* p *is positive semidefinite on* $\mathcal{D}_A$ *(or a projection thereof) iff it has a sum of squares representation with optimal degree bounds:*

$$p = s^* s + \sum f_j^* \, L_A \, f_j, \tag{1}$$

where s, f_j *are matrix-valued polynomials of degree no greater than* $\frac{1}{2}\deg(p)$*.*

Theorem 2 is a remarkable strengthening of analogous commutative results (e.g. Putinar's theorem), where strict positivity is required and obtained degree bounds are terrible in that they depend exponentially on $\big(\min\big\{p(x) : x \in \mathcal{S}_A\big\}\big)^{-1}$. The proof of Theorem 2 introduced two novel ideas to the standard separating hyperplane argument. This standard argument starts with p, assumes it is not in the cone $\mathcal{C}$ of polynomials of the form on the right hand side of (1) and focuses on a linear functional ℓ separating p from $\mathcal{C}$. Then it does a variant of the Gelfand-Naimark-Segal (GNS) construction to produce the contradiction (provided serious assumptions hold). What we did was figure out a way to change the ℓ to get a new $\tilde{\ell}$, which still separates and which has excellent properties even absent most of the assumptions. This uses free truncated moment matrices analogous to the commutative ones for which there is a good theory originated by Curto-Fialkow. The second novelty is the application of the Effros-Winkler [3] Hahn-Banach separation argument after the GNS construction to derive the contradiction. When applied to linear p, Theorem 2 reduces (cf. [6, 10]) to the finite-dimensional case of the classical Stinespring-Arveson results [9] on completely positive maps in operator algebra. Thus this convex Positivstellensatz is a nonlinear extension of the main results of complete positivity.

Theorem 2 is also effective in that given L_A and p, verifying positivity of p on L_A can be done with a single semidefinite program (SDP). In the linear case this makes $\mathcal{D}_A \subseteq \mathcal{D}_B$ a convenient relaxation for the inclusion $\mathcal{S}_A \subseteq \mathcal{S}_B$; clearly, $\mathcal{D}_A \subseteq \mathcal{D}_B$ implies $\mathcal{S}_A \subseteq \mathcal{S}_B$. Dilation theory [8]provides information on the quality of this relaxation. We explain this for the case where L_A is a diagonal pencil describing the unit cube $[-1,1]^g$. The NP-hard problem of verifying the containment $[-1,1]^g \subseteq \mathcal{S}_B$ is the matrix cube problem of Ben-Tal and Nemirovskii.

Theorem 3 ([8]). *If L_B is a $d \times d$ linear pencil with $[-1,1]^g = \mathcal{S}_A \subseteq \mathcal{S}_B$, then*

$$\mathcal{D}_A \subseteq \vartheta(d)\ \mathcal{D}_B,$$

where

$$\frac{1}{\vartheta(d)} = \min_{\substack{a\in\mathbb{R}^d \\ |a_1|+\cdots+|a_d|=d}} \int_{S^{d-1}} \left|\sum_{i=1}^d a_i\xi_i^2\right| d\xi = \min_{\substack{B\in\mathbb{S}_d \\ \mathrm{trace}|B|=d}} \int_{S^{d-1}} |\xi^* B\xi|\ d\xi. \tag{2}$$

Moreover, $\vartheta(d)$ is the smallest number with this property: if $\vartheta' < \vartheta(d)$, then there is a $d \times d$ pencil L_B such that $[-1,1]^g \subseteq \mathcal{S}_B$ but $\mathcal{D}_A \nsubseteq \vartheta'\ \mathcal{D}_B$.

Probabilistic results on the binomial distribution can be used to vastly simplify the expression (2) for $\vartheta(d)$. For instance, if d is even, then

$$\frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{d-1}}{d} \le \frac{1}{\vartheta(d)} = \frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{d}{4}+\frac{1}{2}\right)}{\Gamma\left(\frac{d}{4}+1\right)} \le \frac{2}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{d+1}}$$

The bound $\vartheta(d)$ arises from the following dilation-theoretic result; its proof is constructive and explicitly identifies the dilations acting on an L^2-space. Recall: a matrix C dilates to T if

$$T = \begin{pmatrix} C & * \\ * & * \end{pmatrix}.$$

Equivalently, there is an isometry V such that $C = V^*TV$.

Theorem 4 (Simultaneous Dilation [8]). *Let $d \in \mathbb{N}$. There is a Hilbert space $\mathcal{H}$, a family $\mathscr{C}_d$ of commuting self-adjoint contractions on $\mathcal{H}$, and an isometry $V : \mathbb{R}^d \to \mathcal{H}$ such that for each symmetric $d \times d$ contraction matrix X there exists a $T \in \mathscr{C}_d$ such that $X = \vartheta(d)\ V^*TV$. Moreover, $\vartheta(d)$ is the smallest such constant.*

We conclude this note by showing how Theorem 3 follows from the Simultaneous Dilation theorem.

Proof of Theorem 3. Let the pencil L_B be of size d with $[-1,1]^s \subseteq \mathcal{S}_B$. The claim is $\mathcal{D}_A \subseteq \vartheta(d)\mathcal{D}_B$. First of all, it suffices to prove $\mathcal{D}_A \cap \mathbb{S}_d^g \subseteq \vartheta(d)\mathcal{D}_B \cap \mathbb{S}_d^g$; this follows from complete positivity considerations.

For $X = (X_1, \ldots, X_s) \in \mathcal{D}_A \cap \mathbb{S}_d^g$, by the Simultaneous Dilation theorem,

$$\frac{1}{\vartheta(d)}X = V^*TV = (V^*T_1V, \ldots, V^*T_sV), \qquad T_j \in \mathscr{C}_d.$$

Because the T_j commute and are contractions, $\mathcal{S}_A \subseteq \mathcal{S}_B$ implies

$$L_A(T) \succeq 0 \qquad \text{(spectral theorem aka simultaneous diagonalization).}$$

Hence, $L_A\left(\frac{1}{\vartheta(d)}X\right) = (I \otimes V)^*\ L_A(T)\ (I \otimes V) \succeq 0.$ □

The inclusion factor $\vartheta(d)$ has been identified for a few other interesting classes of including spectrahedra (e.g. balls, polytopes) in [2, 4].

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Hyperbolicity in Higher Codimension

Mario Kummer
(joint work with E. Shamovich)

Hyperbolic subvarieties $X \subseteq \mathbb{P}^n_{\mathbb{R}}$ are those that admit a finite surjective *real fibered* linear projection to some $\mathbb{P}^k_{\mathbb{R}}$. This means that there is a linear subspace $E \subseteq \mathbb{P}^n_{\mathbb{R}}$ of dimension $n-1-\dim X$ with $X \cap E = \emptyset$ such that the linear projection $\pi_E : X \to \mathbb{P}^k_{\mathbb{R}}$ from center E has the property that $x \in X(\mathbb{R}) \Leftrightarrow \pi_E(x) \in \mathbb{P}^k_{\mathbb{R}}(\mathbb{R})$ for all $x \in X(\mathbb{C})$. Hyperbolic varieties have been introduced by Shamovich and Vinnikov in [18] as a generalization hyperbolic polynomials. A homogeneous multivariate polynomial $h \in \mathbb{R}[x_1, \ldots, x_n]$ is *hyperbolic with respect to* $e \in \mathbb{R}^n$ if for every $v \in \mathbb{R}^n$ the univariate polynomial $h(te+v)$ has only real roots and $h(e) \neq 0$. Hyperbolic polynomials appear in many areas of mathematics. They were first studied in the context of partial differential equations, more precisely hyperbolic partial differential equations, see, for example, [6, 10]. Gårding [7] discovered strong convexity properties of hyperbolic polynomials. This led to applications of the theory of hyperbolic polynomials in the field of convex optimization [1, 8, 17]. In the last decades interest in hyperbolic polynomials also arose from the field of combinatorics, more precisely from matroid theory [5, 4, 19]. Very recently, hyperbolic polynomials have been the key tool in a celebrated pair of articles by Marcus, Spielman and Srivastava where they show the existence of bipartite Ramanujan graphs of all degrees [14] and solve the Kadison–Singer problem [15].

It was shown in [11] that if the real zero set of a hyperbolic polynomial is smooth, then it consists of some connected components that are homeomorphic to the sphere and at most one that is homeomorphic to the real projective space. Here is a similar result for hyperbolic varieties.

Theorem ([12])**.** *Let $X \subseteq \mathbb{P}^n_{\mathbb{R}}$ be a smooth irreducible hyperbolic variety. Then $X(\mathbb{R})$ is a disjoint union of s connected components that are homeomorphic to the sphere $\mathbb{S}^k$ and r connected components homeomorphic to $\mathbb{RP}^k$ where $k = \dim X$. If $k \geq 2$, then $2s + r = \deg X$.*

The proof of the theorem heavily relies on the fact that real fibered morphisms are unramified at smooth real point as shown in [12]. This restrictive property was also used in [3] to show that there are no torically maximal hypersurfaces. However, it is not yet clear whether there can be more than one connected component that is homeomorphic to the real projective space.

Problem. *Find a smooth irreducible hyperbolic variety of dimension at least two with more than one connected component of its real part is homeomorphic to $\mathbb{RP}^k$.*

A classical result of Nuij [16] tells us that the set of hyperbolic polynomials of fixed degree is closed and connected. Moreover, it is the closure of its interior which consists exactly of those that define a hypersurface without real singularities. Some of these properties extend to hyperbolic varieties. The proper parameter space to work in is the Hilbert scheme. For the following we refer to [13]. We showed that the set of varieties that are hyperbolic with respect to a fixed linear space E is closed and connected inside the (open subset of the) Hilbert scheme of all varieties of some given Hilbert polynomial that do not intersect E. Here we take the euclidean topology on the real points of the Hilbert scheme. Connectivity is shown by deforming a hyperbolic variety to a highly non-reduced subscheme that is supported on a linear space via the limit of linear transformations. Then one proceeds as in Hartshorne's proof of the connectivity of the Hilbert scheme [9]. It is also true that hyperbolic varieties that have smooth real part are in the interior of this set. In general, the interior of the set of hyperbolic varieties is not connected. This can be seen for example by looking at the set of real twisted cubics in projective three-space which is not connected [2]. There are also examples of hyperbolic varieties that are not in the closure of the set of smooth hyperbolic varieties as the example of generic reciprocal linear spaces shows.

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Sums of Squares on Projective Varieties

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(joint work with Grigoriy Blekherman, Mauricio Velasco)

1. An Algebraic Perspective on Positive Semidefinite Matrix Completion

A projective variety $X \subset \mathbb{P}^n$ is the zero set of any radical ideal $I \subset \mathbb{R}[x_0, x_1, \ldots, x_n]$. For every real point $p \in X(\mathbb{R})$, the sign of a quadratic form $Q \in \mathbb{R}[x_0, \ldots, x_n]_2$ at p is well-defined. In fact, the same is true for every residue class $\overline{Q} \in \mathbb{R}[x_0, \ldots, x_n]/I_2$ in the homogeneous part of degree 2 of the homogeneous coordinate ring of X, which we denote by R_2. So we consider the two convex cones

$$P_X = \{\overline{Q} \in R_2 \colon \overline{Q}(p) \geq 0 \text{ for all } p \in X(\mathbb{R})\}$$

$$\Sigma_X = \left\{\overline{Q} \in R_2 \colon \overline{Q} = \sum_{i=1}^{r} \overline{\ell_i}^2 \text{ for some linear forms } \ell_1, \ldots, \ell_r \in \mathbb{R}[x_0, x_1, \ldots, x_n]_1\right\}$$

As usual, we get the inclusion $\Sigma_X \subset P_X$.

Example 1.1. (a) Consider the twisted cubic $X = \nu_3(\mathbb{P}^1) = \{(s^3 : s^2t : st^2 : t^3) \colon (s : t) \in \mathbb{P}^1\} \subset \mathbb{P}^3$ and the quadratic form $Q = x_0^2 + 2x_1x_2 + x_3^2$. Restricted to X, this quadratic form gives the bivariate sextic $s^6 + 2s^3t^3 + t^6$ via the parametrization of X. Indeed, the residue class $\overline{Q}$ is a square in R_2, namely

$$x_0^2 + 2x_1x_2 + x_3^2 + I_2 = x_0^2 + 2x_0x_3 + x_3^2 + I_2 = (x_0 + x_3)^2 + I_2$$

because $x_1x_2 - x_0x_3$ vanishes on $\nu_3(\mathbb{P}^1)$. This is equivalent to the Gram matrix method, which is well-known in real algebraic geometry. See [5] for a recent survey and the Gram matrix method in the context of toric varieties.

(b) The central example of this report is coordinate subspace arrangements in $\mathbb{P}^{n-1}$. The relevant combinatorics is best expressed in terms of graphs: Let $G = ([n], E)$ be a finite simple graph on n vertices. We write $I_G = \langle x_ix_j \colon \{i,j\} \notin E\rangle \subset \mathbb{R}[x_1, \ldots, x_n]$ for the non-edge ideal associated to G. Its vanishing set in $\mathbb{P}^{n-1}$ is the subspace arrangement

$$X_G = \bigcup_{K \subset G \text{ clique}} \mathbb{P}(\text{span}\{e_i \colon i \in K\}) \subset \mathbb{P}^{n-1}.$$

The canonical projection $\mathbb{R}[x_1, \ldots, x_n]_2 \to R_2$ takes the symmetric $n \times n$-Gram matrix M of a quadratic form Q_M to its residue class modulo the monomials x_ix_j for the non-edges of G. In other words, we forget the coefficients of these monomials. The set of all quadratic forms with the same residue class in R_2 is the set of all completions of a G-partial matrix.

In [2], we characterize when the two cones P_X and Σ_X coincide on reduced schemes over $\mathbb{R}$.

Theorem 1.2 (Blekherman-Sinn-Velasco, 2017)**.** *Let $I \subset \mathbb{R}[x_0, \ldots, x_n]$ be a real radical ideal which does not contain any linear forms. Then $P_X = \Sigma_X$ if and only if I has Castelnuovo-Mumford regularity 2.*

Remark 1.3. (a) A prime ideal has regularity 2 if and only if X is a variety of minimal degree, i.e. $\deg(X) = \text{codim}(X) + 1$, see [6]. In this special case, Theorem 1.2 recovers the main theorem of [3].

(b) A square-free monomial ideal I_G has Castelnuovo-Mumford regularity 2 if and only if G is a chordal graph by a result of Fröberg, see [7]

2. The Existence of Positive Definite Completions

Using results from Stanley-Reisner theory, we study the positive definite matrix completion problem in [2]. Here is one main result.

Theorem 2.1 (Blekherman-Smith-Velasco, 2017)**.** *Let $G = ([n], E)$ be a simple graph. A G-partial matrix $\overline{Q} \in R_2$ has a positive definite completion if and only if*

(a) $\overline{Q}$ is strictly positive on $X(\mathbb{R})$; and

(b) $\overline{Q}$ has a representation as a sum of at least $n - m + 3$ linear forms, which cannot be shortened. Here, m is the shortest length of a chordless cycle of G.

The rank bound given in (b) is best-possible, i.e. there exists a G-partial matrix that can be written as a sum of $n - m + 2$ squares but which does not admit a positive definite completion. For applications in statistics, it is of interest to find the best possible lower bound such that we can find a positive definite completion with probability 1. This is formally expressed in the following definition.

Definition 2.2. The *maximum likelihood threshold* of a graph G is the smallest rank r such that for every generic positive semidefinite matrix A of rank r, there exists a positive definite matrix P such that $\pi_G(A) = \pi_G(P)$, where $\pi_G \colon \mathbb{R}[x_1, \ldots, x_n] \to R_2$ is the canonical projection. In other words, the residue class $\overline{Q_A}$ is an interior point of Σ_X.

This is a semi-algebraic invariant in nature. It has a natural algebraic relaxation.

Definition 2.3. The *generic completion rank* of G is the smallest rank r such that the projection of the variety of matrices of rank r is full dimensional, i.e. $\pi_G(V_r) = \dim(R_2) = n + \#E$.

Proposition 2.4 (Uhler, 2012)**.** *The maximum likelihood threshold of G is at most the generic completion rank of G.*

It was an open question in the field whether these two invariants could be different. The first example where they differ is given by complete bipartite graphs.

Theorem 2.5 (Blekherman-Sinn, 2017)**.** *Let $m \geq 3$. The generic completion rank of the complete bipartite graph $K_{m,m}$ is m. Its maximum likelihood threshold is the smallest k such that $\binom{k+1}{2} \geq 2m$. In particular, for $m = 5$, the generic completion rank is 5, whereas the maximum likelihood threshold is 4.*

There are many open questions related to the graph parameters introduced above. Explicitly, I want to mention a central open question about the relation among these two graph parameters also raised in [1].

Question 2.6. *Is there a function $f \colon \mathbb{N} \to \mathbb{N}$ such that its value $f(\mathrm{mlt}(G))$ at the maximum likelihood threshold of any finite simple graph G is an upper bound on the generic completion rank of G?*

Also, the generic completion rank of graphs, is not very well understood. The best known bounds in terms of graph invariants are related to the clique number, which is the dimension of the corresponding coordinate subspace arrangement as an algebraic set, and the treewidth of the graph, see [4, 8]. They can be far apart and quite different from the actual generic completion rank. The special case of bipartite graphs, which we introduced here in the context of completion of partial symmetric matrices, is also related to the generic completion rank of partial matrices without symmetry constraints (or even requiring the matrices to be square) by an additive constant, see [1, Proposition 2.14]. This indicates that it is a central graph parameter that deserves attention in the future.

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Truncated Moment Problems: An Introductory Survey

Raúl E. Curto

We present an introduction to the truncated moment problem, based on joint work with L.A. Fialkow, H.M. Möller and S. Yoo. Our lecture is organized around the following topics:

- A Bit of History
- TCMP and TRMP
- Basic Positivity Condition
- Definition of the Algebraic Variety
- First Existence Criterion for TMP
- The Flat Extension Theorem
- Localizing Matrices
- A Version of Riesz-Haviland for TMP
- The Quartic MP
- The Extremal MP
- Cubic Column Relations
- The Extremal Sextic MP (Division Algorithm)
- The Non-Extremal Sextic Moment Problem (Rank Reduction)

Inverse problems naturally occur in many branches of science and mathematics. An inverse problem entails finding the values of one or more parameters using the values obtained from observed data. A typical example of an inverse problem is the inversion of the Radon transform. Here a function (for example of two variables) is deduced from its integrals along all possible lines. This problem is intimately connected with image reconstruction for X-ray computerized tomography.

Moment problems are a special class of inverse problems. While the classical theory of moments dates back to the beginning of the 20th century, the systematic study of *truncated* moment problems began only a few years ago. In this talk we will first survey the elementary theory of truncated moment problems, and then focus on those problems with cubic column relations.

For a degree $2n$ complex sequence $\gamma \equiv \gamma^{(2n)} = \{\gamma_{ij}\}_{i,j \in Z_+, i+j \leq 2n}$ to have a representing measure μ, it is necessary for the associated moment matrix $M(n)$ to be positive semidefinite, and for the algebraic variety associated to γ, $\mathcal{V}_\gamma \equiv \mathcal{V}(M(n)) := \bigcap_{M(n)\hat{p}=0} \mathcal{Z}(p)$, to satisfy rank $M(n) \leq$ card $\mathcal{V}_\gamma$. (Here p is a polynomial of degree at most $2n$, $\mathcal{Z}(p)$ its zero set, and $\hat{p}$ the vector of coefficients of p.) Additionally, in the presence of a representing measure the following consistency condition must hold: if a polynomial $p(z, \bar{z}) \equiv \sum_{ij} a_{ij} \bar{z}^i z^j$ of degree at most $2n$ vanishes on $\mathcal{V}_\gamma$, then the *Riesz functional* $\Lambda(p) \equiv p(\gamma) := \sum_{ij} a_{ij} \gamma_{ij} = 0$.

Positive semidefiniteness, recursiveness, and the variety condition of a moment matrix are necessary and sufficient conditions to solve the quadratic ($n = 1$) and quartic ($n = 2$) moment problems. Also, positive semidefiniteness, combined with the above mentioned consistency condition, is a sufficient condition in the case of *extremal* moment problems, i.e., when the rank of the moment matrix (denoted by r) and the cardinality of the associated algebraic variety (denoted by v) are equal. However, these conditions are not sufficient for *non*-extremal (i.e., $r < v$) sextic ($n = 3$) or higher-order truncated moment problems.

For the sextic moment problem, we first consider cubic column relations in $M(3)$ of the form (in complex notation) $Z^3 = itZ + u\bar{Z}$, where u and t are real numbers. For (u, t) in the interior of a real cone, we prove that the algebraic variety V_γ consists of exactly 7 points, and we then apply the above mentioned solution of the extremal moment problem to obtain a necessary and sufficient condition for the existence of a representing measure. This requires a new representation theorem for sextic polynomials in z and $\bar{z}$ which vanish in the 7-point set V_γ. Our proof of this representation theorem relies on two successive applications of the Fundamental Theorem of Linear Algebra.

For general extremal sextic moment problems, verifying consistency amounts to having good representation theorems for sextic polynomials in two variables vanishing on the algebraic variety of the moment sequence. We obtain such representation theorems using the Division Algorithm from algebraic geometry. As a consequence, we are able to complete the analysis of extremal sextic moment problems.

Assume now that $M(3) \geq 0$, and that it satisfies the variety condition $r \leq v$ as well as consistency. Also assume that $M(3)$ admits at least one *cubic* column relation. We prove the existence of a related matrix $\widetilde{M(3)}$ with rank $\widetilde{M(3)} <$ rank $M(3)$ and such that each representing measure for $\widetilde{M(3)}$ gives rise to a representing measure for $M(3)$. As a concrete application, we discuss the case when rank $M(3) = 8$ and card $\mathcal{V}(M(3)) \leq 9$.

Along the way, we settle three key instances of the non-extremal sextic moment problem, as follows: when $r = 7$, positive semidefiniteness, consistency and the variety condition guarantee the existence of a 7-atomic representing measure; when $r = 8$ we construct two determining algorithms, corresponding to the cases $v = 9$ and $v = +\infty$. To accomplish this, we generalize the above mentioned rank-reduction technique, which was used in previous work to find an explicit solution of the nonsingular quartic moment problem.

We now give a short list of outstanding open problems in TMP.

Problem 1. *(Flat Extension Problem) Given a recursively generated moment matrix $M(n)$, find necessary and sufficient conditions to guarantee that $M(n)$ admits a flat extension $M(n+1)$.*

Problem 2. *Let p be irreducible, and assume that $\mathcal{V} = \mathcal{Z}(p)$. Does it follow that $M(n)$ admit a r_n-atomic representing measure? How is $\mathcal{V}$ affected by passage from $M(n)$ to $M(n+1)$?*

Problem 3. *Study the solubility of TMP on the irreducible algebraic set $y^2 - x^3 = 0$.*

Problem 4. *For rank $\mathcal{M}(3) = 8$ and card $\mathcal{V}(\beta) = 9$, find necessary and sufficient conditions for the existence of a representing measure for β.*

While we now have a good understanding of the sextic TMP with finite algebraic variety, the case of $v = \infty$ still requires much work. The proof of the solubility of the Quartic MP used affine planar transformations to reduce a general quadratic column relation to one of five canonical types: $y = x^2$, $xy = 1$, $xy = 0$, $x^2 + y^2 = 1$ and $x^2 = x$. We approach the Sextic MP in an analogous manner. First, we recall that every *singular*, irreducible cubic can be converted, using affine planar transformations, into one of three possible forms: **(i)** $y^2 = x^3$ (which we already mentioned in Problem 3), **(ii)** $y^2 = x^2(x+1)$ or **(iii)** $y^2 = x^2(x-1)$. Similarly, *nonsingular* irreducible cubics can be transformed into one of two types: **(iv)** $y^2 = x(x-1)(x-w)$ (for $w > 1$), or **(v)** $y^2 = x(x^2 + kx + 1)$ (for $-2 < k < 2$). Thus, the TMP for irreducible cubics amounts to the analysis of five distinct cases.

Problem 5. *For each of the above mentioned five cases of irreducible cubics, characterize the solubility of TMP in terms of the initial data.*

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Moment problems in statistical mechanics: kinetic hierarchies and effective equations

Tobias Kuna

The infinite dimensional moment problem is a key-tool for several techniques in applications; we consider here as an example the treatment of dynamics in statistical physics, which is an important sub-area of physics treating the joint and maybe cooperative behaviour of a large number of identical components. One of the aims of the talk was to point out why and where the moment problem and techniques for the moment problem are relevant for the description of dynamics in statistical physics. In general, dynamics can be described on several different levels. For example a dynamics may be given in terms of an ODE or the associated Liouville equation. Whenever the functional dependence in the ODE is polynomial then the Liouville equation will preserve the set of formal power series. Hence pairing the Liouville equation with a measure one can derive a dynamics in terms of moments. The case of particle systems was considered, the so-called BBGKY hierarchy and the idea of moment closure was introduced. The latter is behind the derivation of classical PDE's like the Boltzmann equation, Vlasov equation, Euler equation, Navier-Stokes equation, (Reaction-)Diffusion equation.

Let us give a short introduction into the infinite dimensional moment problem. Consider two (topological) vector spaces V and V' in duality $(\cdot,\cdot)$. A linear function on V' is a linear mapping from $b_1 : V' \to \mathbb{R}$, which due to the duality can be represented as $w \mapsto b_1(w) = (a^{(1)}, w)$ for some $a^{(1)} \in V$. A homogenous quadratic polynomial is given by a bilinear form $b_2 : V' \times V' \to \mathbb{R}$, $w \mapsto b_2(w,w)$. For an appropriate choice of the topology on V' and the topology on tensor products one may write

$$w \mapsto b_2(w,w) = (a^{(2)}, w^{\otimes 2}) \qquad \text{for some } a^{(2)} \in V \otimes V. \tag{1}$$

Higher order monomials are defined analogously. Finite linear combinations of such monomials are infinite dimensional polynomials. The classical form of polynomials can be recovered for finite dimensional V choosing a basis $(e_i)_{i\in I}$ and we denote the associated dual basis by $(e_i')_{i\in I}$. Therefore, if one expands $w = \sum_{i\in I} w_i e_i'$ then one can expand the homogenous polynomials of second degree (1) and obtain that in the basis they take the form

$$w = \sum_{i\in I} w_i e_i' \mapsto \sum_{i,j\in I} a^{(2)}_{i,j} w_i w_j. \tag{2}$$

The coefficients of the polynomials can be seen as elements of the closure (in a suitable topology) of the symmetric algebra of V. Note that the norm of b_2 as a bilinear form and the norm in the tensor product are typically not equivalent.

To any given measure μ on V' one can associate moments, e.g. the second moment $m^{(2)}$ is defined by the following equality for all $a^{(2)} \in V$

$$(a^{(2)}, m^{(2)}) := \int_{V'} (a^{(2)}, w^{\otimes 2}) \mu(dw) \tag{3}$$

and it is in general an element of the dual space of bilinear-forms on V'. For an appropriate topology on the tensor space it can be seen as an element on $V' \otimes V'$ as well. In this way the moments give also rise to a linear form L, the Riesz functional, on the symmetric algebra of V.

Given a sequence $m^{(n)}$ of elements of $(V')^{\otimes n}$ and a closed subset K of V', the moment problem asks whether there exists a measure μ on K such that each $m^{(n)}$ is the n-th moment of μ (defined analogously to (3)). We call the moment problem infinite dimensional if K is not locally compact and hence V' is infinite dimensional.

A solution to the moment problem is a characterization of the existence of the representing measure only in terms of conditions on the moments. Usually, they are of the following four different types:

(I) positivity conditions on the moment sequence or the Riesz functional;
(II) conditions on the asymptotic behaviour of the moments as a sequence of their degree;
(III) properties of the putative support of the representing measure or in other words the extensiveness of the class of allowed sets K;
(IV) properties of the moments as elements in $(V')^{\otimes n}$.

If V is a function space, Condition IV can be split into local regularity and growth properties as generalized functions. As well Condition III contains that the representation of K as a basic semi-algebraic set may naturally include polynomials with coefficients $a^{(k)}$ not only from an algebraic tensor product but from a completion of the algebraic tensor product in a suitable topology extending the class of K. Also sets K represented using an infinite number of polynomial equalities are often natural. Condition II and IV can be at least partially formulated as continuity properties of the Riesz functional.

The general aim in the theory of moments is to construct a solution which is as weak as possible with respect to some combination of the above types of conditions, since it seems unfeasible to get one solution which is optimal in all types simultaneously.

An example of this effect was presented for the case of the moment problem for point processes, that is K is the set of all Radon measures of the form $\sum_{i \in I} \delta_{x_i}$ for I countable index set and $x_i \in \mathbb{R}^d$. A Riesz-Haviland type result for point process is proved in [13]. Here the only assumption is non- negativity on non-negative polynomials, which is the strongest positivity condition but no conditions of the other types are assumed. All the following works assume positive semi-definiteness of the Riesz-functional. In [10] and [3] the moment problem was solved under the additional assumption that the factorial moment measures (correlation functions) are non-negative and that the d-th moment is bounded by $d!$, which corresponds to an analytic Laplace transform of the measure in a neighbourhood of zero. We call the polynomials corresponding to the factorial moment measures factorial monomials. In [11] the positivity of the factorial moments was replaced by the assumption that the factorial moments have a density which is in L^∞ (the usual moments never have this property). In [9] the bound on the d-th moment was weakend to $(d!)^2$, but additionally the Riesz functional shifted by the aforementioned factorial monomials need to be positive semi-definite. The result is based on [17] which extends the ideas of [6], [19], [12] and [2], [5].

For the convenience of the reader, we add a few further references about the infinite dimensional moment problem just as a starting point for further reading on the topic, but far from being an exhaustive list in any sense.

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Invariant Nonnegative Forms as Sums of Squares

Charu Goel

The relationship between the cone of positive semidefinite (psd) real forms and its subcone of sums of squares (sos) of forms is of fundamental importance in real algebraic geometry and optimization, and has been studied extensively. The study of this relationship goes back to the 1888 seminal paper of Hilbert [16], where he gave a complete characterisation of the pairs $(n, 2d)$ for which a psd n-ary $2d$-ic form can be written as sos. In this note, we discuss how this relationship changes under the additional assumptions of invariance on the given forms.

A real form f is *nonnegative* or *positive semidefinite* (psd) if $f(x) \geq 0$ for all $x \in \mathbb{R}^n$ and is a *sum of squares* (sos) if there exist other forms h_j such that $f = h_1^2 + \cdots + h_k^2$. Every sos is automatically psd, but, as we shall see, not the converse. In general it is difficult to determine whether a particular form is psd, however an easy way to check this is if it can be written as sos. This can be done using algebraic techniques (like the Gram matrix method [7, 18]) as well as numerical optimization techniques (like semidefinite programming [11]).

Hilbert [16] studied the inclusion $\mathcal{P}_{n,2d} \supseteq \Sigma_{n,2d}$, where $\mathcal{P}_{n,2d}$ and $\Sigma_{n,2d}$ are respectively the cones of psd and sos forms of degree $2d$ in n variables. He proved that:

$$\mathcal{P}_{n,2d} = \Sigma_{n,2d} \textit{ if and only if } n = 2, d = 1, \textit{ or } (n, 2d) = (3, 4).$$

In order to establish that $\Sigma_{n,2d} \subsetneq \mathcal{P}_{n,2d}$, he demonstrated that $\Sigma_{3,6} \subsetneq \mathcal{P}_{3,6}$, $\Sigma_{4,4} \subsetneq \mathcal{P}_{4,4}$, thus reducing the problem to these two basic cases using an argument to increase the number of variables and degree of a given psd not sos form while simultaneously preserving the psd not sos property. The first explicit examples of psd not sos forms in these two cases were found by Motzkin [17] and Robinson [20], in the late 1970's. Subsequently more examples were given by Choi-Lam [2, 3, 4], Reznick [19] and Schmüdgen [21].

In 1976, Choi and Lam [3] considered the same inclusion for *symmetric forms* (i.e forms invariant under the action of the symmetric group S_n). As an analogue of Hilbert's approach, they demonstrated that establishing the strict inclusion for all $n \geq 3, 2d \geq 4$ and $(n, 2d) \neq (3, 4)$ reduces to show it just for the pairs $(n, 2d) = (3, 6), (n, 4)_{n \geq 4}$, by using a trick that increases the degree – however not the number of variables – of a given psd not sos symmetric form by simultaneously preserving the psd not sos symmetric property. Assuming the existence of psd not sos symmetric n-ary quartics for $n \geq 5$, they showed that Hilbert's characterisation above remains unchanged. Recently, we [12] constructed explicitly these quartic forms, thus completing their proof. For this we used test set for (positivity of) symmetric quartics, that was originally given by Choi-Lam-Reznick [5] and later generalized by Timofte [22] for symmetric polynomials of degree $2d$ in n variables.

Recently, we studied systematically the above inclusion of cones for *even symmetric forms* (i.e. forms invariant under the action of the group $S_n \times \mathbb{Z}_2^n$). The idea was to develop an analogue of reduction to basic cases, in the same spirit as Hilbert and Choi-Lam. Choi-Lam-Reznick [6] and Harris [14, 15] established that Hilbert's characterisation is no longer true for even symmetric forms; indeed equality of these cones holds also for the pairs $(n, 4)_{n \geq 4}$ and $(3, 8)$. Moreover, they gave psd not sos even symmetric examples for the pairs $(n, 6)_{n \geq 3}, (3, 10)$ and $(4, 8)$. Building up on their work, we [13] established strict inclusion for the pairs $(3, 2d)_{d \geq 6}, (n, 8)_{n \geq 5}, (n, 2d)_{n \geq 4, d \geq 5}$, and proved that it suffices for all the remaining cases [i.e. for all $n \geq 3, 2d \geq 6$ and $(n, 2d) \neq (3, 8)$]. For this we introduced as our leading tool a "Degree Jumping Principle"(that increases the degree of a given psd not sos even symmetric form while simultaneously preserving the psd not sos even symmetric property) and constructed explicit counterexamples for the pairs $(n, 8)_{n \geq 5}, (n, 10)_{n \geq 4}, (n, 12)_{n \geq 4}$. This let us (S. Kuhlmann, B. Reznick and I) to a complete resolution of all remaining open cases, thus providing [13] a complete analogue of Hilbert's theorem for even symmetric forms, namely,

an even symmetric n-ary $2d$-ic psd form is sos if and only if
$n = 2$ *or* $d = 1$ *or* $(n, 2d) = (n, 4)_{n \geq 3}$ *or* $(n, 2d) = (3, 8)$.

Remark. It would be interesting to see how the symmetric and even symmetric analogues of Hilbert's theorem presented above can be used computationally and in other problems related to sums of squares.

As discussed above, taking invariance under a bigger group results in equality of the cones of invariant psd and invariant sos forms for more number of pairs. This naturally opens up an idea to investigate a wider generalization of analogues of Hilbert's theorem for forms invariant under other group actions (i.e. other than S_n and $S_n \times \mathbb{Z}_2^n$). A major achievement in this direction would be a generalization of Timofte's degree principle [22] to invariant forms, since test sets for positivity of symmetric polynomials played an important role in establishing the analogues of Hilbert's theorem for symmetric and even symmetric forms. Further, more sophisticated arguments and tools (like degree jumping principle and reduction to basic cases) have to be developed for the invariant forms under consideration, along the same lines as Hilbert, Choi-Lam and Goel-Kuhlmann-Reznick.
The problem of finding sos decompositions of a polynomial invariant under the action of a finite group was studied in [11] and in [8] (for reductive groups). Further, it is interesting as well as important to find the explicit description of the cone of invariant sos forms; this is done for invariance under a finite group generated by pseudo reflections in [9].

It is noteworthy that the first counterexamples (i.e. psd not sos ternary sextics and quaternary quartics) substantiating Hilbert's 1888 theorem were given almost 80 years later in 1967. Moreover, the results on equality and strict inclusions of the cones of psd and sos forms (respectively symmetric, even symmetric forms) by Hilbert (respectively Choi-Lam-Reznick, Harris and Goel-Kuhlmann-Reznick) were building stones in establishing Hilbert's theorem (respectively its analogue for symmetric and even symmetric forms). Thus, given a finite group G, establishing equality or strict inclusion of cones of invariant psd and invariant sos forms for any (n, 2d) will be a novel contribution in this research area and would have a strong impact on the applications of sums of squares. In this spirit and as a starting point we are investigating the inclusion of cones for forms invariant under the action of finite reflection groups and Lie groups, using a recent generalization of Timofte's degree principle for these groups given by Acevedo-Velasco [1] and Friedl-Riener-Sanyal [10].

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Tracial polynomial optimization for bounding matrix factorization ranks

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(joint work with Sander Gribling, David de Laat)

We consider factorizations of matrices by nonnegative vectors or positive semidefinite matrices and offer a unified framework - based on tracial polynomial optimization - for designing tractable bounds for the smallest dimension where such factorizations can be found. We begin with introducing different types of matrix factorizations that fall within this common framework, depending whether the factors are vectors or matrices and whether we search for asymmetric factorizations (allowing different factors for the rows and the columns) or for symmetric factorizations (asking for the same factors for rows and columns). This leads to various rank notions for which no efficient algorithms are known for their exact computation.

Consider first a rectangular matrix $A \in \mathbb{R}^{m\times n}$. A *nonnegative factorization* of A is provided by a set of nonnegative vectors $u_i, v_j \in \mathbb{R}^d_+$ $(i \in [m], j \in [n])$ (for some $d \in \mathbb{N}$) such that $A = (u_i^T v_j)$. A *positive semidefinite (psd) factorization* of A consists of a set of positive semidefinite matrices $X_i, Y_j \in \mathcal{S}^d_+$ $(i \in [m], j \in [n])$ (for some $d \in \mathbb{N}$) such that $A = (\mathrm{Tr}(X_iY_j))$. The smallest d for which such factorization exist are called the *nonnegative rank* and *the positive semidefinite rank* of A, denoted by $\mathrm{rank}_+(A)$ and $\mathrm{rank}_{psd}(A)$, respectively. Clearly,

$$\mathrm{rank}(A) \le \mathrm{rank}_+(A),\ \mathrm{rank}_{psd}(A) \le \mathrm{rank}_+(A) \le \min\{m, n\}.$$

These notions of ranks have applications in communication complexity and for the study of the extension complexity of polytopes (see, e.g., [5]). Consider now a symmetric matrix $A \in \mathcal{S}^n$. Then A is *completely positive* (cp) (resp., *completely positive semidefinite* (cpsd)) if it admits a Gram factorization $A = (u_i^T u_j)$ by nonnegative vectors $u_i \in \mathbb{R}^d_+$ (resp., $A = (\mathrm{Tr}(X_iX_j))$ by psd matrices $X_i \in \mathcal{S}^d_+$); the smallest such d are the *cp-rank* cp-rank(A) and the *cpsd-rank* cpsd-rank(A) of A, respectively. Clearly the cone $\mathcal{CP}^n$ of cp matrices is contained in the cone $\mathcal{CS}^n_+$

of cpsd matrices and cpsd-rank$(A) \leq$ cp-rank(A) for matrices in $\mathcal{CP}^n$. One can easily upper bound the cp-rank using Caratheodory's theorem:

$$\text{cp-rank}(A) \leq \binom{n+1}{2}$$

(sharper bounds exist). We refer to [7] for a class of matrices whose cp-rank is quadratic in n while their cpsd-rank is linear in n, and to [7, 10] for a class of matrices whose cpsd-rank grows exponentially fast in n (like $2^{\sqrt{n}}$). On the other hand, no upper bound (depending only on the matrix size n) is known for matrices in $\mathcal{CS}_+^n$. A first fundamental open question is

Question 1: Does there exist an upper bound (depending only on n) for the cpsd rank of matrices in $\mathcal{CS}_+^n$?

While the cone $\mathcal{CP}^n$ is well studied (see, e.g., [1]), the cone $\mathcal{CS}_+^n$ was introduced recently in [8], motivated by applications in quantum information. It is used, e.g., to model quantum analogues of classical graph parameters as conic optimization problems over the cone $\mathcal{CS}_+^n$ and to represent the set of quantum correlations as an affine section of $\mathcal{CS}_+^n$. The structure of the cone $\mathcal{CS}_+^n$ (for $n \geq 5$) is still largely unknown. A second fundamental open question is

Question 2: Is the cone $\mathcal{CS}_+^n$ closed?

A positive answer to Question 1 implies a positive answer to Question 2. Moreover, a positive answer to Question 2 implies that the set of quantum correlations is closed, an open problem in quantum information (see [11]), as well as disprove Connes' embedding conjecture (which follows by combining results of [9, 12]).

We may define the larger cone $\mathcal{CS}_{vN+}^n$ consisting of all matrices $A = (\tau(x_i x_j))$ for positive elements x_i in a finite von Neumann algebra with trace τ. This cone is closed and contains the closure of the cone $\mathcal{CS}_+^n$ [2]. Moreover, it is shown in [2] that, if Connes' conjecture holds, then equality $cl(\mathcal{CS}_+^n) = \mathcal{CS}_{vN+}^n$ holds.

In addition, the amount of entanglement needed to realize a given (synchronous) quantum correlation correponds to the cpsd-rank of an associated cpsd matrix. This motivates having tools to compute good lower bounds for this rank.

We now present our new method for deriving hierarchical lower bounds for the cpsd rank (and other ranks as well). Consider a matrix $A \in \mathcal{CS}_+^n$ with cpsd-rank$(A) = d$, and let $\mathbf{X} = (X_1, \cdots, X_n) \in (\mathcal{S}_+^d)^n$ be an optimal factorization of A, i.e., $A = (\mathrm{Tr}(X_i X_j))$. Define the linear form $L_{\mathbf{X}} \in \mathbb{R}\langle \mathbf{x} \rangle^*$ on the ring $\mathbb{R}\langle \mathbf{x} \rangle$ of noncommmutative polynomials in the variables $\mathbf{x} = (x_1, \cdots, x_n)$ by

$$L(p) = Re(\mathrm{Tr}(p(X_1, \cdots, X_n))) \qquad \text{for } p \in \mathbb{R}\langle \mathbf{x} \rangle.$$

Then $L_{\mathbf{X}}$ satisfies the folowing properties:

(1) L is tracial ($L(pq) = L(qp)$) and symmetric ($L(p^*) = L(p)$);
(2) $L(1) = d$ and $L(x_i x_j) = A_{ij}$ for $i, j \in [n]$;
(3) $L \geq 0$ on the quadratic module generated by the polynomials $x_i, A_{ii} - x_i^2$ ($i \in [n]$);
(4) The associated moment matrix $M(L) = (L(u^* v))_{u,v \in \langle \mathbf{x} \rangle}$ has finite rank.

This suggests defining lower bounds for cpsd-rank(A) by minimizing $L(1)$ for linear forms L acting on *truncated* polynomial subspaces of $\mathbb{R}\langle\mathbf{x}\rangle$: For any integer $t \geq 2$, let $\xi_t(A)$ be the minimum value of $L(1)$ where $L \in \mathbb{R}\langle\mathbf{x}\rangle_{2t}^*$ satisfies the analogues of conditions (1)-(3) truncated at degree $2t$. Each parameter $\xi_t(A)$ can be expressed as a semidefinite program and thus computed efficiently (for fixed t). Further, $\xi_\infty(A)$ is obtained by taking $t = \infty$ and $\xi_*(A)$ by adding the finite rank condition (4) to it. We have

$$\xi_2(A) \leq \cdots \leq \xi_t(A) \leq \cdots \leq \xi_*(A) \leq \xi_\infty(A) \leq \text{cpsd-rank}(A).$$

The localizing constraint (3) ensures that the quadratic module is Archimedean. Using general results about trace optimization (see [3, 6]) one can show that the bounds $\xi_t(A)$ converge to $\xi_\infty(A)$ as $t \to \infty$. Moreover, equality $\xi_t(A) = \xi_*(A)$ holds if the program $\xi_t(A)$ has an optimal solution satisfying a 'flatness criterion'. In addition we can give an interpretation of the parameters $\xi_\infty(A)$ and $\xi_*(A)$ in terms of C^*-algebras with tracial states. The inequality $\xi_*(A) \leq$ cpsd-rank(A) is strict in general, however one can add additional localizing constraints to the program $\xi_t(A)$ in order to get stronger bounds.

This method applies to the other factorization ranks. For the cp-rank simply ask that L should act on *commutative* polynomials. One may also strengthen (3) and ask nonnegativity on the quadratic module generated by all $x_i, A_{ij} - x_i x_j$, as well as the constraints $L(u), L(u(A_{ij} - x_i x_j)) \geq 0$ for all monomials u, and $A^{\otimes l} - (L(uv))_{u,v \in \langle\mathbf{x}\rangle_l} \succeq 0$ for any $l \geq 1$. This yields a hierarchy of lower bounds for the cp-rank, whose first bound (roughly) corresponds to the previously best known lower bound in [4]. One may also adapt the method for the nonnegative and psd ranks. Roughly the idea is that we now deal with a 'partial' matrix whose entries are known only at the off-diagonal blocks. The key to get again localizing constraints like (3) is to note there is a factorization where all entries of u_i, v_j are bounded (and analogously for the psd factors X_i, Y_j after rescaling A).

For example, the first bound in the hierarchy finds the exact value of the cp-rank for matrices $\begin{pmatrix} qI_p & J \\ J & pI_q \end{pmatrix}$ and it improves the bound of [4] for the nonnegative rank of the slack matrix corresponding to the problem of deciding existence of a triangle nested between two parallelograms.

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The van der Waerden and Kadison-Singer conjectures with symmetries

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One of the key ingredients in the recent solution [2] of the Kadison-Singer problem was to bound the zeros of so called mixed characteristic polynomials. In this note we discuss two conjectures on upper sharp bounds on the largest zero of such polynomials. These conjectures are similar in nature to the van der Waerden type theorem of Gurvits [1].

A homogeneous polynomial $P(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$, $\mathbf{x} = (x_1, \dots, x_m)$, of degree d is called *doubly stochastic* if

$$\frac{\partial P}{\partial x_i}(\mathbf{1}) = \frac{d}{m}, \quad \text{for all } 1 \le i \le m,$$

where $\mathbf{1} = (1, \dots, 1)$. The *mixed characteristic polynomial* of P is the univariate polynomial

$$\chi_P(t) = \left(1 - \frac{\partial}{\partial x_1}\right)\left(1 - \frac{\partial}{\partial x_2}\right)\cdots\left(1 - \frac{\partial}{\partial x_m}\right) P\Big|_{x_1 = \cdots = x_m = t}.$$

A polynomial $P \in \mathbb{R}[\mathbf{x}]$ is called *stable* if $P(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}^m$. Let $\mathcal{D}_{m,d}$ denote the space of all doubly stochastic and stable polynomials in $\mathbb{R}[x_1, \dots, x_m]$ of degree d. Since the operator $1 - \partial/\partial x_i$ preserves stability it follows that all zeros of $\chi_P(t)$ are real if $P \in \mathcal{D}_{m,d}$. Let

$$\lambda(P) : \lambda_1(P) \ge \lambda_2(P) \ge \cdots \ge \lambda_d(P)$$

be the zeros of $\chi_P(t)$. The following conjecture (essentially) appeared in [2].

Conjecture 1. *If $P \in \mathcal{D}_{m,d}$, then $\lambda_1(P) \le \lambda_1(P_\infty)$ where*

$$P_\infty := \frac{(x_1 + \cdots + x_m)^d}{m^d}.$$

A version of Gurvits theorem [1] which implies the van der Waerden conjecture for doubly stochastic matrices may be phrased as

$$\frac{\partial}{\partial x_1}\cdots\frac{\partial}{\partial x_m}P \geq \frac{\partial}{\partial x_1}\cdots\frac{\partial}{\partial x_m}P_\infty = \frac{m!}{m^m} \tag{1}$$

for all $P \in \mathcal{D}_{m,m}$.

Conjecture 1 may be strengthened as follows. Recall that if $\alpha, \beta \in \mathbb{R}^d$ are such that $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_d$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_d$, then β *majorizes* α, written $\alpha \preceq \beta$, if

$$\begin{aligned} \alpha_1 &\leq \beta_1, \\ \alpha_1 + \alpha_2 &\leq \beta_1 + \beta_2, \\ &\ddots \\ \alpha_1 + \cdots + \alpha_{d-1} &\leq \beta_1 + \cdots + \beta_{d-1}, \\ \alpha_1 + \cdots + \alpha_d &= \beta_1 + \cdots + \beta_d. \end{aligned}$$

Conjecture 2. *If $P \in \mathcal{D}_{m,d}$, then $\lambda(P) \preceq \lambda(P_\infty)$.*

It is not hard to deduce (1) from Conjecture 2. Hence Conjecture 2 may be regarded as a strong Van der Waerden conjecture.

Consider the linear operator $T : \mathbb{R}[x_1, \ldots, x_m] \to \mathbb{R}[x_1, \ldots, x_m]$ defined by

$$T(P) = \frac{1}{md}(x_1 + \cdots + x_m)\left(\frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_m}\right)P.$$

Define a flow Ψ_s, $s \geq 0$, by $\Psi_s = \exp(s(I-T))$ where I is the identity operator.

Lemma 3. *If $s \geq 0$, then $\Psi_s : \mathcal{D}_{m,d} \to \mathcal{D}_{m,d}$. Moreover*

$$\lim_{s\to\infty} \Psi_s(P) = P_\infty,$$

where the limit is uniform on compacts.

Let Sym_m denote the space of symmetric polynomials in $\mathbb{R}[x_1, \ldots, x_m]$, i.e., polynomials that are invariant under permuting the variables. The proof of the following theorem will appear elsewhere.

Theorem 4. *If $P \in \mathcal{D}_{m,d} \cap \mathrm{Sym}_m$, then the map*

$$s \mapsto \lambda_1(\Psi_s(P))$$

is monotone decreasing. In particular $\lambda_1(P) \leq \lambda_1(P_\infty)$.

Conjecture 5. *If $P \in \mathcal{D}_{m,d} \cap \mathrm{Sym}_m$ and $0 \leq r \leq s$, then $\lambda(\Psi_r(P)) \preceq \lambda(\Psi_s(P))$.*

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Moment Determinacy of Probability Distributions

Jordan Stoyanov

The goal is to present in a compact form almost all significant results about uniqueness and/or non-uniqueness of probability distributions in terms of their moments. We use the language of Modern Probability Theory and describe classical and recent results, provide illustrations, outline open questions, and give references.

1. Basics notions and notations: We deal with one of the main objects in Probability theory, *random variables* (r.v.), defined on a standard *probability space* $(\Omega, \mathcal{F}, \mathbf{P})$. A r.v. X has a range of values $\mathbb{R}$, $\mathbb{R}_+$ of a subset. Other objects are: *distribution function* (d.f.) $F(x) = \mathbf{P}[\omega : X(\omega) \leq x], x \in \mathbb{R}$; *density* $f(x) = F'(x)$, $x \in \mathbb{R}$; *distribution* $\mu = \mu_F$, a measure on $(\mathbb{R}, \mathcal{B})$, generated by F or X; *characteristic function* (ch.f.) $\psi(t) = \mathbf{E}[\mathrm{e}^{\mathrm{i}tX}], t \in \mathbb{R}$; *moment generating function* (m.g.f.) $M(t) = \mathbf{E}[\mathrm{e}^{tX}] < \infty$, $t \in (-t_0, t_0)$, $t_0 > 0$. Given a r.v. $X \sim F$, its *expectation* is: $\mathbf{E}[X] = \int_\Omega X(\omega)\mathrm{d}\mathbf{P}(\omega) = \int x\mathrm{d}F(x) = \int x\, f(x)\mathrm{d}x = \int x\,\mu(\mathrm{d}x)$. Let $\mathbf{E}[|X|^k] < \infty$, $k = 1, 2, \ldots$, then $m_k = \mathbf{E}[X^k]$ is the *moment of order k* and $\{m_k, k = 1, 2, \ldots\}$ a *moment sequence* of F and X. **Question:** Is there a d.f. $G \neq F$ with the same moments? **Answer:** Sometime 'yes', sometime 'no'.

Fact 1: Any X, F, f, μ with finite moments $\{m_k\}$ is either *M-determinate*, unique with these moments (M-det), or *M-indeterminate* (M-indet).
Fact 2: (C. Berg) For M-indet F, there are infinitely many absolutely continuous and infinitely many discrete distributions, all with the same moments as F.
Here are names of the moment problem, depending on the range of values of X:

$[0, 1]$ (**Hausdorff**); $\mathbb{R}_+ = [0, \infty)$ (**Stieltjes**); $\mathbb{R} = (-\infty, \infty)$ (**Hamburger**).

2. Example: *Log*$\mathcal{N}$ *distribution:* $Z \sim \mathcal{N}(0, 1), \varphi(x) = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-x^2/2}$, $x \in \mathbb{R}$. Then $X = \mathrm{e}^Z \sim Log\mathcal{N}(0, 1)$, $f(x) = \frac{1}{\sqrt{2\pi}}\,\frac{1}{x}\,\exp\left[-\frac{1}{2}(\ln x)^2\right], x > 0$; $f(x) = 0, x \leq 0$. About X: no m.g.f., heavy tail, however finite are all moments, $m_k = \mathbf{E}[X^k] = \mathrm{e}^{k^2/2}$, $k = 1, 2, \ldots$ Let us write two infinite sets of r.v.s called ***Stieltjes classes***, one *absolutely continuous* (Stieltjes, Heyde) and one *discrete* (Chihara, Leipnik):

X_ε, $\varepsilon \in [-1, 1]$: density $f_\varepsilon(x) = f(x)\,[1 + \varepsilon \sin(2\pi \ln x)]$, $x > 0$;

Y_a, $a > 0$: $\mathbf{P}[Y_a = a\mathrm{e}^n] = a^{-n}\,\mathrm{e}^{-n^2/2}/A$, $n = 0, \pm 1, \pm 2, \ldots$

Property: $\mathbf{E}[X_\varepsilon^k] = \mathbf{E}[Y_a^k] = \mathbf{E}[X^k] = \mathrm{e}^{k^2/2}$, $k = 1, 2, \ldots \Rightarrow Log\mathcal{N}$ is M-indet!

3. Conditions for (in)determinacy: There are classical conditions of the type 'iff', see [1], [2], [15], they are uncheckable. Thus, we focus on checkable conditions.

• ***Cramér's condition:*** r.v. $X \sim F$ on $\mathbb{R}$, the m.g.f. $M(t) = \mathbf{E}[\mathrm{e}^{tX}] < \infty$, $t \in (-t_0, t_0)$, some $t_0 > 0$ (*light tails*) ($\equiv$ to the analyticity of the ch.f. of X). Two claims: (i) X has all moments finite; (ii) X, i.e. F, is M-det.

• ***Hardy's condition*** (**"new" condition, 100 years old):** (G.H. Hardy, *Math. Messenger* **46** (1917), 175–182, and **47** (1918), 81–88.) Given is a r.v. $X > 0$, $X \sim F$. Suppose $\sqrt{X}$ has a m.g.f.: $\mathbf{E}[\mathrm{e}^{t\sqrt{X}}] < \infty$ for $t \in [0, t_0)$, $t_0 > 0$ (Hardy's condition = (H)). Then all moments $m_k = \mathbf{E}[X^k]$, $k = 1, 2, \ldots$, are finite and F is the only d.f. with the moment sequence $\{m_k\}$.

Remark 1: See [19] for details. (H) is a condition on $\sqrt{X}$, but the conclusion is for X. Moreover, $\frac{1}{2}$ is the best possible power of X in (H), in order X itself to be M-det. (H) is sufficient but not necessary for X to be M-det.

• ***Carleman's condition:*** Known are all moments $m_k = \mathbf{E}[X^k]$, $k = 1, 2, \ldots$, $\mathrm{C} = \sum_{k=1}^{\infty}(m_{2k})^{-1/2k}$ (Hamburger case), $\mathrm{C} = \sum_{k=1}^{\infty}(m_k)^{-1/2k}$ (Stieltjes case). Then $\mathrm{C} = \infty \Rightarrow F$ is M-det. Notice, $\mathrm{C}= \infty$ is only sufficient for F to be M-det.

• ***Krein's condition:*** Let $X \sim F$, finite moments, density $f > 0$. For X in $\mathbb{R}$ or $\mathbb{R}_+$, define: $\mathrm{K}[f] = \int_{-\infty}^{\infty} \frac{-\ln f(y)}{1+y^2} \mathrm{d}y$ and $\mathrm{K}[f] = \int_{a}^{\infty} \frac{-\ln f(y^2)}{1+y^2} \mathrm{d}y$ for some $a \geq 0$. Then, in both cases, $\mathrm{K}[f] < \infty \;\Rightarrow\; F$ is M-indet.

Remark 2: If $\mathrm{K}[f] = \infty$ and we add ***Lin's condition*** (recall: $f > 0$, f is smooth and $L(x) := \frac{-x\,f'(x)}{f(x)} \nearrow \infty$ as $x_0 < x \to \infty$), then F is M-det; see [7].

• ***Rate of growth of the moments:*** r.v. $X \sim F$ with unbounded support and finite $m_k = \mathbf{E}[X^k], k = 1, 2, \ldots$ Then, as $k \to \infty$, $m_k \nearrow \infty$ and $\{m_k\}$ is log-convex for $X > 0$, while $m_{2k} \nearrow \infty$ and $\{m_{2k}\}$ is log-convex for $X \in \mathbb{R}$. Define

$\Delta_k = \frac{m_{k+1}}{m_k}$ (Stieltjes case), $\Delta_k = \frac{m_{2k+2}}{m_{2k}}$ (Hamburger case).

$\{\Delta_k\}$ is strictly increasing, its unique $\lim_{k\to\infty} \Delta_k = \infty$. Suppose there is a number $\delta \geq 0$ and a slowly varying sequence ℓ_k such that $\Delta_k \approx k^{\delta}\,\ell_k$ for large k. The number δ is said to be the ***rate of growth of the moments*** of X. We have $0 \leq \delta \leq \infty$. E.g., $\delta = 0$ for $X \in [0, 1]$ or $[-1, 1]$; $\delta = \infty$ for $X \sim Log\mathcal{N}$.

Statement 1: If $\delta \leq 2$, then X is M-det; $\delta = 2$ is the best possible rate for which X is M-det.

Statement 2: If $\delta > 2$ **and** Lin's condition holds, then X is M-indet.

Remark 3: Most of the above probabilistic conditions have their counterparts in Theory of Operators; for details see [16, p. 145].

4. More results: Here we deal with functional transformations of random data.

Result 1: Suppose $X \sim F, F' = f > 0$, smooth f, satisfies Lin's condition. Then Lin's condition holds for any of the r.v.s: X^r and $\ln X$, if $X > 0$; $|X|^r$ and $\ln|X|$, if X is in $\mathbb{R}$. Under mild conditions, e^X also obeys this property; see [20].

Result 2: The product of $n \geq 2$ independent arbitrarily distributed r.v.s also satisfies Lin's condition.

Result 3: Let X satisfy Lin's condition and have rate δ. Then for any n, the power X^n and the product $X_1 \cdots X_n$ are both M-det or both are M-indet.

5. More illustrations: It is useful to see applications of the above 'tools'.

Example 1: Let $X \sim F$ where $F \in DGG(a, b, c)$, double generalized gamma distribution. Its density is $f(x) = \tilde{c}\,|x|^{a-1} \exp(-b|x|^c)$, $a, b, c > 0$. If $X \in \mathbb{R}$ (Hamburger), X is M-det for $c \geq 1$ and M-indet for $c \in (0, 1)$. If $X \in \mathbb{R}_+$ (Stieltjes), X is M-det for $c \geq \frac{1}{2}$ and M-indet for $c \in (0, \frac{1}{2})$. A few ways to do this.

Example 2: Take a r.v. $Z \sim \mathcal{N}(0, 1) : |Z|^r$ is M-det for $0 \leq r \leq 4$, and M-indet for $r > 4$. To apply twice Cramér's and twice Hardy's, is the shortest way to prove that Z^4 is M-det. Strange case: $X = Z^3$ is M-indet, but $|X|$ is M-det. Reason: $X \in \mathbb{R}$ (Hamburger) and its rate is $\delta_X = 3$, while $|X| \in \mathbb{R}_+$ (Stieltjes) has rate $\delta_{|X|} = 3/2$. We have a simple proof that the product of $n \geq 3$ normals is M-indet.

Example 3: Given $\xi \sim Exp(1)$, its density is $\mathrm{e}^{-x},\ x > 0$ (Stieltjes case). For $X = \xi^3$, $\mathbf{E}[X^k] = (3k)!$, density $f(x) = \frac{1}{3}\, x^{-2/3}\, \mathrm{e}^{-x^{1/3}},\ x > 0 \ \Rightarrow\ X$ is M-indet, by Krein. We write a Stieltjes class $\mathbf{S}(f,h) = \{f_\varepsilon = f[1+\varepsilon h], \varepsilon \in [-1,1]\}$ with $h(x) = \sin\left(\frac{\pi}{6} - \sqrt{3}x^{1/3}\right)$. In $\mathbf{S}(f,h)$, f_ε is a density $\Rightarrow$ there is a r.v. $X_\varepsilon \sim f_\varepsilon$ and for any $\varepsilon \in [-1,1]$ one holds $\mathbf{E}[X_\varepsilon^k] = \mathbf{E}[X^k] = (3k)!,\ k = 1,2,\dots$

6. Final comments: For more on these and related topics, see the references.
Conjecture: Let f be the density of $Log\mathcal{N}$, Exp^3 or $\mathcal{N}^3$. They are M-indet, we write Stieltjes classes $\{f_\varepsilon = f[1+\varepsilon h_j], \varepsilon \in [-1,1]\}$. It is known that, in each of these three cases, f is infinitely divisible. Show that $f_\varepsilon, \varepsilon \neq 0$, is not.
Open Q1: Find discrete r.v.s on $\mathbb{R}_+$ with $m_k = (3k)!,\ k = 1,2,\dots$
Open Q2: Find discrete r.v.s on $\mathbb{R}$ with $m_{2k-1} = 0, m_{2k} = (6k-1)!!, k = 1,2,\dots$
Open Q3: How to write Lin's condition for discrete distributions?
Open Q4: How to write Krein's condition in dimension 2 or more?

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The core variety and representing measures in the truncated moment problem

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(joint work with Lawrence Fialkow)

Let S be a topological space, let V be a finite dimensional vector space of continuous real-valued functions on S, and let V^* be the dual space of linear functionals on V. The main question of the truncated moment problem is:

given $L \in V^*$ does there exist a Borel measure μ on S such that $L(f) = \int_S f\, d\mu$ for all $f \in V$?

For results and application of the truncated moment problem see a recent survey by the second author [1]. Our approach is similar to a recent preprint of di Dio and Schmüdgen [2] who independently obtained some of the same results, e.g. Proposition 6. The main difference is that we also handle the case of linear functionals which do not come from a measure.

Let P be the cone of nonnegative functions on V:

$$P = \{f \in V \quad | \quad f(s) \geq 0 \ \text{ for all } \ s \in S\}.$$

We equip V with the Euclidean topology and note that P is a closed convex cone in V. For a subset $T \subseteq V$ of V we will denote by $\mathcal{Z}(T) \subset S$ the set of common zeroes of functions in T:

$$\mathcal{Z}(T) = \{s \in S \ | \ f(s) = 0 \ \text{ for all } \ f \in T\}.$$

We will assume that $\mathcal{Z}(V) = \emptyset$. Notice that this does not restrict us in generality since otherwise we can simply replace S with $S \setminus \mathcal{Z}(V)$.

Let M be the cone of functionals in V^* which have a representing measure

$$M = \{L \in V^* \mid \text{there exists Borel measure } \mu \text{ s.t. } L(f) = \int_S f\, d\mu \, \text{for all} \, f \in V\}.$$

We observe that M is a convex cone in V^* but M may fail to be closed. For a point $s \in S$ let $L_s \in V^*$ denote the point evaluation functional of S on V: $L_s(f) = f(s)$. Let C be the cone of all functionals coming from finite atomic measures on S:

$$C = \text{ConicalHull}\{L_s \ | \ s \in S\}.$$

It is clear that C is a convex cone in V^* and $C \subseteq M$. Note that by Caratheodory's theorem the support of a finite atomic measure may be chosen such that it consists of at most $\dim V$ atoms, or in other words we use at most $\dim V$ point evaluations.

We first show the following Lemma:

Lemma 1. *The closure $\bar{C}$ of C is equal to the dual cone of P:*

$$\bar{C} = P^*.$$

Proof. We observe that the dual cone C^* by definition is

$$C^* = \{f \in V \quad | \quad L_s(f) = f(s) \geq 0 \ \text{ for all } \ s \in S\} = P.$$

By bipolarity theorem we have $\bar{C} = (C^*)^* = P^*$. □

From the above Lemma it follows that the interiors on C and M agree and they consist of strictly positive functionals on P:

$$\operatorname{int}(C) = \operatorname{int}(M) = \{L \in V^* \mid L(f) > 0 \text{ for all } f \in P\}.$$

In particular any linear functional strictly positive on P has a finite atomic representing measure.

For a linear functional $L \in V^*$ we define the following iterative construction: Let $S = S_0$ and let S_1 be the zero set of all nonnegative functions in the kernel of L:

$$S_1 = \mathcal{Z}(p \in P \quad | \quad L(p) = 0).$$

Then we iteratively define:

$$S_{i+1} = \mathcal{Z}(p \in V \quad | \quad L(p) = 0 \text{ and } p \text{ is nonnegative on } S_i).$$

We note that if $S_1 = S$ then it follows from Lemma 1 that either L or $-L$ has a representing measure. We now make the following crucial observation:

Lemma 2. *If there exists $g \in V$ such that $L(g) = 0$ and g vanishes identically on S_1 then there exists a function $f \in V$ strictly positive on S_1 such that $L(f) = 0$ and $S_2 = \emptyset$.*

Proof. Since S_1 is nonempty we know that L does not vanish on all of P and there a strictly positive $p \in P$ such that $L(p) \neq 0$. We will consider the case $L(p) > 0$, the proof of the other case is identical. We may assume without loss of generality that $L(g) < 0$, otherwise we can consider $-g$. Then there exists a positive $\lambda \in \mathbb{R}$ such that $L(p + \lambda g) = 0$ and $p + \lambda g$ is strictly positive on S_1. □

From Lemma 2 we see that either $S_2 = \emptyset$ or the kernel of L contains subspace $U_1 \subset V$ of all functions vanishing on S_1. Therefore, if S_2 is non-empty we can treat L as a functional on the quotient vector space $V_1 = V/U_1$, which is the vector space of functions in V restricted to S_1.

Inside V_1 we may similarly define the cone P_1 of functions nonnegative on S_1 and then S_2 becomes

$$S_2 = \mathcal{Z}(p \in P_1 \quad | \quad L(p) = 0).$$

Therefore we may repeat the above argument and similarly define vector spaces U_i and V_i. Since we have $\dim V_{i+1} \leq \dim V_i - 1$ and $V = V_0$ is finite dimensional we see that this procedure terminates and for some $k \in \mathbb{N}$ we have either $S_k = S_{k+1}$ or $S_k = \emptyset$. We call the terminal set S_k the *core variety* of L and denote it by $\mathcal{CV}(L)$.

If the core variety is non-empty, then at the step $S_k = S_{k+1}$ we see that L lies in the interior of the cone P_K^*. From Lemma 1 it follows that L comes from a measure and moreover S_k is equal to the union of all supports of finite atomic measures representing L.

Therefore we have shown the following Theorem:

Theorem 3. *For any $L \in V^*$ exactly one of the following holds:*

(1) The core variety of L is empty and neither L nor $-L$ has a representing measure.

(2) The core variety of L is non-empty and then either L or $-L$ has a representing measure and the core variety of L equals the union of all supports of finite atomic measure representing L or $-L$.

As a Corollary of the second case we obtain the following Theorem, usually referred to as Bayer-Teichmann Theorem:

Corollary 4 (Bayer-Teichmann Theorem)**.** *Suppose that $L \in V^*$ has a representing measure. Then L has a finite atomic representing measure.*

We also show that core varieties give facial decomposition the cone M of linear functionals with a representing measure:

Proposition 5. *Let $L \in M$. Let F_L be the set of all linear functionals in M whose core variety is contained in the core variety of L:*

$$F_L = \{m \in M \ \mid \ \mathcal{CV}(m) \subseteq \mathcal{CV}(L)\}.$$

Then F_L is a face of M. Moreover $m \in \operatorname{relint} F_L$ in and only if $\mathcal{CV}(m) = \mathcal{CV}(L)$.

Proof. Suppose that there exist $\ell_1, \ell_2 \in M$ such that $\ell_1 + \ell_2 \in F_L$ but at least one of the functionals ℓ_i doesn't lie in F_L. We may assume that $\ell_1 \notin F_L$. Then by Theorem 3 ℓ_1 has a finite atomic representing measure whose support is not contained in $\mathcal{CV}(L)$. Since ℓ_2 also has a finite atomic representing measure we see that $\ell_1 + \ell_2$ has a finite atomic representing measure whose support is not contained in $\mathcal{CV}(L)$. This is a contradiction by Theorem 3 part (2) since $\ell_1 + \ell_2 \in F_L$. It follows that F_L is a face of M.

Now suppose that $\mathcal{CV}(m) = \mathcal{CV}(L)$. Then we know that F_m is also a face of M and $F_m = F_L$. It follows that $m \in \operatorname{relint} F_L$. Finally, suppose that $m \in \operatorname{relint} F_L$. Then F_L is the minimal face of M containing m. Since F_m also contains m it follows that $F_L \subseteq F_m$, but by definition we also have $F_m \subseteq F_L$ and the Proposition follows. □

There is a geometric termination criterion for the termination of iterative construction of the core variety for functionals in M:

Proposition 6. *Let $L \in V^*$ be a linear functional with a representing measure. Then $\mathcal{CV}(L) = S_1 = \mathcal{Z}(p \in P \ \mid L(p) = 0)$ if and only if F_L is an exposed face of M.*

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Some open questions on the truncated K-moment problem

Lawrence Fialkow

Let K denote a closed subset of $\mathbb{R}^n$, and for $m > 0$ let $\beta \equiv \beta^{(m)} := \{\beta_i\}_{i\in\mathbb{Z}_+^n, |i|\le m}$, $\beta_0 > 0$, denote an n-dimensional real multisequence of degree m. The *Truncated K-Moment Problem* (TKMP) asks for conditions on β which insure that there exists a positive Borel measure μ, supported in K, such that $\beta_i = \int_K x^i d\mu$ $(|i| \le m)$ (where $x = (x_1, \ldots, x_n)$, $i = (i_1, \ldots, i_n)$, $|i| = i_1 + \cdots + i_n$, and $x^i = x_1^{i_1} \cdots x_n^{i_n}$); μ is a *K-representing measure* for β. The *Truncated Moment Problem* (TMP) is the special case of TKMP with $K = \mathbb{R}^n$; μ is then called a *representing measure* for β. We will discuss some open questions related to two distinct but interrelated solutions to TKMP, one involving flat extensions of positive moment matrices, the other involving positive extensions of Riesz functionals.

1. Questions concerning the Flat Extension Theorem

Let $M_d \equiv M_d(\beta)$ denote the *moment matrix* associated with $\beta \equiv \beta^{(2d)}$. The rows and columns of M_d are denoted by X^i and are indexed (in degree-lexicographic order) by the monomials x^i in $\mathcal{P}_d \equiv \{p \in \mathbb{R}[x_1, \ldots, x_n] : \ deg\ p \le d\}$; the entry in row X^i, column X^j is β_{i+j}. Corresponding to

$$p \equiv \sum_{i\in\mathbb{Z}_+^n, |i|\le d} a_i x^i \in \mathcal{P}_d$$

is the element $p(X) \equiv \sum a_i X^i$ of $Col\ M_d$, the column space of M_d. Let $V \equiv V(M_d)$ denote the *algebraic variety* corresponding to M_d, i.e.,

$$V = \bigcap_{p\in\mathcal{P}_d, p(X)=0} \mathcal{Z}_p,$$

where $\mathcal{Z}_p = \{x \in \mathbb{R}^n : p(x) = 0\}$. In the sequel, we set $r := rank\ M_d$ and $\nu := card\ V(M_d)$. Consider the following solution to TMP, expressed in terms of flat extensions of positive moment matrices; for a TKMP analogue, see [3].

Theorem 1.1. *(Flat Extension Theorem* [3],[4]*) $\beta^{(2d)}$ has a representing measure if and only if there is an integer $k \ge 0$ such that M_d admits a positive moment matrix extension M_{d+k} which in turn admits a flat extension M_{d+k+1}, i.e., rank $M_{d+k+1} =$ rank M_{d+k}. In this case, M_{d+k+1} has a unique representing measure μ, with supp $\mu = V(M_{n+k+1})$ and card supp $\mu =$ rank M_{d+k+1}.*

The advantage of this result is that when a flat extension M_{d+k+1} can be found, Theorem 1.1 permits the explicit calculation of a finitely atomic representing measure for β; but there are difficulties in applying Theorem 1.1.

Problem 1.2. *Given β, find a good estimate for the smallest value of k that would be required to achieve a flat extension M_{d+k+1}.*

A theorem of Bayer and Teichmann [1], which generalizes a theorem of Tchakaloff [9] for the case when K is compact, implies that if $\beta \equiv \beta^{(m)}$ has a K-representing measure, then it has a finitely atomic K-representing measure with at most $dim\ \mathcal{P}_m$ atoms. This estimate is based on Caratheodory's Theorm, and an example of Tchakaloff shows that there are cases of K in which $dim\ \mathcal{P}_m$ is actually the minimal number of atoms in any K-representing measure. As noted in [4], the result of [1] can be used to show that in Theorem 1.1, we may always take

$$k \leq \Delta_1 := dim\ \mathcal{P}_{2d} - rank\ M_d, \tag{1}$$

but as we will see shortly, in general this is not a good estimate. Consider a sequence of positive moment matrix extensions leading to a flat extension,

$$M_d \to M_{d+1} \to \cdots \to M_{d+k} \to M_{d+k+1}, \tag{2}$$

where $k \geq 0$, $rank\ M_{d+i} < rank\ M_{d+i+1}$ if $k > 0$ and $0 \leq i \leq k-1$, and $rank\ M_{d+k} = rank\ M_{d+k+1}$; we refer to (2) as a *convergent extension sequence.* For each i, let $r_{d+i} = rank\ M_{d+i}$ and $\nu_{d+i} = card\ V(M_{d+i})$. Since $V(M_{d+i+1}) \subseteq V(M_{d+i})$, Theorem 1.1 implies $\nu \geq \nu_{d+k+1} = r_{d+k+1} = r_{d+k} \geq r + k$, so

$$k \leq \Delta_2 := \nu - r. \tag{3}$$

The estimate in (3) is usually much better than that in (1); examples in [6] illustrate a case where $n = 2$, $d = 5$, $\Delta_1 = 47$, $\Delta_2 = 6$, and a case where $n = 3$, $d = 4$, $\Delta_1 = 138$, $\Delta_2 = 7$. Although Δ_2 is generally an improvement over Δ_1, Δ_2 is itself not a definitive estimate. For example, in [6] it is proved that there exist flat extensions of M_d, i.e., $k = 0$, in cases where Δ_2 is arbitrarily large. On the other hand, the length of a convergent extension sequence can be arbitrarily large. Indeed, let G denote a $d \times d$ rectangular grid in the plane, let μ be a measure satisfying $supp\ \mu = G$, and let $M_d \equiv M_d([\mu])$. We have $V(M_d) = G$ and $rank\ M_d = \frac{(d+1)(d+2)}{2} - 2$, so $\Delta_2 = \frac{(d-1)(d-2)}{2}$. In this case, [5] shows that the unique convergent extension sequence for M_d has $k = d - 2$. In this example, as the rank steadily increases between M_d and M_{2d-2}, the variety is unchanged at each stage of the extension sequence. We have never observed the variety drop in size more than twice in a convergent extension sequence.

Question 1.3. *In a convergent extension sequence, at most how many times can we have $\nu_{i+1} < \nu_i$?*

One of the goals of *multivariable cubature theory* is to discover representing measures with the fewest atoms; classical Gaussian quadrature on the real line illustrates this theory. In general, for any representing measure μ for $\beta^{(2d)}$, we have $card\ supp\ \mu \geq rank\ M_d$, so flat extensions M_{d+1} give rise to representing measures with the fewest possible atoms. In cases where there is no flat extension M_{d+1}, we may have have convergent extension sequences of varying lengths.

Question 1.4. *Does the shortest convergence sequence for $\beta^{(2d)}$ always correspond to a representing measure with the fewest atoms?*

2. Problems concerning positive Riesz functionals

For $\beta \equiv \beta^{(m)}$, let $L_\beta : \mathcal{P}_m \mapsto \mathbb{R}$ denote the *Riesz functional*, defined by $L_\beta(\sum a_i x^i) = \sum a_i \beta_i$. Note that for $m = 2d$, $L_\beta(fg) = \langle M_d \widehat{f}, \widehat{g} \rangle$ $(f, g \in \mathcal{P}_d)$. A classical theorem of M. Riesz ($n = 1$) and E.K. Haviland ($n > 1$) shows that $\beta \equiv \beta^{(\infty)}$ has a K-representing measure in the *Full K-Moment Problem* if and only if the corresponding functional $L_\beta : \mathbb{R}[x_1, \ldots, x_n] \mapsto \mathbb{R}$ is *K-positive*, i.e., if $p|K \geq 0$ implies $L_\beta(p) \geq 0$. (For $K = \mathbb{R}^n$, we say that L_β is *positive.*) K-positivity is clearly a necessary condition for the existence of a K-representing measure for $\beta^{(m)}$ in TKMP, and the proof of Tchakaloff's Theorem shows that it is also sufficient if K is compact. For $n = 1$, $d = 2$, and $K = \mathbb{R}$, positivity of L_β is equivalent to positive semidefiniteness of M_2, but this is not always sufficient for the existence of a measure. A result of [4] shows that for $m = 2d$ or $m = 2d+1$, $\beta \equiv \beta^{(m)}$ has a K-representing measure if and only if L_β admits a K-positive extension $L_{\tilde{\beta}^{(2d+2)}}$. In view of this result, the following problem is fundamental.

Problem 2.1. *Find concrete conditions on $\beta^{(m)}$ which insure that L_β is positive.*

Let $\beta \equiv \beta^{(2d)}$. If M_d is positive semidefinite and *flat*, i.e., *rank* $M_d =$ *rank* M_{d-1}, then $\beta^{(2d)}$ has a measure by Theorem 1.1, so clearly L_β is positive. Let $\mathcal{F} \equiv \mathcal{F}_{n,d}$ denote the set of flat positive n-dimensional moment matrices of degree $2d$. Since the cone of positive Riesz functionals is closed, if $M_d \in \overline{\mathcal{F}}$ (the closure of $\mathcal{F}$), then L_β is positive. It follows from lower-semicontinuity of rank that if $M_d(\beta) \in \overline{\mathcal{F}}$, then *rank* $M_d \leq$ *dim* P_{d-1}, and membership in $\overline{\mathcal{F}}$ clearly also implies $M_d \succeq 0$. In [8] we proved that the preceding two concrete conditions imply membership in $\overline{\mathcal{F}}$ in the cases covered by Hilbert's theorem on sums of squares, i.e., $n = 1$, $d = 1$, $n = d = 2$ (though the proof is independent of Hilbert's Theorem). In [7] we proved the same result for $n = 2$, $d = 3$. In the latter case, positivity of L_β is also a special case of the following remarkable result of G. Blekherman [2].

Theorem 2.2. *For $n \geq 1$, $d \geq 3$, if rank $M_d \leq 3d - 3$, then L_β is positive.*

Moreover, Blekherman's results in [2] imply that the results of [8] and [7] cannot be extended beyond the cases noted above.

Problem 2.3. *Give a concrete characterization of $\overline{\mathcal{F}}$.*

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Robust control, structured uncertainty, and Linear Matrix Inequalities

Joseph A Ball

Introduction. One feature of the current hot area of Free Noncommutative Analysis (see [8]) is to extend functions f defined on a domain in $\mathbb{C}^d$ to a function defined on larger noncommutative domain consisting of d-tuples of freely noncommuting square matrices or even Hilbert-space operators. When this is done, some results in the commuting-variable domain (e.g., Inverse and Implicit Function Theorem, Oka Approximation Theorem, the theory of Reproducing Kernel Hilbert Spaces and Nevanlinna-Pick interpolation theory—see [2, 1, 5, 3, 6]) have a strikingly simpler form in the free noncommutative domain. We offer another such example coming from Robust Control Theory (see [7]).

1. The standard H^∞-control problem. Consider the discrete-time stationary linear input/state/output (i/s/o) linear system

$$G\colon \begin{cases} x(n+1) &= Ax(n) + Bw(n), \quad x(0) = x_0 \\ z(n) &= Cx(n) + Dw(n) \end{cases}$$

induced by the system matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{W} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Z} \end{bmatrix}$$

where we take all of the spaces $\mathcal{X}$ (the **state space**), $\mathcal{W}$ (the **input space**), and $\mathcal{Z}$ (the **output space**) to be finite-dimensional linear spaces. We say that the system in **internally stable** if

$$w(n) = 0 \text{ for all } n \geq 0,\ x(0) \text{ arbitrary } \Rightarrow x(n) \to 0 \text{ as } n \to \infty. \tag{1}$$

and has (normalized strict) **performance** if

$$x(0) = 0,\ \mathbf{w} \in \ell^2_{\mathcal{W}} \Rightarrow \mathbf{z} \in \ell^2_{\mathcal{Z}} \text{ with } \|\mathbf{z}\|_{\ell^2} \leq \rho \|\mathbf{w}\|_{\ell^2 \mathcal{W}} \text{ for some } \rho < 1$$

where $\mathbf{w} = \{w(n)\}_{n \geq 0}$ and $\mathbf{z} = \{z(n)\}_{n \geq 0}$ are the input- and output-signal trajectories. The **Analysis Problem** is to characterize via conditions intrinsic to the matrices A, B, C, D when internal stability and performance occurs. The solution to the Analysis Problem is straightforward: *internal stability occurs if and only if* $\sigma(A) \subset \mathbb{D}$ (A has spectrum or eigenvalues inside the unit disk), *also characterized by an LMI condition:* $\exists X \succ 0$ so that $A^* X A - X \prec 0$. Performance is characterized by an LMI condition:

$$\exists X \succ 0 \text{ with } \begin{bmatrix} A & B \\ C & D \end{bmatrix}^* \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \prec 0. \tag{2}$$

For the design problem we suppose that we have an augmented plant of the form

$$G_a = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & C_2 & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{W} \\ \mathcal{U} \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{W} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Z} \\ \mathcal{Y} \end{bmatrix}$$

inducing a system of the form

$$\begin{bmatrix} x(n+1) \\ z(n) \\ y(n) \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} x(n) \\ w(n) \\ u(n) \end{bmatrix} \tag{3}$$

where $x(n)$ is the **state**, $w(n)$ is the **disturbance**, $u(n)$ is the **control**, $z(n)$ is the **disturbance**, $y(n)$ is the **measurement** at time n, with ambient spaces $\mathcal{X}$, $\mathcal{W}$, $\mathcal{U}$, $\mathcal{Z}$, $\mathcal{Y}$ all finite-dimensional. The **Design Problem** is to solve for a control system matrix

$$\begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} : \begin{bmatrix} \mathcal{X}_K \\ \mathcal{Y} \end{bmatrix} \to \begin{bmatrix} \mathcal{X}_K \\ \mathcal{U} \end{bmatrix}$$

so that the closed-loop system given by the coupling of the augmented plant system (3) with the control system

$$\begin{bmatrix} x_K(n+1) \\ u(n) \end{bmatrix} = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} \begin{bmatrix} x_K(n) \\ y(n) \end{bmatrix} \tag{4}$$

is first of all **well-posed**, by which is meant that one can eliminate $u(n)$ and $y(n)$ from the coupling of the systems (3) and (4) (as well as from the coupling of sufficiently nearby systems) to arrive at the equivalent closed-loop system

$$\begin{bmatrix} \begin{bmatrix} x(n+1) \\ x_K(n+1) \end{bmatrix} \\ z(n) \end{bmatrix} = \begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} x(n) \\ x_K(n) \end{bmatrix} \\ w(n) \end{bmatrix}. \tag{5}$$

It turns out that well-posedness holds exactly when $\overline{D} = \left[\begin{smallmatrix} I & -D_K \\ -D_{22} & I \end{smallmatrix}\right]$ is invertible and then the matrices A_{cl}, B_{cl}, C_{cl}, D_{cl} are given explicitly by

$$A_K = \left[\begin{smallmatrix} A & 0 \\ 0 & A_K \end{smallmatrix}\right] + \left[\begin{smallmatrix} B_2 & 0 \\ 0 & B_K \end{smallmatrix}\right] \overline{D}^{-1} \left[\begin{smallmatrix} 0 & C_K \\ C_2 & 0 \end{smallmatrix}\right], \quad B_{cl} = \left[\begin{smallmatrix} B_1 \\ 0 \end{smallmatrix}\right] + \left[\begin{smallmatrix} B_2 & 0 \\ 0 & B_K \end{smallmatrix}\right] \overline{D}^{-1} \left[\begin{smallmatrix} 0 \\ D_{21} \end{smallmatrix}\right],$$

$$C_{cl} = \left[\begin{smallmatrix} C_1 & 0 \end{smallmatrix}\right] + \left[\begin{smallmatrix} D_{12} & 0 \end{smallmatrix}\right] \overline{D}^{-1} \left[\begin{smallmatrix} 0 & C_K \\ C_2 & 0 \end{smallmatrix}\right], \quad D_{cl} = D_{11} + \left[\begin{smallmatrix} D_{12} & 0 \end{smallmatrix}\right] \overline{D}^{-1} \left[\begin{smallmatrix} 0 \\ D_{21} \end{smallmatrix}\right].$$

Note that these formulas are affine in the unknown system matrices A_K, B_K, C_K, D_K for the controller K to be designed (with the exception of the term $\overline{D}_K^{-1}$, but this can be taken care of by normalizing D_{22} to be 0). The complete **Design Problem** is to find K which generates a well-defined closed-loop system $\left[\begin{smallmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{smallmatrix}\right]$ which is internally stable and has performance. One can then apply the criteria for internally stability and performance mentioned in Section 1 above, namely: *there should exist a* $X_{cl} = \left[\begin{smallmatrix} X & X_{12} \\ X_{21} & X_2 \end{smallmatrix}\right] \succ 0$ *on* $\mathcal{X} \oplus \mathcal{X}_K$ *so that* $\left[\begin{smallmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{smallmatrix}\right]^* \left[\begin{smallmatrix} X & 0 \\ 0 & I \end{smallmatrix}\right] \left[\begin{smallmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{smallmatrix}\right] - \left[\begin{smallmatrix} X_{cl} & 0 \\ 0 & I \end{smallmatrix}\right] \prec 0$. The difficulty is that the resulting LMI still involves A_K, B_K, C_K, D_K, unknown quantities we wish to solve for. It is possible to find necessary and sufficient conditions for such A_K, B_K, C_K, D_K to exist (see the result of Skelton-Iwasaki-Grigoridis as recounted in [7]). The result is the following. Given a matrix Γ, we use the notation $\Gamma_\perp$ for any matrix whose columns form a basis for any subspace complementary to the kernel of Γ. Also for convenience we take $\mathcal{X} = \mathcal{X}_K$.

Theorem 1. Given an augmented system matrix as in (3), the standard H^∞ problem has a solution if and only if there exist $X \succ 0$ and $Y \succ 0$ so that

$$\begin{bmatrix} ([\, B_2^* \; D_{12}^* \,]_\perp)^* & 0 \\ 0 \; 0 & I \end{bmatrix} \begin{bmatrix} AYA^*-Y & AYC_1^* & B_1 \\ C_1YA^* & C_1YC_1^*-I & D_{11} \\ B_1^* & D_{11}^* & -I \end{bmatrix} \begin{bmatrix} [\, B_2^* \; D_{12}^* \,]_\perp & \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \\ 0 & I \end{bmatrix} \prec 0, \tag{6}$$

$$\begin{bmatrix} ([\, C_2 \; D_{21} \,]_\perp)^* & 0 \\ 0 \; 0 & I \end{bmatrix} \begin{bmatrix} A^*XA-X & A^*XB_1 & C_1^* \\ B_1^*XA & B_1^*XB_1-I & D_{11}^* \\ C_1 & D_{11} & -I \end{bmatrix} \begin{bmatrix} [\, C_2 \; D_{21} \,]_\perp & \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \\ 0 & I \end{bmatrix} \prec 0, \tag{7}$$

$$\left[\begin{smallmatrix} X & I \\ I & Y \end{smallmatrix}\right] \succeq 0. \tag{8}$$

2. Robust control with respect to structured uncertainty. To introduce the more general problem, we need the notion of **structured singular value** (see [7]. We let $Q_{\boldsymbol{\Delta}}$ be a subspace of $M \times N$ matrices parametrized by a matrix pencil $Q_{\boldsymbol{\Delta}}(z) = \sum_{s=1}^d Q_{\boldsymbol{\Delta},s} z_s$ where the free parameter $z = (z_1, \dots, z_d) \in \mathbb{C}^d$. Our main example is to parametrize the set $\boldsymbol{\Delta} = \{\mathrm{diag}_{1 \le r \le k}[Z^{(r)\otimes I_{n_r}}] \colon Z^{(r)} = [z_{ij}^{(r)}]$ where $z = (z_{ij}^{(r)}$ where $1 \le r \le k$, $1 \le i \le \ell_r$, $1 \le j \le m_r$ so $z \in \mathbb{C}^d$ where $d = \sum_{r=1}^k \ell_r \cdot m_r\}$, and $Q_{\boldsymbol{\Delta}}(z) = \mathrm{diag}_{1 \le r \le k}[Z^{(r)\otimes I_{n_r}}]$ as a function of the variable $z = \{z_{ij}^{(k)}\} \in \mathbb{C}^d$. The matrix M (of an appropriate size) is said to be $\boldsymbol{\Delta}$-robustly stable if $I - Q_{\boldsymbol{\Delta}}(z)M$ is invertible for all z with $\|Q_{\boldsymbol{\Delta}}(z)\| \le 1$. There is a notion of **μ-singular value** $M \mapsto \mu_{\boldsymbol{\Delta}}(M)$ so that this happens exactly one $\mu_{\boldsymbol{\Delta}}(M) < 1$. The criterion of performance (2) can be expressed in terms of μ-singular values: $\mu_{\boldsymbol{\Delta}_{\mathrm{scalar}} \oplus \boldsymbol{\Delta}_{\mathrm{full}}}([\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}]) < 1$. The problem of **robust control** is to design a controller K with closed-loop stability and performance not only for the nominal plant but for all perturbations of the nominal plant caused by structured uncertainties (e.g., variations of the plant parameters in certain blocks of the entries rather than in all of the entries). This engineering problem leads to the general math problem: *given* G_a *as in* (3)*, design* K *as in* (4) *so that there is a resulting well-posed closed-loop system* (5) *such that* $\mu_{\boldsymbol{\Delta}}([\begin{smallmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{smallmatrix}]) < 1$. It turns that there is a LMI condition for this problem of the same form and flavor as in (6)–(8) which is sufficient but in general not necessary for a solution of the problem to exist.

The **free relaxation** of this problem is as follows. Rather than $\left[\begin{smallmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{smallmatrix}\right]$ we look at $\left[\begin{smallmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{smallmatrix}\right] \otimes I_{\ell^2}$ as an operator from $\left[\begin{smallmatrix} \mathcal{X} \oplus \mathcal{X}_K \\ \mathcal{W} \end{smallmatrix}\right] \otimes \ell^2$ to $\left[\begin{smallmatrix} \mathcal{X} \oplus \mathcal{X}_K \\ \mathcal{Z} \end{smallmatrix}\right] \otimes \ell^2$ and we change the free independent scalar variables $z_{ij}^{(k)}$ to freely noncommutative operator variables $Z_{ij}^{(k)} \in \mathcal{L}(\ell^2)$. Fortunately this relaxation still has a control interpretation, namely: robustness of the design with respect to structured time-varying uncertainty (see [7]). It turns out that the modified LMI conditions analogous to (6)–(8) mentioned above are necessary and sufficient for a solution of this problem to exist—see [7, 4] for complete details.

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Reciprocal linear spaces and their Chow forms

Cynthia Vinzant
(joint work with Mario Kummer)

Real polynomials that are hyperbolic or stable have been studied by for over 50 years in the context of differential equations, optimization, and combinatorics. Stable polynomials have especially strong connections with combinatorics through the theory of matroids. Recently Shamovich and Vinnikov generalized the notion of hyperbolicity to general varieties. Extending this definition to a notion of stability, we again find that stability comes with rich combinatorial structure and explore this structure in the case of reciprocal linear spaces.

An irreducible real variety $X \subseteq \mathbb{P}^{n-1}(\mathbb{C})$ of dimension $d-1$ is called **hyperbolic** with respect to a linear space L of codimension d if L does not intersect X and for every real linear space $L' \supset L$ of codimension $d-1$, all the intersection points of L' with X are real.

To generalize this notion to *stability*, we need to define an "orthant" of linear spaces L. Let $k < n \in \mathbb{Z}_+$. Given a linear space $L \in \mathrm{Gr}(k,n)$, let $\sigma(L)$ be the length-$\binom{n}{k}$ vector of the sign patterns of the Plücker coordinates of L. For any sign pattern $\sigma = (\sigma_I : I \in \binom{[n]}{k}) \in \{\pm 1\}^{\binom{n}{k}}$, let $\mathrm{Gr}(k,n)^\sigma$ denote the open subset of the Grassmannian $\mathrm{Gr}(k,n)$ of k-dimensional linear spaces $L \subset \mathbb{R}^n$ for which $\sigma(L) = \pm\sigma$. We say that the variety X is σ-**stable** if it is hyperbolic with respect to every linear space $L \in \mathrm{Gr}(n-d,n)^\sigma$.

Given a linear space $\mathcal{L} \in \mathrm{Gr}(d,n)$ we define its reciprocal linear space to be

$$\mathcal{L}^{-1} = \overline{\left\{(x_1^{-1},\ldots,x_n^{-1}) \ : \ x \in \mathcal{L}\right\}}.$$

Reciprocal linear spaces have been studied in the contexts of matroids [4], interior points methods for linear programming [3, 5] and entropy maximization for log-linear models [6], among others. Varchenko [8] proved that a reciprocal linear space is hyperbolic with respect to the orthogonal linear space $\mathcal{L}^{\perp}$. In fact, this variety is σ-stable, where $\sigma = \sigma(\mathcal{L}^{\perp})$.

The stability of $\mathcal{L}^{-1}$ is certified by a determinantal representation of its *Chow form*, which is a polynomial in $\mathbb{C}[\mathrm{Gr}(n-d,n)]$ describing when a linear space $\mathcal{M} \in \mathrm{Gr}(n-d,n)$ has non-trivial intersection with $\mathcal{L}^{-1}$. A classical example of a stable hypersurface is the variety of a determinant $\det(\sum_i x_i A_i)$ where the matrices A_i are positive semidefinite, real symmetric matrices. As developed in [7], one way to generalize this example to stable varieties is to give positive semidefinite matrices $\{A_I : I \in \binom{[n]}{d}\}$ so that a linear space $\mathcal{M} \in \mathrm{Gr}(n-d,d)$ has non-trivial intersection with $\mathcal{L}^{-1}$ if and only if the determinant of the matrix $\sum_I p_I(\mathcal{M}^{\perp})A_I$ vanishes.

1. A determinantal representation

We construct the matrices A_I as follows. For $d < n \in \mathbb{Z}_+$, define the following $\binom{n}{d}$ vectors $\{v_I : I \in \binom{[n]}{d}\}$ in $\mathbb{R}^{\binom{n-1}{d-1}}$. We index the basis vectors of $\mathbb{R}^{\binom{n-1}{d-1}}$ by size $d-1$ subsets of $[n-1]$. For $I = \{i_1, \ldots, i_d\} \in \binom{[n]}{d}$ with $i_1 < \ldots < i_d$, define the vector $v_I \in \mathbb{R}\{e_K : K \in \binom{[n-1]}{d-1}\}$ to be

$$v_I = \begin{cases} e_{I\setminus\{n\}} & \text{if } \ n \in I \\ \sum_{k=1}^{d}(-1)^k e_{I\setminus\{i_k\}} & \text{if } \ n \notin I. \end{cases}$$

Theorem 1. *Let $\mathcal{L} \in \mathrm{Gr}(d,n)$ with all Plücker coordinates nonzero. If $\mathcal{M} \in \mathrm{Gr}(n-d,n)$ intersects $\mathcal{L}^{-1}$ non-trivially, then the determinant of the real symmetric matrix*

$$\sum_{I\in\binom{[n]}{d}} p_I(\mathcal{M}^{\perp}) \cdot p_I(\mathcal{L})^{-1} \cdot v_I v_I^T \tag{1}$$

is zero. This gives a symmetric Livšic-type determinantal representation of $\mathcal{L}^{-1}$ in the sense of [7] *that is definite at every linear space $\mathcal{M}$ in $\mathrm{Gr}(n-d,n)^{\sigma(\mathcal{L}^{\perp})}$.*

In the case $d = 2$, the matrix (1) represents the Laplacian of a graph. For $d > 2$, this matrix is the generalized Laplacian discussed in [1]. The vectors v_I are the columns of a boundary operator of a simplicial complex, namely the complete simplicial complex K_n^{d-1} of dimension $d-1$ on n vertices. The terms in the expansion of the determinant (1) correspond to spanning forests of K_n^{d-1} (see [1, Def. 3]).

Corollary 2. *Expanding the determinant* (1) *and multiplying by $\prod_I p_I(\mathcal{L})$ yields*

$$\sum_{\substack{F \text{ is a spanning} \\ \text{forest of } K_n^{d-1}}} c_F \cdot \prod_{I\in F} p_I(\mathcal{M}^{\perp}) \cdot \prod_{I\notin F} p_I(\mathcal{L}),$$

for some positive integers $c_F \in \mathbb{Z}_+$.

These expansions are particularly interesting because they are multiaffine in the Plücker coordinates $p_I(\mathcal{M}^\perp)$ (*i.e.* every monomial is square-free). Multiaffine stable polynomials have special combinatorial structure. Choe, Oxley, Sokal, and Wagner show that the support $\{I : c_I \neq 0\}$ of a homogeneous multiaffine stable polynomial $f = \sum_I c_I \prod_{i\in I} x_i$ forms the bases of a matroid [2]. Corollary 2 suggests that the support of the Chow form of a stable variety will have similar combinatorial structure, especially when it is multiaffine.

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Moment Problems and PSD symmetric functions

Bruce Reznick

In the early 1980s, I wrote [2], a paper about inequalities satisfied by quotients of products of power-sums. By early March 2017, this paper had amassed a total of zero citations on MathSciNet, so I felt I could justifiably talk about its contents at this Oberwolfach Workshop. (The paper is freely available from Project Euclid: https://projecteuclid.org/euclid.pjm/1102723674.)

For a positive integer r and $x \in \mathbb{R}^n$, define the r-th power sum:

$$M_r(x) = M_{r,n}(x) = \sum_{j=1}^{n} x_j^r.$$

There are some standard inequalities for products of power sums, such as the Hölder and Jensen inequalities. Using elementary methods, [2] considers cases which are not covered by these; in particular, by determining the maximum and minimum (as a function of n) of these three homogeneous symmetric rational functions $f_i = f_{i,n}$:

$$f_1(x) = \frac{M_1(x)M_3(x)}{M_4(x)}; \qquad f_2(x) = \frac{M_1(x)M_3(x)}{M_2^2(x)}; \qquad f_3(x) = \frac{M_1^3(x)M_3(x)}{M_2^3(x)}.$$

Each of these achieves its extrema at points $x \in \mathbb{R}^n$ with at most two different coordinates: $x = (1, \dots, 1, t, \dots, t)$, with $n-k$ 1's and k t's.

At the time, Choi, Lam and I were working on writing psd symmetric forms as a sum of squares (see e.g. [1]) and these seemed like a potentially interesting source of examples, although it didn't work out back then. In view of the title of this workshop, I wanted to revisit the subject, especially since $\inf f_1$ can be more easily analyzed via the Moment Problem.

Here are the relevant answers:

$$\max f_1(x) = n, \quad \min f_1(x) = -\alpha_n n, \quad \text{where} \quad \alpha_n < \frac{1}{8},\ \alpha_n \to \frac{1}{8};$$

$$\max f_2(x) = \frac{3\sqrt{3}}{16} n^{1/2} + \frac{5}{8} + \mathcal{O}(n^{-1/2}),$$
$$\min f_2(x) = -\frac{3\sqrt{3}}{16} n^{1/2} + \frac{5}{8} + \mathcal{O}(n^{-1/2});$$
$$\max f_3(x) = \frac{(\sqrt{n-1}+1)^4}{8\sqrt{n-1}}, \qquad \min f_3(x) = -\frac{(\sqrt{n-1}-1)^4}{8\sqrt{n-1}}.$$

In the case of f_1, the convergence of (α_n) is not monotone:

$$\alpha_{15} \approx .124999536, \qquad \alpha_{16} \approx .124905705.$$

The reason for all of this, in some sense, is that the global minimum of $\frac{M_1 M_3}{n M_4}$ occurs when $\frac{n-k}{k} = 7 + 4\sqrt{3} \approx 13.93$, which is of course irrational. (The maximum is easily derived from the usual inequalities.) As $n \to \infty$, there exist k_n so that $\frac{n-k_n}{k_n} \to 7 + 4\sqrt{3}$, and this implies that $\inf \frac{M_1(x)M_3(x)}{nM_4(x)} = -\frac{1}{8}$.

From the moment theory point of view, note that

$$\frac{M_1(x)M_3(x)}{nM_4(x)} = \frac{(\int_{-\infty}^{\infty} t\, d\mu)(\int_{-\infty}^{\infty} t^3\, d\mu)}{(\int_{-\infty}^{\infty} d\mu)(\int_{-\infty}^{\infty} t^4\, d\mu)},$$

where μ is the measure with unit point masses at $t = x_1, \dots, x_n$. As is well known, a special case of the Hamburger moment problem says that, on writing

$$a_j = \int_{-\infty}^{\infty} t^j\, d\mu,$$

for any non-negative measure, the resulting Hankel matrix

$$H := \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix}$$

is positive semidefinite, and for any choice of a_j's making H psd, there exists a nonnegative measure satisfying

$$a_j = \int_{-\infty}^{\infty} t^j\, d\mu, \quad 0 \le j \le 3; \qquad a_4 \ge \int_{-\infty}^{\infty} t^4\, d\mu.$$

The computation of the lower bound becomes a nice undergraduate optimization problem: minimize $\frac{a_1 a_3}{a_0 a_4}$ given that $\det H \geq 0$. One finds that the extremal measure gives $-\frac{1}{8}$ for this minimum, and the measure has two atoms whose masses have ratio of $7 + 4\sqrt{3}$, at positions with a ratio of $-(2 + \sqrt{3})$. (See [2, p.462] for details.)

One application is that the symmetric quartic form $8M_1M_3 + nM_4$ is positive definite for all n. It's obviously then sos for $n = 2, 3$. What about larger n? Thirty years ago, this seemed beyond the reach of hand-computation. Here is a hand-computed sos representation for this symmetric quartic for all n, which I should have been able to find in 1981!

$$n^3(8M_1M_3 + nM_4) = \sum_{j=1}^{n}(-8M_1^2 + 4nx_jM_1 + n^2x_j^2)^2.$$

This representation can also be used to show that $8M_1M_3 + nM_4$ is a strictly positive definite n-ary quartic for all n.

Another new result was announced (but not proved) in the talk: if $a, b \in \mathbb{N}$ and a is odd, then

$$\frac{M_1^a M_2^{2b} M_3^a}{n^{a+b} M_4^{a+b}} = -\frac{1}{2^a} \cdot \frac{a^a(a+2b)^{a+2b}}{(2a+2b)^{2a+2b}};$$

note that $a = 1, b = 0$ recovers the bound $-\frac{1}{8}$.

Neither f_2 nor f_3 seems susceptible to the moment method, because the largest index r occurs in the numerator, not the denominator. But calculus shows that the extreme values of f_2 and f_3 must occur at a point with at most two different coordinates, and cubic equations arise (see [2] for details.) In particular, the extreme values for f_2 occur when $k = 1$ and $t = 1 + 2\sqrt{n}\cos\theta$, where $\cos 3\theta = n^{-1/2}$. There are three such values of $\cos\theta$: the one with $\theta \approx \frac{\pi}{6}$ gives the maximum, and the one with $\theta \approx \frac{5\pi}{6}$ gives the minimum. For f_3, the cubic has a linear factor and $t = \pm\sqrt{n-1}$ at the extrema.

I can report no progress in computing λ_n so that $M_2^2 + \lambda_n M_1 M_3$ is sos. On the other hand, a new result is that $\inf \frac{M_1(x)M_5(x)}{nM_6(x)}$ seems to be $-\frac{1}{4}$.

As I said at the Workshop, the standard American cultural approach to reviving [2] would involve a "reboot" with perhaps a more popular cast. I'm in talks to have Arnold Schwarzenegger make a special guest appearance as the Nullstellensatz.

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Real free loci of linear matrix pencils

Jurij Volčič
(joint work with Igor Klep)

Let $A_1,\dots,A_g \in \mathrm{M}_d(\mathbb{C})$. The formal affine linear combination $L = I - \sum_j A_j x_j$, where x_j are freely noncommuting variables, is called a **(monic) linear pencil** of size d. If all A_j are hermitian matrices, then L is a **hermitian** pencil. Linear pencils appear in various areas, from matrix theory and real algebraic geometry to convex optimization and control theory. In the spirit of free real algebraic geometry and free analysis, the evaluation of L at a point $X = (X_1,\dots,X_g) \in \mathrm{M}_n(\mathbb{C})^g$ is defined using the (Kronecker) tensor product

$$L(X) = I \otimes I - \sum_{i=1}^{g} A_i \otimes X_i \in \mathrm{M}_{nd}(\mathbb{C}),$$

giving rise to the **free (singular) locus**,

$$\mathscr{Z}(L) = \bigcup_{n\in\mathbb{N}} \mathscr{Z}_n(L), \qquad \text{where} \quad \mathscr{Z}_n(L) = \{X \in \mathrm{M}_n(\mathbb{C})^g \colon \det(L(X)) = 0\}.$$

Clearly, each $\mathscr{Z}_n(L)$ is an algebraic subset of $\mathrm{M}_n(\mathbb{C})^g$. If L is a hermitian pencil, $\mathscr{Z}_n(L)$ is closed under conjugate transposition and thus has a natural real structure. In this case we also consider the real points of $\mathscr{Z}_n(L)$, namely the set of tuples of hermitian matrices in $\mathscr{Z}_n(L)$, denoted $\mathscr{Z}_n^{\mathrm{h}}(L)$. The set

$$\mathscr{Z}^{\mathrm{h}}(L) = \bigcup_{n\in\mathbb{N}} \mathscr{Z}_n^{\mathrm{h}}(L)$$

is called the **free real locus** of L. Zariski closures of boundaries of free spectrahedra [2], singularity sets of noncommutative rational functions [3, 4], and "free real algebraic hypersurfaces" are examples of free real loci.

In [6, Theorems 3.6 and 5.4] we solved the inclusion problem for free real loci in terms of algebras generated by the coefficients of the corresponding pencils. For $L = I - \sum_i A_i x_i$ and $L' = I - \sum_i A_i' x_i$ of sizes d and d', respectively, let $\mathcal{A} \subseteq \mathrm{M}_d(\mathbb{C})$ and $\mathcal{A}' \subseteq \mathrm{M}_{d'}(\mathbb{C})$ be the $\mathbb{C}$-algebras generated by $A_1,\dots,A_g$ and $A_1',\dots,A_g'$, respectively. Let $\operatorname{rad}\mathcal{A}$ denote the Jacobson radical of $\mathcal{A}$.

Theorem 1. *Let L and $\widetilde{L}$ be as above. Then $\mathscr{Z}(L) \subseteq \mathscr{Z}(\widetilde{L})$ if and only if there exists a homomorphism $\widetilde{\mathcal{A}}/\operatorname{rad}\widetilde{\mathcal{A}} \to \mathcal{A}/\operatorname{rad}\mathcal{A}$ induced by $\widetilde{A}_i \mapsto A_i$.*

If L and $\widetilde{L}$ are hermitian, then $\mathscr{Z}^{\mathrm{h}}(L) \subseteq \mathscr{Z}^{\mathrm{h}}(\widetilde{L})$ if and only if there exists a $*$-homomorphism $\widetilde{\mathcal{A}} \to \mathcal{A}$ induced by $\widetilde{A}_i \to A_i$.

The proof proceeds in three steps. First we show that the Jacobson radical of the coefficient algebra is irrelevant for the free locus using the theory of trace identities on $n \times n$ matrices. Then we use an algebraization trick to relate the multiplicative structure of the coefficient algebra with points in the free locus. Finally, the second part of Theorem 1 follows by the properties of hyperbolic polynomials, i.e., the real variety $\mathscr{Z}_n(L)$ is Zariski dense in $\mathscr{Z}_n(L)$. In particular, we see that for a hermitian pencils, the inclusion of free loci $\mathscr{Z}(L) \subseteq \mathscr{Z}(\widetilde{L})$ is equivalent to the inclusion of free real loci $\mathscr{Z}^{\mathrm{h}}(L) \subseteq \mathscr{Z}^{\mathrm{h}}(\widetilde{L})$.

Theorem 1 presents the foundation for a more precise analysis of free loci. If the coefficients of L generate $\mathrm{M}_d(\mathbb{C})$, then we say that L is an **irreducible pencil**. If L and L' are irreducible pencils and $\mathscr{Z}(L) \subseteq \mathscr{Z}(L')$, then $\mathscr{Z}(L) = \mathscr{Z}(L')$. In this case L' and L are similar and moreover unitarily similar if they are hermitian [6, Theorem 3.11 and Corollary 5.5]. A free locus is **irreducible** if it is not a union of smaller free loci. By [6, Proposition 3.12], a free locus is irreducible if and only if it is a free locus of some irreducible pencil.

By applying Burnside's theorem on existence of subspaces to the coefficient algebra of the pencil it follows that every monic pencil L is similar to a pencil of the form

$$\begin{pmatrix} L_1 & \star & \cdots & \star \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & L_m & \star \\ 0 & \cdots & 0 & I \end{pmatrix}, \tag{1}$$

where L_k are irreducible pencils.

From the definition it does not follow that an irreducible free locus restricts to an irreducible hypersurface in $\mathrm{M}_n^g(\mathbb{C})$. Indeed, let

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then one can check that A_1, A_2 generate the full algebra of 3×3 matrices and

$$\det(I - \xi_1 A_1 - \xi_2 A_2) = (1 - \xi_1 + \xi_2)(1 - \xi_1 - \xi_2).$$

Hence $L = I - A_1 x_1 - A_2 x_2$ is an irreducible pencil but the surface $\mathscr{Z}_1(L)$ is not irreducible. However, in a forthcoming paper [5] we show that if $\mathscr{Z}(L)$ is an irreducible locus, then $\mathscr{Z}_n(L)$ is irreducible in $\mathrm{M}_n^g(\mathbb{C})$ for large enough n. Moreover, we prove that the determinant of the pencil is irreducible in the following sense. For $n \in \mathbb{N}$ let $\Xi^{(n)} = (\Xi_1^{(n)}, \dots, \Xi_g^{(n)})$ be the tuple of $n \times n$ generic matrices, i.e., matrices whose entries are independent commuting variables.

Theorem 2. *If L is an irreducible pencil, then there exists $n_0 \in \mathbb{N}$ such that $\det L(\Xi^{(n)})$ is an irreducible polynomial for all $n \geq n_0$.*

Let us say a few words about the proof. We apply the first fundamental theorem for the action of $\mathrm{GL}_n(\mathbb{C})$ on $\mathrm{M}_n^g(\mathbb{C})$ with simultaneous conjugation and the algebraization trick to establish the following: if $\det L'(\Xi^{(n)})$ is irreducible for all $n \geq n_1$, where $L' = I - \sum_j A_j x_j - A_{j'} A_{j''} x_{g+1}$, then $\det L(\Xi^{(n)})$ is irreducible for all $n \geq 2n_1$. Theorem 2 is then proved by induction on generation of $d \times d$ matrices by the coefficients of L and using the fact that the determinant of a generic matrix is irreducible. While the bound on n_0 constructed through the proof is exponential with respect to the size of L, one might believe that there exists a linear bound on n_0.

As an application of Theorem 2 we can inspect smooth points on the boundary of a free spectrahedron (LMI domain). Let L be a hermitian monic pencil of size d and let $\mathcal{D}_n(L)$ be the set of tuples of $n \times n$ hermitian matrices X making $L(X)$ positive semidefinite. The set

$$\mathcal{D}(L) = \bigcup_{n\in\mathbb{N}} \mathcal{D}_n(L)$$

is the **free spectrahedron** of L. Also denote

$$\partial\mathcal{D}(L) = \bigcup_{n\in\mathbb{N}} \partial\mathcal{D}_n(L), \qquad \partial\mathcal{D}_n(L) = \mathcal{D}_n(L) \cap \mathscr{Z}_n^{\mathrm{h}}(L),$$
$$\partial^1\mathcal{D}(L) = \bigcup_{n\in\mathbb{N}} \partial^1\mathcal{D}_n(L), \qquad \partial^1\mathcal{D}_n(L) = \{X \in \partial\mathcal{D}_n(L)\colon\ \dim\ker L(X) = 1\}.$$

Hence $\partial\mathcal{D}(L)$ is the boundary of the free spectrahedron $\mathcal{D}(L)$ and it is easy to see that if $\partial^1\mathcal{D}_n(L) \neq \emptyset$, then $\partial^1\mathcal{D}_n(L)$ are precisely the smooth points of $\partial\mathcal{D}_n(L)$. However, it is not a priori clear that $\partial^1\mathcal{D}(L)$ is nonempty and this question is related to a matrix theory problem known as Kippenhahn's conjecture.

A hermitian monic pencil L is **LMI-minimal** if it is of minimal size among all hermitian pencils L' satisfying $\mathcal{D}(L') = \mathcal{D}(L)$. Note that if L and L' are irreducible hermitian pencils, then $\mathscr{Z}(L) = \mathscr{Z}(L')$ implies $\mathcal{D}(L) = \mathcal{D}(L')$. Using Burnside's theorem and the hermitian structure of L we see that L is unitarily similar to $L_1 \oplus \cdots \oplus L_m$, where L_i are pairwise non-similar irreducible hermitian pencils.

For LMI-minimal hermitian pencils we can prove the following density result on smooth points.

Corollary 3. *Let L be a LMI-minimal hermitian pencil. Then there exists $n_0 \in \mathbb{N}$ such that $\partial^1\mathcal{D}_n(L)$ is Zariski dense in $\mathscr{Z}_n(L)$ for all $n \geq n_0$.*

The proof of Corollary 3 applies Theorem 2 and properties of hyperbolic polynomials. The density of $\partial^1\mathcal{D}(L)$ in $\mathcal{D}(L)$ is important in the study of free analytic maps between free spectrahedra [1].

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Truncated Moment Problem: Set of Atoms and Carathéodory Numbers

Konrad Schmüdgen
(joint work with Philipp J. di Dio)

The aim of this exposé is to summarize some results presented at the Oberwolfach meeting *Real Algebraic Geometry With a View Toward Moment Problems and Optimization*, March 2017. These results have been obtained in joint work with P. dD. Theorems 1–2 are from [1], while Theorems 3–6 are proved in [2].

Suppose that $\mathcal{X}$ is locally compact topological Hausdorff space, $\mathcal{A}$ is a **finite-dimensional** real linear subspace of $C(\mathcal{X};\mathbb{R})$ and $\mathsf{A} = \{a_1, \ldots, a_m\}$ is a vector space basis of $\mathcal{A}$. The truncated moment problem is the following question:

Let $s = (s_j)_{j=1}^m$ be a real sequence. Does there exist a positive Radon measure on $\mathcal{X}$ such that

$$s_j = \int_{\mathcal{X}} a_j(x)\, d\mu \quad \text{for} \quad j = 1, \ldots, m? \tag{1}$$

Let L_s be the linear functional on $\mathcal{A}$ defined by $L_s(a_j) = s_j, j = 1, \ldots, m$. It is clear that (1) is satisfied if and only if $L_s(a) = \int_{\mathcal{X}} a(x)\, d\mu(x)$ for all $a \in \mathcal{A}$.

In the affirmative case, we say that s is a *moment sequence*, L_s is a *moment functional* and μ is a representing measure of s resp. L_s.

In the case, where $\mathsf{A} = \{x^\alpha : \alpha \in \mathbb{N}_0^n,\ |\alpha| \le d\}$ and $\mathcal{X} \subseteq \mathbb{R}^n$, the vector space $\mathcal{A}$ is formed by the polynomials in n variables of degree at most d and the preceding gives the "usual" truncated $\mathcal{X}$-moment problem for polynomials, see e.g. [4], [6].

From now on **we suppose that $L \neq 0$ is a moment functional on $\mathcal{A}$**.

Let $\mathcal{M}_L$ denote the set of representing measures of L. Then the set of atoms of all representing measures of L is

$$\mathcal{W}(L) := \{x \in \mathcal{X} : \mu(\{x\}) > 0 \text{ for some } \mu \in \mathcal{M}_L\}.$$

For $x \in \mathcal{X}$ let $s_{\mathsf{A}}(x)$ denote the vector $s_{\mathsf{A}}(x) := (a_1(x), \ldots, a_m(x))^T \in \mathbb{R}^m$.

Theorem 1. *A moment functional L has a unique representing measure if and only if the set $\{s_{\mathsf{A}}(x) : x \in \mathcal{W}(L)\}$ is linearly independent.*

Another important notion is the core variety $V(L)$ which was invented by L. Fialkow [5]. Define inductively $N_k(L)$, $k \in \mathbb{N}$, and $V_j(L)$, $j \in \mathbb{N}_0$, by $V_0(L) = \mathcal{X}$,

$$N_k(L) := \{p \in \mathcal{A} : L(p) = 0,\ p(x) \ge 0 \text{ for } x \in V_{k-1}(L)\},$$
$$V_j(L) := \{t \in \mathcal{X} : p(t) = 0 \text{ for } p \in N_j(L)\}.$$

Then the *core variety* $V(L)$ of L is defined by $V(L) := \bigcap_{k=0}^\infty V_k(L)$.

Theorem 2. *Each representing measure of L is supported on $V(L)$ and we have $\mathcal{W}(L) = V(L)$.*

The set $\mathcal{S}$ of all moment sequences is a cone in $\mathbb{R}^m$ such that $\mathbb{R}^m = \mathcal{S} - \mathcal{S}$. By the Richter–Tchakaloff Theorem [7] each moment sequence and moment functional has a k-atomic representing measure, where $k \le m$.

The *Carathéodory number* $\mathcal{C}_{\mathsf{A}}(s)$ of $s \in \mathcal{S}$ is the smallest $k \in \mathbb{N}$ such that s has a k-atomic representing measure. The *Carathéodory number* of the moment cone is the number $\mathcal{C}_{\mathsf{A}} := \max\{\mathcal{C}_{\mathsf{A}}(s) : s \in \mathcal{S}\}$. The *signed Carathéodory number* $\mathcal{C}_{\mathsf{A},\pm}$ is the smallest $n \in \mathbb{N}$ such that each $s \in \mathbb{R}^m$ has a signed k-atomic representing measure with $k \leq n$.

Theorem 3. *Suppose that* $\mathcal{A} \subset C(\mathcal{X}, \mathbb{R})$ *contains a function* e *s.t.* $e(x) > 0$ *for all* $x \in \mathcal{X}$. *If* $\dim \mathcal{A} \geq 2$ *and* $\mathcal{X}$ *has at most* $\dim \mathcal{A} - 1$ *path-connected components, then* $\mathcal{C}_{\mathsf{A}} \leq \dim \mathcal{A} - 1$.

From now on suppose that $\mathsf{A} \subseteq C^1(\mathbb{R}^d; \mathbb{R})$. Let $C = (c_1, ..., c_k)$, $X = (x_1, ..., x_k)$, where $c_j > 0$ and $x_j \in \mathbb{R}^d$. Define

$$(C, X) := \mu_{(C,X)} := \sum_{j=1}^{k} c_j \delta_{x_j},$$

$$S_{k,\mathsf{A}} : (\mathbb{R}_{\geq 0})^k \times \mathbb{R}^k \to \mathbb{R}^m, \ (C, X) \mapsto S_{k,\mathsf{A}}(C, X) := \sum_{i=1}^{k} c_i \cdot s_{\mathsf{A}}(x_i).$$

Here δ_x denotes the delta measure at the point $x \in \mathcal{X}$. We denote by $DS_{k,\mathsf{A}}$ the total derivative of $S_{k,\mathsf{A}}$. Another important number is defined by

$$\mathcal{N}_A := \min\{k \in \mathbb{N} \,|\, \operatorname{rank} DS_{k,\mathsf{A}} = m\}.$$

Thus, $\mathcal{N}_A$ is the smallest number of atoms such that $DS_{k,\mathsf{A}}$ has full rank $m = |\mathsf{A}|$. It is not difficult to show that $\left\lceil \frac{|\mathsf{A}|}{n+1} \right\rceil \leq \mathcal{N}_A$.

Theorem 4. *Suppose that* $\mathsf{A} \subset C^r(\mathbb{R}^n; \mathbb{R})$ *and* $r > \mathcal{N}_{\mathsf{A}} \cdot (n+1) - m$. *Then*

$$\mathcal{N}_{\mathsf{A}} \leq \mathcal{C}_{\mathsf{A}} \quad \textit{and} \quad \mathcal{N}_{\mathsf{A}} \leq \mathcal{C}_{\mathsf{A},\pm} \leq 2\mathcal{N}_{\mathsf{A}}.$$

Now we specialize to polynomials and set

$$\mathsf{A}_{n,d} := \{x^\alpha : \alpha \in \mathbb{N}_0^n, \ |\alpha| \leq d\}, \quad \mathsf{B}_{n,d} := \{x^\alpha : \alpha \in \mathbb{N}_0^n, \ |\alpha| = d\}.$$

For polynomials of $\mathsf{A}_{n,d}$ we consider the moment problem on $\mathcal{X} = \mathbb{R}^n$, while for homogeneous polynomials of $\mathsf{B}_{n,d}$ the moment problem is treated on the real projective space $\mathcal{X} = \mathbb{P}(\mathbb{R}^{n-1})$.

The Carathéodory numbers of $\mathsf{A}_{2,2k-1}$ have been studied in [8]. A classical result on this matter is Möller's lower bound $\text{Mö}(2, 2k-1) := \binom{k+1}{2} + \lfloor \frac{k}{2} \rfloor$. The following theorem improves this *lower bound*.

Theorem 5.

$$\text{Mö}(2, 2k-1) \leq \left\lceil \frac{|\mathsf{A}_{2,2k-1}|}{3} \right\rceil \leq \mathcal{C}_{\mathsf{A}_{2,2k-1}} \quad \text{for} \quad k \in \mathbb{N},$$

$$\left\lceil \frac{|\mathsf{A}_{2,2k-1}|}{3} \right\rceil - \text{Mö}(2, 2k-1) \geq \frac{(k-2)^2 - 4}{6} \quad \textit{for} \quad k \geq 4.$$

Now we turn to *upper bounds* for Caratheodory numbers.

Let $d \in \mathbb{N}$. Then $\mathcal{B}_{3,2d} := \text{Lin } \mathsf{B}_{3,2d}$ are the homogeneous polynomials in 3 variables of degree $2d$. For $f \in \mathcal{B}_{3,2d}$, let $\mathcal{Z}_{\mathbb{P}}(f)$ be the set of zeros of f in $\mathbb{P}(\mathbb{R}^2)$.

Let $\beta(2d)$ denote the maximum of numbers $|\mathcal{Z}_{\mathbb{P}}(f)|$, where $f \in \mathcal{B}_{3,2d}$, $\mathcal{Z}_{\mathbb{P}}(f)$ is finite and $f \geq 0$ on $\mathbb{R}^3$. By a classical result of Choi, Lam, and Reznick [3], we have $\beta(2d) \leq \alpha(2d) := \frac{3}{2}d(d-1)+1$, $d \in \mathbb{N}$. We abbreviate $\mathcal{C}_{2d} := \mathcal{C}_{\mathsf{B}_{3,2d}}$.

Theorem 6.

$$\mathcal{C}_{2d} \leq \max_{k=0,\dots,d} \left\{ \binom{2d+2}{2} - \binom{2d+2-k}{2} + \beta(2(d-k)) \right\} + 1 \quad \text{for } d \in \mathbb{N},$$

$$\mathcal{C}_{2d} \leq \frac{3}{2}d(d-1) + 2 = \alpha(2d) + 1 \quad \textit{for} \quad d \geq 5.$$

Theorem 7. $\mathcal{C}_{B_{4,4}} \leq 16$.

Example 8. $(d, n) = (3, 5)$: *W.R. Harris (1999) has discovered a polynomial $h \in \mathcal{B}_{3,10}$ such that $\mathcal{Z}_{\mathbb{P}}(h) = 30$ and $h \geq 0$ on $\mathbb{R}^3$. Then*

$$\mathcal{N}_{\mathsf{B}_{3,10}} = 15 < 30 \leq \mathcal{C}_{\mathsf{B}_{3,10}} \leq \alpha(10) + 1 \leq 32.$$

The above results and the corresponding proof indicate that a link between Carathéodory numbers $\mathcal{C}_{2d}$ and the maximal number $\beta(2d)$ of projective zeros of polynomials $f \in \mathcal{B}_{3,2d}$ for which $\mathcal{Z}_{\mathbb{P}}(f)$ is finite and $f \geq 0$ on $\mathbb{R}^3$.

We close this research exposé by formulating the following

Conjecture: $\beta(2d) \leq \mathcal{C}_{2d} \leq \beta(2d) + 1$ *for* $d \geq 3$.

and the

Open Problem: *Let $f \in \mathcal{B}_{n,2d}$ be such that $f \geq 0$ on $\mathbb{R}^n$ and $\mathcal{Z}_{\mathbb{P}}(f) = \{z_1, \dots, z_k\}$. Is the set $\{s_{\mathsf{B}_{n,2d}}(z_i) : i = 1, \dots, k\}$ linearly independent?*

The latter is true for a number of classical polynomials (Motzkin, Robinson, Harris, Choi–Lam–Reznick).

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Moment problems on $\mathbb{T}^2$

Greg Knese

Given $c_j \in \mathbb{C}$ for $j = -N, \dots, N$, there exists a measure μ on the unit circle $\mathbb{T}$ such that

$$c_j = \int_{\mathbb{T}} z^j \, d\mu$$

if and only if the matrix

$$T = (c_{j-k})_{j,k=0,\dots,N}$$

is positive semidefinite. This is the solution to the classical truncated trigonometric moment problem. A slightly modified version asks for $\#\mathrm{supp}(\mu) \geq N+1$ and this requires T above to be positive definite. In this situation we can solve the problem with a specific type of measure called a Bernstein-Szegő measure. These are measures of the form

$$d\mu_p = \frac{1}{|p|^2} d\sigma$$

where σ is normalized Lebesgue measure on $\mathbb{T}$ and $p \in \mathbb{C}[z]$ has degree at most N and no zeros in the closed unit disk $\overline{\mathbb{D}}$. Such measures have an entropy maximizing property that makes them natural. See [6] for more on this.

One could view this as a characterization of the moments of Bernstein-Szegő measures and Geronimo and Woerdeman were able to generalize this characterization to the 2-torus $\mathbb{T}^2$.

Their theorem requires some setup. Let $c_j \in \mathbb{C}$ where $j = (j_1, j_2) \in \mathbb{Z}^2$ and $(|j_1|, |j_2|) \leq N = (N_1, N_2)$. Assume

$$T = (c_{j-k})_{0 \leq j,k \leq N}$$

is positive definite. Note we are using the componentwise partial order on tuples. We can then define an inner product on

$$\mathcal{P}_N = \{q \in \mathbb{C}[z_1, z_2] : \deg q \leq N\}$$

via

$$\langle P, Q \rangle = \sum P_j \bar{Q}_k c_{j-k}$$

where $P = \sum_{0 \leq j \leq N} P_j z^j$, $Q = \sum_{0 \leq j \leq N} Q_j z^j$. Note we are using multi-index notation and $\deg q$ refers to the bidegree of q.

Theorem (Geronimo and Woerdeman [1]**)** Given the above data there exists $p \in \mathcal{P}_N$ with no zeros in the closed bidisk $\overline{\mathbb{D}}^2$ such that

$$c_j = \int_{\mathbb{T}^2} z^j \frac{1}{|p|^2} d\sigma$$

if and only if

$$(\mathcal{P}_{N-(1,0)} \ominus \mathcal{P}_{N-(1,1)}) \perp (\mathcal{P}_{N-(0,1)} \ominus \mathcal{P}_{N-(1,1)})$$

using the inner product $\langle \cdot, \cdot \rangle$ defined above.

Problem: Generalize this theorem to three or more variables.

This theorem has several interesting applications (bivariate Fejér-Riesz factorizations, autoregressive filters). Of particular interest to this workshop is that this result and its proof have applications to determinantal representations for stable polynomials. Thus, the result can be turned around and used to study bivariate polynomials with no zeros on $\overline{\mathbb{D}}^2$. However, polynomials merely with no zeros on $\mathbb{D}^2$ are also of interest.

In the paper [5], we generalized the Geronimo-Woerdeman theorem to the setting of Bernstein-Szegő measures where p is allowed to have zeros on $\mathbb{T}^2$. This requires a rethinking of the problem because now moments are no longer finite and one must instead work with the ideal:

$$I_p := \{q \in \mathbb{C}[z_1, z_2] : q/p \in L^2(\mathbb{T}^2)\}$$

and its truncations $I_p \cap \mathcal{P}_N$. In the paper [4], we constructed explicit generators of the ideal I_p and computed the dimension of $I_p \cap \mathcal{P}_N$ in terms of certain intersection multiplicities. However the ideal I_p is still mysterious.

Problem: Give a local characterization of the ideal I_p.

The stability requirement of the Geronimo-Woerdeman theorem can be relaxed in a different way. In the paper [2], we were able to relax the theorem to $p \in \mathbb{C}[z_1, z_2]$ with no zeros on $\mathbb{T} \times \overline{\mathbb{D}}$ and this produced a corresponding (and more complicated) condition on the moments of associated Bernstein-Szegő measures. This stability requirement looks unnatural but actually translates into a hyperbolicity condition if one employs a Cayley transform. This can be leveraged to give a proof of the (weakened) self-adjoint version of the Helton-Vinnikov determinantal representation (see [3]).

Problem: Is it possible to generalize the Geronimo-Woerdeman theorem to p with no zeros on $\mathbb{T}^2$?

Problem: Do these constrained moment problems on $\mathbb{T}^2$ help us say anything about the general truncated trigonometric moment problem on $\mathbb{T}^2$?

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Moments of random discrete measures

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(joint work with Yuri Kondratiev, Tobias Kuna)

Let X be a locally compact Polish space (e.g. $X = \mathbb{R}^d$). Let $\mathcal{B}(X)$ denote the associated Borel σ-algebra. A measure σ on X is called a *Radon measure* if $\sigma(\Lambda) < \infty$ for each compact $\Lambda \subset X$. Let $\mathbb{M}(X)$ denote the space of all Radon measures. The space $\mathbb{M}(X)$ is equipped with the vague topology. Let $\mathcal{B}(\mathbb{M}(X))$ denote the Borel σ-algebra on $\mathbb{M}(X)$.

A *random measure on* X is a measurable mapping $\xi : \Omega \to \mathbb{M}(X)$, where $(\Omega, \mathcal{F}, P)$ is a probability space. Often we think of a random measure as a probability measure on $\mathbb{M}(X)$.

An important characteristic of a random measure is its moment sequence. We say that a random measure ξ has finite moments of all orders if, for each $n \in \mathbb{N}$ and all compact $A \subset X$,

$$\mathbb{E}[\xi(A)^n] < \infty.$$

Then, the *n-th moment measure of* ξ is the unique symmetric measure $M^{(n)} \in \mathbb{M}(X^n)$ defined by the following relation

$$\forall A_1, \dots, A_n \in \mathcal{B}(X) : \qquad M^{(n)}(A_1 \times \cdots \times A_n) := \mathbb{E}[\xi(A_1) \cdots \xi(A_n)].$$

Then $(M^{(n)})_{n=1}^{\infty}$ is called the *moment sequence of the random measure* ξ.

We may consider an important subclass of the class of random measures. The *cone of discrete Radon measures on* X is defined by

$$\mathbb{K}(X) := \left\{ \eta = \sum_i s_i \delta_{x_i} \in \mathbb{M}(X) \,\middle|\, s_i > 0,\, x_i \in X \right\}.$$

A random measure taking values in $\mathbb{K}(X)$ is called a *random discrete measure.* Note that, for many important example of random discrete measure, with probability one, the set of atoms, $\{x_i\}$, is *dense in* X.

A random measure ξ is called *completely random* if, for any mutually disjoint sets $A_1, \dots, A_n \in \mathcal{B}_0(X)$, the random variables $\xi(A_1), \dots, \xi(A_n)$ are independent. Kingman's theorem [1] states that every completely random measure ξ can be represented as $\xi = \xi_d + \xi_f + \xi_r$. Here ξ_d, ξ_f, ξ_r are independent completely random measures such that: ξ_d is a deterministic measure on X without atoms; ξ_f is a random measure with fixed (non-random) atoms, i.e., there exists a deterministic countable collection of points $\{x_i\}$ in X and non-negative independent random variables $\{s_i\}$ with $\xi_f = \sum_i s_i \delta_{x_i}$; finally the most essential part ξ_r is an extended marked Poisson process which has no fixed atoms:

$$\mathbb{E}\left[e^{\langle \xi, f \rangle}\right] = \exp\left[\int_{X \times (0,\infty)} (e^{sf(x)} - 1)\, d\nu(x,s)\right], \quad f \in C_0(X).$$

In particular, ξ_r is a random discrete measure. Thus, a completely random measure is a random discrete measure up to a non-random component.

If one drops the assumption that the random measure is completely random, one cannot expect anymore to concretely characterize the distribution of ξ. Thus, a natural questions to ask is: Is a given random measure a random discrete measure? In this talk, we will solve this problem by looking at the moments of a random measure. The talk is based on paper [2]. Our approach is significantly influenced by the paper by Rota and Wallstrom [3].

A *partition of a nonempty set* Z is any finite collection $\pi = \{A_1, \dots, A_k\}$, where $A_1, \dots, A_k$ are mutually disjoint nonempty subsets of Z such that $Z = \bigcup_{i=1}^{k} A_i$. The sets $A_1, \dots, A_k$ are called blocks of the partition π. We denote by $\Pi(n)$ the collection of all partitions of the set $\{1, 2, \dots, n\}$. For each partition $\pi = \{A_1, \dots, A_k\} \in \Pi(n)$, we denote by $X_\pi^{(n)}$ the subset of X^n which consists of all $(x_1, \dots, x_n) \in X^n$ such that $x_i = x_j$ if and only if i and j belong to the same block of the partition π. For example, for the so-called zero partition $\widehat{0} = \big\{\{1\}, \{2\}, \dots, \{n\}\big\}$, the set $X_{\widehat{0}}^{(n)}$ consists of all points $(x_1, \dots, x_n) \in X^n$ whose coordinates are all different. For the so-called one partition $\widehat{1} = \big\{\{1, 2, \dots, n\}\big\}$, the set $X_{\widehat{1}}^{(n)}$ consists of all points $(x_1, \dots, x_n) \in X^n$ such that $x_1 = x_2 = \cdots = x_n$. Clearly, the collection of sets $X_\pi^{(n)}$ with π running over $\Pi(n)$ forms a partition of X^n.

Let $m^{(n)}$ be any Radon measure on X^n. For each partition $\pi \in \Pi(n)$, we denote by $m_\pi^{(n)}$ the restriction of the measure $m^{(n)}$ to the set $X_\pi^{(n)}$. Note that we may also consider $m_\pi^{(n)}$ as a measure on X^n by setting $m_\pi^{(n)}(X^n \setminus X_\pi^{(n)}) := 0$. Then we get

$$m^{(n)} = \sum_{\pi \in \Pi(n)} m_\pi^{(n)}.$$

Let us fix a partition $\pi = \{A_1, A_2, \dots, A_k\} \in \Pi(n)$. We assume that the blocks of the partition are enumerated so that $\min A_1 < \min A_2 < \cdots < \min A_k$. We construct a measurable, bijective mapping $B_\pi : X_\pi^{(n)} \to X_{\widehat{0}}^{(k)}$ as follows. For any $(x_1, \dots, x_n) \in X_\pi^{(n)}$, we set $B_\pi(x_1, \dots, x_n) = (y_1, \dots, y_k)$, where, for $i = 1, 2, \dots, k$, $y_i = x_j$ for a $j \in A_i$ (recall that $x_j = x_{j'}$ for all $j, j' \in A_i$). Note that, if $\pi = \widehat{0}$, then B_π is just the identity mapping. We denote by $B_\pi(m_\pi^{(n)})$ the push-forward of the measure $m_\pi^{(n)}$ under B_π.

Let us now additionally assume that the initial measure $m^{(n)}$ is symmetric, i.e., the measure $m^{(n)}$ remains invariant under the natural action of permutations $\sigma \in \mathfrak{S}_n$ on X^n. For a partition π as in the above paragraph, we set $i_l := |A_l|$, the number of elements of the block A_l. Note that $i_1 + i_2 + \cdots + i_k = n$. Since $m^{(n)}$ is symmetric, it is clear that the measure $B_\pi(m_\pi^{(n)})$ is completely identified by the numbers $i_1, \dots, i_k$. Hence, we will denote $m_{i_1, \dots, i_k} := B_\pi(m_\pi^{(n)})$.

Thus, a given sequence of symmetric Radon measures $m^{(n)}$ on X^n, $n \in \mathbb{N}$, uniquely identifies a sequence of Radon measures $m_{i_1, \dots, i_k}$ on $X_{\widehat{0}}^{(k)}$, where $i_1, \ldots, i_k \in \mathbb{N}$, $k \in \mathbb{N}$. Note that this sequence is symmetric in the entries $i_1, \ldots, i_k$, i.e., for any permutation $\sigma \in \mathfrak{S}_k$,

$$dm_{i_{\sigma(1)},\ldots,i_{\sigma(k)}}(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) = dm_{i_1,\ldots,i_k}(x_1, \ldots, x_k).$$

Theorem 1. *Let μ be a random measure on X, i.e., a probability measure on $\mathbb{M}(X)$. Assume that μ has finite moments, and let $(M^{(n)})_{n=1}^{\infty}$ be its moment sequence. Further assume that the following growth conditions are satisfied:*

(C1) *For each $\Lambda \in \mathcal{B}_0(X)$, there exists a constant $C_\Lambda > 0$ such that $M^{(n)}(\Lambda^n) \leq C_\Lambda^n\, n!\,, \quad n \in \mathbb{N}$.*

(C2) *For each $\Lambda \in \mathcal{B}_0(X)$, there exists a constant $C'_\Lambda > 0$ such that $M^{(n)}(\Lambda_{\widehat{0}}^{(n)}) \leq (C'_\Lambda)^n n!$ for all $n \in \mathbb{N}$, and for any sequence $\{\Lambda_k\}_{k=1}^{\infty} \in \mathcal{B}_0(X)$ such that $\Lambda_k \downarrow \emptyset$, we have $C'_{\Lambda_k} \to 0$ as $k \to \infty$.*

Then μ is a random discrete measure, i.e., $\mu(\mathbb{K}(X)) = 1$, if and only if the moment sequence $(M^{(n)})_{n=0}^{\infty}$ satisfies the following conditions:

(i) *For any $n \in \mathbb{N}$, $\Delta \in \mathcal{B}_0(X_{\widehat{0}}^{(n)})$, and $(i_1, \ldots, i_n) \in \mathbb{N}_0^n$, denote $\xi^{\Delta}_{i_1,\ldots,i_n} := \frac{1}{n!} M_{i_1+1,\ldots,i_n+1}(\Delta)$. Then the sequence $(\xi^{\Delta}_{i_1,\ldots,i_n})$ is positive definite.*

(ii) *Let $\Delta = (\Lambda)_{\widehat{0}}^{(n)}$ with $\Lambda \in \mathcal{B}_0(X)$. Set $r_i^\Delta := \xi^{\Delta}_{i,0,0,\ldots,0}\,, \quad i \in \mathbb{N}_0$. Then, for any finite sequence of complex numbers, $(z_n)_{n=0}^{N}$, we have*

$$\sum_{i,j=0}^{N} r^{\Delta}_{i+j+1}\, z_i\, \overline{z_j} \geq 0,$$

and furthermore

$$\sum_{k=1}^{\infty} (D^{\Delta}_{k-1} D^{\Delta}_{k})^{-1} \det \begin{bmatrix} r_1^\Delta & r_2^\Delta & \cdots & r_k^\Delta \\ r_2^\Delta & r_3^\Delta & \cdots & r_{k+1}^\Delta \\ \vdots & \vdots & \vdots & \vdots \\ r_k^\Delta & r_{k+1}^\Delta & \cdots & r_{2k-1}^\Delta \end{bmatrix}^2 = \infty,$$

where

$$D_k := \det \begin{bmatrix} r_0^\Delta & r_1^\Delta & \cdots & r_k^\Delta \\ r_1^\Delta & r_2^\Delta & \cdots & r_{k+1}^\Delta \\ \vdots & \vdots & \vdots & \vdots \\ r_k^\Delta & r_{k+1}^\Delta & \cdots & r_{2k}^\Delta \end{bmatrix}.$$

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Bianalytic maps between free spectrahedra

Scott McCullough
(joint work with Meric Augat, Bill Helton, Igor Klep)

The aim of this project is to determine, up to affine linear equivalence, when the fully matricial solution sets of two linear matrix inequalities are freely bianalytic. Besides the obvious analogy with rigidity results from several complex variables, motivation also comes from systems theory where one would like to understand the degree of non-uniqueness in mapping the fully matricial solution set of a matrix inequality bianalytically to the fully matricial solution set of a linear matrix inequality.

Fix throughout a positive integer g. Let $x = (x_1, \dots, x_g)$ denote a tuple of freely non-commuting intederminates. Let $\langle x \rangle$ denote the *words* in x. Thus an $\alpha \in \langle x \rangle$ of length n has the form

$$\alpha = x_{i_1} \cdots x_{i_n}.$$

Let $\mathbb{C}\langle x \rangle$ denote the *polynomials* in x and let $M_d(\mathbb{C}\langle x \rangle)$ denote the $d \times d$ matrix polynomials in x. Thus an element $p \in M_d(\langle x \rangle)$ takes the form

$$p = \sum p_\alpha \alpha$$

where the sum is finite, $p_\alpha \in M_d(\mathbb{C})$ and $\alpha \in \langle x \rangle$.

Let $M_n = M_n(\mathbb{C})^g$ denote the set of g-tuples of $n \times n$ matrices and M^g the sequence, or graded set, $(M_n^g)_n$. A $p \in M_d(\langle x \rangle)$ is naturally evaluated at an $X \in M_n^g$ as

$$p(X) = \sum p_\alpha \otimes \alpha(X) = \sum p_\alpha \otimes X^\alpha,$$

where

$$\alpha(X) = X_{i_1} \cdots X_{i_g} =: X^\alpha.$$

The output is a $dn \times dn$ matrix, $p(X) \in M_d \otimes M_n$.

The graded set $\mathfrak{P}_p = (\mathfrak{P}_p(n))_n$, where

$$\mathfrak{P}_p(n) = \{X \in M_n^g : I + p(X) + p(X)^* \succ 0\}$$

(often it is assumed that $p(0) = 0$), is the *feasibility set* or *fully matricial solution set* of the *matrix inequality* $p(X) \succ 0$. For optimization purposes, one would like each $\mathfrak{P}_p(n)$ to be convex or at least be able to map $\mathfrak{P}_p$ in a reasonable fashion to a convex $\mathfrak{P}_q$.

Of course if q is a linear polynomial,

$$q = \sum A_j x_j,$$

where $A_j \in M_d^g$, then $\mathfrak{P}_q$ is convex. The resulting inequality $I + \sum A_j \otimes X_j + \sum A_j^* \otimes X_j^* \succ 0$ is a *linear matrix inequality* and its solution set $\mathfrak{P}_q$ is the *fully matricial solutions set* of the linear matrix inequality, also known as a *free spectrahedron.* Indeed, $\mathfrak{P}_q(1)$ is itself a version of a spectrahedron.

A tuple $\Xi \in M_g^g$ (so a g-tuple of $g \times g$ matrices) is *convexotonic* if

$$\Xi_k \Xi_j = \sum_{s=1}^{g} (\Xi_j)_{ks} \Xi_s.$$

A straightforward computations shows, for words α,

$$\Xi_k \Xi^\alpha = \sum_{s=1}^{g} (\Xi^\alpha)_{ks} \Xi_s.$$

Convexotinic tuples arise naturally in the context of finite dimensional algebras. If $R = (R_1, \dots, R_g)$ is a linearly independent tuple of $n \times n$ matrices that spans a g-dim algebra, then

$$R_k R_i = \sum_{s=1}^{g} (\Xi_j)_{ks} R_s$$

determines uniquely a convexotonic tuple Ξ, referred to as the *structure constants* for the algebra (given the particular choice of basis). Conversely, if Ξ is a convexotonic g-tuple (not necessarily linearly independent), then Ξ arises as the structure constants for a (many) g-dimensional algebra(s).

A *convexotonic map* is a map of the form

$$\begin{aligned} f_\Xi(x) = f(x) =& x(I - \Lambda_\Xi(x))^{-1} \\ =& \begin{pmatrix} x_1 & \dots & x_g \end{pmatrix} \left(I - \sum_{j=1}^{g} \Xi_j x_j\right)^{-1}. \end{aligned}$$

where Ξ is a convexotonic tuple. It turns out that $f_{-\Xi}(x) = x(I + \Lambda_\Xi(x))^{-1}$ is the inverse of f_Ξ. Further f is a polynomial if and only if Ξ is nilpotent. For $g \leq 5$ the indecomposable g dimensional algebras are classified. In the case $g = 2$ there are four types.

I: R_1 is nilpotent of order 3 and $R_2 = R_1^2$ and otherwise 0. In this case,

$$\Xi_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \Xi_2 = 0.$$

Hence

$$f_\Xi = \begin{pmatrix} x_1 & x_2 + x_1^2 \end{pmatrix}, \quad f_\Xi^{-1} = f_{-\Xi} = \begin{pmatrix} x_1 & x_2 - x_1^2 \end{pmatrix}.$$

II: $R_1^2 = R_1$ and $R_1 R_2 = R_2$ and otherwise 0. For instance

$$R_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \Xi = R,$$

$$f_\Xi = \begin{pmatrix} (1 - x_1)^{-1} x_1 & (1 - x_1)^{-1} x_2 \end{pmatrix}, \quad f_\Xi^{-1} = \begin{pmatrix} (1 + x_1)^{-1} x_1 & (1 + x_1)^{-1} x_2 \end{pmatrix}.$$

III: Take transposes in Type II.

IV: $R_1^2 = R_1$; $R_1 R_2 = R_2 R_1$ and otherwise 0. For instance,

$$R_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \Xi = R,$$

$$f_\Xi = \big((1-x_1)^{-1}x_1 \quad (1-x_1)^{-1}x_1(1-x_1)^{-1}\big)$$

Suppose now R is g-dimensional subalgebra of M_d spanned by $\{R_1,\dots,R_g\}$ with structure constants Ξ. Let C be a $d\times d$ unitary matrix such that $C-I$ is invertible. Let $A=(C-I)^{-1}R$ and $B=CA$. Let $\mathcal{D}_A=\mathfrak{P}_q$ where $q=\sum A_jx_j$ and similarly for $\mathcal{D}_B$.

Theorem. *The convexotonic map f_Ξ is a binanlytic mapping from $\mathcal{D}_A$ to $\mathcal{D}_B$. In particular, each convexotonic map bianalytically identifies many different pairs of free spectrahedra.*

Our main result is a partial converse. The *generic* condition on a tuple $A\in M_d^g$ appearing in the following theorem is a bit stronger than irreducibility.

Theorem. *Assuming $\mathcal{D}_A$ and $\mathcal{D}_B$ are bounded, if $f:\mathcal{D}_A\to\mathcal{D}_B$ is bianalytic and is defined on $t\mathcal{D}_A$ (for some $t>1$) and A and B are* **generic**, *then f is a convexotonic map:*

(a) $A\overset{u}{\sim}CB$, for a unitary C (in particular A and B have the same size);
(b) there is a convexotonic tuple Ξ of $g\times g$ matrices such that
  (i) $A_j(C-I)A_k=\sum_{j=1}^{g}(\Xi_k)_{j,s}A_s$;
  (ii) in particular $R=(C-I)A$ spans an algebra with structure constants Ξ;
  (iii) $f=f_\Xi$.

We conjecture that the generic condition is not needed. While the condition is in a sense generic in the category of free spectrahedra, it is not in important subcategories. In particular, A generic includes the requirement $\cap\ker(A_j)=(0)$, a condition never satisfied when $\mathcal{D}_A$ is a ball; i.e., invariant under left multiplication by unitary matrices.

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