Arbeitsgemeinschaft mit aktuellem Thema: Higher Gross Zagier Formulas

Mathematisches Forschungsinstitut Oberwolfach

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Organizers

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Introduction

For an elliptic curve E over a global field K, the conjecture of Birch and Swinnerton-Dyer asserts a deep relationship between the arithmetic invariants (Mordell–Weil groups and Tate–Shafarevich groups) and the analytic invariants (the complex L-function $L(E/K,s)$). The rank part of the conjecture asserts that the vanishing order of $L(E/K,s)$ at its center $s=1$ coincides with the rank of Mordell–Weil group $E(K)$. The refined part of the conjecture is an identity of the leading term of $L(E/K,s)$ at $s=1$,

$$\frac{L^{(r)}(E/K,1)}{r!} \sim \det\left((P_i,P_j)_{\mathrm{NT}}\right) \tag{1}$$

where $(P_i,P_j)_{\mathrm{NT}}$ is the matrix of Néron–Tate height pairings of a $\mathbb{Z}$-basis $\{P_1,...,P_r\}$ of $E(K)/E(K)_{\mathrm{tor}}$, and $\sim$ means the two sides are equal up to some explicit terms such as the order of Tate–Shafarevich group, the local Tamagawa numbers and the real periods. Equivalently, the Néron–Tate height pairing induces a metric on the determinant of the Mordell–Weil group $E(K)\otimes_{\mathbb{Z}}\mathbb{R}$, and the RHS of the conjectural formula is the norm of a generator of the determinant of the lattice $E(K)/E(K)_{\mathrm{tor}}$. Beilinson and Bloch also formulated a generalization of the B-SD conjecture to higher dimensional varieties.

The Gross–Zagier formula provides an evidence to the B-SD conjecture for elliptic curves E over $\mathbb{Q}$ when the L-function has a zero of order at most one.

Let f be the weight two newform associated to E by the theorem of Wiles, Taylor–Wiles, and Breuil–Conrad–Diamond–Taylor. Let $\phi : X_0(N) \to E$ be a modular parameterization. Let K be an imaginary quadratic extension of $\mathbb{Q}$, with discriminant D. Under suitable hypotheses, the theory of complex multiplication and the map ϕ allow us to define the *Heegner point* $y_K \in E(K)$. The Gross–Zagier formula is the following identity on the first order derivative of the base-changed L-function $L(E/K, s) = L(f/K, s)$ at the center $s = 1$ ([4, 14])

$$\frac{L'(f/K,1)}{(f,f)} = \frac{1}{\sqrt{|D|}} \frac{(y_K, y_K)_{\mathrm{NT}}}{\deg(\phi)},$$

where (f, f) is the Petersson inner product. A similar formula, but for the central value of the L-function, was also discovered around the same time by Waldspurger [13].

What about higher order derivatives of L-function at the center? In [15] a formula for arbitrary order derivative is proved for unramified cuspidal automorphic representation π of PGL_2 over a *function field* $F = k(X)$, where X is a curve over a finite field k. The r-th central derivative of the L-function (base changed along a quadratic extension F'/F) is expressed in terms of the self-intersection number of the *Heegner–Drinfeld cycle* Sht_T^r (or rather its π-isotypic component) on the moduli stack Sht_G^r:

$$\mathcal{L}^{(r)}(\pi_{F'}, 1/2) \sim ([\mathrm{Sht}_T^r]_\pi, [\mathrm{Sht}_T^r]_\pi). \tag{2}$$

The moduli stack Sht_G^r is closely related to the moduli stack of Drinfeld Shtukas of rank two with r modifications. One important feature of this stack is that it admits a natural fibration over the r-fold self-product X^r of the curve X over $\mathrm{Spec}\, k$

$$\mathrm{Sht}_G^r \longrightarrow X^r \,.$$

In the number field case, the analogous spaces only exist (at least for the time being) when $r \leq 1$. When $r = 0$, the moduli stack Sht_G^0 is the constant groupoid over k

$$\mathrm{Bun}_G(k) \simeq G(F) \backslash \left(G(\mathbb{A}_F)/K \right), \tag{3}$$

where $\mathbb{A}_F$ is the ring of adèles of F, and K a maximal compact open subgroup of $G(\mathbb{A})$. The double coset in the RHS of (3) remains meaningful for a number field F. When $r = 1$ the counterpart of Sht_G^1 in the case $F = \mathbb{Q}$ is the moduli stack of elliptic curves, which lives over $\mathrm{Spec}(\mathbb{Z})$. Therefore the formula

(2) can be viewed as a simultaneous generalization (for function fields) of the Waldspurger formula [13] (in the case of $r = 0$) and the Gross–Zagier formula [4] (in the case of $r = 1$). Moreover, there is a way to rewrite the RHS of the formula (2) so that it looks just like (1). The formula (2) opens the possibility of relating higher derivatives of automorphic L-functions to geometric invariants in the function fields case.

The basic strategy of the proof of (2) is to compare two relative trace formulae (abbreviated as RTF), an "analytic" one for the L-functions, and a "geometric" one for the intersection numbers. The strategy of using RTF initiated by Jacquet in 1980s has been successful in related and similar questions on higher rank reductive groups when $r = 0$ (e.g.,[7, 18]) and $r = 1$ (e.g., [16]).

The aim of the workshop is to carefully define the relevant objects that appear in the formula (2), especially the moduli stack of Shtukas and the Heegner–Drinfeld cycle; to review Jacquet's RTF; and to sketch the geometric ideas used in the comparison of the two RTFs. We will follow [15] closely.

Talks

Day 1

1. **An overview of Gross–Zagier and Waldspurger formula.**
   Description: References [17, §3.1,§2.2] and [4, I.1-I.3, I.6, V.2].

   Recall the modular curve $X_0(N)$ and the Heegner points. State the Gross–Zagier formula [4, Thm 6.3 in I.6] (for the trivial character), [17, Thm 3.1]. If time permits, state the Waldspurger formula [17, §2.2].

2. **The stack $\mathrm{Bun}_{\mathrm{GL}_n}$ and Hecke stacks.**
   Description: References [6, §2] and [15, §5.1.1-5.1.3].

   Define the stack $\mathrm{Bun}_{\mathrm{GL}_n}$ of rank n vector bundles on a curve X over a field k. State basic properties (representability and smoothness, e.g. [6, §2]). Discuss the special case $n = 1$ (the Picard stack). Adelic description of the k-points $\mathrm{Bun}_{\mathrm{GL}_n}(k)$ in terms of the double cosets $\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A})/\prod_{x\in|X|}\mathrm{GL}_n(\mathcal{O}_x)$. Define the Hecke stack of elementary modification and state basic properties (representability, smoothness and dimension formula, e.g. [15, §5.1.3]).

3. **Moduli of Shtukas (I).**
   Description: References [15, §5,1,5.2] and [1, §1, §3] (cf. [8, Chap. (I.1, I.2)], [11, §2]).

   Define the stack $\mathrm{Sht}^r_{\mathrm{GL}_n}$ of Shtukas of rank n with r elementary modification. Discuss its basic properties: representability, smoothness [8, Prop. 1, p.29] and dimension. Discuss two simple special cases: when $r = 0$ [1, Prop. 1.1], and when $n = 1$ [1, §3, Prop. 3.1].

4. **Moduli of Shtukas (II).**
   Description: Reference [15, §5.3-5.5] (cf. [8, Chap. (I.4)]).

   Define Hecke correspondence $\mathrm{Sht}^r_G(h_D)$ associated to h_D in the spherical Hecke algebra [15, §5.3]. Discuss its properties [15, Lemma 5.8, 5.9, Prop. 5.10]. Define the Heegner–Drinfeld cycle [15, §5.4], and state the definition of $\mathbb{I}_r(f)$ given in [15, §5.5, Definition 5.15].

   Day 2

5. **Automorphic forms over function fields.**
   Description: Reference [9, §9.1,§9.2].

   Give a formulation of automorphic forms as functions on $\mathrm{Bun}_G(k)$. Characterize cuspidal representations as sub-Hecke-modules of compactly supported functions. Reduction theory on Bun_G using Harder-Narashimhan filtration and Eisenstein series.

6. **The work of Drinfeld.**
   Description: Reference [5] (cf. [2, §3]).

   An overview of Drinfeld work on global Langlands correspondence over function field, cf. [5] (which only deals with the Ramanujan conjecture but suffices for the illustrative purpose). The goal of this talk is to justify the central role played by the moduli of Shtukas in the Langlands program for function fields.

7. **Analytic RTF: the geometric side.**
   Description: References [15, §2] and [7, §2, §4, §5].

   Set up the relative trace formula (RTF) for the diagonal torus A inside $G = \mathrm{PGL}_2$ [15, §2.2]. Describe the orbit classification [15, §2.1]. State RTF for the anisotropic torus T inside G, and the orbit matching ([7, §2]), the fundamental lemma [7, Prop. 5.1], and the existence of matching functions (smooth transfer) [7, Prop. 4.1]. If time permits, state the formula of $\mathbb{J}(f, s)$ when f is the identity element in the Hecke algebra [15, Prop. 2.4].

8. **Analytic RTF: the spectral side.**
Description: Reference [15, §4]

Hecke theory (for L-functions of cuspidal automorphic representation of GL_2, e.g., the book of Jacquet–Langlands). Eisenstein ideal and the kernel function [15, §4.1,4.2]. State the definition of the spherical character $\mathbb{J}_\pi(f, s)$, and [15, Prop, 4.5]. The main goal for the rest of the workshop is then to show the identity $\mathbb{I}_r(f) = \mathbb{J}_r(f)$.

Day 3

9. **Definition of $\mathcal{N}_d$; geometric interpretation of orbital integrals.**
Description: Reference [15, §3.2,§3.3].

Introduce the moduli space $\mathcal{N}_d$ and rewrite it using symmetric powers of the curve. Introduce the fibration $f_{\mathcal{N}_d} : \mathcal{N}_d \to \mathcal{A}_d$. The key result is [15, Prop 3.2], which expresses the orbital integral $\mathbb{J}_r(h_D)$ appearing in the RTF as a weighted sum of traces of Frobenius acting on the cohomology of fibers of $f_{\mathcal{N}_d}$ (with certain local systems as coefficients).

10. **Definition and properties of $\mathcal{M}_d$.**
Description: Reference [15, §6.1].

Introduce the moduli space $\mathcal{M}_d$ and the fibration $f_{\mathcal{M}_d} : \mathcal{M}_d \to \mathcal{A}_d$. Give a description of $\mathcal{M}_d$ in terms of symmetric powers of the double covering curve X'. Prove properness of $f_{\mathcal{M}_d}$. The key result is [15, Prop 6.1].

Day 4

11. **LTF for cohomological correspondences**
Description: References [12, §1.1-1.2, §2.1] and [15, §A.4].

Review the basic functors for étale sheaves. Introduce the notion of a cohomological correspondence. Give basic examples such as when the correspondence is a graph or both projections are finite étale. Define local terms of the Lefschetz trace formula (LTF). State [12, Prop 1.2.5, Cor 1.2.6] and a special case of [12, Thm 2.1.3] when the correspondence is the graph of Frobenius. State [15, Prop A.12].

12. **Intersection theory on stacks.**
Description: References [3, Ch. 6], [10] and [15, §A.1-A.3].

Review intersection theory in [3], especially the refined Gysin map [3, §6.2]. Definition of Chow group of proper cycles and the intersection product on a smooth Deligne-Mumford stack. State the key result [15, Thm A.10]. Mention the reason why derived intersection and comparison between cycle-theoretic and K-theoretic intersection products are used in the proof.

13. **Definition and description of $\mathrm{Hk}^{\mu}_{\mathcal{M},d}$; Expressing $\mathbb{I}_r(h_D)$ as a trace.**
Description: Reference [15, §6.2].

Introduce the Hecke correspondence $\mathrm{Hk}^{\mu}_{\mathcal{M},d}$ for $\mathcal{M}_d$. Describe $\mathrm{Hk}^{\mu}_{\mathcal{M},d}$ in concrete terms over an open subset of $\mathcal{A}_d$ ([15, Lemma 6.3]). State the key theorem [15, Thm 6.5] which expresses $\mathbb{I}_r(h_D)$ as a trace on the cohomology of fibers of $f_{\mathcal{M}_d}$. Prove that theorem modulo [15, Thm 6.6], which will be sketched in Talk 14.

14. **Alternative calculation of $\mathbb{I}_r(h_D)$.**
Description: Reference [15, §6.3].

The goal of this talk is to sketch a proof of [15, Thm 6.6]. Define each space that appears in the diagram in the beginning of [15, §6.3]. Compute that fiber products of each row and column of that diagram. State [15, Thm A.10] and briefly explain why it is applicable to the current situation (modulo the dimension calculations in [15, §6.4])

Day 5

15. **Comparison of $\mathcal{M}_d$ and $\mathcal{N}_d$; the weight factors.**
Description: Reference [15, §8].

Determine the direct image complexes of $f_{\mathcal{M}_d}$ and $f_{\mathcal{N}_d}$ on the base $\mathcal{A}_d$ in terms of middle extension perverse sheaves ([15, Prop 8.2, Prop 8.5]). Compute the eigenvalues of the Hecke correspondence $\mathrm{Hk}^{\mu}_{\mathcal{M}}$ acting on the the direct image complexes of $f_{\mathcal{M}_d}$ ([15, Prop 8.3]). Show that these eigenvalues match the weight factors in the derivatives of the orbital integrals $\mathbb{J}(h_D, s)$. This proves that $\mathbb{I}_r(f) = \mathbb{J}_r(f)$ for most f in the Hecke algebra.

16. **Horocycles.**
Description: References [15, §7.2-7.3] and [1, §4].

Introduce the notion of horocycles and describe their geometry ([15, Lemma 7.5]). Introduce the cohomological constant term map (restricting to generic fibers). The key result is [15, Lemma 7.8], which states that the Hecke algebra acts on the cohomological constant terms via the Satake transform. A consequence is that for functions f in the Eisenstein ideal, it acts trivially on the cohomological constant terms.

17. **Cohomological spectral decomposition; finish the proof.**
Description: Reference [15, §7.4, §9].

Use the results in Talk 16 to sketch proofs of the key finiteness results [15, Lemma 7.11, Lemma 7.13], which state that the Hecke algebra acts on the cohomology of Sht_G^r by a finite-type quotient, and the cohomology of Sht_G^r is finitely generated under any local Hecke algebra. State the cohomological spectral decomposition theorem [15, Thm 7.14]. Finish the proof of $\mathbb{I}_r(f) = \mathbb{J}_r(f)$ for all f ([15, §9.1]) which implies the main theorem.

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Participation:

The idea of the Arbeitsgemeinschaft is to learn by giving one of the lectures in the program.

If you intend to participate, please send your full name and full postal address to

`ag@mfo.de`

by **4 December 2016** at the latest.

You should also indicate which talk you are willing to give:
First choice: talk no. ...
Second choice: talk no. ...
Third choice: talk no. ...

You will be informed shortly after the deadline if your participation is possible and whether you have been chosen to give one of the lectures.

The Arbeitsgemeinschaft will take place at Mathematisches Forschungsinstitut Oberwolfach, Lorenzenhof, 77709 Oberwolfach-Walke, Germany. The institute offers accomodation free of charge to the participants. Travel expenses cannot be covered. Further information will be given to the participants after the deadline.