Mathematisches Forschungsinstitut Oberwolfach

Report No. 18/2017

DOI: 10.4171/OWR/2017/18

Arbeitsgemeinschaft: Higher Gross Zagier Formulas

Organised by
Zhiwei Yun, Stanford
Wei Zhang, New York

2 April – 8 April 2017

Abstract. The aim of this Arbeitsgemeinschaft is to go over the proof of the higher Gross–Zagier formula established in the paper [YZ15]. The formula relates arbitrary order central derivative of the base change L-function of an unramifed automorphic representation of PGL_2 over a function field to the self-intersection number of a certain algebraic cycle on the moduli stack of Shtukas.

Mathematics Subject Classification (2010): 11R39, 11S37, 22E57.

Introduction by the Organisers

For an elliptic curve E over a global field K, the conjecture of Birch and Swinnerton-Dyer asserts a deep relationship between the arithmetic invariants (Mordell–Weil groups and Tate–Shafarevich groups) and the analytic invariants (the complex L-function $L(E/K, s)$). The rank part of the conjecture asserts that the vanishing order of $L(E/K, s)$ at its center $s = 1$ coincides with the rank of Mordell–Weil group $E(K)$. The refined part of the conjecture is an identity of the leading term of $L(E/K, s)$ at $s = 1$,

$$\frac{L^{(r)}(E/K, 1)}{r!} \sim \det((P_i, P_j)_{\mathrm{NT}}) \tag{1}$$

where $(P_i, P_j)_{\mathrm{NT}}$ is the matrix of Néron–Tate height pairings of a $\mathbb{Z}$-basis $\{P_1, ..., P_r\}$ of $E(K)/E(K)_{\mathrm{tor}}$, and $\sim$ means the two sides are equal up to some explicit terms such as the order of Tate–Shafarevich group, the local Tamagawa numbers and the real periods. Equivalently, the Néron–Tate height pairing induces a metric on the determinant of the Mordell–Weil group $E(K) \otimes_{\mathbb{Z}} \mathbb{R}$, and the RHS of the conjectural formula is the norm of a generator of the determinant of the lattice $E(K)/E(K)_{\mathrm{tor}}$. Beilinson and Bloch also formulated a generalization of the B-SD conjecture to higher dimensional varieties.

The Gross–Zagier formula provides an evidence to the B-SD conjecture for elliptic curves E over $\mathbb{Q}$ when the L-function has a zero of order at most one. Let f be the weight two newform associated to E by the theorem of Wiles, Taylor–Wiles, and Breuil–Conrad–Diamond–Taylor. Let $\phi : X_0(N) \to E$ be a modular parameterization. Let K be an imaginary quadratic extension of $\mathbb{Q}$, with discriminant D. Under suitable hypotheses, the theory of complex multiplication and the map ϕ allow us to define the *Heegner point* $y_K \in E(K)$. The Gross–Zagier formula is the following identity on the first order derivative of the base-changed L-function $L(E/K, s) = L(f/K, s)$ at the center $s = 1$ ([4, 6])

$$\frac{L'(f/K,1)}{(f,f)} = \frac{1}{\sqrt{|D|}} \frac{(y_K, y_K)_{\mathrm{NT}}}{\deg(\phi)},$$

where (f, f) is the Petersson inner product. A similar formula, but for the central value of the L-function, was also discovered around the same time by Waldspurger [5].

What about higher order derivatives of L-function at the center? In [YZ15] a formula for arbitrary order derivative is proved for unramified cuspidal automorphic representation π of PGL_2 over a *function field* $F = k(X)$, where X is a curve over a finite field k. The r-th central derivative of the L-function (base changed along a quadratic extension F'/F) is expressed in terms of the self-intersection number of the *Heegner–Drinfeld cycle* Sht_T^r (or rather its π-isotypic component) on the moduli stack Sht_G^r:

$$\mathcal{L}^{(r)}(\pi_{F'}, 1/2) \sim ([\mathrm{Sht}_T^r]_\pi, [\mathrm{Sht}_T^r]_\pi). \tag{2}$$

The moduli stack Sht_G^r is closely related to the moduli stack of Drinfeld Shtukas of rank two with r modifications. One important feature of this stack is that it admits a natural fibration over the r-fold self-product X^r of the curve X over $\mathrm{Spec}\, k$

$$\mathrm{Sht}_G^r \longrightarrow X^r \ .$$

In the number field case, the analogous spaces only exist (at least for the time being) when $r \le 1$. When $r = 0$, the moduli stack Sht_G^0 is the constant groupoid over k

$$\mathrm{Bun}_G(k) \simeq G(F)\backslash \left(G(\mathbb{A}_F)/K\right), \tag{3}$$

where $\mathbb{A}_F$ is the ring of adèles of F, and K a maximal compact open subgroup of $G(\mathbb{A})$. The double coset in the RHS of (3) remains meaningful for a number field F. When $r = 1$ the counterpart of Sht_G^1 in the case $F = \mathbb{Q}$ is the moduli stack of elliptic curves, which lives over $\mathrm{Spec}\,(\mathbb{Z})$. Therefore the formula (2) can be viewed as a simultaneous generalization (for function fields) of the Waldspurger formula [5] (in the case of $r = 0$) and the Gross–Zagier formula [4] (in the case of $r = 1$). Moreover, there is a way to rewrite the RHS of the formula (2) so that it looks just like (1). The formula (2) opens the possibility of relating higher derivatives of automorphic L-functions to geometric invariants in the function fields case.

The basic strategy of the proof of (2) is to compare two relative trace formulae (abbreviated as RTF), an "analytic" one for the L-functions, and a "geometric" one for the intersection numbers. The strategy of using RTF initiated by Jacquet in 1980s has been successful in related and similar questions on higher rank reductive groups when $r = 0$ (e.g.,[7, 18]) and $r = 1$ (e.g., [16]).

The aim of the workshop is to carefully define the relevant objects that appear in the formula (2), especially the moduli stack of Shtukas and the Heegner–Drinfeld cycle; to review Jacquet's RTF; and to sketch the geometric ideas used in the comparison of the two RTFs. The talks (except those providing general background) roughly correspond to various parts of the main reference [YZ15].

References

[1] V.Drinfeld, *Moduli varieties of F-sheaves*, Funktsional. Anal. i Prilozhen. 21 (1987), no. 2, 23-41.

[2] E. Frenkel, *Recent advances in the Langlands program.* Bull. Amer. Math. Soc. (N.S.) 41 (2004), no. 2, 151–184.

[3] W.Fulton, *Intersection theory, 2nd ed.*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3.

[4] B.Gross, D.Zagier, *Heegner points and derivatives of L-series*, Invent. Math. 84 (1986), no. 2, 225-320.

[5] Harder, G.; Kazhdan, D. A. *Automorphic forms on GL2 over function fields (after V. G. Drinfeld).* Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, pp. 357–379, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979

[6] J. Heinloth, *Lectures on the moduli stack of vector bundles on a curve.* Affine flag manifolds and principal bundles, 123–153, Trends Math., Birkhuser/Springer Basel AG, Basel, 201

[7] H. Jacquet, *Sur un résultat de Waldspurger*, Ann. Sci. École Norm. Sup. (4) 19 (1986), no. 2, 185–229

[8] L. Lafforgue, *Chtoucas de Drinfeld et conjecture de Ramanujan-Petersson.* Astrisque No. 243 (1997), ii+329 pp.

[9] Laumon, *Cohomology of Drinfeld modular varieties.* Part II. Automorphic forms, trace formulas and Langlands correspondence. With an appendix by Jean-Loup Waldspurger. Cambridge Studies in Advanced Mathematics, 56. Cambridge University Press, Cambridge, 1997. xii+366 pp.

[10] A.Kresch, *Cycle groups for Artin stacks*, Invent. Math. 138 (1999), no. 3, 495–536.

[11] Y. Varshavsky, *Moduli spaces of principal F-bundles*, Selecta Math. (N.S.) 10 (2004), no. 1, 131–166

[12] Y.Varshavsky, *Lefschetz-Verdier trace formula and a generalization of a theorem of Fujiwara*, Geom. Funct. Anal. 17 (2007), no. 1, 271-319.

[13] J.-L. Waldspurger, *Sur les valeurs de certaines fonctions L automorphes en leur centre de symétrie*, Compositio Math. 54 (1985), no. 2, 173–242.

[14] X. Yuan, S. Zhang, W. Zhang. *The Gross–Zagier formula on Shimura curves.* Annals of Mathematics Studies. #184, Princeton University Press, 2012, ISBN: 9781400845644.
[15] Z. Yun, W. Zhang, *Shtukas and the Taylor expansion of L-functions*, arXiv:1512.02683
[16] W. Zhang, *On arithmetic fundamental lemmas*, Invent. Math. 188 (2012), no. 1, 197–252.
[17] W. Zhang, *The Birch–Swinnerton-Dyer conjecture and Heegner points: a survey.* Current Developments in Mathematics, Volume 2013, 169–203.
(Available on http://math.columbia.edu/∼wzhang/math/pub.html)
[18] W. Zhang, *Automorphic period and the central value of Rankin-Selberg L-function*, J. Amer. Math. Soc. 27 (2014), no. 2, 541–612.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, "US Junior Oberwolfach Fellows".

Arbeitsgemeinschaft: Higher Gross Zagier Formulas

Table of Contents

Abstracts

Talk 1: An overview of the Gross–Zagier and Waldspurger formulas

Yunqing Tang

In this talk, we will state the Gross–Zagier formula, which relates the Néron–Tate height of Heegner points to the central derivative of the L-function of certain weight 2 cusp forms. We will also state Waldspurger formula on the central value of L-function of certain cuspidal automorphic representation of quaternion algebra.

1. The modular curve $X_0(N)$ and Heegner points

1.1. **The modular curve.** The *modular curve* $Y_0(N)$ over $\mathbf{Q}$ is defined to be the moduli space of $\phi : E \to E'$ where E, E' are elliptic curves and ϕ is an isogeny with kernel isomorphic to $\mathbf{Z}/N\mathbf{Z}$.

The set of complex points $Y_0(N)(\mathbf{C})$ is the locally symmetric space $\Gamma_0(N)\backslash\mathbf{H}$, where

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z})\colon c \equiv 0 \pmod N \right\}.$$

One defines $X_0(N)$ to be the compactification of $Y_0(N)$ obtained by adding cusps (the set of cusps admits a bijection to $\Gamma_0(N)\backslash\mathbf{P}^1(\mathbf{Q})$). The modular curve $X_0(N)$ is defined over $\mathbf{Q}$ and parametrizes isogenies of generalized elliptic curves $\phi\colon E \to E'$ such that $\ker\phi$ is isomorphic to $\mathbf{Z}/N\mathbf{Z}$ and meets every component of E. (See [1].) The cusp on $X_0(N)$ given by $\infty \in \mathbf{P}^1(\mathbf{Q})$ corresponds to the nodal cubic curve. We will use ∞ to denote this cusp.

1.2. **CM points and Heegner points.** (See [2].)
In terms of the uniformization of $X_0(N)(\mathbf{C})$ by $\mathbf{H}$, complex multiplication (CM) points $E_\tau \cong \mathbf{C}/\mathbf{Z} + \tau\mathbf{Z}$ correspond to $\tau \in \mathbf{H}$ such that there exist $a, b, c \in \mathbf{Z}$ such that $a\tau^2 + b\tau + c = 0$. If $\gcd(a, b, c) = 1$, the discriminant $D = b^2 - 4ac$ is the discriminant of $\mathrm{End}_{\mathbf{C}}(E_\tau)$. CM points are defined over $\overline{\mathbf{Q}}$.

Let K be an imaginary quadratic field of odd discriminant D over $\mathbf{Q}$. The *Heegner condition* says that for all $p \mid N$, one has p split or ramified in K, and that $p^2 \nmid N$ if p is ramified. The Heegner condition is equivalent to the existence of a point $(\phi\colon E \to E') \in X_0(N)(\overline{\mathbf{Q}})$ satisfying $\mathrm{End}_{\overline{\mathbf{Q}}}(E) = \mathrm{End}_{\overline{\mathbf{Q}}}(E') = \mathcal{O}_K$. Such point is called a *Heegner point.* The theory of complex multiplication implies that Heegner points are defined over the Hilbert class field H of K.

Let $J_0(N)$ denote the Jacobian of $X_0(N)$. One constructs $P \in J_0(N)(K)$ from the Heegner point by letting $P = \sum_{\sigma\in\mathrm{Gal}(H/K)}((\sigma(x)) - (\infty))$.

2. Gross–Zagier formula

2.1. **Néron-Tate height.** Let $\mathcal{L}$ be a line bundle on $J_0(N)$ corresponding to twice a theta divisor Θ. Since $\mathcal{L}$ is ample, there exists $n \in \mathbf{N}$ such that $\mathcal{L}^{\otimes n}$ is very ample and hence induces a projective embedding of $J_0(N)$. One restricts the Weil height on K-points of the projective space to $J_0(N)(K)$ to obtain a height function $h^K_{\mathcal{L}^{\otimes n}}$. One defines $h^K_{\mathcal{L}}$ on $J_0(N)(K)$ by $\frac{1}{n}h^K_{\mathcal{L}^{\otimes n}}$.

The *Néron-Tate height* for $J_0(N)(K)$ is defined to be $\widehat{h} := \lim_{n\to\infty} \dfrac{h^K_{\mathcal{L}}(2^n x)}{4^n}$. It induces a quadratic pairing on $J_0(N)(K)$ and also on $J_0(N)(K) \otimes_{\mathbf{Z}} \mathbf{C}$. Néron's theory interprets the Néron–Tate height as the sum of local intersection numbers on $X_0(N)$ (see for example [3]).

2.2. **L-functions.** Let f be a weight 2 newform for $\Gamma_0(N)$, that is, a cuspidal Hecke eigenform, orthogonal to modular forms coming from smaller level.

One may view f as an automorphic form for $\mathrm{GL}_2/\mathbf{Q}$ and f_K denotes its base change to K. We have

$$L(f_K, s) = L(f, s)L(f \otimes \eta_{K/\mathbf{Q}}, s),$$

where $\eta_{K/\mathbf{Q}}$ is the quadratic character associated to $K/\mathbf{Q}$ by class field theory. Explicity, if the Fourier expansion of f is given by $\sum_{n\geq 1} a_n q^n$, then

$$L(f, s) = \sum_{n\geq 1} a_n q^n; \quad L(f \otimes \eta_{K/\mathbf{Q}}, s) = \sum_{n\geq 1} \eta(n) a_n q^n.$$

The Heegner condition implies that $L(f_K, 1) = 0$.

2.3. **The Hecke algebra.** The *Hecke algebra* is the double coset

$$\Gamma_0(N)\backslash \mathrm{GL}_2(\mathbf{Q})/\Gamma_0(N).$$

It is the algebra of correspondences on $X_0(N)$ generated by

$$T_m \colon [E \xrightarrow{\phi} E'] \mapsto \sum_C [E/C \to E'/C],$$

where C runs through order m subgroups of E which intersect $\ker\phi$ trivially.

The Hecke algebra acts on $X_0(N)$, hence also on $J_0(N)(K)$. We will use $P(f)$ to denote the projection of P to the f-isotypic component of $J_0(N)(K) \otimes_{\mathbf{Z}} \mathbf{C}$.

2.4. **The formula.**

Theorem 2.1 (Gross–Zagier [4])**.** *Given a weight 2 new form f of $X_0(N)$, we have*

$$\widehat{h}(P(f)) \sim \frac{L'(f_K, 1)}{||f||_{\mathrm{Pet}}},$$

where $\sim$ means that both sides are equal up to a constant independent of f and $||f||_{\mathrm{Pet}} := \int_{\Gamma_0(N)\backslash \mathbf{H}} f(z)\overline{f(z)}\, dxdy$.

Remark 2.1. The work of Zhang [7] and Yuan-Zhang-Zhang [6] shows an analogous formula under the generalized Heegner condition. For simplicity, assume $(N, D) = 1$ and write $N = N^+N^-$, where N^+ (resp. N^-) is the product of powers of split (resp. inert) primes. The *generalized Heegner condition* assumes N^- is squarefree and its number of prime factors is even. The (generalized) Heegner points are constructed as CM points on the Shimura curve associated to the quaternion algebra B ramified exactly at primes dividing N^-.

3. Waldspurger formula

Let F be a number field, B a quaternion algebra over F, and G the algebraic group associated to $B^\times$. Let K/F be a quadratic extension with a given embedding $K \hookrightarrow B$. Let T to be $\mathrm{Res}_{K/F}\,\mathbf{G}_m$. One naturally views T as a subgroup of G. Let $\eta : F^\times\backslash\mathbf{A}_F^\times \to \mathbf{C}^\times$ be the quadratic character associated to K/F.

Let π be an irreducible cuspidal automorphic representation of G, $\widetilde{\pi}$ its contragredient, and ω_π the central character. Let π_K denote the base change of π to K. Let $\chi\colon T(F)\backslash T(\mathbf{A}_F) \to \mathbf{C}^\times$ be a unitary character, such that $\omega_\pi \cdot \chi|_{\mathbf{A}_F^\times} = 1$. The center of $L(\pi_K \otimes \chi, s)$ is normalized to be $1/2$.

One defines *period integral* $P_\chi\colon \pi \to \mathbf{C}$ by

$$f \mapsto P_\chi(f) = \int_{T(F)\backslash T(\mathbf{A}_F)/\mathbf{A}_F^\times} f(t)\chi(t)\,dt.$$

Theorem 3.1 (Waldspurger [5], see also, for example, [8])**.** *For $f_1 \in \pi$ and $f_2 \in \widetilde{\pi}$,*

$$P_\chi(f_1)P_\chi(f_2) \sim \frac{L(\pi_K \otimes \chi, 1/2)}{L(\pi, \mathrm{Ad}, 1)}\alpha(f_1 \otimes f_2)$$

where $\sim$ means that both sides are equal up to a constant independent of π, χ, f_1, f_2, and $\alpha = \prod_v \alpha_v$ is a product of local terms

$$\alpha_v \in \mathrm{Hom}_{K_v^\times}(\pi_v \otimes \chi_v, \mathbf{C}) \otimes \mathrm{Hom}_{K_v^\times}(\widetilde{\pi}_v \otimes \chi_v^{-1}, \mathbf{C}),$$

normalized by Waldspurger (so that, in particular, α_v is 1 in the spherical case).

References

[1] P. Deligne and M. Rapoport, *Les schémas de modules de courbes elliptiques*, Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), Springer, Berlin, 1973, pp. 143–316. Lecture Notes in Math., Vol. 349 (French).

[2] Benedict H. Gross, *Heegner points on $X_0(N)$*, Modular forms (Durham, 1983), Ellis Horwood Ser. Math. Appl.: Statist. Oper. Res., Horwood, Chichester, 1984, pp. 87–105.

[3] ———, *Local heights on curves*, Arithmetic geometry (Storrs, Conn., 1984), Springer, New York, 1986, pp. 327–339.

[4] Benedict H. Gross and Don B. Zagier, *Heegner points and derivatives of L-series*, Invent. Math. **84** (1986), no. 2, 225–320.

[5] J.-L. Waldspurger, *Sur les valeurs de certaines fonctions L automorphes en leur centre de symtrie*, Compositio Math. **54** (1985), no. 2, 173–242 (French).

[6] Xinyi Yuan, Shou-Wu Zhang, and Wei Zhang, *The Gross-Zagier formula on Shimura curves*, Annals of Mathematics Studies, vol. 184, Princeton University Press, Princeton, NJ, 2013.

[7] Shouwu Zhang, *Heights of Heegner points on Shimura curves*, Ann. of Math. (2) **153** (2001), no. 1, 27–147.
[8] Wei Zhang, *The Birch–Swinnerton-Dyer conjecture and Heegner points: a survey*, Current developments in mathematics 2013, Int. Press, Somerville, MA, 2014, pp. 169–203.

Talk 2: The stacks $\mathbf{Bun}_n$ and Hecke

Timo Richarz

1. Why stacks?

In algebraic geometry one would like to have a classifying space BGL_n for vector bundles, such that

$$\mathrm{Hom}(S, \mathrm{BGL}_n) = \{\text{vector bundles of rank } n \text{ on } S\} \,/ \sim .$$

Such an object can not be represented by a scheme, since a vector bundle is locally trivial, so any map $S \to \mathrm{BGL}_n$ would need to be locally constant which for schemes would imply constant. There are several possible ways to wriggle out of this situation.

(1) Add extra data (e.g. level structure) in order to eliminate automorphisms!
(2) Do not pass to isomorphism classes!

Stacks are the result of the second option.

2. Bun_n as a stack

We refer to Heinloth [He10, §1-2] for further details. Fix a field k in this section.

Definition 2.1. A *stack* $\mathcal{M}$ is a sheaf (in the appropriate topology) of groupoids

$$\mathcal{M}\colon \mathrm{Sch}_k^{\mathrm{op}} \to \mathbf{Grp} \subset \mathbf{Cat}.$$

Example 2.2. The classifying stack

$$\mathrm{BGL}_n := [\mathrm{pt}\,/\mathrm{GL}_n]$$

takes S to the groupoid of vector bundles of rank n on S.

Example 2.3. Let X be a smooth, projective, geometrically connected curve over k. We define the stack Bun_n taking S to the groupoid of vector bundles of rank n on $X \times S$.

How to make this geometric? We have a map $\mathrm{pt} \to \mathrm{BGL}_n$ corresponding to the trivial bundle. If $\mathcal{E}$ is a rank n vector bundle on S, then we get by definition a classifying map

$$f_{\mathcal{E}}\colon S \to \mathrm{BGL}_n.$$

Consider the fibered product

$$\begin{tikzcd} S \times_{\mathrm{BGL}_n} \mathrm{pt} \arrow[r] \arrow[d] & \mathrm{pt} \arrow[d] \\ S \arrow[r, "f_{\mathcal{E}}"] & \mathrm{BGL}_n \end{tikzcd}$$

Its T-valued points are

$$\{(f, \varphi\colon \mathrm{Triv} \circ p \xrightarrow{\sim} f_{\mathcal{E}} \circ f\} = \underline{\mathrm{Isom}}(\mathcal{O}_S^{\oplus n}, \mathcal{E})(T),$$

which is the frame bundle of $\mathcal{E}$. Let us think about what this means.

(1) We can recover $\mathcal{E} = \mathcal{O}_S^n \times^{\mathrm{GL}_n} \underline{\mathrm{Isom}}(\mathcal{O}_S^{\oplus n}, \mathcal{E})$, i.e. the map $\mathrm{pt} \to \mathrm{BGL}_n$ is the universal vector bundle!

(2) The map $\mathrm{pt} \to \mathrm{BGL}_n$ is a smooth surjection after every base change!

Definition 2.4. A stack $\mathcal{M}$ is called *algebraic* if

(1) For all maps $S \to \mathcal{M}$ and $S' \to \mathcal{M}$ from schemes S, S', the fibered product $\mathcal{S} \times_{\mathcal{M}} \mathcal{S}'$ is a scheme.

(2) There exists a scheme U together with a smooth surjection $U \to \mathcal{M}$ called an *atlas*.

(3) The map $U \times_{\mathcal{M}} U \to U \times U$ is quasi-compact and quasi-separated.

An algebraic stack $\mathcal{M}$ is *smooth* (resp. locally of finite type, ...) if there is an atlas $U \twoheadrightarrow \mathcal{M}$ such that U is smooth (resp. locally of finite type, ...).

Example 2.5. (Picard stack) We define $\mathrm{Pic}_X = \mathrm{Bun}_{X,1}$. Let Jac_X be the Jacobian of X. This is the coarse moduli space of Pic_X, so we have a map $\mathrm{Pic}_X \to \mathrm{Jac}_X$, which preserves the labelling of connected components by degree. Suppose you have $x \in X(k) \neq \emptyset$. Then we actually have an isomorphism

$$\mathrm{Pic}_X \xrightarrow{\sim} \mathrm{Jac}_X \times B\mathbf{G}_m,$$

where the map $\mathrm{Pic}_X \to B\mathbf{G}_m$ corresponds to the restriction of the universal line bundle on $X \times \mathrm{Pic}_X$ to $\{x\} \times \mathrm{Pic}_X$. This shows that Pic_X is a smooth algebraic stack locally of finite type of dimension $g(X) - 1$.

Theorem 2.6. *The stack* Bun_n *is a smooth algebaic stack locally of finite type over* k*, of dimension* $n^2(g(X) - 1)$*, and* $\pi_0(\mathrm{Bun}_n) = \mathbf{Z}$*.*

3. Adelic uniformization of Bun_n

We refer to the notes of a course of Yun [Yun15, §2.4] for more details. Let $k = \mathbf{F}_q$ be a finite field. Let F be a the function field of X, and $|X|$ the set of closed points. For $x \in |X|$, denote by $\mathcal{O}_x$ the completed local ring at x. This is non-canonically isomorphic to $k_x[\![\varpi_x]\!]$. We also set $F_x = \mathrm{Frac}(\mathcal{O}_x)$, which is non-canonically isomorphic to $k_x(\!(\varpi_x)\!)$. Recall the ring of adeles

$$\mathbf{A} = \prod_{x \in |X|}{}' (F_x, \mathcal{O}_x) = \{(a_x) \in \prod F_x \mid a_x \in \mathcal{O}_x \text{ for almost all } x \in |X|\}.$$

Theorem 3.1 (Weil)**.** *There is a canonical isomorphism of groupoids*

$$\mathrm{GL}_n(F) \backslash \left(\mathrm{GL}_n(\mathbf{A}) / \prod_{x \in |X|} \mathrm{GL}_n(\mathcal{O}_x) \right) \xrightarrow{\sim} \mathrm{Bun}_n(k).$$

Here if S is a set with a (left) group action of G, then $G\backslash S$ can be considered as a groupoid, whose objects are orbits and automorphisms are stabilizers.

3.1. **Level structure.** Given $D = \sum d_x \cdot x$ an effective divisor, we can look at the double quotient

$$\mathrm{GL}_n(F)\backslash\left(\mathrm{GL}_n(\mathbf{A})/K_D\right) \cong \left\{(\mathcal{E},\alpha) \mid \alpha\colon \mathcal{E}|_D \cong \mathcal{O}_D^{\oplus n}\right\}$$

where $K_D = \ker\left(\prod_{x\in|X|}\mathrm{GL}_n(\mathcal{O}_x) \to \prod_{x\in|X|}\mathrm{GL}_n(\mathcal{O}_x/\varpi_x^{d_x})\right)$.

3.2. **Split groups.** If G is any (not necessarily reductive) connected algebraic group which splits over k, then

$$G(F)\backslash\left(G(\mathbf{A})/\prod_{x\in|X|}G(\mathcal{O}_x)\right) \cong \mathrm{Bun}_G(k).$$

4. Hecke stacks

We refer to Yun and Zhang [YZ15, §5.1] for further details. Let $r \geq 0$ and $\mu = (\mu_1,\dots,\mu_r)$ a sequence of dominant coweights of GL_n such that μ_i is either $\mu_+ = (1,0,\dots,0)$ or $\mu_- = (0,\dots,0,-1)$.

Definition 4.1. The *Hecke stack* Hk_n^μ is the stack defined by $\mathrm{Hk}_n^\mu(S)$ is the groupoid classifying the following data:

- a sequence $(\mathcal{E}_0,\dots,\mathcal{E}_r)$ of rank n vector bundles on $X\times S$.
- a sequence $(x_1,\dots,x_r)$ of morphisms $x_i\colon S\to X$, with graphs $\Gamma_{x_i}\subset X\times S$,
- maps $(f_1,\dots,f_r)$ with

$$f_i\colon \mathcal{E}_{i-1}|_{X\times S\backslash\Gamma_{x_i}} \xrightarrow{\sim} \mathcal{E}_i|_{X\times S\backslash\Gamma_{x_i}}$$

  such that if $\mu_i = \mu_+$, then f_i extends to $\mathcal{E}_{i-1}\hookrightarrow\mathcal{E}_i$ whose cokernel is an invertible sheaf on Γ_{x_i}, and if $\mu_i = \mu_-$ then f_i^{-1} extends to $\mathcal{E}_i\hookrightarrow\mathcal{E}_{i-1}$ whose cokernel is an invertible sheaf on Γ_{x_i}.

For $i = 0,\dots,r$ we have a map $p_i\colon \mathrm{Hk}_n^\mu\to\mathrm{Bun}_n$ sending $(\underline{\mathcal{E}},\underline{x},\underline{f})\mapsto\mathcal{E}_i$, and $p_X\colon \mathrm{Hk}_n^\mu\to X^r$ sending $(\underline{\mathcal{E}},\underline{x},\underline{f})\mapsto\underline{x}$.

Lemma 4.2. *The morphism*

$$(p_0,p_X)\colon\ \mathrm{Hk}_n^\mu\to\mathrm{Bun}_n\times X^r$$

is representable by a proper smooth morphism of relative dimension $r(n-1)$, whose fibers are iterated $\mathbf{P}^{n-1}$-bundles.

References

[He10] J. Heinloth: *Lectures on the moduli stack of vector bundles on a curve*, in Affine flag manifolds and principal bundles, 123-153, Trends Math., Birkhäuser/Springer Basel AG, Basel, 2010.

[Yun15] Z. Yun: *The Langlands correspondence for global function fields*, Lecture notes by Tony Feng, available at http://web.stanford.edu/ tonyfeng/FFL.pdf.

[YZ15] Z. Yun and W. Zhang: *Shtukas and the Taylor expansion of L-functions*, preprint available at arXiv:1512.02683.

Talk 3: Moduli of Shtukas I

Doug Ulmer

1. Review of Hecke stacks

Let X be a (smooth, projective, geometrically connected) curve over $\mathbf{F}_q$. Let $F = \mathbf{F}_q(X)$. Fix integers $n \geq 1$ and $r \geq 0$. Let $\mu = (\mu_1, \dots, \mu_r)$ with each $\mu_i = \pm 1$. Usually we require that r is even, and moreover that $\sum \mu_i = 0$.

In the previous talk we met the Hecke stack Hk_n^μ, parametrizing modifications of type μ of rank n vector bundles. If S is an $\mathbf{F}_q$-scheme, then $\mathrm{Hk}_n^\mu(S)$ is the groupoid of

- vector bundles $(\mathcal{E}_0, \dots, \mathcal{E}_r)$ on $X \times S$.
- S-valued points $x_i : S \to X$, $i = 1, \dots, r$.
- If $\mu_i = +1$, a map $\phi_i \colon \mathcal{E}_{i-1} \hookrightarrow \mathcal{E}_i$ with cokernel an invertible sheaf supported on Γ_{x_i}. If $\mu_i = -1$, a map $\phi_i \colon \mathcal{E}_i \hookrightarrow \mathcal{E}_{i-1}$ with cokernel an invertible sheaf supported on Γ_{x_i}.

We have a map $\mathrm{Hk}_n^\mu \to \mathrm{Bun}_n \times X^r$ sending $(\underline{\mathcal{E}}, \underline{x}, \underline{\phi}) \to (\mathcal{E}_0, \underline{x})$, which is an iterated $\mathbf{P}^{n-1}$-bundle and thus smooth of fiber dimension $r(n-1)$, so Hk_n^μ is smooth of dimension $n^2(g-1) + nr$.

2. Moduli of shtukas for GL_n

2.1. **Definition.**

Definition 2.1. A *shtuka* of type μ and rank n is a "Hecke modification" plus a Frobenius structure. More precisely, $\mathrm{Sht}_n^\mu(S) = \{(\underline{\mathcal{E}}, \underline{x}, \underline{\phi})\}$ together with an isomorphism $\iota : \mathcal{E}_r \xrightarrow{\sim} {}^\tau\mathcal{E}_0 := (\mathrm{Id}_X \times \mathrm{Frob}_S)^*\mathcal{E}_0$.

We have a cartesian diagram

$$\begin{tikzcd} \mathrm{Sht}_n^\mu \arrow[r] \arrow[d] & \mathrm{Hk}_n^\mu \arrow[d, "p_0 \times p_r"] \\ \mathrm{Bun}_n \arrow[r, "\mathrm{Frob} \times \mathrm{Id}"] & \mathrm{Bun}_n \times \mathrm{Bun}_n \end{tikzcd}$$

Example 2.2. For $n = 1$, the choice of points x_i determines the higher $\mathcal{E}_i$ from $\mathcal{E}_0$, namely $\mathcal{E}_i = \mathcal{E}_{i-1} \otimes \mathcal{O}(\mu_i x_i)$. So $\mathrm{Hk}_1^\mu \cong \mathrm{Pic}_X \times X^r$. For a point of Hk_1^μ to be an element of Sht_1^μ, we also need $\mathcal{E}_r \cong {}^\tau\mathcal{E}_0$, i.e.

$${}^\tau\mathcal{E}_0 \otimes \mathcal{E}_0^{-1} \cong \mathcal{O}(\sum \mu_i x_i).$$

Thus Sht_1^μ is a familiar object: its fibers over X^r are torsors for the kernel of the Lang isogeny, $\mathrm{Pic}_X \to \mathrm{Pic}_X$, i.e., torsors for $\mathrm{Pic}_X^0(\mathbf{F}_q)$.

Example 2.3. For $r = 0$, $\mathrm{Sht}_n^\mu(S)$ is a vector bundle $\mathcal{E}$ on $X \times S$ and an isomorphism $\mathcal{E} \cong {}^\tau\mathcal{E}$. This looks like part of descent data. If $S = \operatorname{Spec} \overline{\mathbf{F}}_q$, then such $\mathcal{E}$ come from $\mathcal{E}$ on X itself via pullback. More generally, for $r = 0$

$$\mathrm{Sht}_n^\mu = \coprod_{\mathcal{E}} [\operatorname{Spec} \mathbf{F}_q / \operatorname{Aut} \mathcal{E}].$$

where the union is over objects $\mathcal{E}$ of $\mathrm{Bun}_n(\mathbf{F}_q)$. What exactly does this mean? Concretely, an element of Sht_n^μ is an $\mathrm{Aut}(\mathcal{E})$-torsor on S, which we can think of as a twisted form of $p_X^*(\mathcal{E})$ on $X \times S$.

2.2. **Basic geometric facts about** Sht_n^μ**.**

1. Sht_n^μ is a Deligne-Mumford stack, smooth and locally of finite type.
2. There is a morphism $\mathrm{Sht}_n^\mu \to X^r$ which is separated, smooth, and of relative dimension $r(n-1)$.

2.3. **Level structures.**

Definition 2.4. Let $D \subset X$ be a finite closed subscheme (in this case, just a finite collection of points with multiplicities). A *level D structure* on $(\underline{\mathcal{E}}, \underline{x}, \underline{\phi})$ is an isomorphism

$$\mathcal{E}_0|_{D \times S} \xrightarrow{\sim} \mathcal{O}_{D \times S}^{\oplus n}$$

such that $|D| \cap \{x_1, \ldots, x_r\} = \emptyset$, which is compatible with Frobenius in the sense that the following diagram commutes:

$$\begin{tikzcd} \mathcal{E}_0|_{D \times S} \arrow[r, "\sim"] \arrow[d, "\sim"] & \mathcal{O}_{D \times S}^{\oplus n} \arrow[d, equal] \\ {}^\tau\mathcal{E}_0|_{D \times S} \arrow[r, "\sim"] & {}^\tau\mathcal{O}_{D \times S}^{\oplus n} \end{tikzcd}$$

Note that there is an action of $\mathrm{GL}_n(\mathcal{O}_D)$ on the set of level structures.

In practice, we'll introduce level structure in order to rigidify the objects.

2.4. **Stability conditions.** The components of Bun_n are indexed by $\mathbf{Z}$, via

$$\mathcal{E} \mapsto \deg \det \mathcal{E}.$$

We need to fix this to get something of finite type. But that still won't be enough, since we have things like $\mathcal{O}(ap) \oplus \mathcal{O}(-ap)$. For a vector bundle $\mathcal{E}$, let

$$M(\mathcal{E}) := \max\{\deg \mathcal{L} \mid \mathcal{L} \hookrightarrow \mathcal{E}\}.$$

This is enough to cut down to something of finite type.

Definition 2.5. Define $\mathrm{Sht}_{n,D,d,m}^\mu$ to be the stack whose S-points consist of data $(\underline{\mathcal{E}}, \underline{x}, \underline{\phi}, \iota : \mathcal{E}_r \xrightarrow{\sim} {}^\tau\mathcal{E}_0)$ and a level D structure such that $\deg(\det \mathcal{E}_0) = d$ and $M(\mathcal{E}_0) \leq m$.

Facts:

1. If $\deg(D) \gg 0$ (with respect to n, m, d) then $\mathrm{Sht}_{n,D,d,m}^\mu$ is represented by a quasi-projective variety.

(2) The map $[\mathrm{Sht}^{\mu}_{n,D,d,m}/\mathrm{GL}_n(\mathcal{O}_D)] \hookrightarrow \mathrm{Sht}^{\mu}_n$ is an open embedding.
(3) Sht^{μ}_n is the union of these substacks for varying d, m.

This is enough to check that Sht^{μ}_n is a DM stack locally of finite type over $\mathbf{F}_q$.

2.5. **Smoothness.** Recall the cartesian square

$$\begin{tikzcd} \mathrm{Sht}^{\mu}_n \arrow[r] \arrow[d] & \mathrm{Hk}^{\mu}_n \arrow[r] \arrow[d, "p_0 \times p_r"] & X^r \\ \mathrm{Bun}_n \arrow[r, "\mathrm{Frob}\times \mathrm{Id}"] & \mathrm{Bun}_n \times \mathrm{Bun}_n & \end{tikzcd}$$

Note that $d\,\mathrm{Frob} = \mathrm{Frob}_* = 0$, and $\mathrm{Id}_* = \mathrm{Id}$. On the other hand, p_{0*} and p_{r*} are both surjections.

Corollary 2.6. *The maps* $(\mathrm{Frob}, \mathrm{Id})\colon \mathrm{Bun}_n \to \mathrm{Bun}_n \times \mathrm{Bun}_n$ *and* $(p_0, p_r)\colon \mathrm{Hk}^{\mu}_n \to \mathrm{Bun}_n \times \mathrm{Bun}_n$ *are transverse.*

Corollary 2.7. *The map* $\mathrm{Sht}^{\mu}_n \to X^r$ *is smooth, and so has relative dimension* $(n-1)r$.

3. Moduli of Shtukas for PGL_2

Let $G = \mathrm{PGL}_2 = \mathrm{GL}_2/\mathbf{G}_m$, and let Bun_G be the stack of G-torsors on X, which is isomorphic to $\mathrm{Bun}_2/\mathrm{Bun}_1$, with the action being $\otimes$. This action lifts to Hk^{μ}_2, by

$$(\underline{\mathcal{E}}, \underline{x}, \underline{\phi}) \mapsto (\underline{\mathcal{E}} \otimes \mathcal{L}, \underline{x}, \underline{\phi} \otimes \mathrm{Id}).$$

This action doesn't restrict to Sht^{μ}_2 unless $\mathcal{L} \cong {}^{\tau}\mathcal{L}$. Therefore, only the subgroup $\mathrm{Pic}_X(\mathbf{F}_q)$ acts on Sht^{μ}_2. We have cartesian diagrams

$$\begin{tikzcd} \mathrm{Pic}_X(\mathbf{F}_q) \arrow[r] \arrow[d] & \mathrm{Pic}_X \arrow[d] \\ \mathrm{Pic}_X \arrow[r, "\mathrm{Frob}\times \mathrm{Id}"] & \mathrm{Pic}_X \times \mathrm{Pic}_X . \end{tikzcd} \tag{1}$$

and

$$\begin{tikzcd} \mathrm{Sht}^{\mu}_n \arrow[r] \arrow[d] & \mathrm{Hk}^{\mu}_n \arrow[d, "p_0 \times p_r"] \\ \mathrm{Bun}_n \arrow[r, "\mathrm{Frob}\times \mathrm{Id}"] & \mathrm{Bun}_n \times \mathrm{Bun}_n \end{tikzcd} \tag{2}$$

and the objects for $G = \mathrm{PGL}_2$ are obtained by quotienting the second diagram (2) by the action of the corresponding groups in the first diagram (1).

3.1. **Independence of signs when $n = 2$.** If μ, μ' are r-tuples of signs and $n = 2$, then there is a canonical isomorphism $\mathrm{Sht}_G^{\mu} \xrightarrow{\sim} \mathrm{Sht}_G^{\mu'}$. We'll show this by giving an explicit isomorphism between Sht_G^{μ}, for any μ, and $\mathrm{Sht}^{\mu'}$ where $\mu' = (+1, \dots, +1)$.

Suppose we are given $(\underline{\mathcal{E}}, \underline{x}, \underline{\phi}, \iota) \in \mathrm{Sht}_G^{\mu}$. The key idea is that we can transform an injection $\mathcal{E}_{i-1} \hookleftarrow \mathcal{E}_i$ with deg 1 cokernel into $\mathcal{E}_{i-1} \hookrightarrow \mathcal{E}_i \otimes \mathcal{O}(x_i)$. So we take every instance of $\mathcal{E}_{i-1} \hookleftarrow \mathcal{E}_i$, which is a modification of type $\mu_i = -1$, into $\mathcal{E}_{i-1} \hookrightarrow \mathcal{E}_i \otimes \mathcal{O}(x_i)$, which is a modification of type $\mu_i = +1$. Given $(\underline{\mathcal{E}}, \underline{x}, \underline{\phi})$ let

$$D_i := \sum_{\substack{1 \le j \le i \\ \mu_j = \mu_-}} \Gamma_{x_i}.$$

Let $\mathcal{E}_i' = \mathcal{E}_i \otimes \mathcal{O}_{X \times S}(D)$, and note that

$$\mathcal{E}_0' \hookrightarrow \mathcal{E}_1' \hookrightarrow \dots \hookrightarrow \mathcal{E}_r'$$

inherits the structure of an element of $\mathrm{Sht}_G^{\mu'}$.

References

[1] V. G. Drinfel′d. Moduli varieties of F-sheaves. *Funktsional. Anal. i Prilozhen.*, 21:23–41, 1987.

[2] L. Lafforgue. Chtoucas de Drinfeld et conjecture de Ramanujan-Petersson. *Astérisque*, (243):ii+329, 1997.

[3] Y. Varshavsky. Moduli spaces of principal F-bundles. *Selecta Math. (N.S.)*, 10:131–166, 2004.

[4] Z. Yun and W. Zhang. Shtukas and the Taylor expansion of L-functions. Preprint, arXiv:1512.02683, 2015.

Talk 4: Moduli of Shtukas II

Brian Smithling

1. Goal

The main goal of this talk was to define the intersection number

$$\mathbb{I}_r(f) := \left\langle \theta_*^{\mu}[\mathrm{Sht}_T^{\mu}], f * \theta_*^{\mu}[\mathrm{Sht}_T^{\mu}] \right\rangle_{\mathrm{Sht}_G'^r} \in \mathbf{Q} \tag{1}$$

for $r \ge 0$ an even integer and f an element of the spherical Hecke algebra $\mathcal{H}$. Here $\theta_*^{\mu}[\mathrm{Sht}_T^{\mu}] \in \mathrm{Ch}_{c,r}(\mathrm{Sht}_G'^r)_{\mathbf{Q}}$ is a class in the "rational Chow group of dimension r cycles proper over the base field," and the pairing is the natural one between proper cycle classes of complementary dimension. This Chow group is acted on by ${}_c\mathrm{Ch}_{2r}(\mathrm{Sht}_G'^r \times \mathrm{Sht}_G'^r)_{\mathbf{Q}}$, the "rational Chow group of dimension $2r$ cycles proper over the first factor," which has a $\mathbf{Q}$-algebra structure. We refer the reader to Rapoport's talk on intersection theory (or to [YZ15, §A.1]) for the Chow-theoretic generalities. The key points for us in making sense of the right-hand side of (1) were to define the stacks Sht_T^{μ} and $\mathrm{Sht}_G'^r$, to define a ring homomorphism $H' \colon \mathcal{H} \to {}_c\mathrm{Ch}_{2r}(\mathrm{Sht}_G'^r \times \mathrm{Sht}_G'^r)$, and to define the morphism $\theta^{\mu} \colon \mathrm{Sht}_T^{\mu} \to \mathrm{Sht}_G'^r$. We closely follow [YZ15, §§5.3–5.5].

2. The Hecke algebra

Let k be a finite field with q elements, and keep the notation of the previous two talks. Let $G := \mathrm{PGL}_2$. For $x \in |X|$, we set $K_x := G(\mathcal{O}_x)$ and $K := \prod_{x\in|X|} K_x$.

Definition 2.1. The *spherical Hecke algebra* for G over X is the convolution algebra

$$\mathcal{H} := C_c^\infty\big(K\backslash G(\mathbf{A}_F)/K, \mathbf{Q}\big) = \bigotimes_{x\in|X|}{}' \mathcal{H}_x,$$

where $\mathcal{H}_x := C_c^\infty(K_x\backslash G(F_x)/K_x, \mathbf{Q})$ is the local Hecke algebra at x.

For $n \geq 0$, let $M_{x,n}$ denote the image of $\mathrm{Mat}_2(\mathcal{O}_x)_{\mathrm{val}_x(\det)=n}$ in $G(F_x)$, and set $h_{nx} := \mathbf{1}_{M_{x,n}} \in \mathcal{H}_x$. It is easy to see from the Cartan decomposition that the functions h_{nx} form a $\mathbf{Q}$-basis for $\mathcal{H}_x$. For $D = \sum_{x\in|X|} n_x x$ an effective divisor on X, we set $h_D := \otimes_{x\in X} h_{nx} \in \mathcal{H}$. Then the h_D's for varying D form a $\mathbf{Q}$-basis for $\mathcal{H}$.

3. Hecke correspondences

Let μ be an r-tuple with the same number of μ_+'s and μ_-'s, as in Ulmer's talk (a *balanced* r-tuple). Let D be an effective divisor on X. We define $\mathrm{Sht}_2^\mu(h_D)$ to be the stack over k whose S-points classify the data of an r-tuple $(x_1, \dots, x_r)$ of morphisms $S \to X$ together with a commutative diagram

$$\begin{tikzcd}
\mathcal{E}_0 \arrow[r, dashed] \arrow[d, hook, "\phi_0"] & \mathcal{E}_1 \arrow[r, dashed] \arrow[d, hook, "\phi_1"] & \cdots \arrow[r, dashed] & \mathcal{E}_r \arrow[r, "\sim"] \arrow[d, hook, "\phi_r"] & {}^\tau\mathcal{E}_0 \arrow[d, hook, "{}^\tau\phi_0"] \\
\mathcal{E}_0' \arrow[r, dashed] & \mathcal{E}_1' \arrow[r, dashed] & \cdots \arrow[r, dashed] & \mathcal{E}_r' \arrow[r, "\sim"] & {}^\tau\mathcal{E}_0'
\end{tikzcd} \tag{2}$$

such that the top and bottom rows, together with the common tuple $(x_1, \dots, x_r)$, both form points on Sht_2^μ, and such that the map of line bundles $\det\phi_i \colon \det\mathcal{E} \hookrightarrow \det\mathcal{E}'$ has divisor $D \times S \subset X \times S$ for all i. The groupoid $\mathrm{Pic}_X(k)$ acts naturally on Sht_2^μ (by tensoring everything in (2)), and we define

$$\mathrm{Sht}_G^\mu(h_D) := \mathrm{Sht}_2^\mu(h_D)/\operatorname{Pic}_X(k).$$

Both Sht_2^μ and Sht_G^μ are independent of μ up to canonical isomorphism over X^r, and therefore we replace μ by r in the notation. By definition, there is a commutative diagram

$$\begin{tikzcd}
& \mathrm{Sht}_G^r(h_D) \arrow[dl, "\overleftarrow{p}"'] \arrow[dr, "\overrightarrow{p}"] & \\
\mathrm{Sht}_G^r \arrow[dr] & & \mathrm{Sht}_G^r \arrow[dl] \\
& X^r &
\end{tikzcd} \tag{3}$$

where $\overleftarrow{p}$ and $\overrightarrow{p}$ are induced by projecting off the top and bottom rows of (2), respectively.

Lemma 3.1 ([YZ15, Lem. 5.8]). *The morphisms $\overleftarrow{p}$ and $\overrightarrow{p}$ are representable and proper, and the morphism $(\overleftarrow{p}, \overrightarrow{p})\colon \mathrm{Sht}^r_G(h_D) \to \mathrm{Sht}^r_G \times \mathrm{Sht}^r_G$ is representable and finite.*

Sketch proof. For $\overrightarrow{p}$, its fibers are closed subschemes in a product of Quot schemes. For $\overleftarrow{p}$, we dualize the diagram (2) and apply the same argument. For representability and affineness of $(\overleftarrow{p}, \overrightarrow{p})$, its fibers are closed in a product of Hom schemes of vector bundles. Properness, and hence finiteness, then follows from Sht^r_G being separated and $\overleftarrow{p}$ being proper. □

We refer to [YZ15] for proofs of the next two results.

Lemma 3.2 ([YZ15, Lem. 5.9]). *The geometric fibers of $\mathrm{Sht}^r_G(h_D) \to X^r$ have dimension r, and hence $\dim \mathrm{Sht}^r_G(h_D) = 2r$.* □

By the lemmas, we may define a $\mathbf{Q}$-linear map

$$H\colon \mathcal{H} \longrightarrow {}_c\mathrm{Ch}_{2r}(\mathrm{Sht}^r_G \times \mathrm{Sht}^r_G)_{\mathbf{Q}}, \quad h_D \longmapsto (\overleftarrow{p}, \overrightarrow{p})_*[\mathrm{Sht}^r_G(h_D)].$$

Proposition 3.3 ([YZ15, Prop. 5.10]). *The map H is a ring homomorphism.* □

In fact we need a variant of the above discussion for the definition of $\mathbb{I}_r(f)$. Let $\nu\colon X' \to X$ be an étale cover of degree 2, with the curve X' also geometrically connected. We base change the diagram (3) under $(X')^r \times_{X^r} -$, and systematically use a prime for the objects in the new diagram. Then we analogously get a map

$$H'\colon \mathcal{H} \longrightarrow {}_c\mathrm{Ch}_{2r}(\mathrm{Sht}'^r_G \times \mathrm{Sht}'^r_G)_{\mathbf{Q}}, \quad h_D \longmapsto (\overleftarrow{p}', \overrightarrow{p}')_*[\mathrm{Sht}'^r_G(h_D)],$$

which is again a ring homomorphism.

Definition 3.4. For $f \in \mathcal{H}$, we write $f * -$ for the operator on $\mathrm{Ch}_{c,*}(\mathrm{Sht}'^r_G)_{\mathbf{Q}}$ induced by $H'(f)$ under the action of ${}_c\mathrm{Ch}_{2r}(\mathrm{Sht}'^r_G \times \mathrm{Sht}'^r_G)_{\mathbf{Q}}$.

4. The Heegner-Drinfeld cycle

We continue with μ a balanced r-tuple. Let $\widetilde{T} := \mathrm{Res}_{X'/X}\,\mathbf{G}_m$ and $T := \widetilde{T}/\mathbf{G}_m$. We define $\mathrm{Sht}^\mu_{\widetilde{T}} := \mathrm{Sht}^\mu_{1,X'}$, the moduli stack of rank 1 shtukas over X' of type μ (where now we interpret $\mu_\pm$ as the coweight ± 1 for $\mathbf{G}_m$). The Picard groupoid $\mathrm{Pic}_{X'}(k)$ acts naturally on $\mathrm{Sht}^\mu_{\widetilde{T}}$, and hence so does $\mathrm{Pic}_X(k)$ via the pullback map $\nu^*\colon \mathrm{Pic}_X(k) \to \mathrm{Pic}_{X'}(k)$. We set $\mathrm{Sht}^\mu_T := \mathrm{Sht}^\mu_{\widetilde{T}}/\mathrm{Pic}_X(k)$. Then the forgetful map

$$\pi^\mu_T\colon \mathrm{Sht}^\mu_T \longrightarrow (X')^r$$

is a torsor under the finite Picard groupoid $\mathrm{Pic}_{X'}(k)/\mathrm{Pic}_X(k)$. Hence Sht^μ_T is proper smooth of dimension r over k. Furthermore Sht^μ_T is canonically independent of μ over X^r (but not as a stack over $(X')^r$).

Quite generally, for $\mathcal{L}$ a line bundle on $X' \times S$, the pushforward $\nu_*\mathcal{L}$ is a vector bundle of rank 2 on $X \times S$. This induces a morphism

$$\overline{\theta}^\mu\colon \mathrm{Sht}^\mu_T \longrightarrow \mathrm{Sht}^r_G, \quad (\underline{x}', \underline{\mathcal{L}}, \underline{f}, \iota) \longmapsto \big(\nu(\underline{x}'), \nu_*\underline{\mathcal{L}}, \nu_*\underline{f}, \nu_*\iota\big).$$

We then define

$$\theta^\mu\colon \mathrm{Sht}^\mu_T \xrightarrow{(\pi^\mu_T, \overline{\theta}^\mu)} (X')^r \times_{X^r} \mathrm{Sht}^r_G = \mathrm{Sht}'^r_G.$$

Since Sht_T^μ is proper over k of dimension r, we may make the following definition.

Definition 4.1. The *Heegner–Drinfeld cycle* is the class $\theta_*^\mu[\mathrm{Sht}_T^\mu] \in \mathrm{Ch}_{c,r}(\mathrm{Sht}_G'^r)_{\mathbf{Q}}$.

Since $\dim \mathrm{Sht}_G'^r = 2r$ by Ulmer's talk, the cycle classes $\theta_*^\mu[\mathrm{Sht}_T^\mu]$ and $f * \theta_*^\mu[\mathrm{Sht}_T^\mu]$, for $f \in \mathcal{H}$, are of complementary dimensions. With this the definition of $\mathbb{I}_r(f)$ in (1) now makes sense. The value is independent of μ by [YZ15, Lem. 5.16].

Let us conclude by stating a rough version of the main result of [YZ15].

Theorem 4.2. *If π is an everywhere unramified cuspidal automorphic representation of $G(\mathbf{A}_F)$, then there is an equality up to some explicit constant factors,*

$$\mathcal{L}^{(r)}(\pi_{F'}, 1/2) \sim \left([\mathrm{Sht}_T^\mu]_\pi, [\mathrm{Sht}_T^\mu]_\pi\right)_\pi.$$

Here the left-hand side is the central value of the rth derivative of a modified L-function for the base change $\pi_{F'}$ of π to $F' = k(X')$. The right-hand side is, roughly speaking, the self intersection of the π-isotypic component $[\mathrm{Sht}_T^\mu]_\pi$ of the Heegner–Drinfeld cycle under the pairing induced by $\langle\ ,\ \rangle_{\mathrm{Sht}_G'^r}$.

References

[1] Z. Yun and W. Zhang, *Shtukas and the Taylor expansion of L-functions*, to appear in Ann. of Math.

Talk 5: Automorphic forms over function fields

Ye Tian

1. Cuspidal automorphic forms

1.1. **Goal.** Let X/k be a curve over a finite field of genus g and $F = k(X)$. Let $\mathbf{A} = \mathbf{A}_F$ and $\mathcal{O} = \prod_{x \in |X|} \mathcal{O}_x$, where $\mathcal{O}_x$ is the completed local ring of X at x.

Let $G = \mathrm{GL}_d$ and Z be the center of G. Then $G(\mathcal{O})$ is a maximal compact subgroup of $G(\mathbf{A})$.

Definition 1.1. A function $\varphi \colon G(F)\backslash G(\mathbf{A}) \to \mathbf{C}$ is called *smooth* if it factors through $G(F)\backslash G(\mathbf{A})/K$ for some open subgroup K of $G(\mathbf{A})$; it is called *cuspidal* if for any proper standard parabolic $P \subset G$, with unipotent N, the *constant term* along P

$$\varphi_P(g) = \int_{N(F)\backslash N(\mathbf{A})} \varphi(ng)\, dn$$

vanishes.

The main goal of the talk is to prove:

Theorem 1.2 (Harder)**.** *For a compact open subgroup $K \subseteq G(\mathcal{O})$, cuspidal functions $\varphi : G(F)\backslash G(\mathbf{A})/K \to \mathbf{C}$ have support uniformly finite modulo $Z(\mathbf{A})$.*

The main reference is [1] Chapter 9, §9.1-9.2, and Appendices D.6 and E.0-E.1;

1.2. **Cuspidal Automorphic representations.**

Definition 1.3. A smooth function $\varphi\colon G(F)\backslash G(\mathbf{A}) \to \mathbf{C}$ is called *automorphic* if the space spanned by right translations by $G(\mathbf{A})$ of φ is admissible. (Recall that a smooth representation of a totally disconnected locally compact Hausdorff topological group is *admissible* if the subspace of fixed vectors under any compact subgroup is finite dimensional.)

Corollary 1.4. *A cuspidal smooth function $\varphi\colon G(F)\backslash G(\mathbf{A}) \to \mathbf{C}$ is automorphic if and only if φ is Z-finite, i.e.* $\dim_{\mathbf{C}}\langle\{\varphi(\cdot z), z \in Z(\mathbf{A})\}\rangle < \infty$.

Definition 1.5. Let χ_G denote the group of complex valued smooth characters on $Z(F)\backslash Z(\mathbf{A})$. A function $\varphi\colon G(\mathbf{A}) \to \mathbf{C}$ has *central character* $\chi \in \chi_G$ if $\varphi(zg) = \chi(z)\varphi(g)$ for all $z \in Z(\mathbf{A})$.

Remark 1.6. If φ is cuspidal automorphic form with a central character, after twisting by $\mu \circ \det$ for some idele character μ, we may view φ as a function on $G(F)\backslash G(\mathbf{A})/Ka^{\mathbf{Z}}$, for some open compact subgroup K and some $a \in Z(\mathbf{A}) = \mathbf{A}^\times$ with $\deg a = 1$.

Let $\mathcal{A}_{\text{cusp}}(G(F)\backslash G(\mathbf{A})/Ka^{\mathbf{Z}})$ denote the space of cuspdial automorphic forms $\varphi : G(F)\backslash G(\mathbf{A})/Ka^{\mathbf{Z}} \to \mathbf{C}$. Harder's theorem implies that

$$\mathcal{A}_{\text{cusp}}(G(F)\backslash G(\mathbf{A})/Ka^{\mathbf{Z}})$$

has a finite support and therefore is of finite dimension.

Definition 1.7. We define $\mathcal{A}_{G,\text{cusp},\chi}$ to be the space of automorphic cuspidal forms on $G(\mathbf{A})$ of central character χ. This has an action of $G(\mathbf{A})$ by right translation.

Theorem 1.8. *For any $\chi \in \chi_G$, $\mathcal{A}_{G,\text{cusp},\chi}$ is an admissible representation of $G(\mathbf{A})$. Moreover, it has a countable direct sum decomposition*

$$\mathcal{A}_{G,\text{cusp},\chi} = \bigoplus_{\pi\in\Pi_{G,\text{cusp},\chi}} \pi.$$

Here $\Pi_{G,\text{cusp},\chi}$ is the set of equivalence classes of irreducible automorphic cuspidal representations of central character χ.

What is the content of this statement? By definition that $\pi \in \Pi_{G,\text{cusp},\chi}$ occurs as a subquotient. The theorem says that it actually occur as an honest subrepresentation with multiplicity one.

Proof. Admissibility follows from Harder's theorem.

Semisimplicity: after twisting $\mathcal{A}_{G,\text{cusp},\chi} \otimes (\mu \circ \det)$, we can assume that is χ is unitary. Then

$$\langle\varphi_1, \varphi_2\rangle := \int_{G(F)Z(\mathbf{A})\backslash G(\mathbf{A})} \overline{\varphi_1}\varphi_2 \, dg$$

defines a $G(\mathbf{A})$-invariant positive definite Hermitian scalar product on $\mathcal{A}_{G,\text{cusp},\chi}$. This implies a direct sum decomposition of an admissible representation

$$\mathcal{A}_{G,\text{cusp},\chi} = \bigoplus \pi^{m(\pi)}$$

with $m(\pi) := \dim_{\mathbf{C}} \mathrm{Hom}_{G(\mathbf{A})}(\pi, \mathcal{A}_{G,\mathrm{cusp},\chi}) \geq 1$ for every $\pi \in \Pi_{G,\mathrm{cusp},\chi}$. Since $G(\mathbf{A})$ has a countable open basis at e, the decomposition is countable.

To see that $m(\pi) = 1$, we use that the space of Whittaker functional is 1-dimensional. If $\psi\colon F\backslash\mathbf{A}_F \to \mathbf{C}^\times$ is a non-trivial unitary character, and U is the unipotent radical of the Borel, then ψ defines a character on $U(\mathbf{A})$ and we have that

$$\mathrm{Hom}_{U(\mathbf{A})}(\pi, \psi) \cong \mathrm{Hom}_{G(\mathbf{A})}(\pi, \mathrm{Ind}_{U(\mathbf{A})}^{G(\mathbf{A})} \psi)$$

is one-dimensional, which is proved by passing to the local Whittaker model.

If $\xi\colon \pi \hookrightarrow \mathcal{A}_{G,\mathrm{cusp},\chi}$ then we get a functional $W_\xi \in \mathrm{Hom}_{G(\mathbf{A})}(\pi, \mathrm{Ind}_{U(\mathbf{A})}^{G(\mathbf{A})} \psi)$, sending

$$\varphi \longmapsto \left(W_{\xi(\varphi)} : g \mapsto \int_{U(F)\backslash U(\mathbf{A})} \xi(\varphi)(ng)\psi(n)^{-1} dn \right).$$

On the other hand, we can"recover" the automorphic form $\xi(\varphi)$ via its Foruier expansion

$$\xi(\varphi)(g) = \sum_{\gamma \in U_{d-1}(F)\backslash G_{d-1}(F)} W_{\xi(\varphi)}[\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g]$$

so the 1-dimensionality of the Whittaker model for π implies $m(\pi) = 1$. □

2. Reduction theory on Bun_G

Definition 2.1. For a non-zero vector bundle $\mathcal{E}$ over X, the slope of $\mathcal{E}$ is defined to be $\mu(\mathcal{E}) := \frac{\deg \mathcal{E}}{\mathrm{rank}\, \mathcal{E}}$. We have $\deg \mathcal{E} = \deg(\det \mathcal{E})$.

By Riemann-Roch,

$$\chi(\mathcal{E}) = \deg \mathcal{E} + \mathrm{rank}\, \mathcal{E} \cdot (1 - g_X).$$

Definition 2.2. A (non-zero) vector bundle $\mathcal{E}$ over X is said to be *semistable* if for all sub-bundles

$$0 \subsetneq \mathcal{F} \subset \mathcal{E},$$

we have $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$. There is an equivalent formulation in terms of quotients, i.e. for any quotient $\mathcal{E} \twoheadrightarrow \mathcal{G} \neq 0$, $\mu(\mathcal{G}) \geq \mu(\mathcal{E})$.

Definition 2.3. A filtration of a vector bundle $\mathcal{E}$ on X

$$0 = F_0\mathcal{E} \subset F_1\mathcal{E} \subset \ldots \subset F_s\mathcal{E} = \mathcal{E}$$

is a *Harder-Narasimhan (HN) filtration* if $F_j\mathcal{E}/F_{j-1}\mathcal{E}$ are semistable with slopes μ_j satisfying

$$\mu_1 > \mu_2 > \ldots > \mu_s.$$

Example 2.4. Let $X = \mathbf{P}^1/k$. Then $\mathcal{E} = \bigoplus_{i=1}^s \mathcal{O}(n_i)^{r_i}$ with $n_1 > n_2 > \ldots > n_s$ and $r_i \geq 1$ integers, and the HN filtration of $\mathcal{E}$ is

$$0 \subset \mathcal{O}(n_1)^{r_1} \subset \mathcal{O}(n_1)^{r_1} \oplus \mathcal{O}(n_2)^{r_2} \subset \ldots \subset .$$

Theorem 2.5 (Harder-Narasimhan)**.** *Any non-zero vector bundle over X admits a unique HN filtration.*

Proof. Let μ_1 be the maximal slope of sub-bundles $\mathcal{F} \subset \mathcal{E}$. By Riemann-Roch, we know this to be $< \infty$. We claim that in any HN filtration, $F_1\mathcal{E}$ is the maximal subbundle $\mathcal{E}_1$ with $\mu(\mathcal{E}_1) = \mu_1$. (The uniqueness would then follow by induction.) In fact, for any non-zero sub bundle $\mathcal{F} \subset \mathcal{E}$, let $F_i\mathcal{F} = F_i\mathcal{E} \cap \mathcal{F}$. Then we have

$$\mu(\mathcal{F}) \leq \max_i \left(\mu(F_j\mathcal{F}/F_{j-1}\mathcal{F}), F_j\mathcal{F}/F_{j-1}\mathcal{F} \neq 0\right) \leq \max_i \left(\mu(F_j\mathcal{E}/F_{j-1}\mathcal{E})\right) = \mu(F_1\mathcal{E}),$$

and the equality holds only if $F_j\mathcal{F}/F_{j-1}\mathcal{F} = 0$ for all $j = 2, \cdots, s$, i.e. $\mathcal{F} \subset F_1\mathcal{E}$.

To see that $\mathcal{E}_1$ exists, suppose you have distinct $\mathcal{E}_1', \mathcal{E}_1''$ which both have maximal rank r_1 with slope μ_1. Consider $\mathcal{F} := \langle \mathcal{E}_1' + \mathcal{E}_1'' \rangle$, the saturation of the subsheaf of $\mathcal{E}$ spanned by $\mathcal{E}_1'$ and $\mathcal{E}_1''$. Then

$$\deg \mathcal{F} \geq 2r_1\mu_1 - \deg(\mathcal{E}_1' \cap \mathcal{E}_1'')$$

(the inequality comes from the saturation) while

$$\operatorname{rank} \mathcal{F} = 2r_1 - \operatorname{rank}(\mathcal{E}_1' \cap \mathcal{E}_1'') > r_1.$$

So $\mu(\mathcal{F}) \geq \mu_1$ and dominates both $\mathcal{E}_1'$ and $\mathcal{E}_1''$. □

Write the Borel $B = TU$ with T diagonal and U its unipotent. By Weil's adelic uniformization, we can interpret

$$B(F)\backslash B(\mathbf{A})/B(\mathcal{O}) \leftrightarrow \text{ isomorphism classes of flags of rank } (1, \ldots, 1).$$

Let Δ be the set of simple roots of G.

Theorem 2.6 (Siegel Domain)**.** *Let $c_2 \geq 2g$ be an integer. Then*

$$G(\mathbf{A}) = G(F)U(\mathbf{A})T(\mathbf{A})_{c_2}^{\Delta}G(\mathcal{O})$$

where $T(\mathbf{A})_{c_2}^{\Delta} = \{t \in T(\mathbf{A})\colon \deg \alpha(t) \leq c_2, \forall \alpha \in \Delta\}$. In other words (by Iwasawa decomposition), for every $\mathcal{E}$ of rank d over X, there is at least one flag

$$0 \subset \mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_d = \mathcal{E}$$

such that $\deg(\mathcal{E}_{j+1}/\mathcal{E}_j) - \deg(\mathcal{E}_j/\mathcal{E}_{j-1}) \leq c_2$ *for all j.*

Proof. Take a line bundle $\mathcal{L}$ over X such that

$$1 \leq \deg \mathcal{E} - d \deg \mathcal{L} + d(1-g) \leq d. \tag{1}$$

By Riemann-Roch, the lower bound implies that $H^0(X, \mathcal{E} \otimes \mathcal{L}^\vee)$ is non-zero, and therefore there exists $\mathcal{L} \hookrightarrow \mathcal{E}$. Let $\mathcal{E}_1$ be the saturation of the image of $\mathcal{L}$. The upper bound gives rise to an inequality

$$\deg \mathcal{E}_1 \geq \deg \mathcal{L} \geq \frac{\deg \mathcal{E}}{d} - g.$$

By induction, we can lift a filtration of $\mathcal{E}/\mathcal{E}_1$ with this property to one of $\mathcal{E}$, say $(0) \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots$. The only question is to check the desired inequality for $i = 1$. If $\mathcal{E}$ is semistable, then $\frac{\deg \mathcal{E}_2}{2} \leq \frac{\deg \mathcal{E}}{d}$, together with the inequality $\deg \mathcal{E}_1 \geq \frac{\deg \mathcal{E}}{d} - g$, which implies

$$\deg(\mathcal{E}_2/\mathcal{E}_1) - \deg \mathcal{E}_1 = 2\left(\frac{\deg \mathcal{E}_2}{2} - \deg \mathcal{E}_1\right) \leq 2g \leq c_2.$$

If $\mathcal{E}$ is not semistable, take an HN filtration, whose associated subquotients are semistable by definition. We apply the conclusion from the semistable case to each subquotient. The only issue is to check that the inequality still holds at the endpoints. The desired inequalities end up following from the semistability.

$\square$

Theorem 2.7. *Let $K \subset G(\mathcal{O})$ be a compact open subgroup. There exists an open subset $C_K \subset G(\mathbf{A})$ satisfying*

(1) *$Z(\mathbf{A})G(F)C_K K = C_K$, i.e. C_K is left invariant under $Z(\mathbf{A})G(F)$ and right invariant under K, and*
(2) *$Z(\mathbf{A})G(F)\backslash C_K/K$ is finite.*

Moreover, $\operatorname{supp}\varphi \subset C_K$ *for all cuspidal $\varphi : G(F)\backslash G(\mathbf{A})/K \to \mathbf{C}$.*

Proof. Let $c_2 \geq 2g$. It follows from the compactness of $U(F)\backslash U(\mathbf{A})$ and $F^\times\backslash \mathbf{A}^{\times 0\leq \deg \leq \deg(x)}$ with $x \in |X|$ a closed point, that

$$G(\mathbf{A}) = G(F)U(\mathbf{A})T(\mathbf{A})^{\Delta}_{c_2}G(\mathcal{O}) = G(F)T(\mathbf{A})^{\Delta}_{c_2}\Omega$$

where Ω is an open compact subset with $\Omega K = \Omega$. Then there exists an open subgroup $K' \subset K$ such that $\bigcup_{u\in\Omega} u^{-1}K'u \subset K$. We claim that there exits $c_1 < 0$ such that for any $\alpha \in \Delta$, any $t \in T(\mathbf{A})^{\Delta,\alpha}_{c_2,c_1} := \{t \in T(\mathbf{A})^{\Delta}_{c_2} \mid \deg\deg\alpha(t) < c_1\}$, we have

$$N^\alpha(\mathbf{A}) = N^\alpha(F)\cdot(N^\alpha(\mathbf{A}) \cap tK't^{-1}).$$

This is based on the fact that $\mathbf{A} = F + a\mathcal{O}$ for any $a \in \mathbf{A}^\times$ with degree sufficient small. Take $C_K = G(F)T(A)^{\Delta}_{[c_1,c_2]}\Omega$, where

$$T(A)^{\Delta}_{[c_1,c_2]} := \{t \in T(\mathbf{A}) \mid c_1 \leq \deg(\alpha(t)) \leq c_2\}.$$

Note that $Z(\mathbf{A}) \subset T(\mathbf{A})^{\Delta}_{[c_1,c_2]}$ since $c_1 < 0$. Suppose that $g \in G(\mathbf{A}) \setminus C_K$ is an element with

$$g = \gamma tu, \quad \gamma \in G(F),\ t \in T(\mathbf{A})^{\Delta}_{c_2},\ u \in \Omega.$$

Namely, there is $\alpha \in \Delta$ such that $\deg(\alpha(t)) < c_1$. Thus for any given cuspidal φ

$$0 = \varphi_{P^\alpha}(tu) = \int_{N^\alpha(F)\backslash N^\alpha(\mathbf{A})} \varphi(ntu)dn = \varphi(tu) = \varphi(g).$$

Here we have used the facts that $N^\alpha(\mathbf{A}) = N^\alpha(F)\cdot(N^\alpha(\mathbf{A}) \cap tK't^{-1})$, $u^{-1}K'u \subset K$, and the measure dn is chosen such that the volume of $N^\alpha(F)\backslash N^\alpha(\mathbf{A})$ is one.

$\square$

References

[1] G. Laumon, *Cohomology of Drinfeld Modular Varieties*, Part I and Part II, Cambridge studies in advanced mathematics 56.

Talk 6: The work of Drinfeld

Arthur-César Le Bras

The goal of the talk was to justify the central role played by moduli spaces of shtukas in the Langlands program, by giving a brief overview of the work of Drinfeld on the global Langlands correspondence for function fields. This is a big and deep subject and we decided to focus on the results of [3] and on the relation between elliptic modules and shtukas with two legs. Our discussion follows closely [1] and [2].

As usual, let $k = \mathbf{F}_q$, X a smooth projective, geometrically connected curve over k and $F = k(X)$. Choose a point $\infty \in |X|$, and assume for simplicity that $\deg(\infty) = 1$. Let F_∞ be the completion of F at ∞, $\mathbf{C}_\infty$ be the completion of a separable closure $\overline{F}_\infty$ of F_∞, and $A = H^0(X \setminus \{\infty\}, \mathcal{O})$.

1. Elliptic modules

1.1. **Definition.** The seed of shtukas were Drinfeld's *elliptic modules*. Let $\mathbf{G}_a$ be the additive group, and K a characteristic p field. We set $K\{\tau\} = K \otimes_{\mathbf{Z}} \mathbf{Z}[\tau]$, with multiplication given by

$$(a \otimes \tau^i)(b \otimes \tau^j) = ab^{p^i} \otimes \tau^{i+j}.$$

We have an isomorphism $K\{\tau\} \cong \mathrm{End}_K(\mathbf{G}_a)$ sending τ to $X \mapsto X^p$. If a_m is the largest non-zero coefficient, then the *degree* of $\sum_{i=0}^m a_i \tau^i \in K\{\tau\}$ is defined to be p^m. The *derivative* is defined to be the constant term a_0.

Definition 1.1. Let $r > 0$ be an integer and K a characteristic p field. An *elliptic A-module of rank r* is a ring homomorphism

$$\phi\colon A \to K\{\tau\}$$

such that for all non-zero $a \in A$, $\deg \phi(a) = |a|_\infty^r$.

Let S be a scheme of characteristic p. An *elliptic A-module of rank r over S* is a $\mathbf{G}_a$-torsor $\mathcal{L}/S$, with a morphism of rings $\phi\colon A \to \mathrm{End}_S(\mathcal{L})$ such that for all points $s\colon \mathrm{Spec}\, K \to S$, the fiber $\mathcal{L}_s$ is an elliptic A-module of rank r.

Remark 1.2. The function $a \mapsto \phi(a)'$ (the latter meaning the derivative of $\phi(a)$) defines a morphism of rings $i\colon A \to \mathcal{O}_S$, i.e. a morphism $\theta\colon S \to \mathrm{Spec}\, A$.

1.2. **Level structures and moduli space.** Let I be an ideal of A. Let $(\mathcal{L}, \phi)$ be an elliptic module over S. Assume for simplicity that S is an $A[I^{-1}]$-scheme, i.e. the map θ factors through $\theta\colon S \to \mathrm{Spec}\, A \setminus V(I)$.

Let $\mathcal{L}_I$ be the group scheme defined by the equations $\phi(a)(x) = 0$ for all $a \in I$. This is an étale group scheme over S with rank $\#(A/I)^r$. An I-level structure on $(\mathcal{L}, \varphi)$ is an A-linear isomorphism $\alpha\colon (I^{-1}/A)^r_S \xrightarrow{\sim} \mathcal{L}_I$.

Choose $0 \subsetneq I \subsetneq A$. We have a functor

$$F_I^r\colon A[I^{-1}] - \mathbf{Sch} \to \mathbf{Sets}$$

sending S to the set of isomorphism classes of elliptic A-modules of rank r with I-level structure, with θ being the structure morphism.

Theorem 1.3 (Drinfeld). *F_I^r is representable by a smooth affine scheme M_I^r over $A[I^{-1}]$.*

2. Analytic theory of elliptic modules

2.1. **Description in terms of lattices.** Let Γ be an A-lattice in $\mathbf{C}_\infty$ (that is, a discrete additive subgroup of $\mathbf{C}_\infty$ which is an A-module.) Then we define

$$e_\Gamma(x) = x \prod_{x\in\Gamma-0} (1 - x/\gamma).$$

Drinfeld proved that this is well-defined for all $x \in \mathbf{C}_\infty$, and induces an isomorphism of abelian groups $e_\Gamma\colon \mathbf{C}_\infty/\Gamma \xrightarrow{\sim} \mathbf{C}_\infty$. This allows to define a function $\phi_\Gamma\colon A \to \mathrm{End}_{\mathbf{C}_\infty}(\mathbf{G}_a)$, by transporting the A-module structure on the left-hand side to the right-hand side, which only depends on the homothety class of the A-lattice Γ.

The following theorem is reminiscent of the description of elliptic curves over $\mathbf{C}$.

Theorem 2.1 (Drinfeld). *The function $\Gamma \mapsto \phi^\Gamma$ induces a bijection between*

$$\left\{ \begin{array}{c} \textit{rank } r \textit{ projective } A\textit{-lattices} \\ \textit{in } \mathbf{C}_\infty\textit{/homothety} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{rank } r \textit{ elliptic } A\textit{-modules} \\ \textit{over } \mathbf{C}_\infty \textit{ such that } \phi(a)' = a \\ \textit{/isomorphism} \end{array} \right\}$$

Remark 2.2. Under this bijection, an I-level structure equivalent to an A-linear isomorphism $(A/I)^r \cong \Gamma/I\Gamma$ for the lattices.

2.2. **Uniformization.** We now try to parametrize the objects on the left hand side of (2.1). Let Y be a projective A-module of rank r. Then we have a bijection

$$\left\{ \begin{array}{c} \text{homothety classes of } A\text{-lattices in } \mathbf{C}_\infty \\ \text{isomorphic to } Y \text{ as } A\text{-modules} \end{array} \right\} \leftrightarrow \mathbf{C}_\infty^\times \backslash \mathrm{Inj}(F_\infty \otimes_A Y, \mathbf{C}_\infty)/\mathrm{GL}_A(Y).$$

Next we observe that there is a bijection (after fixing an identification $F_\infty \otimes_A Y = F_\infty^r$)

$$\mathbf{C}_\infty^\times \backslash \mathrm{Inj}(F_\infty \otimes_A Y, \mathbf{C}_\infty) \leftrightarrow \mathbf{P}^{r-1}(\mathbf{C}_\infty) \setminus \bigcup (F_\infty\text{-rational hyperplanes}),$$

given by sending $u \in \mathrm{Inj}(F_\infty \otimes_A Y, \mathbf{C}_\infty)$ to $[u(e_1) : \ldots : u(e_r)]$ ($(e_1, \ldots, e_r)$ is the canonical basis of F_∞^r). The right-hand side is the set of $\mathbf{C}_\infty$-points of the famous *Drinfeld upper half-space* Ω^r.

As $\mathrm{Spec}\, A = X \setminus \{\infty\}$, a projective A-module of rank r is the same as a vector bundle of rank r on $X \setminus \{\infty\}$. Using Weil's adélic description of vector bundles, one finally gets

$$M_I^r(\mathbf{C}_\infty) \cong \mathrm{GL}_r(F)\backslash(\Omega^r(\mathbf{C}_\infty) \times \mathrm{GL}_r(\mathbf{A}_F^\infty)/\mathrm{GL}_r(\widehat{A}, I)),$$

where $\mathrm{GL}_r(\widehat{A}, I) := \ker\left(\mathrm{GL}_r(\widehat{A}) := \prod_{v\neq\infty} \mathrm{GL}_r(\mathcal{O}_v) \to \mathrm{GL}_r(A/I)\right)$. This bijection can be upgraded into an isomorphism of rigid analytic spaces :

Theorem 2.3 (Drinfeld)**.** *One has an isomorphism of rigid analytic spaces over F_∞ :*

$$M_I^{r,\mathrm{an}} = \mathrm{GL}_r(F)\backslash(\Omega^r \times \mathrm{GL}_r(\mathbf{A}_F^\infty)/\mathrm{GL}_r(\widehat{A}, I)).$$

3. Cohomology of M_I^2 and global Langlands for GL_2

3.1. **Cohomology of the Drinfeld upper half plane.** We then briefly outlined Drinfeld's proof of global Langlands for GL_2 using the moduli space of elliptic modules. Set $r = 2$, and $\Omega := \Omega^2$. Then one has

$$\Omega(\mathbf{C}_\infty) = \mathbf{P}^1(\mathbf{C}_\infty)\backslash\mathbf{P}^1(F_\infty).$$

There is a map λ from $\Omega(\mathbf{C}_\infty)$ to the Bruhat-Tits tree, sending (z_0, z_1) to the homothety class of the norm on F_∞^2 defined by

$$(a_0, a_1) \in F_\infty^2 \mapsto |a_0 z_0 + a_1 z_1|,$$

and one can think to Ω as being a tubular neighborhood of the Bruhat-Tits tree. Using λ, one gets a quite explicit description of the geometry of the rigid analytic space Ω and proves that there is a $GL_2(F_\infty)$-equivariant isomorphism :

$$H^1_{\text{ét}}(\Omega_{\mathbf{C}_\infty}, \overline{\mathbf{Q}}_\ell) = (\mathcal{C}^\infty(\mathbf{P}^1(F_\infty), \overline{\mathbf{Q}}_\ell)/\overline{\mathbf{Q}}_\ell)^* \cong \mathrm{St}^*.$$

3.2. **Cohomology of M_I^2.** Now we use the uniformization of M_I^2 (theorem 2.3). Rewriting it as follows :

$$M_I^{2,\mathrm{an}} = \left(\Omega \times \mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbf{A}_F)/\mathrm{GL}_2(\widehat{A}, I)\right)/\mathrm{GL}_2(F_\infty).$$

and using the Hochschild-Serre spectral sequence, we deduce a $\mathrm{GL}_2(\mathbf{A}_F) \times \mathrm{Gal}(\overline{F}_\infty/F_\infty)$-equivariant isomorphism[1] :

$$H^1_{\text{ét},!}(M_I^2 \otimes_F \overline{F}, \overline{\mathbf{Q}}_\ell) \cong \mathrm{Hom}_{\mathrm{GL}_2(F_\infty)}(\mathrm{St}, \mathcal{C}_0^\infty(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbf{A}_F)/\mathrm{GL}_2(\widehat{A}, I))) \otimes \mathrm{sp},$$

where sp is a 2-dimensional representation of $\mathrm{Gal}(\overline{F}_\infty/F_\infty)$ corresponding to the Steinberg representation by local Langlands. Drinfeld shows that

$$\varinjlim_I H^1_{\text{ét},!}(M_I^2 \otimes_F \overline{F}, \overline{\mathbf{Q}}_\ell) = \bigoplus_\pi \pi^\infty \otimes \sigma(\pi)$$

where π runs over cuspidal automorphic representations of $\mathrm{GL}_2(\mathbf{A}_F)$ with $\pi_\infty \cong \mathrm{St}$. Here $\sigma(\pi)$ is a degree two $\mathrm{Gal}(\overline{F}/F)$-representation. Moreover, Drinfeld shows that at unramified places, π_v and $\sigma(\pi_v)$ correspond to each other by local Langlands.

Remark 3.1. This result is still quite far from the global Langlands correspondence for GL_2 over F, but it nevertheless allows to construct the local Langlands correspondence for GL_2 over K, a characteristic p local field, as was explained during the talk, by combining this global construction with the decomposition of global L and ϵ-factors as products of local constants (which is known to hold in positive characteristic) and a trick of twisting by a sufficiently ramified character. See [2].

[1] This is cheating a little : one has to apply carefully the Hochschild-Serre spectral sequence and one needs to introduce a compactification of M_I^2 to define the *cuspidal* cohomology of M_I^2 showing up on the left (corresponding to the space of cuspidal functions on the right).

4. From elliptic modules to shtukas

The relation between elliptic modules and shtukas passes through an intermediate object called an *elliptic sheaf.*

Definition 4.1. An *elliptic sheaf of rank* $r > 0$ *with pole at* ∞ is a diagram

$$\begin{tikzcd}
\cdots \arrow[r, hook] & \mathcal{F}_{i-1} \arrow[r, hook, "j_i"] & \mathcal{F}_i \arrow[r, hook, "j_{i+1}"] & \mathcal{F}_{i+1} \arrow[r, hook] & \cdots \\
\cdots \arrow[r, hook] \arrow[ur, hook] & {}^\tau\mathcal{F}_{i-1} \arrow[r, hook, "{}^\tau j_i"] \arrow[ur, hook, "t_i"] & {}^\tau\mathcal{F}_i \arrow[r, hook, "{}^\tau j_{i+1}"] \arrow[ur, hook, "t_{i+1}"] & {}^\tau\mathcal{F}_{i+1} \arrow[r, hook] \arrow[ur, hook] & \cdots
\end{tikzcd}$$

(here as usual $\tau^*\mathcal{F} = (\mathrm{Id}_X \times \mathrm{Frob}_S)^*\mathcal{F})$ with $\mathcal{F}_i$ vector bundles of rank r, such that j and t are $\mathcal{O}_{X\times S}$-linear maps satisfying

1. $\mathcal{F}_{i+r} = \mathcal{F}_i(\infty)$ and $j_{i+r} \circ \ldots \circ j_{i+1}$ is the natural map $\mathcal{F}_i \hookrightarrow \mathcal{F}_i(\infty)$.
2. $\mathcal{F}_i/j_i(\mathcal{F}_{i-1})$ is an invertible sheaf along Γ_∞.
3. For all i, $\mathcal{F}_i/t_i(\tau^*\mathcal{F}_{i-1}) =$ is an invertible sheaf along Γ_z for some $z\colon S \to X \setminus \{\infty\}$ (independent of i).
4. For all geometric points $\overline{s}$ of S, the Euler characteristic $\chi(\mathcal{F}_0|_{X_{\overline{s}}})$ vanishes.

If I is a non-zero ideal of A, there is also a natural notion of I-level structure on an elliptic sheaf over S, at least if S lives over $\operatorname{Spec} A \setminus V(I)$, and Drinfeld proves the following remarkable result.

Theorem 4.2. *Let* $z\colon S \to \operatorname{Spec} A\setminus V(I)$. *Then there exists a bijection, functorial in* S, *between the two sets :*

$$\left\{\begin{array}{c} \text{rank } r \text{ elliptic } A\text{-modules over } S \\ \text{with } I\text{-level structure} \\ \text{such that } \phi(a)' = z(a) \end{array}\right\}\Big/\simeq \leftrightarrow \left\{\begin{array}{c} \text{rank } r \text{ elliptic sheaves over } S \\ \text{with } I\text{-level structure} \\ \text{and zero } z \end{array}\right\}\Big/\simeq$$

The dictionary is explained in detail in [6] (see in particular the enlightening example $r = 1$ and its relation with geometric class field theory discussed there).

One shows that if $(\mathcal{F}_., t_., j_.)$ is an elliptic sheaf, then for all i,

$$t_i(\tau^*\mathcal{F}_{i-1}) = \mathcal{F}_i \cap t_{i+1}(\tau^*\mathcal{F}_i), \text{ viewed as subsheaves of } \mathcal{F}_{i+1}.$$

Hence, one can actually reconstruct the entire elliptic sheaf from the triangle

$$\begin{tikzcd}
\mathcal{F}_0 \arrow[r, hook, "j"] & \mathcal{F}_1 \\
{}^\tau\mathcal{F}_0 \arrow[ur, hook, "t"] &
\end{tikzcd}$$

which is just a shtuka with two legs (one being fixed at ∞)! One can not go in the other direction – shtukas with two legs are more general than elliptic sheaves. There is no direct analogy anymore between shtukas with one pole at ∞ and one zero z and elliptic curves (or abelian varieties) but the family of stalks at closed points of X of such a vector bundle, with their Frobenius, behaves somehow like the family of φ-modules attached to the reduction mod ℓ of the p-divisible group of an abelian variety over a number field, when the prime ℓ varies (the choice of ℓ corresponding roughly to the choice of a closed point and the choice of p corresponding to the choice of z). Shtukas with two legs are the right objects to consider to prove the full Langlands correspondence for GL_r (for all r) over a function field, as demonstrated by [4], [5].

References

[1] H. Carayol-Variétés de Drinfel'd compactes, d'après Laumon, Rapoport et Stuhler. Séminaire Bourbaki, Vol. 1991-92. Astérisque No. 206 (1992), Exp. No. 756, 5, 369-409.
[2] P. Deligne and D. Husemöller-Survey of Drinfel'd modules. Current trends in arithmetical algebraic geometry (Arcata, Calif., 1985), 25-91, Contemp. Math., 67, Amer. Math. Soc., Providence, RI, 1987.
[3] V. Drinfeld-Elliptic modules. Mat. Sb. (N.S.) 94(136) (1974), 594-627, 656.
[4] V. Drinfeld-Langlands' conjecture for GL(2) over functional fields. Proceedings of the International Congress of Mathematicians (Helsinki, 1978), pp. 565-574, Acad. Sci. Fennica, Helsinki, 1980.
[5] L. Lafforgue-Chtoucas de Drinfeld et correspondance de Langlands. Invent. Math. 147 (2002), no. 1, 1-241.
[6] D. Mumford-An algebro-geometric construction of commuting operators and of solutions to the Toda lattice equation, Korteweg deVries equation and related nonlinear equations. Proceedings of the International Symposium on Algebraic Geometry (Kyoto Univ., Kyoto, 1977), pp. 115-153.

Talk 7: Analytic RTF: Geometric Side

Jingwei Xiao

1. Overview

In this lecture, we start to explain the strategy in [1] of comparing two relative trace formulas (RTF). The analytic RTF is defined by the split torus inside PGL_2:

$$\sum_{u\in\mathbf{P}^1(F)-\{1\}} \mathbb{J}_r(u,f) = \mathbb{J}_r(f) = \sum_{\pi} \mathbb{J}_r(\pi,f) \tag{1}$$

And the geometric RTF is defined by the self intersection number of the Heegner-Drinfeld cycle:

$$\sum_{u\in\mathbf{P}^1(F)-\{1\}} \mathbb{I}_r(u,f) = \mathbb{I}_r(f) = \sum_{\pi} \mathbb{I}_r(\pi,f) \tag{2}$$

In both RTF, the middle terms admit two expansions. The spectral expansions on the right are given by linear combinations of $L^{(r)}(\pi,\frac{1}{2})$ (Analytic case) and $\langle \mathrm{Sht}_\pi, \mathrm{Sht}_\pi\rangle$ (Geometric case), with π ranges over cuspidal automorphic unramified representations of PGL_2 and coefficents given by the test function f. The orbit integral expansions on the left are parametrized by certain orbits. To prove the identity $L^{(r)}(\pi,\frac{1}{2}) \approx \langle \mathrm{Sht}_\pi, \mathrm{Sht}_\pi\rangle$, it suffices to prove $\mathbb{J}_r(f) = \mathbb{I}_r(f)$.

In the following, we first define the term $\mathbb{J}_r(f)$ and its orbit integral expansion. Then we define $\mathbb{I}_0(f)$. This case is considered by Jacquet in [3]. Finally we indicate Jacquet's proof of $\mathbb{J}_0(f) = \mathbb{I}_0(f)$.

2. $\mathbb{J}_r(f)$ and the orbit integral expansion

As usual, we denote by F the global field of characteristic p, F' an unramified quadratic extension, η the corresponding quadratic character of $\mathbb{A}_F^\times$. We set $G = \mathrm{PGL}_2$ and A the diagonal torus of G. Let $[G] = G(F)\backslash G(\mathbb{A}_F)$, and similarly for $[A]$.

Consider the orbits $A(F)\backslash G(F)/A(F)$, we have the invariant map:

$$\begin{aligned}\mathrm{inv}\colon A(F)\backslash G(F)/A(F) &\longrightarrow \mathbb{P}^1(F) - \{1\}\\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto \frac{bc}{ad}\end{aligned}$$

We shall say γ is regular semisimple if $\mathrm{inv}(\gamma) \neq 0$ or ∞. The invariant map defines a bijection between $A(F)\backslash G(F)^{r.s.}/A(F)$ and $\mathbb{P}^1(F) - \{0, 1, \infty\}$. When $u = 0$ or ∞, $\mathrm{inv}^{-1}(u)$ consists of three orbits.

Let $f \in C_c^\infty(G(\mathbb{A}_\mathbb{F}))$, define

$$K_f(g_1, g_2) = \sum_{\gamma \in G(F)} f(g_1^{-1}\gamma g_2) \tag{3}$$

$$\mathbb{J}(f, s) = \int_{[A]\times[A]}^{\mathrm{reg}} K_f(h_1, h_2)|h_1 h_2|^s \eta(h_2) dh_1 dh_2 \tag{4}$$

$$\mathbb{J}_r(f) = \left(\frac{d}{ds}\right)^r \bigg|_{s=0} \mathbb{J}(f, s) \tag{5}$$

Some comments are in order. In (3), the summation is in fact finite since f is compactly supported. In (4), the term $|h_1 h_2|^s \eta(h_2)$ gives the base change L-function. This will be explained in detail in the next lecture. In (4), the integral is over a non-compact set and does not converge absolutely, hence one needs to regularize it. We omit details.

We can modify the above definition by only considering γ with a fixed invariant u. In (3), define $K_{f,u}$ by taking summation over γ with a fixed invariant u. In (4), define $\mathbb{J}(u, f, s)$ by replacing K_f by $K_{f,u}$. Finally, define $\mathbb{J}_r(u, f)$ by substituting $\mathbb{J}(u, f, s)$ into (5).

Hence we have the orbit integral expansion of the Analytic RTF:

$$\mathbb{J}_r(f) = \sum_{u \in \mathbf{P}^1(F) - \{1\}} \mathbb{J}_r(u, f) \tag{6}$$

3. $\mathbb{I}_0(f)$ and the orbit integral expansion

For general r, the definition of $\mathbb{I}_r(f)$ involves Shtukas and the Heegner-Drinfeld cycle. When $r = 0$, there is an equivalent defintion given by the RTF of the anisotropic torus inside PGL_2. Set $T = \mathrm{Res}_{F'/F}(\mathbb{G}_m)/\mathbb{G}_m$ and we embed T into $G = \mathrm{PGL}_2$. Consider $T(F)\backslash G(F)/T(F)$.

There exists $\varepsilon \in G(F)$ normalizing T that induces the Galois conjugate on $T(F)$. One checks ε^2 is in $F^\times$ since it is in the center of $G(F)$. Every element in $G(F)$ can be written as $a + b\varepsilon$ for $a, b \in F'$. Define the invariant map:

$$\begin{aligned} \mathrm{inv}\colon T(F)\backslash G(F)/T(F) &\longrightarrow \mathbb{P}^1(F) - \{1\} \\ a + b\varepsilon \qquad &\longmapsto \qquad \frac{b\bar{b}}{a\bar{a}}\varepsilon^2 \end{aligned}$$

It's easy to check that the invariant map is well defined and independent of the choice of ε. We shall say $a + b\varepsilon$ is regular semisimple if $ab \neq 0$. The invariant map induces a bijection between $T(F)\backslash G(F)^{r.s.}/T(F)$ and $\mathrm{Norm}(F')\backslash\{0, 1\} \subset \mathbb{P}^1(F) - \{1\}$. The two orbits which are not regular semisimple map to 0 and ∞.

We have the usual RTF by considering the left and the right translation of T inside G.

Let $f \in C_c^\infty(G(\mathbb{A}_\mathbb{F}))$, define $K_f(g_1, g_2)$ as in (3). Define

$$\mathbb{I}_0(f) = \int_{[T]\times[T]} K_f(h_1, h_2) dh_1 dh_2 \tag{7}$$

As usual, we have the orbit integral expansion:

$$\mathbb{I}_0(f) = \sum_{u \in \mathbf{P}^1(F)-\{1\}} \mathbb{I}_0(u, f) \tag{8}$$

4. $\mathbb{J}_0(f) = \mathbb{I}_0(f)$

In [3], Jacquet proves $\mathbb{J}_0(f) = \mathbb{I}_0(f)$ and uses this to prove the Waldspurger formula. Here, we only consider f when f is a spherical function. For general f, one need to modify the definition of $\mathbb{I}_0(f)$ by also considering division algebras. By (6) and (8), it suffices to prove $\mathbb{J}_0(u, f) = \mathbb{I}_0(u, f)$. We first rewrite both sides as a product of local distributions. Then we compare the local distributions at matching orbits. The comparison for spherical functions is just the following fundamental lemma.

Theorem 4.1 (Fundamental lemma)**.** *Let F'/F be a quadratic unramified extension of non-archimedean local fields. We use the notation G, T, A and define the invariant map as before. Let $\gamma \in A(F)\backslash G(F)^{r.s.}/A(F)$ and $\delta \in T(F)\backslash G(F)^{r.s.}/T(F)$.*

If $\mathrm{inv}(\gamma) = \mathrm{inv}(\delta)$*, then*

$$\int_{A(F)\times A(F)} f(h_1^{-1}\gamma h_2)\eta(h_2) dh_1 dh_2 = \int_{T(F)\times T(F)} f(h_1^{-1}\delta h_2) dh_1 dh_2 \tag{9}$$

If $\mathrm{inv}(\gamma) \neq \mathrm{inv}(\delta)$ *for all* δ*, then*

$$\int_{A(F)\times A(F)} f(h_1^{-1}\gamma h_2)\eta(h_2)dh_1dh_2 = 0 \tag{10}$$

The fundamental lemma is proved in [2] by a direct computation.

References

[1] Zhiwei Yun, Wei Zhang, *Shtukas and the Taylor expansion of L-functions*, to appear in Ann. of Math.

[2] Hervé Jacquet, *Sur un résultat de Waldspurger*, Annales scientifiques de l'.N.S. 4[e] série, tome 19, n[o] 2 (1986), p. 185-229.

Talk 8: Analytic RTF: Spectral Side

Ilya Khayutin

Most of the material presented in this lecture is closely related to the work of Jacquet [3] on Waldspurger's results [8] on central values of L-functions. A major difference from Jacquet's original approach is the simplifications which are obtained by introducing the Eisenstein ideal.

1. Decomposition of the kernel

We have defined

$$\mathbb{J}(f,S) = \int_{[A]\times[A]}^{\mathrm{reg}} \mathbb{K}_f(h_1,h_2)|h_1h_2|^s\eta(h_2)\,dh_1dh_2.$$

First we discuss the decomposition of the kernel function $\mathbb{K}_f(h_1,h_2)$. The necessary theory about automorphic representations for GL_2 is presented in the book of Jacquet and Langlands [2]. The ideas about the decomposition of the kernel are essentially due to Selberg [7].

The vector space $L^2_0([G])$ is a representation of $G(\mathbb{A})$ and it can be decomposed, in a suitable manner, into irreducible representation. The function $\mathbb{K}_f$ is the kernel of the integral operator corresponding to the right action of $C_c^\infty(G(\mathbb{A}))$ on $L^2_0([G])$ by convolutions. Hence it decomposes into a sum (and an integral) of kernels corresponding to the action on irreducible representations.

There is a crude decomposition of the kernel into three parts:

$$\mathbb{K}_f(x_1,x_2) = \mathbb{K}_{f,\mathrm{cusp}} + \mathbb{K}_{f,\mathrm{sp}} + \mathbb{K}_{f,\mathrm{Eis}}$$

corresponding to cuspidal, special, and Eisenstein spectrum.

1.1. **The cuspidal part.** We have

$$\mathbb{K}_{f,\mathrm{cusp}} = \sum_{\pi \text{ cuspidal}} \mathbb{K}_{f,\pi}$$

where

$$\mathbb{K}_{f,\pi}(x,y) = \sum_{\phi} \pi(f)\phi(x)\overline{\phi(y)}.$$

where ϕ runs over an orthonormal basis of the cuspidal representation π.

1.2. **The special part.** Using the determinant map, every character χ factors as

$$\chi\colon [G] \xrightarrow{\det} F^\times\backslash\mathbf{A}^\times/(\mathbf{A}^\times)^2 \to \{\pm 1\}.$$

Then

$$\mathbb{K}_{f,\mathrm{sp},\chi}(x,y) = \pi(f)\chi(x)\overline{\chi(y)}.$$

This is the same expression as for the cuspidal part, but for 1-dimensional representations.

1.3. **The Eisenstein part.** The Eisenstein part is defined similarly for the continuous spectrum. The sum over an orthonormal basis becomes an integral over a continuum of characters. Recall that Eisenstein series corresponds to parabolic induction of automorphic representations of the split torus A.

1.4. **Goals:**

(1) Identify $f \in \mathcal{H}$ such that $\mathbb{K}_{f,\mathrm{Eis}} = 0$.
(2) For such f, show that

$$\mathbb{J}(f) = \sum_{\pi \text{ cuspidal unramified}} \mathbb{J}_\pi(f)$$

$$\mathbb{J}_\pi(f) = \frac{\mathcal{P}(\pi(f)\varphi)\overline{\mathcal{P}_\eta(\varphi)}}{\langle\varphi,\varphi\rangle}.$$

where $\mathcal{P}(\cdot)$ are certain toral periods to be defined later and φ is a spherical vector in the unramified cuspidal representation π.
(3) Explain how the toral periods are related to L-functions.

2. Satake isomorphism

The local Satake transform has been introduced by Satake [6] as a tool to study spherical functions and it is closely related to prior work of Harish-Chandra.

Let $\mathcal{H}_G$ be the spherical Hecke algebra of G. By definition,

$$\mathcal{H}_G = {\bigotimes_{x\in|X|}}' \mathcal{H}_x.$$

There is also a spherical Hecke algebra for the split torus $A \subset G$. As we have $A \cong \mathbf{G}_m$ the toral Hecke algebra is $\mathcal{H}_A = \bigotimes' \mathcal{H}_{A,x}$, and the local Hecke algebras are all isomorphic to

$$\mathcal{H}_{A,x} \cong \mathbf{Q}[F_x^\times/\mathcal{O}_x^\times] \cong \mathbf{Q}[t_x^{-1}, t_x]$$

where $t_x = \mathbf{1}_{\varpi_x^{-1}\mathcal{O}_x^\times}$.

The Weyl group action is, in this normalization,

$$\iota_x(t_x) = q_x t_x^{-1}.$$

The *Satake homomorphism*

$$\mathrm{Sat}_x \colon \mathcal{H}_x \to \mathcal{H}_{A,x}$$

sends $h_x \mapsto t_x + q_x t_x^{-1}$. In fact Sat_x is an isomorphism onto the subring of Weyl invariants.

The local Satake homomorphisms extend to a global one:

$$\mathrm{Sat} \colon \mathcal{H} \to \mathcal{H}_A^\iota.$$

3. Eisenstein ideal

3.1. **Definition of Eisenstein ideal.** The Eisenstein ideal we define is an analogue of the ideal defined by Mazur [5] in the number field setting.

We can identify $\mathbf{A}^\times/\mathbb{O}^\times \cong \mathrm{Div}(X)$. There is a map $\mathrm{Div}(X) \to \mathrm{Pic}(X) \cong F^\times\backslash\mathbf{A}^\times/\mathbb{O}^\times$. Now, $\mathcal{H}_A \cong \mathbf{Q}[\mathrm{Div}(X)] \to \mathbf{Q}[\mathrm{Pic}(X)]$.

The Weyl involution descends to ι_{Pic} on $\mathbf{Q}[\mathrm{Pic}(X)]$,

$$\mathbf{1}_{\mathcal{L}} \mapsto q^{\deg \mathcal{L}} \mathbf{1}_{\mathcal{L}^{-1}}.$$

Thus we have a map

$$a_{\mathrm{Eis}} \colon \mathcal{H} \xrightarrow{\mathrm{Sat}} \mathcal{H}_A^\iota \to \mathbf{Q}[\mathrm{Pic}(X)]^{\iota_{\mathrm{Pic}}}.$$

Definition 3.1. We define the Eisenstein ideal to be $\mathcal{I}_{\mathrm{Eis}} := \ker a_{\mathrm{Eis}}$.

Our main interest in the Eisenstein ideal stems from the following theorem. It is closely related to the work of Lindenstrauss and Venkatesh on the spherical Weyl law [4].

Theorem 3.2. *For $f \in \mathcal{I}_{\mathrm{Eis}}$,*

$$\mathbb{K}_{f,\mathrm{Eis}}(x,y) = 0.$$

Proof. An unramified character of $[A] \cong F^\times\backslash\mathbb{A}^\times$ is constructed in the following way. Begin with an unramified character of the unit idéles $\chi \colon F^\times\backslash\mathbb{A}_1^\times \to \mathbb{C}^\times$. Fix some $\alpha \in \mathbb{A}^\times$ with $|\alpha| = q$, then there is a non-canonical splitting $\mathbb{A}^\times \cong \mathbb{A}_1^\times \times \alpha^{\mathbb{Z}}$ which allows to extend χ to $\mathbb{A}^\times$. For any $u \in \mathbb{C}$ we can define $\chi_u(a) = \chi(a)|a|^u$.

The technical part of the proof is an explicit computation of $\mathbb{K}_{f,\mathrm{Eis}}$ using parabolic induction of unramified characters $\chi_{u+1/2} \colon [A] \cong F^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$. Using the assumption that f is everywhere unramified we arrive at the identity

$$\mathbb{K}_{f,\mathrm{Eis},\chi} = \frac{\log q}{2\pi i} \int_{0+0i}^{0+\frac{2\pi i}{\log q}} (\chi_{u+1/2}(a_{\mathrm{Eis}}(f))\mathbf{1}_K, \mathbf{1}_K) \ldots \, du.$$

Where $\mathbf{1}_K$ is the characteristic function of the fixed maximal compact subgroup. The kernel $\mathbb{K}_{f,\mathrm{Eis}}$ is a sum of $\mathbb{K}_{f,\mathrm{Eis},\chi}$ over all unramified unitary characters $\chi \colon F^\times\backslash\mathbb{A}_1^\times \to \mathbb{C}^\times$.

The kernel $\mathbb{K}_{f,\mathrm{Eis},\chi}$ vanishes for all possible characters exactly when $a_{\mathrm{Eis}}(f) = 0$ as required. $\square$

4. Relation to L-functions

4.1. **Normalization of L-function.** We have

$$L(\pi_{F'}, s) = L(\pi, s)L(\pi \otimes \eta, s).$$

The functional equation reads

$$L(\pi_{F'}, s) = \epsilon(\pi, s)L(\pi_{F'}, 1 - s)$$

where

$$\epsilon(\pi, s) = q^{-8(q-1)(s-1/2)}.$$

Definition 4.1. We define the *normalized L-function*

$$\mathcal{L}(\pi_{F'}, s) = \epsilon(\pi_{F'}, s)^{-1/2} \frac{L(\pi_{F'}, s)}{L(\pi, \mathrm{Ad}, 1)}.$$

For f in the Eisenstein ideal we can write

$$\mathbb{J}(f, s) = \sum_{\pi} \mathbb{J}_\pi(f, s)$$

The Eisenstein part vanishes by the theorem above and the special part vanishes for any unramified f.

The relative trace of a cuspidal representation is equal to

$$\mathbb{J}_\pi(f, s) = \sum_{\varphi \in \text{orthogonal basis of } \pi} \frac{\mathcal{P}(\pi(f)\varphi, s)\mathcal{P}_\eta(\overline{\varphi}, s)}{\langle \varphi, \varphi \rangle}$$

Again because f is unramified all the summands vanish except for the single spherical basis function $\varphi \in \pi^K$.

For any adelic character χ the period integral is defined by

$$\mathcal{P}_\chi(\varphi, s) = \int_{[A]} \varphi \begin{pmatrix} h & \\ & 1 \end{pmatrix} \chi(h)|h|^s \, dh$$

A slightly more comfortable form to work with when calculating L-functions is

$$I(s, \varphi, \chi) = \int_{F^\times \backslash \mathbf{A}^\times} \varphi \begin{pmatrix} h & \\ & 1 \end{pmatrix} \chi(h)|h|^{s-1/2} \, dh \tag{1}$$

A simple change of variables implies the functional equation

$$I(s, \varphi, \chi) = I(1 - s, \widetilde{\varphi}, \chi).$$

where $\widetilde{\varphi}(g) = \varphi({}^t g^{-1})$.

4.2. **Whittaker model.** Our presentation was based on the review of Cogdell [1] and the references within.

To relate $\mathbb{J}_\pi(f)$ with L-functions we use "Whittaker models" which allow us to express the period integral $I(s, \varphi, \chi)$ as an Euler product of local integrals.

Let $\varphi \in \pi$. The Whittaker model is constructed by restricting φ to

$$\varphi\colon U(F)\backslash U(\mathbf{A}) \to \mathbb{C}$$

and taking the Fourier transform on $[U] \cong F\backslash\mathbb{A}$. Cuspidality of π implies that the constant term in the Fourier transform vanishes.

The Fourier coefficients decompose into a product of local factors. Plugging this into the period integral $I(s, \varphi, \chi)$ and unfolding presents the period integral as a product of local integrals — the Euler product.

This is almost the integral definition of the L-function attached to an automorphic representation. However, there is the issue of the dependance on the vector φ. For almost all places the local factor coincides with the local L-function. But at the finitely many bad places, one needs to calculate the correct constant factor.

References

[1] J. W. Cogdell. *Analytic Theory of L-Functions for GL n*, pages 197–228. Birkhäuser Boston, Boston, MA, 2004.

[2] H. Jacquet and R. P. Langlands. *Automorphic forms on* GL(2). Lecture Notes in Mathematics, Vol. 114. Springer-Verlag, Berlin-New York, 1970.

[3] Hervé Jacquet. Sur un résultat de Waldspurger. *Ann. Sci. École Norm. Sup. (4)*, 19(2):185–229, 1986.

[4] Elon Lindenstrauss and Akshay Venkatesh. Existence and Weyl's law for spherical cusp forms. *Geom. Funct. Anal.*, 17(1):220–251, 2007.

[5] B. Mazur. Modular curves and the Eisenstein ideal. *Inst. Hautes Études Sci. Publ. Math.*, (47):33–186 (1978), 1977.

[6] Ichirô Satake. Theory of spherical functions on reductive algebraic groups over $\mathfrak{p}$-adic fields. *Inst. Hautes Études Sci. Publ. Math.*, (18):5–69, 1963.

[7] A. Selberg. Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series. *J. Indian Math. Soc. (N.S.)*, 20:47–87, 1956.

[8] J.-L. Waldspurger. Correspondance de Shimura. *J. Math. Pures Appl. (9)*, 59(1):1–132, 1980.

Talk 9: Geometric Interpretation of Orbital Integrals

Yihang Zhu

The material below is contained in §3.2, 3.3 of the main paper.

1. Interpreting the orbital integrals

Yesterday we introduced the distribution $f \mapsto \mathbb{J}(f, s)$. It has a geometric expansion:

$$\mathbb{J}(f, s) = \sum_{u \in \mathbb{P}^1(F) - \{1\}} \mathbb{J}(u, f, s).$$

For simplicity, in this talk we only discuss the terms $\mathbb{J}(u,f,s)$ with $u\neq 0,\infty$, which are defined as the orbital integrals

$$\mathbb{J}(u,f,s)=\mathbb{J}(\gamma,f,s)=\int_{A(\mathbb{A})\times A(\mathbb{A})} f(h_1^{-1}\gamma h_2)|h_1h_2|^s\eta(h_2)\,dh_1dh_2$$

where $\gamma\in G(F)$ with $\mathrm{inv}(\gamma)=u$. Note that these γ are regular semi-simple and there are no convergence issues. In the following we let $f:=h_D$, for $D=\sum n_x x$ an effective divisor. We let $d:=\deg D$.

Let $\widetilde{\gamma}\in \mathrm{GL}_2(F)$ be a lift of γ. We define $\widetilde{h}:=\bigotimes_x \widetilde{h}_{n_x,x}\in\mathcal{H}(\mathrm{GL}_2)$ where $\widetilde{h}_{n_x,x}:=\mathbf{1}_{\mathrm{Mat}_2(\mathcal{O}_x)_{v_x(\det)=n_x}}\in\mathcal{H}_x(\mathrm{GL}_2)$. As an easy observation, we have the following

Lemma 1.1. *We have*

$$\mathbb{J}(\gamma,h_D,s)=\int_{\Delta(Z(\mathbb{A}))\backslash(\widetilde{A}\times\widetilde{A})(\mathbb{A})}\widetilde{h}_D(h_1^{-1}\widetilde{\gamma}h_2)|\alpha(h_1)\alpha(h_2)|^s\eta(\alpha(h_2))\,dh_1dh_2.$$

Here $\widetilde{A}$ is the diagonal torus in GL_2, and $\alpha\colon \begin{pmatrix} a & \\ & d\end{pmatrix}\mapsto a/d$. □

The function $(h_1,h_2)\mapsto \widetilde{h}_D(h_1^{-1}\widetilde{\gamma}h_2)$ may be viewed as a function on

$$\Delta(Z(\mathbb{A}))\backslash\left(\widetilde{A}(\mathbb{A})/\widetilde{A}(\mathbb{O})\times\widetilde{A}(\mathbb{A})/\widetilde{A}(\mathbb{O})\right)=\Delta(\mathrm{Div}\,X)\backslash(\mathrm{Div}\,X)^4.$$

Moreover, the condition that $\widetilde{h}_D=1$ defines a subset $\mathfrak{N}_{D,\widetilde{\gamma}}$ of $\Delta(\mathrm{Div}\,X)\backslash(\mathrm{Div}\,X)^4$ given as follows.

Definition 1.2. We define $\widetilde{\mathfrak{N}}_{D,\widetilde{\gamma}}\subset(\mathrm{Div}\,X)^4$ to be the set of $(E_1,E_2,E_1',E_2')\in\mathrm{Div}^{\mathrm{eff}}(X)^4$, such that the rational map $\mathcal{O}_X^2\xrightarrow{\widetilde{\gamma}}\mathcal{O}_X^2$ induces a holomorphic map $\varphi_{\widetilde{\gamma}}$ fitting into the following commutative diagram

$$\begin{array}{ccc}
\mathcal{O}_X^2 & \xrightarrow{\quad\widetilde{\gamma}\quad} & \mathcal{O}_X^2\\
\uparrow & & \uparrow\\
\mathcal{O}_X(-E_1)\oplus\mathcal{O}_X(-E_2) & \xrightarrow{\varphi_{\widetilde{\gamma}}} & \mathcal{O}_X(-E_1')\oplus\mathcal{O}_X(-E_2')
\end{array}$$

and satisfying $\mathrm{Div}\,\varphi_{\widetilde{\gamma}}=D$. We define

$$\mathfrak{N}_{D,\widetilde{\gamma}}:=(\Delta(\mathrm{Div}\,X)\cdot\widetilde{\mathfrak{N}}_{D,\widetilde{\gamma}})/\Delta(\mathrm{Div}\,X).$$

The upshot is that $\mathbb{J}(\gamma,h_D,s)$ is a weighted sum over $\mathfrak{N}_{D,\widetilde{\gamma}}$. More precisely:

$$(1)\quad \mathbb{J}(\gamma,h_D,s)=\sum_{(E_1,E_2,E_1',E_2')\in\mathfrak{N}_{D,\widetilde{\gamma}}} q^{-\deg(E_1-E_2+E_1'-E_2')s}\eta(E_1-E_1')\eta(E_2-E_1')$$

2. The moduli spaces

Let $\widehat{X}_d \to \mathrm{Pic}_X^d$ be the moduli stack of sections. Let $X_d = \mathrm{Sym}^d X = X^d//S_d$. We have a natural embedding $X_d \hookrightarrow \widehat{X}_d$ sending

$$(t_1, \dots, t_d) \mapsto (\mathcal{O}_X(t_1 + \ldots + t_d), 1).$$

This is an isomorphism onto the locus in $\widehat{X}_d$ where the section is not the zero section.

Definition 2.1. Let

$$\Sigma_d = \left\{ \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} \mid d_{ij} \in \mathbb{Z}_{\geq 0},\ d_{11} + d_{22} = d_{12} + d_{21} = d \right\}$$

Given $\underline{d} \in \Sigma_d$, we define the moduli space $\widetilde{\mathcal{N}}_{\underline{d}}$ classifying

- $K_1, K_2, K_1', K_2' \in \mathrm{Pic}_X$ such that $\deg K_i' - \deg K_j = d_{ij}$
- A map $\varphi\colon K_1 \oplus K_2 \to K_1' \oplus K_2'$, which we can write as
$$\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}$$
with $\varphi_{ij} : K_j \to K_i'$
- the φ_{ij}'s are required to satisfy a technical condition which we call $(*\underline{d})$, roughly saying that not too many φ_{ij}'s are zero.

We define $\mathcal{N}_{\underline{d}} := \widetilde{\mathcal{N}}_{\underline{d}}/\,\mathrm{Pic}_X$.

Definition 2.2. We define the moduli space $\mathcal{A}_d$ classifying (Δ, a, b) where $\Delta \in \mathrm{Pic}_X^d$ and a, b are global sections of Δ such that they are not both the zero section.

Remark 2.3. The stack $\mathcal{A}_d$ is covered by two pieces $X_d \times_{\mathrm{Pic}_X^d} \widehat{X}_d$ and $\widehat{X}_d \times_{\mathrm{Pic}^d} X_d$ and is therefore a scheme by the representability of $\widehat{X}_d \to \mathrm{Pic}_X^d$.

Definition 2.4. We define a morphism

$$f_{\underline{d}}\colon \mathcal{N}_{\underline{d}} \to \mathcal{A}_d$$

$$(K_1, K_2, K_1', K_2', \varphi) \mapsto (K_1' \otimes K_2' \otimes K_1^{-1} \otimes K_2^{-1}, \varphi_{11} \otimes \varphi_{22}, \varphi_{12} \otimes \varphi_{21}).$$

Here condition $(*\underline{d})$ guarantees that the image indeed lies in $\mathcal{A}_d$.

Proposition 2.5. (1) *$\mathcal{N}_{\underline{d}}$ is a geometrically connected scheme over k.*
(2) *If $d \geq 4g - 3$, $\mathcal{N}_{\underline{d}}$ is smooth of dimension $2d - g + 1$.*
(3) *The morphism $f_{\underline{d}}$ is proper.*

Proof. Using $(*\underline{d})$ we find a covering of $\mathcal{N}_{\underline{d}}$ analogous to the covering of $\mathcal{A}_d$ discussed above. Using that covering it is easy to prove (1) and (3). (Properness reduces to properness of $X_{d_{ij}}$.) For (2), we use $(*\underline{d})$, the above covering, and the following consequence of Riemann-Roch: The map $\widehat{X}_{d_{ij}} \to \mathrm{Pic}_X^{d_{ij}}$ is smooth of relative dimension $1 - g + d_{ij}$ if $d_{ij} \geq 2g - 1$. □

3. Geometrization of the analytic RTF

We now define a crucial local system $L_{\underline{d}}$ on $\mathcal{N}_{\underline{d}}$. By geometric class field theory there is a rank 1 local system L_η on the Picard scheme $\mathrm{Pic}_X^{\mathrm{coarse}}$ whose associated function on $\mathrm{Pic}_X^{\mathrm{coarse}}(k)$ is η, characterized by a compatibility condition with the group scheme structure on $\mathrm{Pic}_X^{\mathrm{coarse}}$. For any $d' \in \mathbb{Z}_{\geq 0}$ we define a local system $L_{d'}$ on $\widehat{X}_{d'}$ as the pullback of L_η via the map $\widehat{X}_{d'} \to \mathrm{Pic}_X \to \mathrm{Pic}_X^{\mathrm{coarse}}$.

There is an open embedding

$$\mathcal{N}_{\underline{d}} \hookrightarrow (\widehat{X}_{d_{11}} \times \widehat{X}_{d_{22}}) \times_{\mathrm{Pic}_X^d} (\widehat{X}_{d_{12}} \times \widehat{X}_{d_{21}}) \tag{2}$$

given by the universal φ_{ij}'s. Finally, we define the rank 1 local system $L_{\underline{d}}$ on $\mathcal{N}_{\underline{d}}$ to be the restriction of the local system

$$L_{d_{11}} \boxtimes \mathbf{Q}_\ell \boxtimes L_{d_{12}} \boxtimes \mathbf{Q}_\ell \text{ on } (\widehat{X}_{d_{11}} \times \widehat{X}_{d_{22}}) \times_{\mathrm{Pic}_X^d} (\widehat{X}_{d_{12}} \times \widehat{X}_{d_{21}})$$

to $\mathcal{N}_{\underline{d}}$ via (2).

Definition 3.1. Let $\delta\colon \mathcal{A}_d \to \widehat{X}_d$, $(\Delta, a, b) \mapsto (\Delta, a-b)$. We also define

$$\mathcal{A}_D := \delta^{-1}(\mathcal{O}_X(D), 1) \cong \Gamma(X, \mathcal{O}_X(D)).$$

and the invariant map

$$\mathrm{inv}_D\colon \mathcal{A}_D(k) \to \mathbb{P}^1(F) - \{1\}, \quad a \mapsto 1 - a^{-1},$$

viewing $a \in \Gamma(X, \mathcal{O}_X(D))$ as a rational function on X, i.e. $a \in F$.

Proposition 3.2. *Let $u \in \mathbb{P}^1(F) - \{1, 0, \infty\}$.*

(1) *If $u \notin \mathrm{Im}\ \mathrm{inv}_D$, then $\mathbb{J}(u, h_D, s) = 0$.*
(2) *If $u = \mathrm{inv}_D(a)$ for $a \in \mathcal{A}_D(k)$, then*

$$\mathbb{J}(u, h_D, s) = \sum_{\underline{d} \in \Sigma_d} q^{(2d_{12}-d)s} \operatorname{Tr}(\mathrm{Frob}_a, Rf_{\underline{d},*} L_{\underline{d}})_{\overline{a}} \tag{3}$$

Proof of (2). Let $\mathcal{N}_a := \coprod_{\underline{d} \in \Sigma_d} f_{\underline{d}}^{-1}(a)$. We have a bijection

$$\mathfrak{N}_{D,\widetilde{\gamma}} \xrightarrow{\sim} \mathcal{N}_a(k)$$

$$(E_1, E_2, E_1', E_2') \mapsto (\mathcal{O}_X(-E_1), \mathcal{O}_X(-E_2), \mathcal{O}_X(-E_1'), \mathcal{O}_X(-E_2'), \varphi_{\widetilde{\gamma}}).$$

Using this bijection and the Lefschetz fixed point formula, the RHS of (3) becomes a weighted sum over $\mathfrak{N}_{D,\widetilde{\gamma}}$, which is equal to the RHS of (1) by the definition of $L_{\underline{d}}$. $\square$

Talk 10: Definition and properties of $\mathcal{M}_d$

Jochen Heinloth

Following [YZ15, Section 6.1], we will construct the analog of the space $\mathcal{N}_d$ introduced in the previous talk for the case of the twisted torus that we saw in Talk 4. Let us recall the basic setup: We fix an étale covering $\nu\colon X' \to X$ of degree 2, where X, X' are smooth connected, geometrically irreducible curves. The non-trivial automorphism of the covering will be denoted by σ.

We denote by $\widetilde{T} := \mathrm{Res}_{X'/X}\,\mathbb{G}_m$ the Weil restriction of $\mathbb{G}_m$ considered as a group scheme over X and $T := (\mathrm{Res}_{X'/X}\,\mathbb{G}_m)/\mathbb{G}_m$ the quotient by the constant group scheme.

Remark 0.3. The torus T can also be viewed as a subgroup of $\widetilde{T}$, because there is an exact sequence

$$(1) \qquad 1 \to \mathbb{G}_m \to \mathrm{Res}_{X'/X}\,\mathbb{G}_m \xrightarrow{id\cdot\sigma^{-1}} \mathrm{Res}_{X'/X}\,\mathbb{G}_m \xrightarrow{\mathrm{Nm}} \mathbb{G}_m \to 1,$$

which induces a sequence

$$1 \to T \to \mathrm{Res}_{X'/X}\,\mathbb{G}_m \xrightarrow{\mathrm{Nm}} \mathbb{G}_m \to 1.$$

1. The starting point

There are probably two motivations for the construction of $\mathcal{M}_d$. You will see that the constructions will be analogs for T of the geometry appearing in the previous talk for the split torus $A = \mathbb{G}_m^2$. Another motivation is the following:

Recall that our final aim is to compute the intersection of the cycle $[\mathrm{Sht}_T^r]$ with an Hecke-translate of itself. Now the space of shtukas Sht_T is itself an intersection of the graph of Frobenius on Bun_T with some auxiliary Hecke correspondence. One of the key steps for the computation will be to reverse the order of these two intersections, i.e., to first intersect Bun_T with a Hecke correspondence and then pass to intersections with the graph of Frobenius.

In this talk we will give a simple modular interpretation of the first intersection and construct the map corresponding to the invariants seen in the previous talk. As usual we will first consider $\widetilde{T}$ and GL_2 instead of T and PGL_2 first and then define the space we need by passing to a quotient by Pic_X.

2. Definition of $\mathcal{M}_d$

First observe that $\mathrm{Bun}_{\widetilde{T}} = \mathrm{Pic}_{X'}$ is the stack of line bundles on X' and thus the push forward ν_* defines a morphism $\mathrm{Bun}_{\widetilde{T}} \to \mathrm{Bun}_2$.

For any $d \in \mathbb{N}$ we denote by $\mathrm{Hecke}_2^d := \langle \mathcal{E}' \subset \mathcal{E} | \mathcal{E}', \mathcal{E} \in \mathrm{Bun}_2^* \times \mathrm{Bun}_2^{\star+d} \rangle$ the stack of modifications of vector bundles, such that the cokernel has length d. This is a smooth stack over Bun_2, because for any vector bundle $\mathcal{E}$ on a curve the scheme of torsion quotients $\mathcal{E} \twoheadrightarrow \mathcal{T}$ of any degree is smooth.

We define $\widetilde{\mathcal{M}}_d^{\heartsuit}$ as the fiber product in the diagram:

$$\begin{tikzcd}
\widetilde{\mathcal{M}}_d^{\heartsuit} \arrow[r] \arrow[d] & \mathrm{Hecke}_2^d \arrow[d] \\
\mathrm{Bun}_{\widetilde{T}} \times \mathrm{Bun}_{\widetilde{T}} \arrow[r, "{(\nu_*,\nu_*)}"] & \mathrm{Bun}_2 \times \mathrm{Bun}_2
\end{tikzcd}$$

By definition we have

$$\widetilde{\mathcal{M}}_d^{\heartsuit}(S) = \langle (\mathcal{L}, \mathcal{L}') \in \mathrm{Pic}^*_{X'} \times \mathrm{Pic}^{*+d}_{X'}, \psi \colon \nu_*\mathcal{L} \hookrightarrow \nu_*\mathcal{L}' \rangle.$$

By adjunction, giving ψ is equivalent to giving a morphism

$$\phi = (\alpha, \sigma^*\beta) \colon \mathcal{L} \oplus \sigma^*\mathcal{L} = \nu^*\nu_*\mathcal{L} \to \mathcal{L}'.$$

Moreover, given $\phi = (\alpha, \sigma^*\beta)$ we have $\det(\psi) = \mathrm{Nm}(\alpha) - \mathrm{Nm}(\beta)$, which one easily checks by computing $\nu^*\psi$ in terms of α, β. Thus ψ is injective if and only if $\mathrm{Nm}(\alpha) \neq \mathrm{Nm}(\beta)$.

Passing to a slightly weaker condition Yun and Zhang define

$$\widetilde{\mathcal{M}}_d := \langle (\mathcal{L}, \mathcal{L}', \alpha, \beta) | (\mathcal{L}, \mathcal{L}') \in \mathrm{Pic}^*_{X'} \times \mathrm{Pic}^{*+d}_{X'}, \alpha \colon \mathcal{L} \to \mathcal{L}', \beta \colon \mathcal{L} \to \sigma^*\mathcal{L}', (\alpha, \beta) \neq (0,0) \rangle$$

$$\mathcal{M}_d := \widetilde{\mathcal{M}}_d / \mathrm{Pic}_X, \mathcal{M}_d^{\heartsuit} := \mathcal{M}_d^{\heartsuit} / \mathrm{Pic}_X.$$

Note that by definition $\widetilde{\mathcal{M}}_d$ has infinitely many connected components all of which are of finite type. Passing to the quotient by Pic_X one obtains a stack with 2 components only and therefore $\mathcal{M}_d$ is of finite type.

Also the Norm map defines a morphism

$$\begin{aligned} f \colon \mathcal{M}_d &\to \mathcal{A}_d := \widehat{X_d} \times_{\mathrm{Pic}_X} \widehat{X_d} - Z_d \\ (\mathcal{L}, \mathcal{L}', \alpha, \beta) &\mapsto (\mathrm{Nm}(\mathcal{L}') \otimes \mathrm{Nm}(\mathcal{L})^{-1}, \mathrm{Nm}(\alpha), \mathrm{Nm}(\sigma^*\mathcal{L}') \otimes \mathrm{Nm}(\mathcal{L})^{-1}, \mathrm{Nm}(\beta)), \end{aligned}$$

where again $\widehat{X}_d$ is the stack classifying line bundles of degree d on X together with a (possibly vanishing) global section and Z_d is the subset where both sections vanish.

Proposition ([YZ15, 6.1]).

(1) The morphism $\iota \colon \mathcal{M}_d \to \widehat{X'}_d \times_{\mathrm{Pic}_X} \widehat{X'}_d - Z'_d$ induced from $(\mathcal{L}, \mathcal{L}', \alpha, \beta) \mapsto (\mathcal{L}' \otimes \mathcal{L}^{-1}, \alpha, \sigma^*\mathcal{L}' \otimes \mathcal{L}^{-1}, \beta)$ is an isomorphism. This stack is smooth if $d > 2g(X') - 1$, it is a DM-stack if $\mathrm{char}(k) \neq 2$.

(2) The morphism $f \colon \mathcal{M}_d \to \mathcal{A}_d$ is proper, as is its restriction $f \colon \mathcal{M}_d^{\heartsuit} \to \mathcal{A}_d^{\heartsuit}$.

Proof. The claim that ι is an isomorphism follows from the isomorphism

$$(\mathrm{Pic}^*_{X'} \times \mathrm{Pic}^{\star+d}_{X'}) / \mathrm{Pic}_X \cong (\mathrm{Pic}^d_{X'} \times_{\mathrm{Pic}_X} \mathrm{Pic}^d_{X'})$$

given by $(\mathcal{L}, \mathcal{L}') \mapsto (\mathcal{L}' \otimes \mathcal{L}^{-1}, \sigma^*\mathcal{L}' \otimes \mathcal{L}^{-1}, \mathrm{can})$, where can is the canonical isomorphism between the norms of the two line bundles. This latter isomorphism follows from applying $H^*(X, \_\_)$ to the sequence (1). The properties of the stack follow form covering the stack on the right hand side by two copies of $\widehat{X}'_d \times_{\mathrm{Pic}_X} X'_d$, i.e. restricting to the subsets where one of the two sections is non-zero. For these one uses that the composition $\widehat{X}'_d \to \mathrm{Pic}'_X \to \mathrm{Pic}_X$ is the composition of a representable morphism and a morphism of Picard stacks whose fibers are torsors under $\ker(\mathrm{Pic}'_X \to \mathrm{Pic}_X)$. This kernel is a μ_2-gerbe over the Prym variety of X'/X, which is a DM-stack if the characteristic is not 2. Pulling this space back to the scheme $X_d \to \mathrm{Pic}_X$ we therefore again obtain a DM-stack. Finally the Abel-Jacobi map $X'_d \to \mathrm{Pic}'_X$ is smooth if $d > 2g' - 1$ by the Riemann-Roch theorem. As X'_d is smooth, this implies that $\widehat{X}'_d \times_{\mathrm{Pic}_X} X'_d$ is smooth for such d.

Part (2) also follows from the above description of $\mathcal{M}_d$. The main ingredient is the observation that the morphism $\widehat{X}'_d \to \widehat{X}_d$ induced by the norm map is proper. This is clear fiberwise, as $X'_d \to X_d$ and $\mathrm{Pic}^d_{X'} \to \mathrm{Pic}^d_X$ are proper. Formally Yun and Zhang prove this by considering the compactification $\overline{X}_d$ of $\widehat{X}_d$ that is given by the stack of line bundles $\mathcal{L}$ together with a homothety class of a non zero section of $\mathcal{L} \oplus \mathcal{O}_X$, which defines a projective stack over Pic_X. □

Remark 2.1.

(1) The stack $\mathcal{A}_d := \widehat{X_d} \times_{\mathrm{Pic}_X} \widehat{X_d} - Z_d$ is proper, as one can check the valuative criterion for line bundles together with pairs of sections. This also implies that $\mathcal{M}_d$ is proper.

(2) The above proof also shows that $\widehat{X'}_d \times_{\mathrm{Pic}_X} \widehat{X'}_d - Z'_d$ is very close to being a scheme: The only points that have non-trivial automorphisms are those where one of the two sections vanishes and for those the automorphism group is μ_2.

References

[1] Z. Yun, W. Zhang. *Shtukas and The Taylor expansion of L-Functions.* arXiv:1512.02683

Talk 11: Intersection Theory on Stacks

Michael Rapoport

The aim of this talk is to introduce intersection theory on Deligne-Mumford stacks which are only locally of finite type over a field k, like the moduli stack of shtukas. Fortunately, we only need the $\mathbf{Q}$-theory, which makes things easier.

1. Definition of $\mathrm{Ch}(X)_{\mathbf{Q}}$

1.1. **Chow groups for finite type.**

Definition 1.1. Let X be a DM stack of finite type over k. Then we define

$$\mathrm{Ch}_*(X)_{\mathbf{Q}} = Z_*(X)_{\mathbf{Q}} / \partial W_*(X)_{\mathbf{Q}}$$

where

- $Z_*(X)_{\mathbf{Q}} = \bigoplus_V \mathbf{Q}$ with V running over irreducible reduced closed substacks of dimension $*$, and
- $W_*(X)_{\mathbf{Q}} = \bigoplus_W k(W)^\times \otimes_{\mathbf{Z}} \mathbf{Q}$ with the same index set, and $k(W)$ viewed as a rational function to $\mathbf{A}^1_k$; the map ∂ to $Z_{*-1}(X)$ is by the "associated divisor map" as in the usual case for schemes.

1.2. **Generalization to locally finite type.** When X is *locally* of finite type over k, we replace $Z_*(X)_{\mathbf{Q}}$ with $Z_{c,*}(X)_{\mathbf{Q}}$ and $W_{c,*}(X)_{\mathbf{Q}}$, where the subscript c indicates that we only take substacks *proper* over $\operatorname{Spec} k$. We have

$$\mathrm{Ch}_c(X)_{\mathbf{Q}} = \varinjlim_{Y \subset X,\ \text{proper}/k} \mathrm{Ch}_*(Y)_{\mathbf{Q}} = \varinjlim_{U \text{ open } \subset X} \mathrm{Ch}_{*,c}(U)_{\mathbf{Q}}.$$

1.3. **Degree map.** We want to define a map

$$\deg\colon \mathrm{Ch}_{c,0}(X)_{\mathbf{Q}} \to \mathbf{Q}.$$

Since we are working with stacks, we need to account for stabilizers.

Definition 1.2. Let $x \in X$ be represented by a geometric point $x^s\colon \operatorname{Spec} k^{\mathrm{sep}} \to X$. We define

$$\deg x := [(k^{\mathrm{sep}})^{\Gamma_x} : k] \cdot \frac{1}{|\operatorname{Aut}(x^s)|}.$$

1.4. **Intersection pairing.** Now let X be smooth, locally of finite type, and of pure dimension n. Then we have an intersection product

$$\mathrm{Ch}_{c,i}(X)_{\mathbf{Q}} \times \mathrm{Ch}_{c,j}(X)_{\mathbf{Q}} \to \mathrm{Ch}_{c,i+j-n}(X)_{\mathbf{Q}} \tag{1}$$

defined as follows. Let Y_1, Y_2 be closed substacks of X, which are proper over k. Then (1) is the colimit of the finite-type intersection products

$$\mathrm{Ch}_i(Y_1)_{\mathbf{Q}} \times \mathrm{Ch}_j(Y_2)_{\mathbf{Q}} \to \mathrm{Ch}_{i+j-n}(Y_1 \cap Y_2) \to \mathrm{Ch}_{c,i+j-n}(X)_{\mathbf{Q}}.$$

The first map is subtle to define: it is the *refined intersection product*

$$(\zeta_1, \zeta_2) \mapsto X \times_{(X\times X)} (\zeta_1 \times \zeta_2).$$

It is a special case of the *refined Gysin morphism*: to define this, start with the fiber product diagram

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{i} & Y \end{array}$$

where i is a regular embedding of codimension e. Then we get a *refined Gysin morphism*

$$i^!\colon \mathrm{Ch}_i(V)_{\mathbf{Q}} \to \mathrm{Ch}_{i-e}(W)_{\mathbf{Q}}.$$

In the expression above

$$X \times_{(X\times X)} (\zeta_1 \times \zeta_2) := \Delta^!(\zeta_1 \times \zeta_2).$$

Thus we have finally constructed the product

$$\mathrm{Ch}_{c,i}(X)_{\mathbf{Q}} \times \mathrm{Ch}_{c,j}(X)_{\mathbf{Q}} \to \mathrm{Ch}_{c,i+j-n}(X)_{\mathbf{Q}}.$$

Then composing with the degree map, we get an intersection pairing

$$\langle , \rangle_X\colon \mathrm{Ch}_{c,j}(X)_{\mathbf{Q}} \times \mathrm{Ch}_{c,n-j}(X)_{\mathbf{Q}} \to \mathbf{Q}.$$

Remark 1.3. (i) We have a cycle class map

$$\mathrm{cl}_X\colon \mathrm{Ch}_{c,j}(X)_{\mathbf{Q}} \to H_c^{2n-2j}(X \otimes_k \overline{k}, \mathbf{Q}_\ell(n-j))$$

and the intersection product is compatible with cup product.

(ii) Consider

$$_c\,\mathrm{Ch}_n(X \times X)_{\mathbf{Q}} = \varinjlim_{Z \subset X \times X} \mathrm{Ch}_*(Z)_{\mathbf{Q}}$$

such that $\mathrm{pr}_1|_Z$ is proper. This is a $\mathbf{Q}$-algebra. It acts on each $\mathrm{Ch}_{c,j}(X)_{\mathbf{Q}}$.

Now that we have a definition, the problem is that we can't really calculate. So instead we pass to K-groups. In the talk, following [YZ15, Appendix A], I introduced two situations (A) and (B) under which there is a compatibility between the refined Gysin homomorphism and the pullback operation in K-theory. This is used in the octahedron lemma [YZ15, Theorem A.10].

2. The octahedron lemma

Consider a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] & B \arrow[l] \arrow[d] \\ U \arrow[r] & S & V \arrow[l] \\ C \arrow[r] \arrow[u] & Y \arrow[u] & D \arrow[l] \arrow[u] \end{tikzcd}$$

Let N be the fiber product as in

$$\begin{tikzcd} N \arrow[r] \arrow[d] & A \times B \times C \times D \arrow[d] \\ X \times_S Y \times_S U \times_S V \arrow[r] & (X \times_S U) \times (X \times_S Y) \times (Y \times_S U) \times (X \times_S V) \end{tikzcd}$$

Lemma 2.1. *There are canonical isomorphisms*

$$(C \times_Y D) \times_{U \times_S V} (A \times_X B) \cong N \cong (C \times_U A) \times_{Y \times_S X} (D \times_V B). \quad \square$$

Theorem 2.2. *Assume everybody is smooth, except B (the "bad" object) of dimension $d_A, d_X, \ldots$. Also assume that the fiber products (on the left) $C \times_Y D$, $U \times_S V$, $C \times_U A, Y \times_S X$ have the expected dimension. Further assume that each of the fiber product diagrams*

$$\begin{tikzcd} A \times_X B \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & X \end{tikzcd}$$

and

$$\begin{array}{ccc} D \times_V B & \longrightarrow & B \\ \big\downarrow & & \big\downarrow \\ D & \longrightarrow & V \end{array}$$

satisfy the compatibility conditions (A) or (B). Finally assume that both fiber product diagrams

$$\begin{array}{ccc} N & \longrightarrow & A \times_X B \\ \big\downarrow & & \big\downarrow \\ C \times_Y D & \longrightarrow & U \times_S V \end{array}$$

and

$$\begin{array}{ccc} N & \longrightarrow & D \times_V B \\ \big\downarrow & & \big\downarrow \\ C \times_U A & \longrightarrow & Y \times_S X \end{array}$$

satisfy the compatibility condition (A). Let $n = \dim N$*. For the diagram*

$$\begin{array}{ccccc} N & \xrightarrow{\alpha} & D \times_V B & \xrightarrow{d} & B \\ \Big\| & & & & \Big\| \\ N & \xrightarrow{\delta} & A \times_X B & \xrightarrow{a} & B \end{array}$$

we have the equality for the highest degree components in the rational Chow groups,

$$\delta^! a^! [B] = d^! \alpha^! [B] \in \mathrm{Ch}_n(N)_{\mathbf{Q}}.$$

Roughly speaking, the proof proceeds by using the relation to K-theory, and lifting the statement of the previous lemma to the level of derived stacks.

References

[1] W. Fulton, *Intersection theory*, Springer Verlag 1998.
[2] Z. Yun, W. Zhang, *Shtukas and the Taylor expansion of L-functions*, to appear in Ann. of Math.

Talk 12: LTF for Cohomological Correspondences

Davesh Maulik

In this lecture, we give some background on the Lefschetz-Verdier trace formula and calculation of local terms, following closely the paper of Varshavsky [1]. We then explain briefly how these are used to prove a key degree calculation in the appendix of [1].

1. Cohomological correspondences

Definition 1.1. Given X_1, X_2 a *correspondence* between X_1, X_2 is a diagram

$$\begin{tikzcd} & C \arrow[dl, "c_1"'] \arrow[dr, "c_2"] & \\ X_1 & & X_2 \end{tikzcd}$$

Given $\mathcal{F}_i \in D(X_i)$, a *cohomological correspondence* is an element

$$u \in \mathrm{Hom}_C(c_1^*\mathcal{F}_1, c_2^!\mathcal{F}_2) = \mathrm{Hom}_{X_2}(c_{2!}c_1^*\mathcal{F}_1, \mathcal{F}_2).$$

If c_1 is proper, then we can define a map

$$R\Gamma_c(u)\colon R\Gamma_c(X_1, \mathcal{F}_1) \to R\Gamma_c(X_2, \mathcal{F}_2).$$

When $X_1 = X_2 = X$ and $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}$, this defines an endomorphism of $R\Gamma_c(X, \mathcal{F})$ and we are interested in calculating its trace.

2. Trace formula and local terms

Consider the cartesian square

$$\begin{tikzcd} \mathrm{Fix}(c) \arrow[r, "\Delta'"] \arrow[d, "c'"] & C \arrow[d, "c=c_1\times c_2"] \\ X \arrow[r, "\Delta"] & X \times X \end{tikzcd}$$

By tracing through a series of adjunctions, we can define a *trace map*

$$\mathrm{R}\mathcal{H}om_C(c_1^*\mathcal{F}, c_2^!\mathcal{F}) \to \Delta'_* K_{\mathrm{Fix}(c)}. \tag{1}$$

Applying H^0 to (1), we get

$$\mathrm{Tr}\colon \mathrm{Corr}_C(\mathcal{F}, \mathcal{F}) \to H^0(\mathrm{Fix}, K_{\mathrm{Fix}(c)}) = H_0^{BM}(\mathrm{Fix}(c)).$$

If β is a connected component of $\mathrm{Fix}(c)$ which is proper over k, we can take the degree of the contribution of β to define *local terms*

$$LT_\beta(u) = \deg(\mathrm{Tr}(u)_\beta) \in \mathbf{Q}_\ell.$$

Varshavsky proves the following:

Theorem 2.1. *The trace map commutes with proper pushforward. In particular, if C, X are proper over k, then*

$$\mathrm{Tr}(R\Gamma_c(u)) = \sum_\beta LT_\beta(u).$$

The second statement is the usual Lefschetz-Verdier trace formula.

3. Local terms for the graph of Frobenius

In order to apply this result, one would like to compute the local terms effectively. When c_2 is quasifinite, given $y \in \mathrm{Fix}(c)$, with $x = c_1(y) = c_2(y)$, the cohomological correspondence u defines an endomorphism

$$u_y \colon F_x \to F_x.$$

The trace $\mathrm{Tr}(u_y)$ is the *naïve local term* associated to y.

In general, these naive terms differ from the true local terms defined earlier; nevertheless, Varshavsky proves that for contracting correspondences, they agree. As a special case, we state his result for the graph of Frobenius.

Let X_0 be a variety over $k = \overline{\mathbf{F}}_q$ and $X = X_0 \times_{\mathbf{F}_q} \overline{\mathbf{F}}_q$. Consider the self-correspondence on X obtained from the graph of the Frobenius morphism Frob. Given a sheaf $\mathcal{E}$ on X, let $u = \mathrm{Frob}^* \mathcal{E} \to \mathcal{E}$ denote an arbitrary cohomological correspondence. Then the local terms coincide with the naïve local terms. In other words,

(1) For all $s \in X_0(\mathbf{F}_q)$, we have

$$LT_s(u) = \mathrm{Tr}(u_s).$$

(2) Furthermore, we have

$$\mathrm{Tr}(R\Gamma_c(u)) = \sum_s \mathrm{Tr}(u_s, E_s).$$

4. Application to the appendix

We now sketch how these ideas are used in the proof of Proposition A.12 in [1]. Consider a correspondence

$$\begin{tikzcd} & C \arrow[dl, "c_1"'] \arrow[dr, "c_2"] & \\ M & & M \end{tikzcd}$$

Assume

- c_1 is proper,
- M is smooth of dimension n, and
- we have a proper map $f \colon C \to S$.

Let $\gamma \in \mathrm{Ch}_n(C)_{\mathbf{Q}}$. Suppose we have a map of cartesian squares

$$\begin{tikzcd}
\mathrm{Sht} \arrow[r, "\Gamma"] \arrow[d] \arrow[ddr] & C \arrow[d] \arrow[ddr, "f"] & \\
M \arrow[r, "\Gamma:=\mathrm{Id}\times\mathrm{Frob}"] \arrow[ddr] & M \times M \arrow[ddr] & \\
 & S(\mathbf{F}_q) \arrow[r] \arrow[d] & S \arrow[d, "\Delta"] \\
 & S \arrow[r, "\mathrm{Id}\times\mathrm{Frob}"] & S \times S
\end{tikzcd}$$

Then we can write

$$\mathrm{Sht} = \coprod_{s \in S(\mathbf{F}_q)} \mathrm{Sht}_s.$$

The Gysin pullback $\Gamma^! \gamma$ can be decomposed into contributions from each $s \in S(\mathbf{F}_q)$; let $(\Gamma^! \gamma)_s$ denote this term. By taking the degree, we obtain a local contribution

$$\langle \gamma, \Gamma_{\mathrm{Frob}} \rangle_s := \deg(\Gamma^! \gamma)_s \in \mathbf{Q}.$$

Theorem 4.1. *We have*

$$\langle \gamma, \Gamma_{\mathrm{Frob}} \rangle_s = \mathrm{Tr}((f_! \mathrm{cl}(\gamma))_s \circ \mathrm{Frob}_s \mid (f_! \mathbf{Q}_\ell)_{\overline{s}}).$$

The argument has two steps: compatibility of trace with proper pushforward, and the identification of naive and actual local terms for cohomological correspondences supported on the graph of Frobenius.

References

[1] Y. Varshavsky, *Lefschetz-Verdier trace formula and a generalization of a theorem of Fujiwara*, Geom. Funct. Anal. 17 (2007), no. 1, 271319.

[2] Z. Yun and W. Zhang, *Shtukas and the Taylor expansion of L-functions*, arxiv:1512.02683.

Talk 13: Definition and description of $\mathrm{Hk}^{\mu}_{\mathcal{M},d}$; expressing $\mathbb{I}_r(h_D)$ as a trace

Liang Xiao

Let $\nu : X' \to X$ be a degree two étale cover of geometrically connected smooth curves over $\mathbb{F}_q$, whose non-trivial automorphism is denoted by σ. Let D be an effective divisor on X of degree d. Let Sht^{μ}_T and Sht^{μ}_G denote the moduli of Shtukas introduced in earlier talks, together with a natural embedding $\theta^{\mu} : \mathrm{Sht}^{\mu}_T \to \mathrm{Sht}'^{\mu}_G := \mathrm{Sht}^{\mu}_G \otimes_{X^r} X'^r$.

The ultimate goal of this talk is to express the geometric side of the relative trace formula

$$\mathbb{I}_r(h_D) = \left\langle \theta^{\mu}_* [\mathrm{Sht}^{\mu}_T],\ h_D * \theta^{\mu}_* [\mathrm{Sht}^{\mu}_T] \right\rangle_{\mathrm{Sht}'^{\mu}_G}$$

as the sum over the Hitchin base $\mathcal{A}_d$ of the traces of certain (twisted) Frobenius operators. For this, we first need to describe the intersection space $\mathrm{Sht}^{\mu}_{\mathcal{M}^{\heartsuit},D}$ defined as the product

(1)
$$\begin{tikzcd} \mathrm{Sht}^{\mu}_{\mathcal{M}^{\heartsuit},D} \arrow[r] \arrow[d] & \mathrm{Sht}'^{\mu}_{G}(h_D) \arrow[d, "h^{\leftarrow}\times h^{\rightarrow}"] \\ \mathrm{Sht}^{\mu}_{T} \times \mathrm{Sht}^{\mu}_{T} \arrow[r, "\theta^{\mu}\times\theta^{\mu}"] & \mathrm{Sht}'^{\mu}_{G} \times \mathrm{Sht}'^{\mu}_{G}. \end{tikzcd}$$

We start with its Hecke version.

Definition 0.2. Let $\widetilde{\mathrm{Hk}}^{\mu}_{\mathcal{M},d}$ denote the moduli space that classifies

(1) points $\underline{x}' := (x'_1, \dots, x'_r) \in X'$, and
(2) a commutative diagram of modifications of line bundles over X'

(2)
$$\begin{tikzcd} \mathcal{L}'_0 \arrow[r, "f'_1"] & \mathcal{L}'_1 \arrow[r, "f'_2"] & \cdots \arrow[r, "f'_r"] & \mathcal{L}'_r \\ \mathcal{L}_0 \arrow[r, "f_1"] \arrow[u, "\alpha_0"] \arrow[d, "\beta_0"'] & \mathcal{L}_1 \arrow[r, "f_2"] \arrow[u, "\alpha_1"] \arrow[d, "\beta_1"'] & \cdots \arrow[r, "f_r"] & \mathcal{L}_r \arrow[u, "\alpha_r"] \arrow[d, "\beta_r"'] \\ \sigma^*\mathcal{L}'_0 \arrow[r, "\sigma(f'_1)"] & \sigma^*\mathcal{L}'_1 \arrow[r, "\sigma(f'_2)"] & \cdots \arrow[r, "\sigma(f'_r)"] & \sigma^*\mathcal{L}'_r, \end{tikzcd}$$

such that

- the first row and the second row each defines a point of $\widetilde{\mathrm{Hk}}^{\mu}_{T}$ over $\underline{x}' \in X'^r$, and
- each column $\mathcal{L}'_i \xleftarrow{\alpha_i} \mathcal{L}_i \xrightarrow{\beta_i} \sigma^*\mathcal{L}'_i$ defines a point of $\widetilde{\mathcal{M}}_d$ (i.e. $\deg \mathcal{L}'_i - \deg \mathcal{L}_i = d$).

We define the quotient

$$\mathrm{Hk}^{\mu}_{\mathcal{M},d} := \widetilde{\mathrm{Hk}}^{\mu}_{\mathcal{M},d} \,/\, \mathrm{Pic}_X.$$

The following is a list of basic properties of $\mathrm{Hk}^{\mu}_{\mathcal{M},d}$.

Lemma 0.3. (1) *We have a natural map $\gamma_i : \mathrm{Hk}^{\mu}_{\mathcal{M},d} \to \mathcal{M}_d$ sending the diagram (2) to its ith column. Then further composition of this γ_i with the natural Hitchin map $f_{\mathcal{M}} : \mathcal{M}_d \to \mathcal{A}_d$ is independent of i; in particular, $f_{\mathcal{M}} \circ \gamma_0 = f_{\mathcal{M}} \circ \gamma_r$.*

(2) *We define* $\mathrm{Sht}^{\mu}_{\mathcal{M},d}$ *by the Cartesian diagram*

$$\begin{tikzcd}
\mathrm{Sht}^{\mu}_{\mathcal{M},d} \arrow[r] \arrow[d] & \mathrm{Hk}^{\mu}_{\mathcal{M},d} \arrow[d, "\gamma_0\times\gamma_r"] \\
\mathcal{M}_d \arrow[r, "1\times\mathrm{Fr}_{\mathcal{M}_d}"] \arrow[d, "f_{\mathcal{M}}"] & \mathcal{M}_d\times\mathcal{M}_d \arrow[d, "f_{\mathcal{M}}\times f_{\mathcal{M}}"] \\
\mathcal{A}_d \arrow[r, "1\times\mathrm{Fr}_{\mathcal{A}_d}"] & \mathcal{A}_d\times\mathcal{A}_d.
\end{tikzcd}$$

Taking the fiber over $\mathcal{A}^{\heartsuit}_d$ *gives* $\mathrm{Sht}^{\mu}_{\mathcal{M}^{\heartsuit},d}$, *which is analogous to* $\mathrm{Sht}^{\mu}_{\mathcal{M}^{\heartsuit},d}$ *introduced in* (1).

(3) *Setting* $\mathcal{H} := \mathrm{Sht}^{1}_{\mathcal{M},D}$, *we can write* $\mathrm{Sht}^{\mu}_{\mathcal{M},D}$ *as an* r*-fold product*

$$\mathrm{Sht}^{\mu}_{\mathcal{M},D} \cong \mathcal{H}\times_{\gamma_1,\mathcal{M}_d,\gamma_0}\mathcal{H}\times_{\gamma_1,\mathcal{M}_d,\gamma_0}\cdots\times_{\gamma_1,\mathcal{M}_d,\gamma_0}\mathcal{H}.$$

Notation 0.4. Consider the following "good" locus of $\mathcal{A}_d$:

$$\mathcal{A}^{\diamondsuit}_d := \{(\Delta,a,b)\mid b\neq 0\}\cong \widehat{X}_d\times_{\mathrm{Pic}^d_X}X_d.$$

We put $\mathcal{M}^{\diamondsuit}_d := \mathcal{M}_d\times_{\mathcal{A}_d}\mathcal{A}^{\diamondsuit}_d$, $\mathrm{Hk}^{\mu}_{\mathcal{M}^{\diamondsuit},d} := \mathrm{Hk}^{\mu}_{\mathcal{M},d}\times_{\mathcal{A}_d}\mathcal{A}^{\diamondsuit}_d$, and $\mathcal{H}^{\diamondsuit} := \mathrm{Hk}^{1}_{\mathcal{M}^{\diamondsuit},d}$.

The following key lemma gives the description of $\mathcal{H}^{\diamondsuit}$ as a correspondence between $\mathcal{M}^{\diamondsuit}_d$.

Lemma 0.5. *We have a Cartesian diagram*

$$\begin{tikzcd}
\mathcal{M}^{\diamondsuit}_d \arrow[d, equal] & \mathcal{H}^{\diamondsuit} \arrow[l, "\gamma_0"'] \arrow[r, "\gamma_1"] \arrow[d, "\cong"] & \mathcal{M}^{\diamondsuit}_d \arrow[d, equal] \\
\widehat{X}'_d\times_{\mathrm{Pic}^d_X}X'_d \arrow[d, "\mathrm{pr}_2"'] & \widehat{X}'_d\times_{\mathrm{Pic}^d_X}I'_d \arrow[l] \arrow[r] \arrow[d, "\mathrm{pr}_2"'] & \widehat{X}'_d\times_{\mathrm{Pic}^d_X}X'_d \arrow[d, "\mathrm{pr}_2"'] \\
X'_d & I'_d \arrow[l, "q_0"'] \arrow[r, "q_1"] & X'_d,
\end{tikzcd}$$

where $I'_d = \{(D',x')\in X'_d\times X'\mid x'\in D'\}$ *is the incidence variety, and the maps* q_0 *and* q_1 *are given by* $q_0(D',x') = D'$ *and* $q_1(D',x') := D'+\sigma(x')-x'$.

In other words, the correspondence $\mathcal{M}^{\diamondsuit}\xleftarrow{\gamma_0}\mathcal{H}^{\diamondsuit}\xrightarrow{\gamma_1}\mathcal{M}^{\diamondsuit}$ *is the smooth pullback (when* d *is sufficiently large) of an explicit correspondence* $X'_d\xleftarrow{q_0}I'_d\xrightarrow{q_1}X'_d$.

Corollary 0.6. *The morphism* $\gamma_i : \mathrm{Hk}^{\mu}_{\mathcal{M}^{\diamondsuit},d}\to\mathcal{M}^{\diamondsuit}_d$ *is finite and surjective and* $\dim\mathrm{Hk}^{\mu}_{\mathcal{M}^{\diamondsuit},d} = \dim\mathcal{M}_d = 2d-g+1$.

Now, let $[\mathcal{H}^{\diamondsuit}]\in\mathrm{Ch}_{2d-g+1}(\mathcal{H})_{\mathbb{Q}}$ denote the class of the closure of $\mathcal{H}^{\diamondsuit}$, which induces an endomorphism

$$f_{\mathcal{M},!}[\mathcal{H}^{\diamondsuit}]:\ \mathbb{R}f_{\mathcal{M},!}\mathbb{Q}_\ell\to\mathbb{R}f_{\mathcal{M},!}\mathbb{Q}_\ell.$$

For a point $a \in \mathcal{A}_d(k)$, we write

$$\big(f_{\mathcal{M},!}[\mathcal{H}^{\diamondsuit}]\big)_a : \ \big(\mathbb{R}f_{\mathcal{M},!}\mathbb{Q}_\ell\big)_{\bar{a}} \to \big(\mathbb{R}f_{\mathcal{M},!}\mathbb{Q}_\ell\big)_{\bar{a}}.$$

for the action on the stalks.

The main theorem of this talk is the following.

Theorem 0.1. *Suppose that D is an effective divisor of degree $\geq \max\{3g-2, 2g\}$. Then*

$$\mathbb{I}_r(h_D) = \sum_{a \in \mathcal{A}_D(k)} \mathrm{Tr}\Big(\big(f_{\mathcal{M},!}[\mathcal{H}^{\diamondsuit}]\big)_a^r \circ \mathrm{Frob}_a;\ \big(\mathbb{R}f_{\mathcal{M},!}\mathbb{Q}_\ell\big)_{\bar{a}}\Big).$$

Proof. Since $\mathrm{Sht}^{\mu}_{\mathcal{M},d}$ is defined over $\mathcal{A}_d(k)$, we may decompose

$$\mathrm{Ch}_0(\mathrm{Sht}^{\mu}_{\mathcal{M},d})_{\mathbb{Q}} = \mathrm{Ch}_0(\mathrm{Sht}^{\mu}_{\mathcal{M},d} - \mathrm{Sht}^{\mu}_{\mathcal{M}^{\heartsuit},d})_{\mathbb{Q}} \oplus \bigoplus_{D \in X_d(k)} \mathrm{Ch}_0(\mathrm{Sht}^{\mu}_{\mathcal{M},D})_{\mathbb{Q}}.$$

We write a subscript D for the corresponding component in this direct sum decomposition.

The first step to prove Theorem 0.1 is to express $\mathbb{I}_r(h_D)$ as

$$\mathbb{I}_r(h_D) = \deg\big((\mathrm{id} \times \mathrm{Fr}_{\mathcal{M}_d})^! \zeta\big)_D \in \mathbb{Q},$$

for some $\zeta \in \mathrm{Ch}_{2d-g+1}(\mathrm{Hk}^{\mu}_{\mathcal{M},d})_{\mathbb{Q}}$ such that $\zeta|_{\mathrm{Hk}^{\mu}_{\mathcal{M}^{\diamondsuit},d}}$ is the fundamental cycle. The existence of such ζ will be explained in the next talk. Then the refined Lefschetz trace formula Proposition A.12 implies that

$$\mathbb{I}_r(h_D) = \sum_{a \in \mathcal{A}_D(k)} \mathrm{Tr}\Big(\big(f_{\mathcal{M},!}\mathrm{cl}(\zeta)\big)_a \circ \mathrm{Frob}_a;\ \big(\mathbb{R}f_{\mathcal{M},!}\mathbb{Q}_\ell\big)_{\bar{a}}\Big).$$

Now, to prove the theorem, it is enough to show that $f_{\mathcal{M},!}\mathrm{cl}(\zeta)$ and $\big(f_{\mathcal{M},!}[\mathcal{H}^{\diamondsuit}]\big)^r$ give the same endomorphism on $\mathbb{R}f_{\mathcal{M},!}\mathbb{Q}_\ell$. But the difference between ζ and $[\mathcal{H}^{\diamondsuit}]^r$ lies in

$$\mathrm{Im}\Big(\mathrm{Hk}^{\mu}_{\mathcal{M},d} \backslash \mathrm{Hk}^{\mu}_{\mathcal{M}^{\diamondsuit},d} \xrightarrow{(\gamma_0,\gamma_r)} \mathcal{M}_d \times \mathcal{M}_d\Big)$$

which has dimension $\leq d + 2g - 1 < 2d - g + 1$ (under our assumption). The theorem follows. $\square$

References

[1] Z. Yun, W. Zhang. *Shtukas and The Taylor expansion of L-Functions.* `arXiv:1512.02683`

Talk 14: Alternative calculation of $\mathbb{I}_r(h_D)$

Yakov Varshavsky

1. Overview

The goal of my talk is to sketch the proof of the following result, which was stated in the previous lecture:

Theorem 1.1. [YZ15, Theorem 6.6] *Let D be an effective divisor on X, of degree $d \geq \max\{2g'-1, 2g\}$. Then there exists $\zeta \in \mathrm{Ch}_{2d-g+1}(\mathrm{Hk}^{\mu}_{\mathcal{M},d})_{\mathbf{Q}}$ such that*

(1) $\zeta|_{\mathrm{Hk}^{\mu}_{\mathcal{M}^{\diamond},d}}$ *is the fundamental class, and*
(2) $\mathbb{I}_r(h_D) = \deg((\mathrm{Id}, \mathrm{Fr}_{\mathcal{M}_d})^{!}\zeta)_D$.

Our strategy will be

- To give a formula for ζ.
- To prove that ζ satisfies properties (1), (2) of Theorem (1.1).

Recall that $\mathbb{I}_r(h_D)$ is the intersection product $\mathbb{I}_r(h_D) = \langle [\mathrm{Sht}^{\mu}_T], h_D * [\mathrm{Sht}^{\mu}_T]\rangle_{\mathrm{Sht}'^{\mu}_G}$, while $[\mathrm{Sht}^{\mu}_T]$ and $h_D * [\mathrm{Sht}^{\mu}_T]$ are defined as intersections of the corresponding objects with Frobenius. Therefore to define the right hand side of (2), we first intersect objects with graphs of Frobenius, and then intersect between them. On the other hand, to define the left hand side of (2), we intersect classes first and then intersect them with graphs of Frobenius. That these coincide is the substance of the "octahedron lemma" from Rapoport's talk, which we briefly recall now.

2. The octahedron lemma

Consider the commutative diagram of algebraic stacks

$$\begin{array}{ccccccc}
A_{11} & \xrightarrow{\alpha} & A_{12} & \longleftarrow & A_{13} & & A_{1*} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A_{21} & \longrightarrow & A_{22} & \longleftarrow & A_{23} & & A_{2*} \\
\uparrow & & \uparrow & & \beta\uparrow & & \delta\uparrow \\
A_{31} & \longrightarrow & A_{32} & \longleftarrow & A_{33} & & A_{3*} \\
 & & & & & & \\
A_{*1} & \xrightarrow{\gamma} & A_{*2} & \longleftarrow & A_{*3}, & &
\end{array}$$

where the right column (resp. bottom row) is the column (resp. row) of fibered products of rows (resp. columns) of our diagram.

Proposition 2.1. *(The octahedron lemma) (a) The two fiber products $A_{1*} \times_{A_{2*}} A_{3*}$ and $A_{*1} \times_{A_{*2}} A_{*3}$ are canonically equivalent.*

(b) Assume that

- *the A_{ij}'s are smooth equidimensional all $(i,j) \neq (1,3)$,*
- *the $A_{2*}, A_{3*}, A_{*1}, A_{*2}$ are smooth of expected dimensions,*

- *the map $\alpha, \beta, \gamma, \delta$ satisfy assumptions (A) and (B) from Rapoport's talk.*

Then we have refined Gysin maps

$$\mathrm{Ch}(A_{13}) \xrightarrow{\alpha^!} \mathrm{Ch}(A_{1*}) \xrightarrow{\delta^!} \mathrm{Ch}(A_{1*} \times_{A_{2*}} A_{3*})$$
$$\mathrm{Ch}(A_{13}) \xrightarrow{\beta^!} \mathrm{Ch}(A_{*3}) \xrightarrow{\gamma^!} \mathrm{Ch}(A_{*1} \times_{A_{*2}} A_{*3}),$$

which moreover satisfy

$$\delta^!\alpha^![A_{13}] = \gamma^!\beta^![A_{13}] \in \mathrm{Ch}(A_{1*} \times_{A_{2*}} A_{3*}) = \mathrm{Ch}(A_{*1} \times_{A_{*2}} A_{*3}).$$

Notice that assertion (a) of the proposition is a straight forward verification, while assertion (b) can be viewed as a version of (a), which involves both "derived fiber products" and "classical fiber products". Therefore, the content of (b) is that our assumptions imply that the "derived fiber products" = "classical fiber products".

3. The fundamental diagram

We are going to apply the octahedron lemma to the following diagram, in which the left two columns were defined in previous lectures, while the right hand column is defined as a Hecke version of the middle one.

$$\begin{tikzcd}
\mathrm{Hk}_T^\mu \times \mathrm{Hk}_T^\mu \arrow[r, "\Pi^\mu \times \Pi^\mu"] \arrow[d] & \mathrm{Hk}_G'^{r} \times \mathrm{Hk}_G'^{r} \arrow[d] & \mathrm{Hk}'^{r}_{G,d} \arrow[l] \arrow[d] & \mathrm{Hk}'^{r}_{G,d} \arrow[d] \\
(\mathrm{Bun}_T)^2 \times (\mathrm{Bun}_T)^2 \arrow[r] & (\mathrm{Bun}_G)^2 \times (\mathrm{Bun}_G)^2 & H_d \times H_d \arrow[l] & \mathcal{M}_d \times \mathcal{M}_d \\
\mathrm{Bun}_T \times \mathrm{Bun}_T \arrow[r, "\Pi \times \Pi"] \arrow[u, "\mathrm{Id} \times \mathrm{Fr}"] & \mathrm{Bun}_G \times \mathrm{Bun}_G \arrow[u, "\mathrm{Id} \times \mathrm{Fr}"] & H_d \arrow[l] \arrow[u, "\mathrm{Id} \times \mathrm{Fr}_{H_d}"] & \mathcal{M}_d \arrow[u, "\mathrm{Id} \times \mathrm{Fr}_{\mathcal{M}_d}"]
\end{tikzcd}$$

$$\mathrm{Sht}_T^\mu \times \mathrm{Sht}_T^\mu \xrightarrow{\theta^\mu \times \theta^\mu} \mathrm{Sht}_G'^{r} \times \mathrm{Sht}_G'^{r} \longleftarrow \mathrm{Sht}'_{G,d}$$

The diagram above satisfies all the assumption of the octahedron lemma. Indeed, the potentially non-smooth objects are $\mathrm{Hk}'^{r}_{G,d}, \mathrm{Sht}'_{G,d}$ and $\mathrm{Hk}'^{r}_{\mathcal{M},d}$, while everything else is smooth. For example, to show the assumption for H_d, we prove that the map $\mathrm{pr}_1 : H_d \to \mathrm{Bun}_G$ is smooth of relative dimension $2d$. For simplicity, let us assume that $d = 1$. In this case, pr_1 factors as $H_d \to \mathrm{Bun}_G \times X \to \mathrm{Bun}_G$, and the first map is a $\mathbf{P}^1$-bundle.

4. Sketch of the proof of Theorem 1.1

Consider class $\zeta := (\Pi^\mu \times \Pi^\mu)^![\mathrm{Hk}_G'^{r}] \in \mathrm{Ch}_{2d-g+1}(\mathrm{Hk}'^{r}_{\mathcal{M}_d})$. We need to show that ζ satisfies properties (1) and (2) from Theorem 1.1.

Property (1): Consider the fiber product

$$\begin{tikzcd} \mathrm{Hk}^{\mu}_{\mathcal{M},d} \arrow[r] \arrow[d] & \mathrm{Hk}'^{r}_{G,d} \arrow[d] \\ \mathrm{Hk}^{\mu}_{T} \times \mathrm{Hk}^{\mu}_{T} \arrow[r] & \mathrm{Hk}'^{r}_{G} \times \mathrm{Hk}'^{r}_{G} \end{tikzcd}$$

The total space $\mathrm{Hk}^{\mu}_{\mathcal{M},d}$ is bad, and hard to understand. However, by dimension estimates, the open substack $\mathrm{Hk}^{\mu}_{\mathcal{M}^{\diamond},d}$ has the expected dimension. This implies that $\zeta|_{\mathrm{Hk}^{\mu}_{\mathcal{M}^{\diamond},d}} \in \mathrm{Ch}(\mathrm{Hk}^{\mu}_{\mathcal{M}^{\diamond},d})$ is the fundamental class.

Property (2): Denote by $\mathrm{Sht}^{\mu}_{\mathcal{M},d}$ the total fiber product of the fundamental diagram. Thanks to Proposition 2.1(2), we thus get the following equality

$$(\mathrm{Id}, \mathrm{Fr}_{\mathcal{M}_d})^{!}(\Pi^{\mu} \times \Pi^{\mu})^{!}[\mathrm{Hk}'^{r}_{G,d}] = (\theta^{\mu} \times \theta^{\mu})^{!}(\mathrm{Id} \times \mathrm{Fr}_{H_d})^{!}[\mathrm{Hk}'^{r}_{G,d}] \in \mathrm{Ch}_0(\mathrm{Sht}^{\mu}_{\mathcal{M},d}).$$

Using definition of ζ and observation $(\mathrm{Id} \times \mathrm{Fr}_{H_d})^{!}[\mathrm{Hk}'^{r}_{G,d}] = [\mathrm{Sht}'^{r}_{G}]$, the latter equality can be rewritten as

$$(1) \qquad (\mathrm{Id}, \mathrm{Fr}_{\mathcal{M}_d})^{!}\zeta = (\theta^{\mu} \times \theta^{\mu})^{!}[\mathrm{Sht}'^{r}_{G}] \in \mathrm{Ch}_0(\mathrm{Sht}^{\mu}_{\mathcal{M},d}).$$

Recall that we have a decomposition

$$\mathrm{Sht}^{\mu}_{\mathcal{M}_d} = \coprod_{D \in X_d(k)} \mathrm{Sht}^{\mu}_{\mathcal{M},D},$$

which implies a decomposition $\mathrm{Ch}_0(\mathrm{Sht}^{\mu}_{\mathcal{M}_d})_{\mathbf{Q}} = \bigoplus \mathrm{Ch}_0(\mathrm{Sht}^{\mu}_{\mathcal{M},D})_{\mathbf{Q}}$. Restricting (1) to D, we conclude that

$$(2) \qquad ((\mathrm{Id}, \mathrm{Fr}_{\mathcal{M}_d})^{!}\zeta)_D = ((\theta^{\mu} \times \theta^{\mu})^{!}[\mathrm{Sht}'^{r}_{G}])_D.$$

Thus, to establish Theorem 1.1 (2), we need to show the equality

$$(3) \qquad \deg(((\theta^{\mu} \times \theta^{\mu})^{!}[\mathrm{Sht}'^{r}_{G}])_D) = \langle \theta^{\mu}_{*}[\mathrm{Sht}^{\mu}_{T}], h_D * \theta^{\mu}_{*}[\mathrm{Sht}^{\mu}_{T}]\rangle_{\mathrm{Sht}'^{\mu}_{G}}.$$

Finally, equality (3) follows from functorial properties of refined Gysin maps.

References

[YZ] Zhiwei Yun and Wei Zhang, *Shtukas and the Taylor expansion of L-functions*, arXiv:1512.02683.

Talk 15: Comparison of $\mathcal{M}_d$ and $\mathcal{N}_d$; the weight factors

Ana Caraiani

1. Outline

The goal of this talk was to present the proof of the *key identity* of [1] for most Hecke functions, i.e. to prove Theorem 8.1 of loc. cit. The proof relies crucially on the so-called "perverse continuation principle", which also played a key role in Ngo's proof of the fundamental lemma.

We consider an effective divisor D on the curve X, of large enough degree

$$d \geq \max\{2g' - 1, 2g\}$$

and the corresponding element h_D in the spherical Hecke algebra $\mathcal{H}$. The key identity is the equality

$$(\log q)^{-r} \mathbb{J}_r(h_D) = \mathbb{I}_r(h_D), \tag{1}$$

where the LHS is the analytic side and the RHS is the geometric side.

To prove the key identity, we rely on the geometrization on both sides, which was developed in previous talks. The idea is to express each of the two distributions as a trace of a correspondence acting on a complex of constructible sheaves over a common base, then to compare the two complexes on a "nice" open set of the base using the theory of perverse sheaves.

The common base is

$$\mathcal{A}_d = (\widehat{X}_d \times_{\mathrm{Pic}^d_X} \widehat{X}_d) \setminus Z(\mathrm{Pic}^d_X),$$

where recall that $\widehat{X}_d$ parametrizes line bundles of degree d on X together with a section and we remove $Z(\mathrm{Pic}^d_X)$, the locus where both sections vanish.

1.1. **The analytic side.** Here, the geometrization takes the form:

$$(\log q)^{-r} \mathbb{J}_r(h_D) = \sum_{\underline{d} \in \Sigma_d} \sum_{a \in \mathcal{A}_D(k)} (2d_{12} - d)^r \cdot \mathrm{Tr}(\mathrm{Frob}_a, (Rf_{\mathcal{N}_{\underline{d}}*} L_{\underline{d}})_{\overline{a}}), \tag{2}$$

where $\underline{d}$ runs over the set Σ_d of quadruples $(d_{11}, d_{12}, d_{21}, d_{22})$ with $d_{11} + d_{22} = d_{12} + d_{21} = d$. The moduli space $\mathcal{N}_{\underline{d}}$ is an open substack of $(\widehat{X}_{d_{11}} \times \widehat{X}_{d_{22}}) \times_{\mathrm{Pic}^d_X} (\widehat{X}_{d_{12}} \times \widehat{X}_{d_{21}})$ and the map

$$f_{\mathcal{N}_{\underline{d}}} : \mathcal{N}_{\underline{d}} \to \mathcal{A}_d$$

is the restriction of the addition (or rather tensor product) map:

$$\mathrm{add}_{d_{11},d_{22}} \times \mathrm{add}_{d_{12},d_{21}} : (\widehat{X}_{d_{11}} \times \widehat{X}_{d_{22}}) \times_{\mathrm{Pic}^d_X} (\widehat{X}_{d_{12}} \times \widehat{X}_{d_{21}}) \to \widehat{X}_d \times_{\mathrm{Pic}^d_X} \widehat{X}_d.$$

The local system $L_{\underline{d}}$ on $\mathcal{N}_{\underline{d}}$ also has a concrete description in terms of geometric class field theory applied to the local system $L := (\nu_* \mathbb{Q}_l)^{\sigma = -1}$ on X obtained from the étale double cover $\nu : X' \to X$ with involution σ.

1.2. **The geometric side.** Here, the geometrization takes the form:

$$\mathbb{I}_r(h_D) = \sum_{a\in\mathcal{A}_D(k)} \mathrm{Tr}((f_{\mathcal{M}!}[\mathcal{H}^\diamond]_a)^r \circ \mathrm{Frob}_a, (Rf_{\mathcal{M}!}\mathbb{Q}_l)_{\overline{a}}). \tag{3}$$

The moduli stack $\mathcal{M}_d$ is an open substack of $\widehat{X'}_d \times_{\mathrm{Pic}_X^d} \widehat{X'}_d$ and the map

$$f_{\mathcal{M}} : \mathcal{M}_d \to \mathcal{A}_d$$

is the restriction of the norm map

$$\widehat{X'}_d \times_{\mathrm{Pic}_X^d} \widehat{X'}_d \xrightarrow{\widehat{\nu}_d\times\widehat{\nu}_d} \widehat{X}_d \times_{\mathrm{Pic}_X^d} \widehat{X}_d.$$

The correspondence $[\mathcal{H}^\diamond]$ is the the fundamental class over the "nice" locus $\diamond$ (corresponding to the non-vanishing of the second section in the moduli description of $\mathcal{A}_d$) and this will allow us to compute the action of $f_{\mathcal{M}!}[\mathcal{H}^\diamond]$ on $Rf_{\mathcal{M}!}\mathbb{Q}_l$.

1.3. **The comparison.** To prove the identity (1), it is enough to match the two traces in (2) and (3).

To compare the two traces, we first compare the complexes $Rf_{\mathcal{N}_{\underline{d}}*}L_{\underline{d}}$ and $Rf_{\mathcal{M}!}\mathbb{Q}_l$ on $\mathcal{A}_d$. We do this by first showing that they each are (up to shift) perverse sheaves and even *middle extensions* of their restrictions to a nice open subset of $\mathcal{A}_d$. On this open subset, each of the complexes is a local system which can be computed explicitly in terms of the representation theory of finite groups. The two representation-theoretic computations give the same result. The two (shifted) middle extension perverse sheaves $Rf_{\mathcal{N}_{\underline{d}}*}L_{\underline{d}}$ and $Rf_{\mathcal{M}!}\mathbb{Q}_l$ can therefore be identified; this is where we use the so-called "perverse continuation principle".

Finally, we compute the action of $f_{\mathcal{M}!}[\mathcal{H}^\diamond]$ on $Rf_{\mathcal{M}!}\mathbb{Q}_l$, which for even r recovers on the geometric side the weight factors $(2d_{12}-d)^r$ seen on the analytic side.

The proof of the key identity can be broken down in the following three steps.

(1) The computation of $Rf_{\mathcal{M}!}\mathbb{Q}_l$.
(2) The computation of $Rf_{\mathcal{N}_{\underline{d}}*}L_{\underline{d}}$.
(3) Computing the action of $f_{\mathcal{M}!}[\mathcal{H}^\diamond]$.

Below, we briefly mention the key ideas going into each of the three steps.

2. The details of the proof

2.1. **The geometric side.** The "nice open subset" of $\mathcal{A}_d$ will be determined by the locus $X_d^\circ \subset X_d \subset \widehat{X}_d$ of multiplicity-free effective divisors of degree d. More precisely, we determine the complex $Rf_{\mathcal{M}!}\mathbb{Q}_l$ restricted to the open subset of $\mathcal{A}_d$ given by $X_d^\circ \times_{\mathrm{Pic}_X^d} X_d^\circ$.

Let S_d be the symmetric group on d elements. For each $i \in \{0,\dots,d\}$ we can define an irreducible representation ρ_i of $\{\pm1\} \rtimes S_d$ (it will have dimension $\binom{d}{i}$). This determines an irreducible local system $L(\rho_i)$ on X_d°. If $j : X_d^\circ \hookrightarrow \widehat{X}_d$, we can define the shifted simple perverse sheaf

$$K_i := j_{!*}(L(\rho_i)[d])[-d]$$

on $\widehat{X}_d$. The following is the main computation on the geometric side.

Proposition 2.1. *Assume $d \geq 2g' - 1$. There is a canonical isomorphism of shifted perverse sheaves on $\mathcal{A}_d$*

$$Rf_{\mathcal{M}!}\mathbb{Q}_l \simeq \oplus_{i,j=0}^{d}(K_i \boxtimes K_j)|_{\mathcal{A}_d}$$

By using proper base change (for the proper map $\widehat{\nu}_d$) and the Künneth formula, we can reduce to showing that there is a canonical isomorphism

$$R\widehat{\nu}_{d!}\mathbb{Q}_l \simeq \oplus_{i=0}^{d} K_i|_{\mathcal{A}_d}$$

The key idea in the proof of Proposition 2.1 is that when $d \geq 2g' - 1$, $\widehat{\nu}_d$ is a small map (the high-dimensional fibers are over a locus of large codimension). Because $\widehat{\nu}_d$ is proper, small and with geometrically irreducible source, one can show that $R\widehat{\nu}_{d!}\mathbb{Q}_l$ is the middle extension of its restriction to X_d°. This restriction is a local system which can be computed explicitly using the representation theory of $\{\pm 1\} \rtimes S_d$.

2.2. **The analytic side.** The main computation on the analytic side is the following.

Proposition 2.2. *Assume $d \geq 2g' - 1$. Let $\underline{d} \in \Sigma_d$. There is a canonical isomorphism of shifted perverse sheaves on $\mathcal{A}_d$*

$$Rf_{\mathcal{N}_{\underline{d}}*}L_{\underline{d}} \simeq (K_{d_{11}} \boxtimes K_{d_{12}})|_{\mathcal{A}_d}$$

The idea of the proof of Proposition 2.2 is the same as for Proposition 2.1, except that the map

$$f_{\mathcal{N}_{\underline{d}}} : \mathcal{N}_{\underline{d}} \to \mathcal{A}_d$$

is not necessarily small. Instead, one uses the explicit description of the local system $L_{\underline{d}}$ to show that the cohomological dimension of the fibers is bounded. Since smallness is used above only to bound the cohomological dimension of the fibers of $f_{\mathcal{N}_{\underline{d}}}$, we can argue as above to prove that $Rf_{\mathcal{N}_{\underline{d}}*}L_{\underline{d}}$ is the middle extension of its restriction to $X_d^\circ \times_{\mathrm{Pic}_X^d} X_d^\circ$. An explicit computation of this restriction follows.

2.3. **The weight factors.** The main result computing the weight factors is the following.

Proposition 2.3. *The correspondence $f_{\mathcal{M}!}[\mathcal{H}^\diamond]$ respects the decomposition*

$$Rf_{\mathcal{M}!}\mathbb{Q}_l \simeq \oplus_{i,j=0}^{d}(K_i \boxtimes K_j)|_{\mathcal{A}_d}$$

and acts on $K_i \boxtimes K_j$ by $(d - 2j)$.

Again, the idea is to use the perverse continuation principle to reduce to considering the open locus $\mathcal{A}_d^\diamond \subset \mathcal{A}_d$ given by $\widehat{X}_d \times_{\mathrm{Pic}_X^d} X_d$. There, the correspondence $\mathcal{H}^\diamond$ is obtained by pullback under a smooth map from an explicit correspondence of X'_d over X_d. After smooth base change, everything can be computed explicitly using the representation theory of $\{\pm 1\} \rtimes S_d$.

References

[1] Z. Yun and W Zhang, *Shtukas and the Taylor expansion of L-functions*, to appear in Annals of Math.

Talk 16: Horocycles

Lizao Ye

1. Outlook

We want to prove

$$\mathbb{J}_r(\pi) = \mathbb{I}_r(\pi).$$

What we have is $\mathbb{J}_r = \mathbb{I}_r$. So we need some spectral decomposition. This has been done for the analytic side. The geometric side rests on spectral decomposition of the cohomology of shtukas, $H_c^{2r}(\mathrm{Sht}_G)$. This is achieved by an analysis of the Hecke action on it.

2. Hecke action

Suppose G is a (split) reductive group. For every $g = \bigotimes g_v \in G(\mathbf{A}_F)$, we get a correspondence $\mathrm{Sht}_G(g)$. It parametrizes pairs of shtukas in relative position g

$$(\mathcal{E}_\cdot \overset{g}{\dashrightarrow} \mathcal{E}'_\cdot).$$

This defines an algebra homomorphism

$$\mathcal{H}_G \to \operatorname{End} H_c^i(\mathrm{Sht}_G).$$

We sketch why this is the case.

Recall that the ring structure on the Hecke algebra is defined by convolution:

$$\mathbf{1}_{Kg_1K} * \mathbf{1}_{Kg_2K} = \sum_{g_3 \in K\backslash G/K} [g_3^{-1}Kg_1K \cap Kg_2^{-1}K : K] \cdot \mathbf{1}_{Kg_3K}$$

The fiber product of $\mathrm{Sht}_G(g_1)$ and $\mathrm{Sht}_G(g_2)$ is "basically" several copies of $\mathrm{Sht}_G(g_3)$:

$$\begin{tikzcd}
& & ? \arrow[dl] \arrow[dr] & & \\
& \mathrm{Sht}_G(g_1) \arrow[dl] \arrow[dr] & & \mathrm{Sht}_G(g_2) \arrow[dl] \arrow[dr] & \\
\mathrm{Sht}_G & & \mathrm{Sht}_G & & \mathrm{Sht}_G
\end{tikzcd}$$

In fact the number $[g_3^{-1}Kg_1K \cap Kg_2^{-1}K : K]$ is the number of copies of $\mathrm{Sht}_G(g_3)$ appearing in the fiber product.

3. The constant term map

Recall here $G = \mathrm{PGL}_2$. Let $B \subset G$ be a Borel, and H the universal Cartan considered as a quotient of B.

Consider the diagram

$$\begin{tikzcd} & \mathrm{Sht}^d_{B,\eta} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathrm{Sht}_{G,\eta} & & \mathrm{Sht}^d_{H,\eta} \end{tikzcd}$$

where the subscript η denotes restriction to the fiber over the (geometric) generic point η of X^r. We have $\dim \mathrm{Sht}_{G,\eta} = r$, $\dim \mathrm{Sht}^d_{B,\eta} = r/2$, and $\dim \mathrm{Sht}^d_{H,\eta} = 0$.

Definition 3.1. The *constant term map* is the composition

$$CT\colon H^r_c(\mathrm{Sht}_{G,\eta}) \xrightarrow{p^*} \prod_d H^r_c(\mathrm{Sht}^d_{B,\eta}) \xrightarrow{\text{trace}} \prod_d H^0_c(\mathrm{Sht}^d_{H,\eta}).$$

Here we need the map p to be proper. This is guaranteed by the following theorem.

Theorem 3.2 (Drinfeld[1],Varshavsky[2])**.** *The maps* $\mathrm{Sht}^d_{B,\eta} \to \mathrm{Sht}_{G,\eta}$ *are finite unramified.*

The key fact we'll need is that the constant term map commutes with the Satake transform

$$\mathrm{Sat} : \mathcal{H}_G \to \mathcal{H}_H, h_x \mapsto t_x + q_x t_x^{-1}.$$

Theorem 3.3. *For* $h \in \mathcal{H}_G$*, we have*

$$CT \circ h = \mathrm{Sat}(h) \circ CT.$$

Proof. Let x be a closed point of the curve X, it's enough to show

$$CT \circ h_x = t_x \circ CT + q_x t_x^{-1} \circ CT$$

We claim that the diagram

$$\begin{tikzcd} \mathrm{Sht}_{B,\eta} \arrow[d] & & \mathrm{Sht}_{B,\eta} \arrow[d] \\ \mathrm{Sht}_{G,\eta} & \mathrm{Sht}_{G,\eta}(h_x) \arrow[l] \arrow[r] & \mathrm{Sht}_{G,\eta} \end{tikzcd}$$

can be completed into

$$\begin{tikzcd} \mathrm{Sht}_{B,\eta} \arrow[d] & C \arrow[l] \arrow[r] \arrow[d] & \mathrm{Sht}_{B,\eta} \arrow[d] \\ \mathrm{Sht}_{G,\eta} & \mathrm{Sht}_{G,\eta}(h_x) \arrow[l] \arrow[r] & \mathrm{Sht}_{G,\eta} \end{tikzcd}$$

such that both squares are cartesian. This follows from the fact that given an inclusion(as sheaves) of rank two vector bundles $\mathcal{E} \to \mathcal{E}'$, giving a line-subbuncle of one of them produces automatically a line-subbundle of the other, by intersection or by saturation.

The stack C is a disjoint union of two (open) substacks C_1 and C_2, where C_1 classifies those whose modification occurs in the sub, and C_2 those whose modification occurs in the quotient. Therefore we can decompose $CT \circ h_x$ as a sum:

$$CT \circ h_x = CT_{x,1} + CT_{x,2}$$

where $CT_{x,1}$ corresponds to C_1, and $CT_{x,2}$ to C_2.

Besides, it's easy to see

$$\mathrm{Sht}_{B,\eta} \xleftarrow{\sim} C_1 \xrightarrow{q_x:1,\text{ étale}} \mathrm{Sht}_{B,\eta}. \tag{1}$$

Standard properties of the trace map then imply

$$CT_{x,1} = t_x \circ CT.$$

Similarly,

$$\mathrm{Sht}_{B,\eta} \xleftarrow{\text{étale},1:q_x} C_2 \xrightarrow{\sim} \mathrm{Sht}_{B,\eta}. \tag{2}$$

Hence

$$CT_{x,2} = q_x t_x^{-1} \circ CT,$$

whence the theorem. □

4. Statement of the main theorem

There's a finite type substack of Sht_G outside of which the map from Sht_B^d is an isomorphism. Thus Sht_B can be viewed as the "infinite part" of Sht_G. So the cohomology of Sht_G on this infinite part admits a filtration by cohomologies of Sht_B^d, which can then be calculated by pushforward to Sht_H^d. Now Sht_H^d is a $\mathrm{Pic}^0(\mathbf{F}_q)$-torsor over X^r, which we understand well. So the issue is in understanding the fibers of $\mathrm{Sht}_B^d \to \mathrm{Sht}_H^d$. They are the so called *horocycles*.

Theorem 4.1. *For large enough degrees d, fibers of $\mathrm{Sht}_B^d \to \mathrm{Sht}_H^d$ are isomorphic to an affine space $\mathbf{G}_a^{r/2}$ divided (in the sense of stacks) by a finite étale group scheme Z.*

Corollary 4.2. *Let $\pi_G\colon \mathrm{Sht}_G \to X^r$. For large d, the cone of*

$$R\pi_{G!}(\mathrm{Sht}_G^{<d}) \to R\pi_{G!}(\mathrm{Sht}_G^{\leq d})$$

is some locally constant sheaf on X^r, concentrated in degree r.

References

[1] V. G. Drinfel'd, *Moduli varieties of F-sheaves*, Funktsional. Anal. i Prilozhen. 21 (1987), no. 2, 23-41 (Russian). MR902291 (89b:11092)

[2] Y. Varshavsky, *Moduli spaces of principal F-bundles*, Selecta Math. (N.S.) 10 (2004), no. 1, 131-166, DOI 10.1007/s00029-004-0343-0. MR2061225 (2005m:14043)

Talk 17: Cohomological spectral decomposition and finish the proof

Chao Li

1. Overview

Our goal today is to finish the proof of the main identity

$$(\log q)^{-r}\mathbb{J}_r(f) = \mathbb{I}_r(f)$$

for all functions in the spherical Hecke algebra $f \in \mathcal{H} = C_c(K\backslash G(\mathbb{A})/K)$ of $G = \mathrm{PGL}_2$. For any π (unramified everywhere) cuspidal automorphic representation of $G(\mathbb{A})$, the LHS via the analytic spectral decomposition and the RHS via the cohomological spectral decomposition (discussed below) would imply the identity

$$\lambda_\pi(f)\cdot\mathcal{L}^{(r)}(\pi_{F'},1/2) \sim \langle[\mathrm{Sht}_T]_\pi, f*[\mathrm{Sht}_T]_\pi\rangle.$$

We now have the wonderful opportunity to apply the identity to simplest element in the Hecke algebra, namely the the unit element $\mathbf{1}_K \in \mathcal{H}$, and obtain our desired Higher Gross–Zagier formula

$$\mathcal{L}^{(r)}(\pi_{F'},1/2) \sim \langle[\mathrm{Sht}_T]_\pi, [\mathrm{Sht}_T]_\pi\rangle.$$

Ana's talk has proved the main identity for many h_D's but we fall short of proving it for the element $\mathbf{1}_K$: in some sense the simplest Hecke function gives the most difficult situation for intersection computation (self-intersection), and considering h_D for sufficiently large D allows us to move away from the self-intersection situation and make the computation easier. What we would like to do is to resolve this tension, and deduce the identity for all Hecke functions from sufficiently many h_D's by just doing commutative algebra. What make the deduction possible are certain key finiteness properties of the Hecke action on the middle cohomology of the moduli of shtukas.

2. Key Finiteness Theorems

Recall the middle cohomology

$$V = H_c^{2r}(\mathrm{Sht}_G^r, \mathbb{Q}_\ell),$$

which admits an action of the Hecke algebra

$$\mathcal{H} = C_c(K\backslash G(\mathbb{A})/K, \mathbb{Q}_\ell) = {\bigotimes_{x\in|X|}}' \mathcal{H}_x.$$

Remark 2.1. V is infinite dimensional caused by the fact that Sht_G^r is only *locally* of finite type. This infinite dimensionality can already be seen when $r = 0$, where we recover the classical Hecke action on the space of automorphic forms of level 1:

$$\mathcal{A} = C_c(G(F)\backslash G(\mathbb{A})/K, \mathbb{Q}_\ell) = \mathcal{A}_{\mathrm{Eis}} \oplus \mathcal{A}_{\mathrm{cusp}}.$$

Here the space of cusp forms $\mathcal{A}_{\mathrm{cusp}}$ is finite dimensional, but the space of Eisenstein series is infinite dimensional.

To kill the Eisenstein part, we again make use of the Eisenstein ideal appeared in Ilya's talk on analytic spectral decomposition. Recall we define the Eisenstein ideal to be

$$I_{\mathrm{Eis}} := \ker(\mathcal{H} \xrightarrow{\mathrm{Sat}} \mathcal{H}_A \cong \mathbb{Q}_\ell[\mathrm{Div}_X(k)] \to \mathbb{Q}_\ell[\mathrm{Pic}_X(k)]),$$

moreover $\mathcal{H}/I_{\mathrm{Eis}} \cong \mathbb{Q}_\ell[\mathrm{Pic}_X(k)]^\iota$, which is 1-dimensional as a ring ($\mathrm{Pic}_X(k)$ is an extension of $\mathbb{Z}$ by the finite group $\mathrm{Jac}_X(k)$).

Definition 2.2. Define $Z_{\mathrm{Eis}} := \mathrm{Spec}\,\mathcal{H}/I_{\mathrm{Eis}}$ a closed subscheme of $\mathrm{Spec}\,\mathcal{H}$, which is reduced and 1-dimensional.

After killing the Eisenstein part, we indeed obtain a finite dimensional vector space.

Theorem 2.1. *$I_{\mathrm{Eis}} \cdot V$ is a finite dimensional $\mathbb{Q}_\ell$-vector space.*

Sketch. Recall that $\mathrm{Sht}_G = \cup_d \mathrm{Sht}_G^{\leq d}$ is a union of open substacks of finite type $\mathrm{Sht}_G^{\leq d}$ with instability index bounded by d. The key point here is that one can understand the difference between the cohomology of

$$H_c^i(\mathrm{Sht}_G^{<d}, \mathbb{Q}_\ell) \text{ and } H_c^i(\mathrm{Sht}_G^{\leq d}, \mathbb{Q}_\ell)$$

using horocycles discussed in Lizao's talk. More precisely, when $d > 2g-2$, the cone of the natural map

$$\mathbf{R}\pi_{G,!}^{<d}\mathbb{Q}_\ell \to \mathbf{R}\pi_{G,!}^{\leq d}\mathbb{Q}_\ell$$

is equal to $\mathbf{R}\pi_{H,!}^{d}\mathbb{Q}_\ell[-r](-r/2)$, which is a local system concentrated in degree r.

In particular, it suffices to work with the generic fiber and show $I_{\mathrm{Eis}} \cdot H_c^r(\mathrm{Sht}_{G,\bar\eta})$ is finite dimensional. Let U be the finite union of $\mathrm{Sht}_G^{\leq d}$, where instability index d is not all $> 2g-2$. Using the compatibility of the cohomological constant map and the Satake transform, we have a commutative diagram

$$\begin{tikzcd}
H_c^r(\mathrm{Sht}_{G,\bar\eta}) \arrow[r, "f*"] \arrow[d, "\prod \gamma_d"] & H_c^r(\mathrm{Sht}_{G,\bar\eta}) \arrow[r] \arrow[d, "\prod \gamma_d"] & H_c^r(\mathrm{Sht}_{G,\bar\eta})/H_c^r(U_{\bar\eta}) \arrow[d, "\prod_{d>2g-2} \gamma_d"] \\
\prod_d H_0(\mathrm{Sht}_{H,\bar\eta}^d) \arrow[r, "\mathrm{Sat}(f)*"] & \prod_d H_0(\mathrm{Sht}_{H,\bar\eta}^d) \arrow[r] & \prod_{d>2g-2} H_0(\mathrm{Sht}_{H,\bar\eta}^d)
\end{tikzcd}$$

For $f \in I_{\mathrm{Eis}}$, by definition $\mathrm{Sat}(f)* = 0$ and so the bottom row is zero. The cohomological constant map on the right is injective since $d > 2g-2$. It follows that the top row is zero, and hence the image of $f*$ is contained in $H_c^r(U_{\bar\eta})$, which is finite dimensional as desired. □

Using a similar argument, one can also prove the following finiteness theorem.

Theorem 2.2. *V is a finitely generated $\mathcal{H}_x$-module for any $x \in |X|$.*

3. Cohomological spectral decomposition

Let $\overline{\mathcal{H}} = \mathrm{Im}(\mathcal{H} \to \mathrm{End}_{\mathbb{Q}_\ell}(V) \times \mathbb{Q}_\ell[\mathrm{Pic}_X(k)])$. We have the following immediate consequence:

Corollary 3.1. *$\overline{\mathcal{H}}$ is a finitely generated $\mathbb{Q}_\ell$-algebra (in particular, a noetherian ring).*

Proof. We have an embedding

$$\overline{\mathcal{H}} \hookrightarrow \mathrm{End}_{\mathcal{H}_x}(V \oplus \mathbb{Q}_\ell[\mathrm{Pic}_X(k)]^\iota).$$

By Theorem 2.2, V is a finite $\mathcal{H}_x$-module. Also $\mathbb{Q}_\ell[\mathrm{Pic}_X(k)]^\iota$ is finite $\mathcal{H}_x$-module (due to the finiteness of $\mathrm{Jac}_X(k)$). It follows that RHS is a finite $\mathcal{H}_x$-module and hence $\overline{\mathcal{H}}$ is a finite $\mathcal{H}_x$-module. Because $\mathcal{H}_x \cong \mathbb{Q}_\ell[h_x]$ is a polynomial algebra, it follows that $\overline{\mathcal{H}}$ is a finitely generated $\mathbb{Q}_\ell$-algebra. □

Now V is a finite module over the noetherian ring $\overline{\mathcal{H}}$ (by Theorem 2.2, V is a finite module even over $\mathcal{H}_x$), we obtain the following cohomological spectral decomposition:

Theorem 3.1 ([YZ15, Theorem 7.14])**.** (1) *There is a decomposition*

$$\mathrm{Spec}\, \overline{\mathcal{H}}^{\mathrm{red}} = Z_{\mathrm{Eis}} \coprod Z_0^r.$$

Here Z_0^r is a finite set of closed points.

(2) *There is a unique decomposition of $\mathcal{H}$-modules*

$$V = V_{\mathrm{Eis}} \oplus V_0$$

such that $\mathrm{supp}\, V_{\mathrm{Eis}} \subseteq Z_{\mathrm{Eis}}$, $\mathrm{supp}\, V_0 = Z_0^r$ *and* V_0 *is finite dimensional over* $\mathbb{Q}_\ell$.

Remark 3.2. The fact that Z_0^r is a finite set of closed points and V_0 is finite dimensional follows from Theorem 2.1. Moreover, when $r = 0$, we exactly recover the automorphic spectral decomposition $V_{\mathrm{Eis}} = \mathcal{A}_{\mathrm{Eis}}$ and $V_0 = \mathcal{A}_{\mathrm{cusp}}$.

4. Finish of the proof of the main identity

Theorem 4.1 ([YZ15, Theorem 9.2])**.** *For any $f \in \mathcal{H}$, we have the identity of rational numbers*

$$(\log q)^{-r}\mathbb{J}_r(f) = \mathbb{I}_r(f).$$

Proof. Define $\tilde{\mathcal{H}}$ to be the image of $\mathcal{H}$ in $\mathrm{End}(V) \times \mathrm{End}(\mathcal{A})$. Then both sides of the identify only depend on the image of f in $\tilde{\mathcal{H}}$. Define $\mathcal{H}' \subseteq \mathcal{H}$ to be the linear subspace spanned by h_D's, where D effective divisor of degree $\geq d_0 = \max\{2g' - 1, 2g\}$. By Ana's Talk, we already proved the identity for $f \in \mathcal{H}'$. So it remains to show that the composition $\mathcal{H}' \hookrightarrow \mathcal{H} \twoheadrightarrow \tilde{\mathcal{H}}$ is surjective.

Using the key finiteness theorem one can prove the following lemma:

Lemma 4.1. *There is an ideal $I \subseteq \tilde{\mathcal{H}}$ such that $\tilde{\mathcal{H}}/I$ is finite dimensional and is generated by the image of $\mathcal{H}'$.*

Besides the key finiteness theorems, we need one additional ingredient concerning the local Hecke algebra. It is a bit magical but completely elementary:

Lemma 4.2. *For any nonzero ideal $I \subseteq \mathcal{H}_x$, and for any $m \geq 1$, we have*

$$I + \mathrm{span}\{h_{nx}\}_{n \geq m} = \mathcal{H}_x.$$

Now we can finish the proof of the main identity using the previous two lemmas. For any $x \in |X|$, look at the commutative diagram

$$\begin{tikzcd}
\mathcal{H}' \arrow[r] & \mathcal{H} \arrow[r] & \tilde{\mathcal{H}} \arrow[r] & \tilde{\mathcal{H}}/I \\
\mathcal{H}' \cap \mathcal{H}_x \arrow[r] \arrow[u] & \mathcal{H}_x \arrow[rr] \arrow[u] & & \mathrm{Im}(\mathcal{H}_x) \arrow[u]
\end{tikzcd}$$

Here the vertical arrows are all natural inclusions.

By Lemma 4.1, $\tilde{\mathcal{H}}/I$ is finite dimensional, so $\mathrm{Im}(\mathcal{H}_x)$ is also finite dimensional. Since $\mathrm{Im}(\mathcal{H}_x)$ is quotient of $\mathcal{H}_x$ by a nonzero ideal, and $\mathcal{H}' \cap \mathcal{H}_x = \{h_{nx} : \deg nx \geq d_0\}$, we know the bottom row is surjective by Lemma 4.2. Now $\{\mathcal{H}_x, x \in |X|\}$ generate $\tilde{\mathcal{H}}$ as an algebra, so the top row is also surjective. By Lemma 4.1, $I \subseteq \mathrm{Im}(\mathcal{H}' \to \tilde{\mathcal{H}})$, so the map $\mathcal{H}' \to \tilde{\mathcal{H}}$ is surjective as desired. $\square$

References

[YZ15] Z. Yun and W. Zhang. Shtukas and the Taylor expansion of L-functions. *ArXiv e-prints*, December 2015.

Reporter: Tony Feng

Participants

Xiaohua Ai
Institut de Mathématiques de Jussieu
Theorie des Nombres, Case 247
Université de Paris VI
4, Place Jussieu
75252 Paris Cedex 05
FRANCE

Dr. Ashay A. Burungale
Département de Mathématiques
Institut Galilee
Université Paris XIII
99 Av. J.-B. Clement
93430 Villetaneuse Cedex
FRANCE

Dr. Li Cai
Yau Mathematical Sciences Center
Tsinghua University
Room 258
Jin Chun Yuan West Building
Beijing 100 084
CHINA

Dr. Ana Caraiani
Department of Mathematics
Princeton University
710 Fine Hall
Washington Road
Princeton NJ 08544-1000
UNITED STATES

Prof. Dr. Huai-Liang Chang
Department of Mathematics
Hong Kong University of Science and Technology
Tower 3 - 3C
Clear Water Bay
Kowloon
HONG KONG

Dr. Thinh Dao Van
Department of Mathematics
National University of Singapore
10 Kent Ridge Cresent
Singapore 119 260
SINGAPORE

Prof. Dr. Christopher Deninger
Mathematisches Institut
Universität Münster
Einsteinstrasse 62
48149 Münster
GERMANY

Gurbir Dhillon
Department of Mathematics
Stanford University
Stanford, CA 94305
UNITED STATES

Dr. Daniel Disegni
Department of Mathematics & Statistics
McGill University
805 Sherbrooke Street West
Montreal QC H3A 0B9
CANADA

Prof. Dr. Laurent Fargues
Institut de Mathématiques de Jussieu
Université de Paris VI
Case 247
4, Place Jussieu
75252 Paris Cedex 05
FRANCE

Tony Feng
Apartment #3
815 Roble Avenue
Menlo Park CA 94025
UNITED STATES

Prof. Dr. Thomas Haines
Department of Mathematics
University of Maryland
College Park, MD 20742-4015
UNITED STATES

Prof. Dr. Urs Hartl
Mathematisches Institut
Universität Münster
Einsteinstrasse 62
48149 Münster
GERMANY

Jochen Heinloth
Fachbereich Mathematik
Universität Duisburg-Essen
Universitätsstrasse 3
45117 Essen
GERMANY

Chi-Yun Hsu
Department of Mathematics
Harvard University
Science Center
One Oxford Street
Cambridge MA 02138-2901
UNITED STATES

Prof. Dr. Brian Hwang
Department of Mathematics
Cornell University
436 Malott Hall
Ithaca, NY 14853-4201
UNITED STATES

Prof. Dr. Yunfeng Jiang
Department of Mathematics
University of Kansas
405 Snow Hall
Lawrence, KS 66045-7567
UNITED STATES

Dr. Vitezslav Kala
Mathematisches Institut
Georg-August-Universität Göttingen
Bunsenstrasse 3-5
37073 Göttingen
GERMANY

Ilya Khayutin
Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton, NJ 08544-1000
UNITED STATES

Prof. Dr. Bruno Klingler
Equipe Formes Automorphes
Université Paris - Diderot
Croisement Ave. de France / Rue Alice
75013 Paris Cedex
FRANCE

Dr. Arthur-César Le Bras
D. M. A.
École Normale Superieure
45, Rue d'Ulm
75230 Paris Cedex 05
FRANCE

Prof. Dr. Chao Li
Department of Mathematics
Columbia University
2990 Broadway
New York, NY 10027
UNITED STATES

Huajie Li
U. F. R. de Mathématiques
Case 7012
Université Paris 7
2, Place Jussieu
75251 Paris Cedex 05
FRANCE

Qirui Li
Department of Mathematics
Columbia University
2990 Broadway
New York, NY 10027
UNITED STATES

Shizhang Li
Department of Mathematics
Barnard College
Columbia University
New York, NY 10027
UNITED STATES

Dr. Yingkun Li
Fachbereich Mathematik
Technische Universität Darmstadt
Schloßgartenstrasse 7
64289 Darmstadt
GERMANY

Prof. Dr. Davesh Maulik
Department of Mathematics
Massachusetts Institute of Technology
77 Massachusetts Avenue
Cambridge, MA 02139-4307
UNITED STATES

Lucia Mocz
Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton, NJ 08544-1000
UNITED STATES

Ilia Nekrasov
Chebyshev Laboratory
St. Petersburg State University
14th Line 29B
Vasilyevsky Island
St. Petersburg 199 178
RUSSIAN FEDERATION

Dr. Vivek Pal
Department o Mathematics
University of Oregon
Fenton Hall
Eugene OR 97403-1222
UNITED STATES

Dr. Xuanyu Pan
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY

Prof. Dr. Michael Rapoport
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Dr. Timo Richarz
Fachbereich Mathematik
Universität Duisburg-Essen
Thea-Leymann-Strasse 9
45127 Essen
GERMANY

Dr. Jyoti Prakash Saha
Mathematisches Forschungsinstitut
Oberwolfach gGmbH
Schwarzwaldstrasse 9-11
77709 Oberwolfach-Walke
GERMANY

Dr. Ariel Shnidman
Department of Mathematics
Boston College
Chestnut Hill, MA 02467
UNITED STATES

Prof. Dr. Brian Smithling
Department of Mathematics
Johns Hopkins University
Baltimore, MD 21218-2689
UNITED STATES

Dr. Changjian Su
Department of Mathematics
Columbia University
RM 408, MC 4406
2990 Broadway
New York NY 10027
UNITED STATES

Matteo Tamiozzo
Fakultät für Mathematik
Universität Duisburg-Essen
45117 Essen
GERMANY

Dr. Yunqing Tang
School of Mathematics
Institute for Advanced Study
1 Einstein Drive
Princeton, NJ 08540
UNITED STATES

Prof. Dr. Ye Tian
Morningside Center of Mathematics
No 55. Zhongguancun East Road
Beijing 100 190
CHINA

Prof. Dr. Douglas Ulmer
School of Mathematics
Georgia Institute of Technology
686 Cherry Street
Atlanta, GA 30332-0160
UNITED STATES

Dr. Yakov Varshavsky
Einstein Institute of Mathematics
The Hebrew University
Givat Ram
Jerusalem 91904
ISRAEL

Jingwei Xiao
Department of Mathematics
Columbia University
2990 Broadway
New York, NY 10027
UNITED STATES

Prof. Dr. Liang Xiao
Department of Mathematics
University of Connecticut
Storrs, CT 06269-3009
UNITED STATES

Cong Xue
Mathématiques
Université de Paris-Sud
Batiment 430
91405 Orsay Cedex
FRANCE

Dr. Zijian Yao
Department of Mathematics
Harvard University
Science Center 428 F
1 Oxford Street
Cambridge MA 02138
UNITED STATES

Lizao Ye
Institut Elie Cartan
-Mathématiques-
Université Henri Poincare, Nancy I
Boite Postale 239
54506 Vandoeuvre-lès-Nancy Cedex
FRANCE

Hongjie Yu
Institut de Mathématiques de Jussieu
Université Paris VII
175, rue du Chevaleret
75013 Paris Cedex
FRANCE

Prof. Dr. Zhiwei Yun
Department of Mathematics
Yale University
10 Hillhouse Avenue
New Haven, CT 06511
UNITED STATES

Prof. Dr. Wei Zhang
Department of Mathematics
Columbia University
2990 Broadway
New York, NY 10027
UNITED STATES

Jianru Zhang
Department of Mathematics
David Rittenhouse Laboratory
University of Pennsylvania
209 South 33rd Street
Philadelphia, PA 19104-6395
UNITED STATES

Yihang Zhu
Mathematics Department
Harvard University
Science Center 242 E
One Oxford Street
Cambridge MA 02138-2901
UNITED STATES