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Analysis, Geometry and Topology of Positive Scalar Curvature Metrics

Organised by
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Abstract. Riemannian manifolds with positive scalar curvature play an important role in mathematics and general relativity. Obstruction and existence results are connected to index theory, bordism theory and homotopy theory, using methods from partial differential equations and functional analysis. The workshop led to a lively interaction between mathematicians working in these areas.

Mathematics Subject Classification (2010): Primary: 53C20 (Global Riemannian Geometry). Secondary: 19K56, 35Q75, 53C15, 53C21, 53C25, 53C27, 53C29, 53C42, 53C44, 57R65.

Introduction by the Organisers

The workshop *Analysis, Geometry and Topology of Positive Scalar Curvature Metrics*, organised by Bernd Ammann (Regensburg), Bernhard Hanke (Augsburg), and André Neves (Chicago) was attended by more than 50 participants from Europe, the US, and Japan, including a number of young scientists on a doctoral or postdoctoral level. Rather than representing a single mathematical discipline the workshop aimed at bringing together researchers from different areas, but working on similar questions. The conference created a stimulating environment for exchange of ideas and methods from topology, from Riemannian and Lorentzian geometry, and from general relativity.

The workshop started with three extended 80 minutes talks by Claude LeBrun, Greg Galloway, and Thomas Schick, introducing to major themes related to the positive scalar curvature problem and appearing again in later talks of the workshop: The notion of mass in Kähler geometry, the role of scalar curvature in

General Relativity, and the application of index theory and differential topology to the classification of positive scalar curvature metrics.

Aspects of General Relativity appeared in research talks dealing with the equality case of the spacetime positive mass theorem, topological investigations and new constructions related to horizon geometry, boundary value problems for the static vacuum equations, and investigations of Lorentzian manifolds in terms of Cauchy problems and boundary value problems for the Dirac operator.

Recent major results in the subject deal with the topology of spaces and moduli spaces of positive scalar curvature metrics, in particular in connection with our now improved understanding of the diffeomorphism groups of smooth manifolds by the use of cobordism categories. In addition, classical methods, such as the Gromoll filtration combined with the use of Toda brackets, remain vital to obtain interesting new results in this direction. Secondary coarse index theory allows a distinction of concordance classes of uniformly positively curved metrics on non-compact manifolds. Further refinements of index theory were shown to be successful to study the positive scalar curvature problem on Thom-Mather stratified and on non-compact smooth manifolds with controlled geometry at infinity.

The positive scalar curvature problem is closely tied to the computation of Yamabe invariants. Some of the talks dealt with the non-uniqueness of solutions to the Yamabe problem on compact and non-compact manifolds, the study of Yamabe invariants on stratified spaces, and the computation of Yamabe invariants by the use of edge-cone Einstein metrics.

An important topic of interest related to positive scalar curvature is the geometry of minimal hypersurfaces, which was the subject of talks concerning minimal hypersurfaces with bounded Morse index and the existence of infinitely many geodesics on asymptotically conical surfaces of non-negative scalar curvature.

The positive scalar problem also has strong connections to various other parts of Riemannian geometry and global analysis. Stability under Ricci flow of Ricci-flat asymptotically locally Euclidean manifolds ties links between Kähler geometry, scalar curvature geometry, geometric partial differential equations and general relativity. Conformal geometry enters the picture when classifying positive scalar curvature metrics on the seven sphere that cannot be the conformal infinity of Poincaré-Einstein metrics on the eight dimensional ball.

A talk about the role of holomorphic sectional curvature in Kähler geometry pointed out an instance when strong curvature assumptions lead to very restrictive classification results. However, the common expectation that the use of stronger curvature notions such as Ricci or sectional curvature always go hand in hand with more restrictive classification results was questioned in three more talks. They dealt with rigidity results in scalar curvature geometry on the one hand, and the application of differential topological methods for a study of moduli spaces of positive Ricci curvature metrics on spheres, and for the construction of multi-parameter families of highly connected seven dimensional manifolds admitting metrics of non-negative sectional curvature, on the other.

Most research talks had a length of 60 minutes with some additional 40 minutes talks contributed by younger participants. It was interesting to see how researchers originating from distinct mathematical communities dealt with similar problems, but referred to different techniques and sometimes arrived at varying views of the same mathematical structures. Once more the positive scalar curvature problem featured itself as an ideal point of reference to take advantage of an exchange of ideas, tools and perspectives for a productive scientific discussion.

Due to the interdisciplinary character of the meeting speakers were asked to keep their lectures at a level accessible to a broad audience with different mathematical backgrounds.

A perfect organization and management by the staff of the Oberwolfach institute created an optimal working environment.

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Workshop: Analysis, Geometry and Topology of Positive Scalar Curvature Metrics

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Abstracts

Mass, Scalar Curvature, Kähler Geometry, and All That

Claude LeBrun

If $n \geq 3$, a complete connected non-compact Riemannian n-manifold (M, g) is said to be *asymptotically Euclidean* (or *AE*) if there is a compact subset $K \subset M$ such that $M - K$ consists of finitely many components, each of which is diffeomorphic to the complement of a closed ball $D^n \subset \mathbb{R}^n$ in such a manner that g becomes the standard Euclidean metric plus terms that fall off (sufficiently rapidly) at infinity. The mild fall-off hypotheses we impose are specifically that, in the given asymptotic coordinates,

$$g_{jk} = \delta_{jk} + O(|x|^{-\frac{n}{2}+1-\varepsilon}), \qquad g_{jk,\ell} = O(|x|^{-\frac{n}{2}-\varepsilon}) \tag{1}$$

for some $\varepsilon > 0$, and that

$$\int_M |s| d\mu < \infty, \tag{2}$$

where s and $d\mu$ are respectively the scalar curvature of g and volume form of g. More generally, a Riemannian n-manifold (M, g) is said to be *asymptotically locally Euclidean* (or *ALE*) if the complement of a compact set K consists of finitely many components, each of which is diffeomorphic to a quotient $(\mathbb{R}^n - D^n)/\Gamma_j$ by some finite subgroup $\Gamma_j \subset O(n)$ which acts freely on the unit sphere, in such a way that g again satisfies (1) and (2). The components of $M - K$ are called the *ends* of M, and, because we have assumed that $n \geq 3$, the Γ_j are just the fundamental groups of the corresponding ends.

The *mass* of an ALE Riemannian n-manifold is an invariant that assigns a real number to each end. This quantity is defined to be

$$\mathfrak{m}(M, g) := \lim_{\varrho \to \infty} \frac{\mathbf{\Gamma}(\frac{n}{2})}{4(n-1)\pi^{n/2}} \int_{S_\varrho/\Gamma} [g_{ij,i} - g_{ii,j}]\, \nu^j \alpha_E \tag{3}$$

where Γ is the fundamental group of the given end, commas represent derivatives in the given asymptotic coordinates, summation over repeated indices is implicit, S_ϱ is the Euclidean coordinate sphere of radius ϱ, α_E is the $(n-1)$-dimensional volume form induced on this sphere by the Euclidean metric, and ν is the outward-pointing Euclidean unit normal vector. With $n = 3$, this concept originated as the so-called ADM mass [1] in general relativity, which reads off the apparent mass of an isolated gravitational source from the asymptotics of its gravitational field. In this picture, the AE (or ALE) m-manifold (M, g) is then imagined to represent a space-like hypersurface in an $(n + 1)$-dimensional space-time. Notice that we take the integral to be over S_ϱ/Γ rather than over S_ϱ, so that the mass, by our conventions, is $1/|\Gamma|$ times the value one might otherwise expect; the normalizing coefficient used above is also of course a matter of convention. Our definition (3) of the mass superficially seems to depend on the choice of asymptotic coordinates. However, Bartnik [2] and Chruściel [5] proved that the mass defined by (3) is finite and independent of the choice of asymptotic coordinates provided we assume the metric g satisfies (1) and (2).

The *positive mass conjecture* states that an AE manifold (M, g) with scalar curvature $s \geq 0$ must have mass $m \geq 0$. The physical motivation for this conjecture originally stemmed from viewing (M, g) as initial data for a time-symmetric space-time solving Einstein's equations for gravitation coupled to physically reasonable matter fields. The $s \geq 0$ hypothesis then means that the local mass density is non-negative, while the $m \geq 0$ conclusion then means that the resulting gravitational system should exert an attractive rather than a repulsive force on distant test particles. This conjecture is now a theorem if mild extra hypotheses are added; if M has dimension $n \leq 7$, a minimal hypersurface argument [12] shows[1] that the conjecture must hold, while a harmonic-spinor argument [10,13] gives an entirely different proof for any n, but assuming that M is *spin*. On the other hand, a proposed extension [6] of the conjecture to ALE manifolds turned out to be incorrect. In fact, there are many ALE Kähler manifolds with $s \geq 0$ that have negative mass; the original counter-examples [9] were all scalar-flat Kähler manifolds of complex dimension 2, and many of these counter-examples are spin.

While the coordinate definition of the mass may seem opaque, my recent joint paper with Hajo Hein [7] shows that it is actually given by a simple, transparent formula (Theorem C below) when the ALE space in question is a *Kähler* manifold. One simplifying feature of the the Kähler case is that such manifolds can only have one end, so the mass becomes an invariant of the manifold rather than of a particular end. Here is an immediate consequence of our formula:

Theorem A. The mass of an ALE scalar-flat Kähler manifold (M, g, J) is a topological invariant, determined entirely by the smooth manifold M, together with the first Chern class $c_1 = c_1(M, J) \in H^2(M)$ of the complex structure and the Kähler class $[\omega] \in H^2(M)$ of the metric.

Our mass formula also puts the examples of [9] in a more general context:

Theorem B. Let (M^4, g, J) be an ALE scalar-flat Kähler surface, and suppose that (M, J) is the *minimal* resolution of a surface singularity. Then $m(M, g) \leq 0$, with equality iff g is Ricci-flat.

We now state our mass formula. If M is a smooth manifold, recall that one can define the compactly supported de Rham cohomology $H^k_c(M)$, as well as the usual de Rham cohomology. There is then a natural map $H^2_c(M) \to H^2(M)$ induced by the inclusion of compactly supported forms into all differential forms, and in the ALE setting, this map is actually an *isomorphism*. This allows us to define

$$\clubsuit : H^2(M) \to H^2_c(M)$$

to be its inverse.

[1] A recent e-print by Schoen and Yau, `arXiv:1704.05490`, gives a modified minimal hypersurface argument that appears to overcome the previous dimensional restrictions.

Theorem C. Any ALE Kähler manifold (M, g, J) of complex dimension m has mass given by

$$\mathfrak{m}(M,g) = -\frac{\langle \clubsuit(c_1), [\omega]^{m-1}\rangle}{(2m-1)\pi^{m-1}} + \frac{(m-1)!}{4(2m-1)\pi^m}\int_M s_g d\mu_g$$

where s_g and $d\mu_g$ are respectively the scalar curvature and volume form of g, while c_1 is the first Chern class, $[\omega]$ is the Kähler class, and $\langle\ ,\ \rangle$ is the duality pairing between $H^2_c(M)$ and $H^{2m-2}(M)$.

Another interesting consequence is the following:

Theorem D (Positive Mass Theorem for Kähler Manifolds)**.** Any AE Kähler manifold with $s \geq 0$ has $\mathfrak{m}(M, g) \geq 0$, with equality iff (M, g) is Euclidean space.

In fact, our proof actually shows that the mass can be bounded from below by the $(2m-2)$-volume of a subvariety. This is reminiscent of the Penrose inequality [3,8,11], which gives a sharp lower bound for the mass of an AE 3-manifold in terms of the area of a minimal surface. Here is our Kähler analog:

Theorem E (Penrose Inequality for Kähler Manifolds)**.** Let (M^{2m}, g, J) be an AE Kähler manifold with scalar curvature $s \geq 0$. Then (M, J) carries a canonical divisor $K = \sum \mathbf{n}_j D_j$, where the D_j are compact complex hypersurfaces, the n_j are positive integers, and $\bigcup_j D_j \neq \varnothing$ iff $(M, J) \neq \mathbb{C}^m$. Consequently,

$$\mathfrak{m}(M,g) \geq \frac{(m-1)!}{(2m-1)\pi^{m-1}} \sum_j \mathbf{n}_j \mathrm{Vol}\,(D_j)$$

with equality iff (M, g, J) is scalar-flat Kähler.

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On the geometry and topology of initial data sets in General Relativity

Gregory J. Galloway

Positive scalar curvature metrics have played an important role in General Relativity ever since the landmark proofs of the positive mass theorem by Schoen and Yau, and by Witten. In this talk we present some further results in General Relativity where scalar curvature plays an important role.

An initial data set in a spacetime (time oriented Lorentzian manifold) consists of a spacelike hypersurface V, together with its induced (Riemannian) metric h and second fundamental form K. After a brief introduction to spacetime geometry and general relativity, we present some topics of recent interest related to the geometry and topology of initial data sets with horizons. Horizons are modeled by *marginally outer trapped surfaces* (MOTSs), which are defined in terms of the initial data.

After reviewing Hawking's theorem on black hole topology, we consider the topology of black holes in higher dimensional gravity inspired by certain developments in string theory and issues related to black hole uniqueness. Natural physical conditions are given to show that black hole horizons must admit metrics of positive scalar curvature, which implies various restrictions on their topology; cf. [3].

We also discuss more recent work on the geometry and topology of the region of space exterior to all black holes, which is closely connected to the notion of *topological censorship*. Topological censorship has to do with the idea that the topology of the region outside of all black holes should be simple, that, somehow, nontrivial topology should end up behind the event horizon. Without going into the rationale for this, there are a number of results supporting this point of view. But these are spacetime results - they involve assumptions global in time. The aim of more recent work with M. Eichmair and D. Pollack [2] was to establish a result supportive of this principle at the *pure initial data level*. The main result shows that for a 3-dimensional asymptotically flat initial data set (V, h, K) with MOTS boundary §, and having no *immersed* MOTSs in the exterior region $V \setminus §$, the topology is as simple as possible: $V \approx \mathbb{R}^3$ minus an open ball.

An entirely different approach to the topology of the exterior region, valid in dimension up to 7, is taken in work with Andersson, Dahl and Pollack [1]. Assuming the dominant energy condition, and the absence of (ordinary) MOTS in the exterior (along with some additional technical condition), it is shown that the one-point compactification of the asymptotically flat exterior admits a metric of positive scalar curvature, which is a product near the MOTS boundary §. In this situation one can then apply certain index theory obstructions, and minimal surface theory obstructions to obtain restrictions on the topology of V. The conclusion also implies that Σ admits a metric of positive scalar curvature. Thus, this result "recovers" the result with Schoen on the topology of black holes discussed above (albeit under stronger assumptions), in addition to giving information about the geometry and topology of the exterior.

All of these results rely on the recently developed theory of marginally outer trapped surfaces, which are natural spacetime analogues of minimal surfaces in Riemannian geometry. Important aspects of this theory (existence, stability) are discussed in the talk.

Slides of this talk, which provide much more detail, are available at the MFO website.

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Spaces and moduli spaces of metrics of positive scalar curvature

Thomas Schick

We surveyed old and recent results on the space of metrics of positive scalar curvature on a given closed smooth manifold M, in particular information on the homotopy groups of these spaces. We focus on the two very different aspects of the task:

- *construction*: tools to construct metrics of positive scalar curvature, or more generally interesting families of such metrics: if they are parametrized by S^n they provide potential candidates for non-trivial elements in homotopy group π_n
- *obstruction/detection*: tools to distinguish non-triviality, or equivalently tools to obstruct the existence, either of a metric of positive scalar curvature or of a homotopy to a constant family (detection of non-triviality is obstruction to triviality).

There are essentially three construction tools, which are all somewhat classical:

- explicit and very specific constructions of metrics like the round metric on the sphere, or homogeneous metrics of positive scalar curvature, which are the starting point for further constructions; including products where one of the factors has positive scalar curvature

- for the construction of families of metrics: pullback by the action of a family of diffeomorphisms
- surgery constructions: if N is obtained from M by suitable surgeries and M has a metric of positive scalar curvature, one constructs one on N (and on the whole trace of the surgery). This goes back to Gromov and Lawson [5] and independently Schoen and Yau [9]; versions for families have been established by Chernysh [2] and Walsh [11] (in increasing generality)

For detection/obstruction there are two basic sets of tools:

- The first is based on minimal hypersurfaces, established by Schoen and Yau. If M has positive scalar curvature and N is a codimension 1 minimal hypersurface, then also N admits a metric of positive scalar curvature. Geometric measure theory in suitable situations guarantees the existence of such minimal hypersurfaces and puts restrictions on them which allow by an iteration to rule out positive scalar curvature e.g. on T^n. This is classical for $n \leq 7$ but due to lack of the required regularity results has only very recently been extended to general dimensions by Schoen and Yau [9,10].
- The second uses index theory and higher index theory and the relative index of the Dirac operator. This is in a certain sense the much more powerful tool, but it requires the existence of a spin structure. Therefore, so far this method has not been used successfully on a manifold whose universal covering does not admit a spin structure.

  For spin manifolds, however, it lead to spectacular results. In particular, Botvinnik, Ebert, and Randall-Williams [1] and independently Perlmutter [8] could show that for any closed spin manifold M of dimension $n \geq 6$ which admits a metric of positive scalar curvature a relative index map $\pi_k(Pos(M)) \to KO_{k+n+1}$ is a non-trivial homomorphism. Recall that $KO_j \cong \mathbb{Z}$ for $j \equiv 0 \pmod 4$ and $KO_j \cong \mathbb{Z}/2$ for $j \equiv 1, 2 \pmod 8$.

  Earlier work of Crowley, Schick, and Steimle [3,4] showed this whenever the target is $\mathbb{Z}/2$, following work of Hitchin for $k = 0, 1$ [7], and of Hanke, Schick, and Steimle [6] when the target is $\mathbb{Z}$ but only if j is much smaller than n. Perlmutter even handles the case $n = 5$.

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The equality case of the spacetime positive mass theorem

Dan A. Lee
(joint work with Lan-Hsuan Huang)

The Riemannian positive mass theorem (PMT) states that any complete asymptotically flat manifold with nonnegative scalar curvature must have nonnegative ADM mass. This was proved by R. Schoen and S.-T. Yau in dimensions less than eight [8,9] and by E. Witten for spin manifolds [12]. A recent preprint of Schoen and Yau extends their argument to all dimensions [11]. If, in addition to the hypotheses of the Riemannian PMT, we also know that the ADM mass is zero, then the space must be isometric to Euclidean space. This latter fact is often called the "equality case" of the Riemannian PMT, and a separate argument of Schoen and Yau showed that it is a direct consequence of the Riemannian PMT. We would like to discuss an analog of this result for initial data sets.

Theorem 1 (Spacetime positive mass theorem). *Assume $n < 8$ or M is spin. If (M^n, g, k) is a complete asymptotically flat initial data set satisfying the dominant energy condition, then the ADM energy-momentum (E, P) satisfies $E \geq |P|$.*

Recall that the *dominant energy condition (or DEC)* is the statement that $\mu \geq |J|_g$, where μ and J are the mass and current densities of (g, k). This theorem generalizes the Riemannian PMT and was proved for spin manifolds by Witten [12] and in dimensions less than eight by M. Eichmair, L.-H. Huang, the author, and Schoen [5]. When $n = 3$, Schoen and Yau had proved that $E \geq 0$ in [10], and that result was later generalized by Eichmair to dimensions less than eight [4]. Our main result, proved in joint work with Lan-Hsuan Huang [6] is the following.

Theorem 2 (Equality case of the spacetime positive mass theorem). *Assume $n < 8$. If (M^n, g, k) is a complete asymptotically flat initial data set satisfying the dominant energy condition, and if the ADM energy-momentum (E, P) satisfies $E = |P|$, then $E = |P| = 0$ and (M, g) isometrically embeds into Minkowski spacetime with second fundamental form k.*

The result was already proved by R. Beig and P. Chruściel for 3-manifolds [1] and by Chruściel and D. Maerten for general spin manifolds [2]. As both of those results used spinors, our goal was to find an argument that relied directly on

Theorem 1 rather than any use of spinors. Technically, we prove that $E = |P| = 0$, and then the second part of the conclusion then follows from work of Eichmair [4].

Let $\Phi(g,k) = (\mu, J)$ be the *constraint operator*, and consider the linearized constraint operator $D\Phi|_{(g,k)}$ at (g,k) as well as its formal adjoint $D\Phi|^*_{(g,k)}$, which takes a function and a vector field as inputs. We will refer to such a pair (f,X) as a *lapse-shift* pair. V. Moncrief [7] observed that if $D\Phi|^*_{(g,k)}(f,X) = (0,0)$, then (f,X) can be used to construct a spacetime Killing vector field on the vacuum spacetime development of (g,k), and vice versa. For this reason, if (g,k) is vacuum, then we refer to an element of the kernel of $D\Phi|^*_{(g,k)}$ as *vacuum Killing initial data.*

We return to the general situation where we do not know whether (g,k) is vacuum. The basic idea behind the spinor proofs in [1,2] is that if $E = |P|$, then Witten's spinors can be used to construct a lapse-shift pair (f,X) which is *asymptotically vacuum Killing initial data* in the sense that $D\Phi|^*_{(g,k)}(f,X)$ decays appropriately at infinity, such that (f,X) is asymptotic to $(E,-P)$. From there they use the following fact.

Theorem 3 (Beig-Chruściel [1]). *If (g,k) is asymptotically flat initial data with asymptotically vacuum Killing initial data (f,X) such that (f,X) is asymptotic to $(E,-P)$, then $E = |P| = 0$.*

Once they know $E = |P| = 0$, they then complete the argument using Witten's spinors again. However, Theorem 3 itself follows from purely asymptotic calculations and has nothing to do with spinors or the DEC. A proof of Theorem 3 also appears in our paper [6], where we make some small improvements to the original argument.

For the proof of Theorem 2, our strategy is to replace the spinor argument with some other way to find the desired asymptotically vacuum Killing initial data (f,X) so that we can apply Theorem 3. Specifically, we want to use a variational argument in conjunction with Theorem 1. We would like to consider deformations that preserve the DEC. It is well-known that a deformation that keeps the constraints (μ, J) fixed need not preserve the DEC $\mu \geq |J|_g$ because the metric g used to compute $|J|_g$ is changing. The basic idea for how to overcome this problem first appeared in [5] and was later formalized by J. Corvino and Huang [3], who introduced the *modified constraint operator at* (g,k), defined by

$$\bar{\Phi}(\gamma,\tau) = \Phi(\gamma,\tau) + (0, \tfrac{1}{2}(\gamma \cdot J)^\sharp),$$

where the $\sharp$ is the index-raising operator from 1-forms to vector fields. Define a modified constraint manifold

$$C_{(g,k)} = \{(\gamma,\tau) \,|\, \bar{\Phi}(\gamma,\tau) = \bar{\Phi}(g,k)\}.$$

The important point is that the DEC holds on a small neighborhood of (g,k) in $\bar{C}_{(g,k)}$, and therefore we can apply Theorem 1 to see that $E(\gamma,\tau) \geq |P(\gamma,\tau)|$ for each (γ,τ) in that neighborhood.[1]

[1]Technically, since these arguments take place in weighted Sobolev spaces, we must prove a version of Theorem 1 for Sobolev regularity [6].

Given a fixed (g,k) with ADM energy-momentum (E,P), choose a lapse-shift pair (f_0,X_0) with the property that (f_0,X_0) identically equals $(E,-P)$ in the asymptotically flat end. We define the *modified Regge-Teitelboim Hamiltonian* at (g,k) to be

$$\mathcal{H}(\gamma,\tau) = E\cdot E(\gamma,\tau) - P\cdot P(\gamma,\tau) - \tfrac{1}{(n-1)\omega_{n-1}}\int_M \bar{\Phi}(\gamma,\tau)\cdot(f_0,X_0)\,dV_g,$$

where dV_g is the volume measure induced by g, and ω_{n-1} is the volume of the standard unit sphere S^{n-1}.

Since the integral term of $\mathcal{H}$ is constant on $C_{(g,k)}$, it is not hard to conclude that (g,k) minimizes $\mathcal{H}$ over a neighborhood of (g,k) in the constraint manifold $C_{(g,k)}$. Indeed, it is fair to think of it as a manifold since one can show that $D\bar{\Phi}|_{(g,k)}$ is surjective as in [5].

Given the constrained minimization and the surjectivity of $D\bar{\Phi}|_{(g,k)}$, we can now apply Lagrange multipliers, which states that there exists some functional λ such that

$$D\mathcal{H}|_{(g,k)}(h,w) = \lambda[D\bar{\Phi}|_{(g,k)}(h,w)],$$

for all (h,w). Or equivalently, there exists (f_1,X_1) such that

$$\begin{aligned} D\mathcal{H}|_{(g,k)}(h,w) &= \int_M (f_1,X_1)\cdot D\bar{\Phi}|_{(g,k)}(h,w)\,dV_g \\ &= \int_M (h,w)\cdot D\bar{\Phi}|^*_{(g,k)}(f_1,X_1)\,dV_g. \end{aligned}$$

On the other hand, it is possible to directly compute

$$D\mathcal{H}|_{(g,k)}(h,w) = \int_M (h,w)\cdot D\bar{\Phi}|^*_{(g,k)}(f_0,X_0)\,dV_g,$$

from the definition of $\mathcal{H}$. This formula is to be somewhat expected because of the natural way that (E,P) arises from the "divergence part" of the constraints. Putting these two computations together, we see that $(f,X) := (f_0-f_1, X_0-X_1)$ lies in the kernel of $D\bar{\Phi}|^*_{(g,k)}$ and is asymptotic to $(E,-P)$ at infinity. Finally, being in the kernel of $D\bar{\Phi}|^*_{(g,k)}$ implies that (f,X) is asymptotically vacuum Killing initial data, and we can now apply Theorem 3 to complete our proof of Theorem 2.

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Harmonic spinors and metrics of positive scalar curvature, via the Gromoll filtration and Toda brackets

Wolfgang Steimle
(joint work with Diarmuid Crowley, Thomas Schick)

Let M^m be a smooth closed spin manifold of dimension $m \geq 6$, and $R(M)$ the space of Riemannian metrics on M, equipped with the C^∞-topology. Denote by $R^+(M) \subset R(M)$ any $\mathrm{Diff}(M)$-invariant subspace such that for each $g \in R^+(M)$, the Dirac operator is invertible. Standard examples are the space of metrics of positive scalar curvature, of positive Ricci curvature, or of positive sectional curvature. We prove that for all $g \in R^+(M)$ and all $n - m \geq 0$ and $n \equiv 0, 1$ modulo 8, there exist elements

$$0 \neq x \in \pi_{n-m}(R^+(M), g)$$

of order two.

Following a proof scheme due to Hitchin [4], the elements are constructed through the action of the diffeomorphism group on the space $R^+(M)$. The main step in the proof is to construct specific elements

$$0 \neq y \in \pi_{n-m} \operatorname{Diff}(D^m, \partial)$$

of order two, by considering Toda brackets in the space PL_m/O_m and applying smoothing theory. Through the canonical homomorphism

$$\pi_{n-m} \operatorname{Diff}(D^m, \partial) \to \pi_0 \operatorname{Diff}(D^n, \partial)$$

and the clutching construction, any of our elements y gives rise to a homotopy $(n+1)$-sphere Σ_y, for which we prove that

$$\alpha(\Sigma_y) \neq 0 \in KO_{n+1} \cong \mathbb{Z}/2.$$

The non-triviality of x and y follows from this, as was already pointed out by Hitchin.

Remarks. (1) It follows that any closed spin manifold M of dimension at least 6 admits a Riemannian metric with a non-trivial harmonic spinor.

(2) The exotic spheres obtained in this way lie extremely deep in the Gromoll filtration [3] of the group of exotic spheres.

(3) Our proof shows that the elements y survive in the homotopy group $\pi_{n-m}\,\mathrm{Diff}(M)$ for any manifold as above, provided $n - m > 0$.

(4) In the case where $R^+(M)$ is the space of metrics of positive scalar curvature, the above result has been proven independently (with a different method) by Botvinnik-Ebert-Randal-Williams [1] and improved to the case $m = 5$ by Perlmutter [5]. We do not know how their classes relate to ours. Classes of infinite order (in the case $n \equiv 3$ modulo 4) have been constructed in [1,5,6].

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Parametrized Morse theory, cobordism categories, and positive scalar curvature

Nathan Perlmutter

Motivated by the recent work of Botvinnik, Ebert, and Randal-Williams [1], we use a cobordism category, together with a parametrized version of the *Gromov-Lawson construction*, to construct a map from the infinite loopspace of a certain Thom spectrum into the space of positive scalar curvature metrics on a closed, Spin-manifold of dimension ≥ 5. Our results yield an alternative proof and extension of the theorem of Botvinnik, Ebert, and Randal-Williams. In particular, we obtain an extension of their main theorem [1, Theorem A] to cover manifolds of dimension five.

Our main construction is a topological category $\mathbf{Cob}^{\mathrm{mf},k}_{\theta,d+1}$. We give the definition below.

Definition 1. Fix an integer $d \in \mathbb{Z}_{\geq 0}$, a fibration $\theta : B \longrightarrow BO(d+1)$, and an integer $k < d/2$. Objects of the topological category $\mathbf{Cob}^{\mathrm{mf},k}_{\theta,d+1}$ are given by pairs $(M, \hat{\ell})$ where:

- $M \subset \mathbb{R}^\infty$ is a closed d-dimensional submanifold, and

- $\hat{\ell} : TM \oplus \epsilon^1 \longrightarrow \theta^*\gamma^{d+1}$ is a bundle map with the property that its underlying map $\ell : M \longrightarrow B$ is k-connected.

For $M, N \in \mathrm{Ob}\,\mathbf{Cob}_{\theta,d+1}^{\mathrm{mf},k}$, a morphism $M \rightsquigarrow N$ is a pair (t, W), where:

- $t \in (0, \infty)$;
- $W \subset [0,t]\times\mathbb{R}^\infty$ is an embedded θ-cobordism between $M\times\{0\}$ and $N\times\{t\}$;
- The height function, $W \hookrightarrow [0,t] \times \mathbb{R}^\infty \xrightarrow{\mathrm{proj}} [0,t]$, is a Morse function with the following property: all critical points $\lambda \in W$ satisfy the inequality,

$$k < \mathrm{index}(\lambda) < d - k + 1.$$

The category $\mathbf{Cob}_{\theta,d+1}^{\mathrm{mf},k}$ is topologized in the standard way following the methods of [3]. The following theorem can be viewed as an analogue of the theorem of Galatius, Madsen, Tillmann, and Weiss but for cobordisms equipped with extra geometric structure, namely the choice of an admissible Morse function.

Theorem 2 (N.P 2017, [6])**.** *Let $k < d/2$ and suppose that the tangential structure $\theta : B \longrightarrow BO$ is chosen so that the space B satisfies Wall's finiteness condition $F(k)$ (see [9]). Then there is a weak homotopy equivalence,*

$$B\mathbf{Cob}_{\theta,d+1}^{\mathrm{mf},k} \simeq \Omega^{\infty-1}\mathbf{hW}_{\theta,d+1}^{k}.$$

In the theorem above, $\mathbf{hW}_{\theta,d+1}^{k}$ is a certain Thom spectrum associated to the space of admissible Morse jets on $\mathbb{R}^{d+1}$. Since $\mathbf{hW}_{\theta,d+1}^{k}$ is a Thom spectrum, its homotopy type can be analyzed using the classical methods of stable homotopy theory. In particular, for certain standard choices of θ, i.e. SO, Spin, String, etc... the rational homotopy groups of $\mathbf{hW}_{\theta,d+1}^{k}$ can be computed completely.

Our next objective is to use the cobordism category $\mathbf{Cob}_{\theta,d+1}^{\mathrm{mf},k}$, together with a parametrized version of the parametrized Gromov-Lawson construction [4], to define a map from the infinite loopspace $\Omega^\infty\mathbf{hW}_{\theta,d+1}^{k}$ into the space of positive scalar curvature metrics, $\mathcal{R}^+(M)$, of a closed manifold M with $\dim(M) \geq 5$. Since the homotopy type of $\mathbf{hW}_{\theta,d+1}^{k}$ is well understood, such a map will enable us to detect many non-trivial homotopy groups in the space of psc metrics $\mathcal{R}^+(M)$. Below we give an outline of how to construct the map, $\Omega^\infty\mathbf{hW}_{\theta,d+1}^{k} \longrightarrow \mathcal{R}^+(M)$.

Construction 3. Let $W : M \rightsquigarrow N$ be a cobordism between two closed d-dimensional manifolds M and N, and let $h : W \longrightarrow [0,1]$ be a proper Morse function with the property that, $k < \mathrm{index}(\lambda) < d - k + 1$, for all critical points $\lambda \in W$ of h. By the work of Chernysh [2] the Morse function h can be used to define a (weak) map,

$$\mathbf{GL}(W,h) : \mathcal{R}^+(M) \longrightarrow \mathcal{R}^+(N), \tag{1}$$

which by the main theorem from [2] is always a weak homotopy equivalence. Furthermore, by the work of Walsh [8] these maps can be shown to vary continuously over parametrized families of Morse functions h and bundles of cobordisms W.

Now, fix $2 \leq k < d/2$. We now use the maps (1) to construct a continuous functor,

$$\mathcal{R}^+ : \mathbf{Cob}^{\mathrm{mf},k}_{\theta,d+1} \longrightarrow \mathrm{Top}\,.$$

On objects, the functor is defined by sending $M \in \mathrm{Ob}\,\mathbf{Cob}^{\mathrm{mf},k}_{\theta,d+1}$ to the space of psc metrics $\mathcal{R}^+(M)$. On morphisms it is defined by sending $W : M \rightsquigarrow N$ to the map $\mathbf{GL}(W, h_W) : \mathcal{R}^+(M) \longrightarrow \mathcal{R}^+(N)$ where h_W is the height function associated to the embedded cobordism $W \subset [0,1] \times \mathbb{R}^\infty$. Consider the *transport category*, $\mathcal{R}^+ \wr \mathbf{Cob}^{\mathrm{mf},k}_{\theta,d+1}$. Since the maps (1) are all homotopy equivalences, it follows that the projection,

$$B(\mathcal{R}^+ \wr \mathbf{Cob}^{\mathrm{mf},k}_{\theta,d+1}) \longrightarrow B\mathbf{Cob}^{\mathrm{mf},k}_{\theta,d+1},$$

is a fibration, and that the fibre over $M \in \mathrm{Ob}\,\mathbf{Cob}^{\mathrm{mf},k}_{\theta,d+1}$ is given by the space, $\mathcal{R}^+(M)$. We define

$$\rho_M : \Omega_M B\mathbf{Cob}^{\mathrm{mf},k}_{\theta,d+1} \longrightarrow \mathcal{R}^+(M) \tag{2}$$

to be a fibre-transport map associated to the above mentioned fibre-sequence. Let us now restrict our attention to the case, $\theta = \mathrm{Spin}(d+1)$. For any closed, d-dimensional Spin-manifold M with $\mathcal{R}^+(M) \neq \emptyset$, the index-difference map of Hitchin [5] is defined, ind-diff : $\mathcal{R}^+(M) \longrightarrow \Omega^{\infty+d+1}\,\mathrm{KO}$.

Theorem 4 (N.P. 2017 [7]). *Fix $d \geq 5$ and $2 \leq k < d/2$. For any closed, d-dimensional,* Spin*-manifold M, the composite map,*

$$\Omega^\infty \mathbf{hW}^k_{\mathrm{Spin},d+1} \simeq \Omega B\mathbf{Cob}^{\mathrm{mf},k}_{\mathrm{Spin},d+1} \xrightarrow{\rho_M} \mathcal{R}^+(M) \xrightarrow{\text{ind-diff}} \Omega^{\infty+d+1}\,\mathrm{KO},$$

induces a non-trivial homomorphism on homotopy groups $\pi_l(-)$ for all $l \in \mathbb{Z}_{\geq 0}$. In particular, the map is surjective on all rational homotopy groups $\pi_l(-) \otimes \mathbb{Q}$.

The above theorem recovers the main theorem of Botvinnik, Ebert, and Randal-Williams. Furthermore it provides an extension of their theorem: our Theorem B above holds in the case that M is five-dimensional, while the $(d = 5)$-case falls out of the range of the techniques of [1].

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Non-Existence and Local Existence of Poincaré-Einstein Metrics

Matthew J. Gursky
(joint work with Qing Han)

In this talk I presented a non-existence result for Poincaré-Einstein metrics. This is joint work with Qing Han (Notre Dame). The motivating example of a Poincaré-Einstein manifold is the unit ball B^n endowed with the hyperbolic metric

$$g_H = \frac{4}{(1-|x|^2)^2} ds^2.$$

Two key properties of this manifold are: (1) it is Einstein (with negative Einstein constant) and (2) it is conformally compact. This latter condition means that there is a defining function $\rho : B^n \to \mathbb{R}^+$ with $\rho = 0$ on $\partial B^n = S^{n-1}$, and $d\rho \neq 0$ on ∂B^n.

More generally, we say that (X, g_+) is a Poincaré-Einstein manifold if X is the interior of a compact manifold $\overline{X}$ with boundary $\partial X = M$; g_+ is an Einstein metric; and there is a defining function $\rho \in C^\infty(X)$ with $\rho > 0$ and $d\rho \neq 0$ on ∂X, and $\rho^2 g_+$ extends to a metric $\overline{g}$ on the compact manifold with boundary $(\overline{X}, \partial X)$. It is easy to check that the Einstein constant of g_+ must be negative, which we normalize so that

$$Ric_{g_+} = -(n-1)g_+. \tag{1}$$

Regularity of the compactified metric up to the boundary can be a delicate issue, but in the setting of our main result it is well understood (and will be assumed to be smooth).

A compactification of a Poincaré-Einstein manifold defines a conformal class of metrics on the boundary: If $\gamma = \overline{g}\big|_M$ is the induced metric, then the conformal class $[\gamma]$ is called the *conformal infinity* of (X, g_+). For the basic example of the Poincaré model for hyperbolic space on the unit ball $B^n \subset \mathbb{R}^n$, the conformal infinity is the standard conformal structure on the round sphere S^{n-1}.

Conversely, given a conformal class of metrics on the boundary $M = \partial X$ one can ask whether the interior admits a Poincaré-Einstein metric whose conformal infinity is the given conformal class. Although there is no general existence theory for this problem, a seminal result was proved by Graham-Lee [2]: Given a metric γ sufficiently close to the round metric γ_0 on the sphere S^{n-1}, there is a Poincaré-Einstein metric g_+ on the ball B^n whose conformal infinity is $[\gamma]$.

In [6], Witten remarks that "...one might ask what is the significance of the fact that the Graham-Lee theorem presumably fails for conformal structures that are sufficiently far from the round one" (see page 263). Our main result is the existence of infinitely many conformal classes on the seven-dimensional sphere S^7 which cannot be the conformal infinity of a Poincaré-Einstein metric on the ball B^8, thus confirming Witten's intuition in this dimension. We rely on the construction of Gromov-Lawson [2], which they used to prove that the space $\mathfrak{R}^+(S^7)$ of positive scalar curvature metrics on S^7 has infinitely many connected components. The precise statement is:

Theorem 1 ([3]). *There are infinitely many connected components of $\mathfrak{R}^+(S^7)$ containing metrics whose conformal class cannot be the conformal infinity of a Poincaré-Einstein metric on the eight-dimensional ball B^8.*

The idea behind the proof is to show that the induced metrics from the Gromov-Lawson construction cannot be extended to Poincaré-Einstein metrics on the unit ball B^8. More precisely, the construction in [2] produces an 8-dimensional manifold with boundary (Y^8, η) (realized as the disk bundle of the total space of a 4-dimensional vector bundle over S^4), with $\partial Y^8 \approx S^7$. Moreover, the metric η has positive scalar curvature, and near the boundary is a product metric:

$$\eta = dt^2 + \eta_0,$$

where η_0 is the induced metric on $\partial Y^8 \approx S^7$. A crucial aspect of this construction is that the manifold $N^8 = Y^8 \cup_{S^7} B^8$ is spin with non-trivial $\widehat{A}$-genus, and therefore does not admit a metric of positive scalar curvature. We want to show that the conformal class of η_0 on S^7 cannot be the conformal infinity of a Poincaré-Einstein metric on the ball.

Assuming to the contrary that g_+ is a P-E metric on B^8 with conformal infinity given by $[\eta_0]$, we first use an observation of J. Qing [5] (based on a result of J. Lee [4]) that one can conformally compactify g_+ to obtain a metric $\overline{g} = \rho^2 g_+$ on B^8 with positive scalar curvature. Moreover, the boundary S^7 is totally geodesic with respect to $\overline{g}$. Therefore, we have metrics η on Y^8 and $\overline{g}$ on B^8, and by definition their induced metrics on the boundary are conformal.

The next step is extend the metric η_0 into B^8 so that it is conformal to $\overline{g}$. This is a straightforward construction: since $\eta_0 = e^{2w_0}\overline{g}$ for some function $w_0 \in C^\infty(S^7)$, we just need to extend w_0 into B^8. If we do this so that the normal derivative of the extension vanishes along the boundary, then the boundary S^7 is totally geodesic with respect to the extended metric (call it $\widetilde{g}$). Consequently, we now have a globally defined metric on $N^8 = Y^8 \cup_{S^7} B^8$ given by

$$\widetilde{g} = \begin{cases} \eta & \text{on } Y^8, \\ \widetilde{g} & \text{on } B^8, \end{cases}$$

Although $\widetilde{g}$ is only C^1, it has positive scalar curvature away from the boundary. However, near the boundary in Y^8 the metric is a product, while in B^8 it is a product up to errors which are quadratic in the distance to the boundary. Using this fact we can perturb $\widetilde{g}$ to obtain a smooth metric whose Yamabe invariant is positive. Since this is a contradiction (N^8 does not admit a metric of PSC), we are done.

An interesting question is whether any boundary conformal class can be extended at least locally (say, in a collar neighborhood of the boundary) to a Poincaré-Einstein metric. This is ongoing work with G. Szekelyhidi (Notre Dame).

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Outermost apparent horizons with nontrivial topology

Eric Larsson

(joint work with Mattias Dahl)

Outermost apparent horizons are the initial data versions of black hole boundaries. When restricting to the important special case of asymptotically Euclidean "time-symmetric" initial data, they are precisely the outermost (compact) minimal hypersurfaces in asymptotically Euclidean manifolds. It is known from work by Galloway and Schoen [3] [4] that if a manifold is given an asymptotically Euclidean metric of nonnegative scalar curvature, then its outermost apparent horizon admits a metric of positive scalar curvature. This gives topological restrictions on the outermost apparent horizon, and it is not known whether these are the only such restrictions.

In joint work with Mattias Dahl [2] we show that every manifold which is the unit normal bundle of a submanifold $S \subset \mathbb{R}^n$ with codimension at least three actually occurs as the outermost apparent horizon in an asymptotically Euclidean manifold with zero scalar curvature. This is done by modifying the Euclidean metric on $\mathbb{R}^n$ by a conformal factor which is based on integrating the Green's function of the Euclidean Laplacian along S, multiplied by a mass parameter ϵ. In the limit $\epsilon \to 0$ the outermost apparent horizon must collapse to S, since its limit would otherwise be a compact minimal surface in Euclidean space (apart from the set S which has sufficiently small dimension for a removable singularity-type theorem to be applicable). The idea is then the following: By choosing ϵ very small, we can confine the outermost apparent horizon to a very small tubular neighborhood of S. In such a neighborhood, the effect of the curvature of S on the conformal factor is very small because S is well approximated by its tangent space. We then expect the outermost apparent horizon to be close to what it would be if S were an affine subspace of $\mathbb{R}^n$. In the affine case, the conformal factor is explicitly computable and we have complete knowledge of the topology of the horizon.

The preceding reasoning can be made rigorous using a convergence argument in rescaled coordinates: By a mean curvature computation in the limit $\epsilon \to 0$ one can show that there are constants C_{inner}, C_{outer}, and R_{outer} such that the tubular neighborhood of S of radius $C_{\text{inner}}\epsilon$ is foliated by hypersurfaces of negative mean curvature, while the annular neighborhood of inner radius $C_{\text{outer}}\epsilon$ and outer radius R_{outer} is foliated by hypersurfaces of positive mean curvature. A maximum principle then confines the outermost apparent horizon to the annular neighborhood of inner radius $C_{\text{inner}}\epsilon$ and outer radius $C_{\text{outer}}\epsilon$. The fact that these bounds both are linear in ϵ then allows us to rescale the coordinates (centered at a point in S) by $1/\epsilon$ and pass to the limit $\epsilon \to 0$. In this limit S converges to its tangent space, and it is easy to find a foliation of the whole space by cylinders of explicitly computable mean curvature. Another application of a maximum principle tells us that the limit of the outermost apparent horizon is a cylinder. Adapting this argument to make it global in S, we see that the outermost apparent horizon is a tubular hypersurface around S.

With this construction, we can produce horizons which are disjoint unions of spheres as in work by Chruściel and Mazzeo [1], and horizons which are products of spheres as in work by Schwartz [5]. We also obtain many new examples. For instance, we see that outermost apparent horizons in 7-dimensional manifolds can have any fundamental group. However, these new examples realize only some of the topologies which are not excluded by the Galloway–Schoen theorem, and more work is needed before we can determine if there are additional restrictions, or if all topologies which have not been ruled out can be realized.

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Compactness of minimal hypersurfaces with bounded Morse index

Ben Sharp

(joint work with Lucas Ambrozio - Alessandro Carlotto and Reto Buzano)

Minimal hypersurfaces are critical points of the volume functional, and the Morse index tells us how many ways one can decrease their volume locally (up to second order); formally it is the number of negative eigenvalues of the Jacobi operator $\mathcal{L}$. We will present a weak compactness and quantisation result for minimal hypersurfaces, in closed manifolds, for which one of the eigenvalues of $\mathcal{L}$ is bounded from below. In particular, when we consider minimal surfaces in three-manifolds with positive scalar curvature, we will show how this leads to a strong compactness result and therefore full analytic and geometric control. The content here builds on the results presented in the 2015 Oberwolfach conference "Partial Differential Equations" [16] and should be read in conjunction with this.

We are primarily interested in closed Riemannian manifolds (N^{n+1}, g) (for $2 \leq n \leq 6$) and the set of all closed, smooth and embedded minimal hypersurfaces in N, denoted $\mathfrak{M}(N)$. Given $M \in \mathfrak{M}(N)$ we denote its volume by $\mathcal{H}^n(M)$, its second fundamental form by A, and we order the eigenvalues of the Jacobi operator $\mathcal{L}$ in the usual way

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i \to \infty \ldots .$$

Given $\Lambda, \mu \geq 0$ and $p \in \mathbb{N}$ we consider the following subspace of $\mathfrak{M}(N)$:

$$\mathfrak{M}_p(\Lambda, \mu) := \{M \in \mathfrak{M}(N) \mid \mathcal{H}^n(M) \leq \Lambda,\ \lambda_p \geq -\mu\}.$$

Due to the work of Almgren [1], Pitts [11] and Schoen-Simon [13] we know that $\mathfrak{M}(N)$ is non empty. Furthermore by results of Marques-Neves [9] we have that $|\mathfrak{M}(N)| \geq n+1$ and when $Ric_N > 0$ then $\mathfrak{M}(N)$ contains countably many distinct elements. Therefore it seems reasonable to try to understand/characterise subsets of $\mathfrak{M}(N)$ by their increasing complexity. This is precisely the motivation for using the spaces $\mathfrak{M}_p(\Lambda, \mu)$.

An important quantity for us is the total curvature of minimal hypersurfaces

$$\mathcal{A}(M) := \int_M |A|^n \, d\mathcal{H}^n,$$

which is scale-invariant, and when $n = 2$ (via the Gauss equations) is essentially the Euler Characteristic of M.

The following theorem can be thought of as a companion to the main results proved in [15] and [2] (see also [16]).

Theorem 1 ([3])**.** *Let $2 \leq n \leq 6$ and N^{n+1} be a smooth closed Riemannian manifold. If $\{M_k^n\} \subset \mathfrak{M}_p(\Lambda, \mu)$ is a sequence, for some fixed constants $\Lambda, \mu \in \mathbb{R}_{\geq 0}$ and $p \in \mathbb{N}$, then up to subsequence, there exist $M \in \mathfrak{M}_p(\Lambda, \mu)$ and $m \in \mathbb{N}$ so that $M_k \to mM$ in the varifold sense. There also exist at most $p-1$ points $\mathcal{Y} = \{y_i\} \subset M$ where the convergence to M is smooth and graphical (with multiplicity m) away from $\mathcal{Y}$.*

Associated with each $y \in \mathcal{Y}$ there are a finite number $0 < J_y \in \mathbb{N}$ of properly embedded minimal hypersurfaces $\{\Sigma_\ell^y\}_{\ell=1}^{J_y} \subset \mathbb{R}^{n+1}$ with finite total curvature for which

$$\lim_{k\to\infty} \mathcal{A}(M_k) = m\mathcal{A}(M) + \sum_{y \in \mathcal{Y}} \sum_{\ell=1}^{J_y} \mathcal{A}(\Sigma_\ell^y),$$

where $\sum_y J_y = L \leq p-1$. Furthermore, for k, k' sufficiently large, M_k is diffeomorphic to $M_{k'}$.

Remark. When $n = 2$, the quantisation of total curvature can be written in terms of the Gauss curvature. Therefore using the Gauss-Bonnet formula we can say that, for k sufficiently large,

$$\chi(M_k) = m\chi(M) + \frac{1}{2\pi}\sum_{y\in\mathcal{Y}}\sum_{\ell=1}^{J_y}\int_{\Sigma_\ell^y} K_{\Sigma_\ell}.$$

By a result of Osserman [10] we know that $\frac{1}{2\pi}\sum_{y\in\mathcal{Y}}\sum_{\ell=1}^{J_y}\int_{\Sigma_\ell^y} K_{\Sigma_\ell}$ is a non-positive even number.

The heuristic picture one should have in mind is that, when the convergence is not smooth (i.e. of multiplicity $m \geq 2$), then the limit must be stable, and therefore the sequence "loses all of its index" in the process of convergence. Indeed, this index is being lost precisely at the points of bad convergence $\mathcal{Y}$. This lost index can be recovered by taking suitable re-scalings of the approaching sequences about points $y \in \mathcal{Y}$, and what we can show is that these re-scalings converge smoothly and locally to one of the $\Sigma_\ell \subset \mathbb{R}^{n+1}$. It is not necessarily the case that all of the lost index can be accounted for in this way, however we show that all of the total curvature is quantised by the limit surface M and $\{\Sigma_\ell\}_{\ell=1}^{L}$ i.e. there is no loss of total curvature in the intermediate neck regions.

The proof heavily uses the curvature estimates developed by Schoen-Simon [13], a beautiful local foliation/maximum principle argument of Brian White [18], and the description of the ends of minimal hypersurfaces with finite total curvature by Rick Schoen [12]. We also crucially require a result of Johan Tysk [17] which tells us that our re-scalings Σ_ℓ have finite total curvature.

We denote $\frac{\mathfrak{M}_p(\Lambda,\mu)}{\simeq}$ to be equal to the set $\mathfrak{M}_p(\Lambda,\mu)$ after identifying elements up to diffeomorphism.

Corollary 2. *There exists $C = C(N,\Lambda,\mu,p)$ such that for all $M \in \mathfrak{M}_p(\Lambda,\mu)$ we have*

$$index(M) + \mathcal{A}(M) + \left|\frac{\mathfrak{M}_p(\Lambda,\mu)}{\simeq}\right| \leq C.$$

Thus we see that the elements of $\mathfrak{M}_p(\Lambda,\mu)$ are geometrically, topologically and analytically well controlled. In other words the measure of complexity induced by controlling the spectrum of $\mathcal{L}$ in this way seems to be a good one. In the case $\mu = 0$, the estimate $|\widehat{\mathfrak{M}}_p(\Lambda,0)| \leq C$ has been obtained independently and using different methods, by Chodosh-Ketover-Maximo [5]. The index bound follows from the upper bound on volume and total curvature and an application of the results of Ejiri-Micallef [6] for $n = 2$ and Cheng-Tysk [4] for $n \geq 3$.

The below is a special case of a more general result, for simplicity we will restrict to the case of $index = 1$ and two-sided surfaces.

Corollary 3. *Suppose that (N^3,g) has positive scalar curvature $R_g > 0$ and consider a sequence of two-sided surfaces $\{M_k^2\} \subset \mathfrak{M}(N)$ such that $index(M_k) = 1$ and $\chi(M_k) < 0$ for all k. Then there exists $M \in \mathfrak{M}(N)$ such that (up to subsequence), $M_k \to M$ smoothly and graphically with multiplicity one.*

Sketch proof. Since the sequence of surfaces are all two-sided and of index one, by Proposition A.1 in [8] we get that $\mathcal{H}^2(M) \leq \Lambda$ for some uniform Λ.

Therefore we may apply the Theorem above to obtain the existence of some M so that $M_k \to mM$. If $m = 1$ then we are done, so for a contradiction we will suppose that $m \geq 2$.

The usual multiplicity analysis gives us that either M is stable, or its two-sided double cover is stable (see e.g. [15], [2], [16]), which by an argument of Schoen-Yau [14, Theorem 5.1] yields that M is a sphere or a projective plane.

Using $index(M_k) = 1$ we find that $L = 1$ and $index(\Sigma_1) = 1$. Applying the characterisation of embedded, index one minimal surfaces in $\mathbb{R}^3$ by Lopez-Ros [7] we see that Σ_1 must be the catenoid, yielding in turn that $\frac{1}{2\pi}\int_{\Sigma_1} K_{\Sigma_1} = -2$.

Using the remark we see that $\chi(M_k) \geq 0$, which contradicts the assumption that $\chi(M_k) < 0$ and we are done. $\square$

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A geometric boundary value problem related to the static vacuum equations in General Relativity

Carla Cederbaum

The Schwarzschild spacetime is one of if not *the* most important example of a spacetime in Mathematical General Relativity. It describes the static, vacuum exterior region of a spherically symmetric, isolated star or black hole. The $n+1$-dimensional Schwarzschild spacetime of mass $m \in \mathbb{R}$ is given by

$$\mathfrak{g} := -u_m^2 dt^2 + \frac{1}{u^2} dr^2 + r^2\, d\Omega^2, \tag{1}$$

$$u_m := u_m(r) := \sqrt{1 - \frac{2m}{r^{n-2}}} \tag{2}$$

on the spacetime manifold $\mathbb{R} \times (r_m, \infty) \times \mathbb{S}^{n-1}$, where $d\Omega^2$ denotes the canonical metric on $\mathbb{S}^{n-1}$ and $r_m := 0$ for $m \leq 0$ and $r_m := (2m)^{\frac{1}{n-2}}$ for $m > 0$. This definition applies whenever $n \geq 3$.

The Schwarzschild spacetime is known to be rigid in various ways:

- *Birkhoff's theorem* [1] asserts that the Schwarzschild spacetime is the only spherically symmetric Lorentzian spacetime $(\mathcal{L}^{n+1}, \mathfrak{g})$ which solves the vacuum Einstein equations $\mathfrak{Ric} = 0$.
- The *static vacuum black hole uniqueness theorem* asserts that the Schwarzschild spacetime is the only asymptotically flat spacetime $(\mathcal{L}^{n+1}, \mathfrak{g})$ with "black hole inner boundary" which is "static" and solves the vacuum Einstein equations $\mathfrak{Ric} = 0$. Here, being *static* means that $(\mathcal{L}^{n+1}, \mathfrak{g}) = (\mathbb{R} \times M^n, \mathfrak{g} = -u^2 dt^2 + g)$, where (M^n, g) is an asymptotically Euclidean Riemannian manifold and $u\colon M^n \to \mathbb{R}^+$ is a function with $u \to 1$ near infinity. Static vacuum black hole uniqueness was proved by many authors under a variety of assumptions, in particular by Bunting and Masood-ul-Alam [2] for $n = 3$, using a very elegant method. Gibbons, Ida, and Shiromizu [6] and Hwang [7] generalized this method to $n \geq 3$ for spin manifolds. In this context, the definition of a *black hole inner boundary* is that ∂M consists of finitely many compact components with vanishing mean curvature, $H = 0$, such that $u = 0$ on ∂M, and such that the normal derivative $\nu(u)$ has a sign on ∂M.
- Analogously, the *static vacuum photon sphere uniqueness theorem* asserts that the Schwarzschild spacetime is the only asymptotically flat spacetime $(\mathcal{L}^{n+1}, \mathfrak{g})$ with "photon sphere inner boundary" which is static and solves the vacuum Einstein equations $\mathfrak{Ric} = 0$. Here, a *photon sphere inner*

boundary is defined as a timelike umbilic hypersurface $\mathcal{P}^n \hookrightarrow (\mathcal{L}^{n+1}, \mathfrak{g})$ on which $u \equiv \text{const}$, see [4]. Static vacuum photon sphere uniqueness was proved by the author and Galloway [5] for $n = 3$, relying on the method suggested by Bunting and Masood-ul-Alam [2].

The goal of this talk was to show that the Schwarzschild spacetime is indeed rigid in a much more general way [3]. Before we discuss the main rigidity theorem, let us briefly recall the symmetry reduced Einstein vacuum equation for static spacetimes $(\mathbb{R}^n \times M^n, \mathfrak{g} = -u^2 dt^2 + g)$, the so-called *static vacuum equations*

$$u\,\mathrm{Ric} = \nabla^2 u, \tag{3}$$

$$\triangle u = 0 \tag{4}$$

on M^n which follow directly from plugging the special form of $\mathfrak{g}$ into the vacuum Einstein equations $\mathfrak{Ric} = 0$. Here, Ric denotes the Ricci tensor of g. A straightforward consequence obtained by tracing (3) is that the scalar curvature of (M^n, g) vanishes, which we denote as $\mathrm{R} = 0$. These equations are used in [2,4–7].

If not discussing vacuum but matter models with non-negative energy density, one finds $\mathrm{R} \geq 0$ — at least in the so-called "Riemannian" case. This condition is related to the *dominant energy condition* in General Relativity. The rigidity theorem we prove does not assume (3) and neither $\mathrm{R} = 0$, only (4) and $\mathrm{R} \geq 0$.

Theorem 1 (Rigidity of Schwarzschild manifold). *Assume $n \geq 3$ and let M^n be a smooth, connected, n-dimensional manifold with non-empty, possibly disconnected, smooth, compact inner boundary $\partial M = \dot\bigcup_{i=1}^{I} \Sigma_i^{n-1}$. Let g be a smooth Riemannian metric on M^n. Assume that (M^n, g) has non-negative scalar curvature $\mathrm{R} \geq 0$ and that it is geodesically complete up to its inner boundary ∂M. Assume in addition that (M^n, g) is asymptotically isotropic with one end of mass $m \in \mathbb{R}$.*

Furthermore, assume that the inner boundary ∂M is umbilic in (M^n, g), and that each component Σ_i^{n-1} has constant mean curvature H_i with respect to the outward pointing unit normal ν_i. Assume that there exists a function $u\colon M^n \to \mathbb{R}$ with $u > 0$ away from ∂M which is smooth and harmonic on (M^n, g), so that $\triangle u = 0$. We ask that u is such that $u|_{\Sigma_i^{n-1}} \equiv: u_i$ is constant on each Σ_i^{n-1} and u is asymptotically isotropic of the same mass m.

Finally, we assume that for each $i = 1, \ldots, I$, we are either in the semi-static horizon case

$$H_i = 0, \quad u_i = 0, \quad \nu_i(u) \neq 0, \tag{5}$$

or we are in the true CMC case *with $H_i > 0$, $u_i > 0$, and such that there exist constants $c_i > \frac{n-2}{n-1}$ so that*

$$\mathrm{R}_{\sigma_i} = c_i H_i^2, \tag{6}$$

$$2\nu(u)_i = \left(c_i - \frac{n-2}{n-1}\right) H_i u_i, \tag{7}$$

where R_{σ_i} denotes the scalar curvature of Σ_i^{n-1} with respect to its induced metric σ_i and $\nu_i(u)|_{\Sigma_i^{n-1}} \equiv: \nu(u)_i$ denotes the normal derivative of u.

Then $m > 0$ and (M^n, g) is isometric to a suitable portion of the spatial Schwarzschild manifold of mass m $((r_m, \infty) \times \mathbb{S}^{n-1}, g_m := \frac{1}{u^2} dr^2 + r^2\, d\Omega^2)$. Moreover, u coincides with the restriction of u_m (up to the isometry).

In Theorem 1, the asymptotic isotropy conditions are defined as follows:

Definition 2. We say that (M^n, g) is *asymptotically isotropic with one end of mass $m \in \mathbb{R}$*, if M^n is diffeomorphic to $\mathbb{R}^n \setminus$ ball outside a compact set, and with respect to the coordinates (y^i) induced by this diffeomorphism, we have

$$g_{ij} = \Big(1 + \frac{m}{2s^{n-2}}\Big)^{\frac{4}{n-2}} \delta_{ij} + \mathcal{O}_2(\frac{1}{s^{n-1}}) \tag{8}$$

as $s := \sqrt{(y^1)^2 + \cdots + (y^n)^2} \to \infty$. We say that a function $u \colon M^n \to \mathbb{R}$ is *asymptotically isotropic of mass m* if, with respect to the same diffeomorphism and coordinates described above, we have

$$u = 1 - \frac{m}{s^{n-2}} + \mathcal{O}_2(\frac{1}{s^{n-1}}) \tag{9}$$

as $s \to \infty$.

We remark that Theorem 1 recovers the static vacuum black hole uniqueness theorem in all dimensions (and dropping the spin assumption in [6,7] and recovers and generalizes the static vacuum photon sphere uniqueness theorem to all dimensions.

Sketch of Proof of Theorem 1. In the talk, we gave a sketch of the proof of Theorem 1 by pictures. For more details, please see [3].

The **first step** is to extend (M^n, g) across each true CMC inner boundary component Σ_i^{n-1} by gluing a suitable, explicitly constructed Riemannian manifold (M_i^n, g_i) into (M^n, g) across Σ_i^{n-1} in a $C^{1,1}$ fashion. The glue-in manifolds (M_i^n, g_i) are constructed such that they give rise to new inner boundary components which are totally geodesic semi-static horizons, i.e. satisfy (5). Also by construction, (M_i^n, g_i) has vanishing scalar curvature. We will also extend the harmonic function u by gluing it to a (positive multiple of a) g_i-harmonic function $u_i \colon M_i^n \to \mathbb{R}$ with $C^{1,1}$-regularity across the gluing surface Σ_i^{n-1}, in a manner that $u_i > 0$ away from the new horizon boundary. This is possible because of the constraint conditions (6), (7). The described argument reduces Theorem 1 to the case where there are only semi-static horizon boundary components, see also Figure 1.

The glue-in manifolds (M_i^n, g_i) are defined as

$$g_i := \frac{1}{u_i(r)^2} dr^2 + \frac{r^2}{r_i^2} \sigma_i, \tag{10}$$

$$u_i(r) := \sqrt{1 - \frac{2\mu_i}{r^{n-2}}}, \tag{11}$$

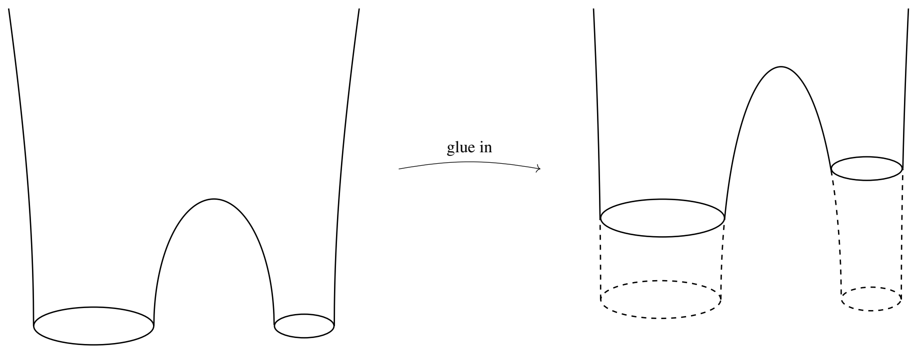

FIGURE 1. Gluing in a suitable, explicitly constructed Riemannian manifold (M_i^n, g_i) into each inner boundary component Σ_i^{n-1}. The new boundary components are totally geodesic semi-static horizons.

on $M_i^n := ((2\mu_i)^{\frac{1}{n-2}}, r_i) \times \Sigma_i^{n-1}$, where r_i is the scalar curvature radius of $(\Sigma_i^{n-1}, \sigma_i)$ given by $\mathrm{R}_{\sigma_i} =: \frac{(n-2)(n-1)}{r_i^2}$ and $\mu_i > 0$ is a suitably chosen mass. This glue-in strategy generalizes that used in [5] to higher dimensions and possibly non-round inner boundary $(\Sigma_i^{n-1}, \sigma_i)$. See [3] for more properties of the manifolds (M_i^n, g_i).

As a **second step**, we adapt [2,6,7] and double the extended manifold constructed above across its umbilic, semi-static horizon boundary (again with $C^{1,1}$-regularity across the doubling surfaces) to obtain a new Riemannian manifold $(\widetilde{M}^n, \widetilde{g}\,)$ which is geodesically complete and has two asymptotically isotropic ends of the same ADM-mass m as (M^n, g), see Figure 2. We denote the original part $M^n \subset \widetilde{M}^n$ as $\widetilde{M}^+$ and the new copy as $\widetilde{M}^-$. At the same time, we extend the function u to $\widetilde{M}^n$ by

$$\widetilde{u} : \widetilde{M}^n \to \mathbb{R} \colon p \mapsto \begin{cases} u(p) & \text{if } p \in \widetilde{M}^+ \\ -u(p) & \text{if } p \in \widetilde{M}^- \end{cases} \tag{12}$$

and observe that $\widetilde{u}$ is smooth away from the gluing surfaces and $C^{1,1}$ across the gluing surfaces. Also, $\widetilde{u}$ is harmonic with respect to $\widetilde{g}$, $\widetilde{u}(\widetilde{M}^n) = (-1,1)$, and $\pm\widetilde{u} \to 1$ as $r \to \infty$ in $\widetilde{M}^\pm$ is also asymptotically isotropic of mass m. This doubling construction first employed by Bunting and Masood-ul-Alam [2] works even though we do not assume the static vacuum equations (3), (4).

The **third step** consists in performing the conformal transformation and one point insertion method from [2,6,7] and ensuring that it makes no use of (3). More precisely, we conformally transform $(\widetilde{M}^n, \widetilde{g}\,)$ to $\widehat{M}^n := \widetilde{M}^n$ via

$$\widehat{g} := \left(\frac{1+\widetilde{u}}{2}\right)^{-\frac{4}{n-2}} \widetilde{g}. \tag{13}$$

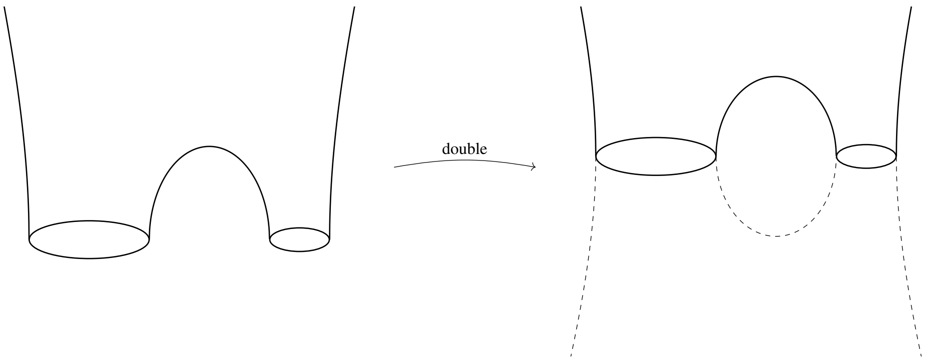

FIGURE 2. Doubling the extended manifold to a geodesically complete one across the totally geodesic boundary.

Exploiting that $\widetilde{u}$ is harmonic with respect to $\widetilde{g}$ and the fact that we chose the magical Yamabe power, we find that $\widehat{\mathrm{R}} \geq 0$. Under this conformal transform, the original asymptotically isotropic end $\widetilde{M}^{+}$ then transforms into an asymptotically isotropic end $\widehat{M}^{n}$ of vanishing ADM-mass $\widehat{m} = 0$, see Figure 3. The asymptotics of $\widetilde{u}$ and $\widetilde{g}$ of the doubled end allows to insert a point p_{∞} in a $C^{1,1}$ fashion so that we obtain a geodesically complete manifold $(\widehat{M}^{n}_{\infty} := \widehat{M}^{n} \cup \{p_{\infty}\}, \widehat{g}_{\infty})$. This manifold satisfies the assumptions of the rigidity case of the positive mass theorem [9,10], except the regularity assumptions across the finitely many gluing hypersurfaces and the point p_{∞}. To remedy this problem, we appeal to McFeron and Székelyhidi [8]. This shows that $(\widehat{M}^{n}_{\infty}, \widehat{g}_{\infty})$ is globally isometric to Euclidean space.

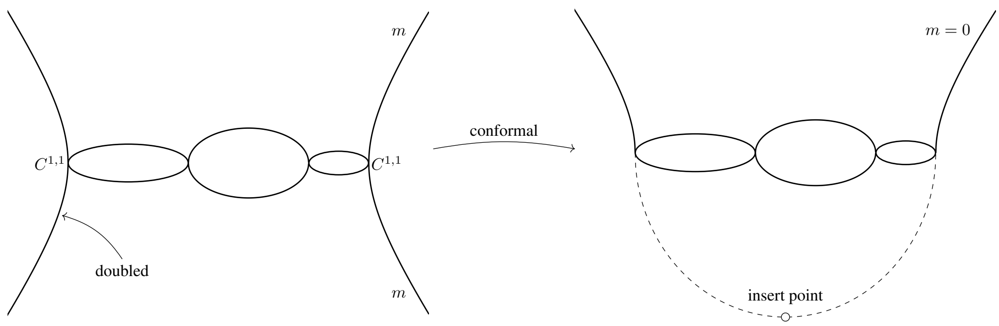

FIGURE 3. Conformal transformation and one-point insertion to a geodesically complete Riemannian manifold with vanishing ADM-mass and non-negative scalar curvature.

In order to conclude that (M^n, g) must have been isometric to a portion of Schwarzschild $(\widetilde{M}^{n}_{m}, \widetilde{g}_{m})$, we proceed as follows: First, recall that each boundary component $\Sigma_i^{n-1} \hookrightarrow (M^n, g)$ is closed and umbilic. As g is conformally equivalent to $\widehat{g}$ and $\widehat{g}$ is isometric to δ, we find that the image of $\Sigma_i^{n-1} \hookrightarrow (\mathbb{R}^n, \delta)$ is a closed, totally umbilic hypersurface and thus necessarily a round sphere and thus in particular a topological sphere by standard arguments. Second, we know that $\widehat{M}^{n}_{\infty}$

is diffeomorphic to $\mathbb{R}^n$ and thus $\widehat{M}^n$ diffeomorphic to $\mathbb{R}^n \setminus \{0\}$. From topological considerations, this shows that the boundary ∂M must have been connected.

Now, let us consider the picture in $(\mathbb{R}^n, \delta)$: A standard computation shows that the conformal factor $\varphi := \left(\frac{1+\widetilde{u}}{2}\right)^{-1}$ is harmonic with respect to $\widehat{g}$ and thus with respect to δ outside the round sphere image of Σ^{n-1}. The boundary value of φ on the round sphere image is a constant by construction, and φ tends to 1 near infinity. Thus by the maximum principle and standard facts on Green's functions, we find that φ is the conformal factor of Schwarzschild of mass m. Because of the boundary data assumptions (5), (6), (7), respectively, $m > 0$.

This finishes the sketch of the proof of Theorem 1.

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Non-uniqueness of solutions to the Yamabe problem on compact and noncompact manifolds

Renato G. Bettiol
(joint work with Paolo Piccione, Bianca Santoro)

The *Yamabe problem* on a (possibly noncompact and incomplete) Riemannian manifold (M, g) is to find a complete metric with constant scalar curvature which is conformal to g. We exploit the geometry of discrete cocompact groups and techniques from Bifurcation Theory to construct large classes of compact and noncompact manifolds on which the Yamabe problem has infinitely many different solutions [1–4].

Regarding the compact case, it can be shown [2] that given compact Lie groups $\mathsf{H} \subset \mathsf{K} \subset \mathsf{G}$ such that $\mathrm{scal}_{\mathsf{K}/\mathsf{H}} > 0$ and either $\mathsf{H} \lhd \mathsf{K}$ or $\mathsf{K} \lhd \mathsf{G}$, the 1-parameter family of homogeneous metrics g_t on G/H obtained by rescaling by $t > 0$ the vertical direction of the homogeneous bundle

$$\mathsf{K}/\mathsf{H} \to \mathsf{G}/\mathsf{H} \to \mathsf{G}/\mathsf{K} \tag{1}$$

has a sequence of *bifurcation instants* accumulating at $t = 0$. In particular, this implies the existence of at least 3 solutions to the Yamabe problem on $(\mathsf{G}/\mathsf{H}, \mathrm{g}_t)$ for infinitely many $t > 0$. For instance, the above holds if (1) is a Hopf bundle

$$\mathbb{S}^3 \to \mathbb{S}^{4n+3} \to \mathbb{H}P^n \quad \text{or} \quad \mathbb{S}^7 \to \mathbb{S}^{15} \to \mathbb{S}^8(1/2), \tag{2}$$

see also [1]. This has been recently generalized by Otoba and Petean [5].

Regarding noncompact manifolds, it can be shown [3] that if (M, g) is a closed manifold with constant positive scalar curvature, and (N, h) is a simply-connected symmetric space of noncompact or Euclidean type, such that $(M \times N, \mathrm{g} \oplus \mathrm{h})$ has positive scalar curvature, then there exist infinitely many solutions to the Yamabe problem on $(M \times N, \mathrm{g} \oplus \mathrm{h})$. As an immediate consequence, there exist infinitely many solutions to the Yamabe problem on $\mathbb{S}^m \times \mathbb{H}^d$ for all $2 \leq d < m$, and on $\mathbb{S}^m \times \mathbb{R}^d$ for all $m \geq 2$, $d \geq 1$. It is easy to see that these infinitely many solutions on $\mathbb{S}^m \times \mathbb{R}$ translate into infinitely many solutions also on $\mathbb{S}^{m+1} \setminus \{\pm p\}$ and on $\mathbb{R}^{m+1} \setminus \{0\}$, which are conformally equivalent to $\mathbb{S}^m \times \mathbb{R}$ via the stereographic projection. These can be seen as simple instances of the so-called *singular Yamabe problem*, which consists of finding solutions to the Yamabe problem on manifolds of the form $M \setminus \Lambda$, where M is a closed manifold and $\Lambda \subset M$ a closed subset. As explained in [3], combining the above result with the conformal equivalence $\mathbb{S}^m \setminus \mathbb{S}^k \cong \mathbb{S}^{m-k-1} \times \mathbb{H}^{k+1}$, it follows that there are infinitely many solutions to the singular Yamabe problem on $\mathbb{S}^m \setminus \mathbb{S}^k$, for all $0 \leq k < (m-2)/2$. This extends the main result in [4], which uses bifurcation techniques that work if and only if $k = 1$, due to the Mostow Rigidity Theorem. Note that $0 \leq k < (m-2)/2$ is the maximal range of dimensions for which multiplicity of solutions is possible, by the asymptotic maximum principle.

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Stability of ALE Ricci-flat manifolds under Ricci flow

Klaus Kröncke

(joint work with Alix Deruelle)

A complete Riemannian manifold (M, g) is called asymptotically locally Euclidean (ALE for short) of order $\tau > 0$ if there exists a compact set $K \subset M$, a radius $R > 0$ and a diffeomorphism $\varphi : M \setminus K \to (\mathbb{R}^n \setminus B_R)/\Gamma$ such that

$$(\varphi_* g - g_{eucl})_{ij} = O_\infty(r^{-\tau}).$$

Here, $\Gamma \subset \mathrm{SO}(n)$ is a discrete subgroup acting freely on S^{n-1}. If Γ is trivial, one recovers the notion of an asymptotically Euclidean (AE) manifold.

If (M, g) is ALE and Ricci-flat, it is ALE of order $n - 1$. If in addition, (M, g) is Kähler or if $n = 4$, it is ALE of order n.

ALE Ricci-flat manifolds are important models in quantum gravity and were extensively studied by Gibbons, Hawking and many other physicists. The first nontrivial example that was discovered in 1979 is the Eguchi-Hanson metric on TS^2. Later, Kronheimer showed for each discrete subgroup $\Gamma \subset \mathrm{SU}(2)$ which acts freely on S^3 the existence of 4-dimensional hyperkähler ALE manifold with fundamental group Γ at infinity. His work provides a large class of interesting examples.

The stability problem for Einstein manifolds and Ricci solitons under Ricci flow was extensively studied in recent years, both in the compact and in the noncompact case. In the case of ALE Ricci-flat four-manifolds, this problem is related to the extension problem of Ricci flows with bounded scalar curvature on compact four-manifolds.

In the context of stability problems, it is more convenient to deal with the Ricci-DeTurck flow instead of the Ricci flow as it has the advantage of being strongly elliptic. The main theorem of this talk is as follows:

Theorem 1. *Let (M^n, g_0) be an ALE Ricci-flat manifold. Assume it is linearly stable and integrable. Then for every $\epsilon > 0$, there exists a $\delta > 0$ such that the* *following holds: for any metric* $g \in \mathcal{B}_{L^2\cap L^\infty}(g_0, \delta)$*, there is a complete Ricci-DeTurck flow* $(M^n, g(t))_{t\geq 0}$ *starting from* g *converging to an ALE Ricci-flat metric* $g_\infty \in \mathcal{B}_{L^2\cap L^\infty}(g_0, \epsilon)$*. Moreover, the* L^∞ *norm of* $(g(t) - g_0)_{t\geq 0}$ *is decaying sharply at infinity:*

$$\|g(t) - g_0\|_{L^\infty(M\setminus B_{g_0}(x_0,\sqrt{t}))} \leq C(n, g_0, \epsilon)\frac{\sup_{t\geq 0}\|g(t) - g_0\|_{L^2(M)}}{t^{\frac{n}{4}}}, \quad t > 0.$$

By linear stability, we mean that the spectrum of the Lichnerowicz operator L_{g_0} lies in $(-\infty, 0]$. By integrability, we mean that the set of stationary points $\mathcal{F}$ of the Ricci-DeTurck flow in $\mathcal{B}_{L^2\cap L^\infty}(g_0, \epsilon)$ is a manifold with $T_{g_0}\mathcal{F} = \ker_{L^2}(L_{g_0})$.

A first important step to prove this theorem is to analyse the structure of the space $\mathcal{F}_{loc} = \mathcal{F} \cap \mathcal{B}_{L^2\cap L^\infty}(g_0, \epsilon)$ for small enough $\epsilon > 0$. It turns out that any $g \in \mathcal{F}_{loc}$ is Ricci flat and satisfies the gauge condition

$$V(g, g_0)^k = g^{ij}(\Gamma(g)^k_{ij} - \Gamma(g_0)^k_{ij}) = 0.$$

In addition, we get fast decay $g - g_0 = O_\infty(r^{-n+1})$. The set $\mathcal{F}_{loc}$ can be embedded as an analytic subset into a finite-dimensional manifold $\mathcal{Z}$ with $T_{g_0}\mathcal{Z} = \ker_{L^2}(L_{g_0})$.

The following theorem provides an important class of examples to which we can apply the main theorem.

Theorem 2. *Let* (M, g_0) *be an ALE Ricci-flat Kähler manifold. Then it is linearly stable and integrable.*

The linear stability follows exactly as in the compact case by relating the Lichnerowicz operator to other operators. To show integrability, one essentially adapts the strategy of the compact case and uses weighted Sobolev spaces. For an integration by parts argument, one needs the fast decay that has been stated above.

For the proof of the main theorem, we also need the following property of the Lichnerowicz operator which follows from work by Devyver [2]:

Theorem 3. *Let* (M, g_0) *be a linearly stable ALE Ricci-flat manifold. Then there exists a constant* $\alpha > 0$ *such that*

$$(-L_{g_0}h, h)_{L^2(g_0)} \geq \alpha \|\nabla h\|^2_{L^2(g_0)}$$

holds for all $h \in C^\infty_{cs}(S^2T^*M) \cap \ker_{L^2}(L_{g_0})^\perp$*.*

This property guarantees a nice L^2 a priori estimate.

Given a Ricci-DeTurck flow $g(t)$ in $\mathcal{B}_{L^2\cap L^\infty}(g_0, \epsilon)$ it is important find a suitably family $g_0(t)$ of reference metrics in $\mathcal{F}_{loc}$ to ensure that it does not escape. It turned out that demanding the property $k(t) = g(t) - g_0(t) \in \overline{L_{g_0(t),g_0}(C^\infty_{cs}(S^2T^*M))}$ yields the best possible choice or reference metrics. Here, $L_{g_0(t),g_0}$ is a slight modification of the Lichnerowicz operator (which is not longer self-adjoint) admitting the property that $\ker_{L^2}(L_{g_0(t),g_0}) = T_{g_0(t)}\mathcal{F}_{loc}$.

The nice effect of this decomposition is that it ensures that $\partial_t g_0(t)$ is quadratic in $k(t)$ because the linear term $L_{g_0(t),g_0}k(t)$ in the expansion of $\partial_t g(t)$ drops after projecting to $T_{g_0(t)}\mathcal{F}_{loc}$. At the end, we get the estimate

$$\int_1^\infty \|\partial_t g_0(t)\|_{L^2}\,dt \le C\int_1^\infty \|\nabla k(t)\|_{L^2}^2\,dt \le C\,\|k(1)\|_{L^2}^2$$

which shows that $g_0(t)$ must converge to some limit g_∞. Elliptic regularity, interpolation inequalities and derivative estimates finally ensure that $g_0(t) \to g_\infty$ and $k(t) \to 0$ in all C^k-norms as $t \to \infty$.

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Classification of almost Kähler four-manifolds of constant holomorphic sectional curvature

Markus Upmeier
(joint work with Mehdi Lejmi)

The study of how the geometry of a manifold governs the global solution theory of a differential equation is an old question. An illustration of this idea concerning the existence of local solutions to the Cauchy–Riemann equations is the following open conjecture:

Conjecture (Goldberg [4]). *Let M be a closed almost Kähler four-manifold. If M is an Einstein manifold, then the almost complex structure must be integrable.*

In the case of non-negative scalar curvature, this conjecture has been verified by Sekigawa [8]. Results under similar curvature assumptions have been obtained my many authors. We mention only those immediately related to our work [2], [3]. The proof of the following theorem was sketched during my talk:

Theorem 1. *Let (M, g, J, ω) be a closed almost Kähler four-manifold of globally constant holomorphic sectional curvature k with respect to the Chern connection. When $k < 0$ assume in addition that the Ricci tensor is J-invariant. Then J is integrable so that M is Kähler, holomorphically isometric to:*

- **($k > 0$):** $\mathbb{C}P^2$.
- **($k = 0$):** *a complex torus or a hyperelliptic curve.*
- **($k < 0$):** *a compact quotient of the complex hyperbolic ball $\mathbb{B}^4$.*

Almost Kähler means that ω is symplectic and $J\colon TM \to TM$ is an almost complex structure $J^2 = -1$ such that $g(X, Y) = \omega(X, JY)$ defines a Riemannian metric. The Chern connection is defined from the Levi-Civita connection D^g by

$$\nabla_X Y = D^g_X Y - \frac{1}{2}J(D^g_X J)Y.$$

From its curvature $R^\nabla(X,Y,Z,W) = g([\nabla_X,\nabla_Y]Z - \nabla_{[X,Y]}Z, W)$ one obtains the holomorphic sectional curvature

$$H(X) = -R^\nabla(X,JX,X,JX), \qquad \forall X \in T_pM,\ \ g(X,X) = 1.$$

Theorem 1 is only formally similar to the Kähler case (see [7, Theorems 7.8, 7.9] and [5,6]). The main difficulty here is to prove the integrability of J. For this we use integral formulae from Chern–Weil theory to show Kählerness under further topological restrictions on the signature and Euler characteristic. When $k \geq 0$ these topological restrictions are then shown to hold using results from Seiberg–Witten theory. The case $k < 0$ follows by applying in our situation a formula for the Bach tensor obtained in [1].

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Secondary large-scale index theory and positive scalar curvature

Rudolf Zeidler

(joint work with Noé Bárcenas)

There is a natural mapping from the positive scalar curvature sequence of Stolz to the analytic surgery sequence of Higson and Roe, which was originally established by Piazza and Schick:

Theorem ([6,9]). *There exists a commutative diagram:*

$$\begin{array}{ccccccccc}
\Omega^{\mathrm{spin}}_{*+1}(\mathrm{B\Gamma}) & \longrightarrow & \mathrm{R}^{\mathrm{spin}}_{*+1}(\mathrm{B\Gamma}) & \longrightarrow & \mathrm{P}^{\mathrm{spin}}_{*}(\mathrm{B\Gamma}) & \longrightarrow & \Omega^{\mathrm{spin}}_{*}(\mathrm{B\Gamma}) & \longrightarrow & \mathrm{R}^{\mathrm{spin}}_{*}(\mathrm{B\Gamma}) \\
\downarrow & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\rho} & & \downarrow & & \downarrow{\scriptstyle\alpha} \\
\mathrm{KO}_{*+1}(\mathrm{B\Gamma}) & \longrightarrow & \mathrm{KO}_{*+1}(\mathrm{C}^*_{\mathrm{r}}\Gamma) & \longrightarrow & \mathrm{S}^{\Gamma}_{*}(\mathrm{E\Gamma}) & \longrightarrow & \mathrm{KO}_{*}(\mathrm{B\Gamma}) & \longrightarrow & \mathrm{KO}_{*}(\mathrm{C}^*_{\mathrm{r}}\Gamma)
\end{array}$$

In particular, for a closed spin manifold M^n with $\pi_1 M = \Gamma$, for every metric g of positive scalar curvature (psc) on M, there exists a higher ρ-invariant $\rho^\Gamma(g) \in \mathrm{S}^\Gamma_n(\mathrm{E}\Gamma)$ in the *analytic structure group*. Moreover, for any two such metrics g_0, g_1, there is the *index difference* $\alpha^\Gamma_{\mathrm{diff}}(g_0, g_1) \in \mathrm{KO}_{n+1}(\mathrm{C}^*_\mathrm{r}\Gamma)$ which satisfies $\partial\alpha^\Gamma_{\mathrm{diff}}(g_0, g_1) = \rho^\Gamma(g_0) - \rho^\Gamma(g_1)$. Non-vanishing of the index difference is an obstruction to concordance of metrics and the ρ-invariant is a psc-bordism invariant.

Results of Botvinnik–Gilkey [3] (together with Higson–Roe [4]) imply that for a finite group Γ the maps $\alpha \otimes \mathbb{Q}\colon \mathrm{R}^{\mathrm{spin}}_n(\mathrm{B}\Gamma) \otimes \mathbb{Q} \to \mathrm{KO}_n(\mathrm{C}^*_\mathrm{r}\Gamma) \otimes \mathbb{Q}$ and $\rho \otimes \mathbb{Q}\colon \mathrm{P}^{\mathrm{spin}}_{n-1}(\mathrm{B}\Gamma) \otimes \mathbb{Q} \to \mathrm{S}^\Gamma_{n-1}(\mathrm{E}\Gamma) \otimes \mathbb{Q}$ are surjective for $n \geq 6$. More generally, results of Weinberger–Yu [7] and Xie–Yu [8] imply a lower bound on the rank of the image of $\alpha \otimes \mathbb{Q}$ (respectively $\rho \otimes \mathbb{Q}$) in even (respectively odd) dimensions based on the number of different orders of torsion elements in Γ.

In the first part of the talk, we have given an overview on the index difference and the ρ-invariant on *non-compact complete manifolds* as we have developed it in [10,11]. This *secondary coarse index theory* allows to treat complete metrics which have uniform positive scalar curvature outside a given subset of a non-compact manifold. Moreover, we have explained how the *secondary partitioned manifold index theorem* allows us to construct various examples of such metrics on non-compact manifolds which can be distinguished up to concordance.

These methods also have consequences on psc metrics on closed manifolds. In particular, we have the following result:

Theorem 1 ([10, Corollary 5.8])**.** *Let Γ be a group. Let N be a closed aspherical spin n-manifold such that there exists k and a proper Lipschitz map $f\colon \tilde N \times \mathbb{R}^k \to \mathbb{R}^{n+k}$ of degree* 1*. Then*

$$\mathrm{S}^\Gamma_*(\mathrm{E}\Gamma) \xrightarrow{\times[N]} \mathrm{S}^{\Gamma\times\pi_1(N)}_{*+n}(\mathrm{E}\Gamma \times \tilde N)$$

is split injective*.*

If $\tilde N$ satisfies the condition of the theorem we call it *stably hypereuclidean.* By a result of Dranishnikov, $\tilde N$ is stably hypereuclidean if N is aspherical and $\pi_1 N$ has finite asymptotic dimension.

The examples of metrics with different ρ-invariants due to Botvinnik–Gilkey and Weinberger–Yu all come from representations of finite subgroups and are of homological degree 0 with respect to the Chern character. The theorem above allows us to construct further examples of metrics of positive scalar curvature in different homological degrees.

Moreover, in joint work with Noé Bárcenas [2], we show that we can systematically obtain everything in homological degree up to two. This has been presented in the second part of the talk. We use the delocalized Chern character [1]

$$\underline{\mathrm{ch}}_\Gamma\colon \mathrm{K}^\Gamma_p(\underline{\mathrm{E}}) \otimes \mathbb{C} \xrightarrow{\cong} \bigoplus_{k\in\mathbb{Z}} \mathrm{H}_{p+2k}(\Gamma; \mathrm{F}\Gamma),$$

where $\mathrm{F}\Gamma$ is the $\mathbb{C}[\Gamma]$-module generated by finite order elements of Γ acted on by conjugation. Matthey [5] has constructed sections of the Chern character in low-degrees:

$$\beta_p\colon \mathrm{H}_p(\Gamma;\mathrm{F}\Gamma) \to \mathrm{K}_p^\Gamma(\underline{\mathrm{E}}\Gamma)\otimes\mathbb{C} \qquad (0\le p\le 2).$$

These maps involve complex K-theory. The real case is obtained by decomposing $\mathrm{F}\Gamma = \mathrm{F}^0\Gamma \oplus \mathrm{F}^1\Gamma$, where $\mathrm{F}^q\Gamma = \{f\in \mathrm{F}\Gamma \mid f(\gamma^{-1}) = (-1)^q f(\gamma)\ \forall\gamma\}$. Matthey's maps then induce maps

$$\beta^{\mathbb{R}}_{p,q}\colon \mathrm{H}_p(\Gamma;\mathrm{F}^q\Gamma)\to \mathrm{KO}^\Gamma_{p+2q}(\underline{\mathrm{E}}\Gamma) \qquad (0\le p\le 2,\ q\in\{0,1\})$$

Our main result is:

Theorem 2 ([2])**.** *Let $k\ge 1$, $q\in\{0,1\}$, $p\in\{0,1,2\}$, $4k+2q\ge 6$. Then there exists a map $\beta^{\mathrm{psc}}_{p,q,k}\colon \mathrm{H}_p(\Gamma;\mathrm{F}^q\Gamma)\to \mathrm{R}^{\mathrm{spin}}_{p+2q+4k}(\mathrm{B}\Gamma)$ such that the following diagram commutes.*

$$\begin{tikzcd}
 & \mathrm{R}^{\mathrm{spin}}_{p+2q+4k}(\mathrm{B}\Gamma)\otimes\mathbb{C} \arrow[r] \arrow[d, "\alpha\otimes\mathbb{C}"] & \mathrm{P}^{\mathrm{spin}}_{p+2q+4k-1}(\mathrm{B}\Gamma)\otimes\mathbb{C} \arrow[d, "\rho\otimes\mathbb{C}"] \\
\mathrm{H}_p(\Gamma;\mathrm{F}^q\Gamma) \arrow[ur, "\beta^{\mathrm{psc}}_{p,q,k}"] \arrow[dr, "\beta^{\mathbb{R}}_{p,q}"'] & \mathrm{KO}_{p+2q}(\mathrm{C}^*_{\mathrm{r}}\Gamma)\otimes\mathbb{C} \arrow[r] & \mathrm{S}^\Gamma_{p+2q-1}(\mathrm{E}\Gamma)\otimes\mathbb{C} \\
 & \mathrm{KO}^\Gamma_{p+2q}(\underline{\mathrm{E}}\Gamma)\otimes\mathbb{C} \arrow[u, "\mu\otimes\mathbb{C}"] &
\end{tikzcd}$$

Here μ denotes the real Baum–Connes assembly map.

Corollary 3 ([2])**.** *Let the rational homological dimension of Γ be at most two and $\mu\otimes\mathbb{Q}$ be an isomorphism. Then for $n\ge 7$ the following maps are surjective:*

$$\alpha\otimes\mathbb{Q}\colon \mathrm{R}^{\mathrm{spin}}_n(\mathrm{B}\Gamma)\otimes\mathbb{Q}\twoheadrightarrow \mathrm{KO}_n(\mathrm{C}^*_{\mathrm{r}}\Gamma)\otimes\mathbb{Q},$$

$$\rho\otimes\mathbb{Q}\colon \mathrm{P}^{\mathrm{spin}}_{n-1}(\mathrm{B}\Gamma)\otimes\mathbb{Q}\twoheadrightarrow \mathrm{S}^\Gamma_{n-1}(\mathrm{E}\Gamma)\otimes\mathbb{Q}.$$

Corollary 4 ([2])**.** *Let $n\ge 7$. If $\mu\otimes\mathbb{Q}$ is injective, then the rank of $\mathrm{R}^{\mathrm{spin}}_n(\mathrm{B}\Gamma)$ is at least the dimension of*

$$\begin{cases} \mathrm{H}_0(\Gamma;\mathrm{F}^0\Gamma)\oplus \mathrm{H}_2(\Gamma;\mathrm{F}^1\Gamma) & n\equiv 0 \mod 4,\\ \mathrm{H}_1(\Gamma;\mathrm{F}^0\Gamma) & n\equiv 1 \mod 4,\\ \mathrm{H}_0(\Gamma;\mathrm{F}^1\Gamma)\oplus \mathrm{H}_2(\Gamma;\mathrm{F}^0\Gamma) & n\equiv 2 \mod 4,\\ \mathrm{H}_1(\Gamma;\mathrm{F}^1\Gamma) & n\equiv 3 \mod 4.\end{cases}$$

Corollary 5 ([2])**.** *Let $n\ge 7$. If $\mu\otimes\mathbb{Q}$ is injective, then the rank of $\mathrm{P}^{\mathrm{spin}}_{n-1}(\mathrm{B}\Gamma)$ is at least the dimension of*

$$\begin{cases} \mathrm{H}_0(\Gamma;\mathrm{F}^0_0\Gamma)\oplus \mathrm{H}_2(\Gamma;\mathrm{F}^1\Gamma) & n\equiv 0 \mod 4,\\ \mathrm{H}_1(\Gamma;\mathrm{F}^0_0\Gamma) & n\equiv 1 \mod 4,\\ \mathrm{H}_0(\Gamma;\mathrm{F}^1\Gamma)\oplus \mathrm{H}_2(\Gamma;\mathrm{F}^0_0\Gamma) & n\equiv 2 \mod 4,\\ \mathrm{H}_1(\Gamma;\mathrm{F}^1\Gamma) & n\equiv 3 \mod 4,\end{cases}$$

where $\mathrm{F}^0_0 = \{f\in\mathrm{F}^0 \mid f(1)=0\}$.

In particular, this allows to systematically obtain non-trivial ρ-classes of positive scalar curvature in all dimensions ≥ 7 depending on low-degree group homology. This extends the previous constructions which only yielded even-dimensional examples.

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Stratified spaces, Dirac operators and Positive Scalar Curvature

Paolo Piazza

(joint work with Pierre Albin, Boris Vertman, Vito Felice Zenobi)

Let ${}^{\mathrm{S}}X$ be a Thom-Mather stratified pseudomanifold and let X^{reg} be its regular part. Assume that X^{reg} is spin. The goal of this talk was to report on recent results concerning analytic, geometric and topological invariants of the spin Dirac operator on X^{reg}. To give a rigorous meaning to this sentence we must

(1) specify which invariants we are interested in
(2) specify which riemannian metrics we allow in X^{reg}.

Regarding the first item, we recalled first of all the main results in the closed case. Let (M, g) be a compact spin manifold without boundary and with fundamental group Γ. We denote by $\not{S}$ the associated spinor bundle and by $\not{D}_g$ the Dirac operator; we denote by $\widetilde{M}$ the universal cover of M and by $\widetilde{\not{D}}_g$ the corresponding Dirac operator. We can first of all consider *numeric* invariants associated to $\not{D}_g$: these are the index in the even dimensional case and the Cheeger-Gromov rho-invariant in the odd dimensional case. The first numeric invariant gives an obstruction to the existence of positive scalar curvature (PSC) metrics while the second gives a tool for distinguishing metrics of PSC that are not path-connected, even modulo the action of the diffeomorphism group; results in this direction were first established by Botvinnik and Gilkey when Γ is finite [2] and by Piazza and Schick [5] when Γ is an arbitrary discrete group containing an element of finite order.

These invariants and these results can be sharpened by passing to K-theory. We therefore explained the Higson-Roe analytic surgery sequence

$$\cdots \to K_{*+1}(D^*(\widetilde{M})^\Gamma)) \to K_{*+1}(D^*(\widetilde{M})^\Gamma/C^*(\widetilde{M})^\Gamma)) \xrightarrow{\delta} K_*(C^*(\widetilde{M})^\Gamma)) \cdots$$

with

$$* = \dim M\,, \quad K_{*+1}(D^*(\widetilde{M})^\Gamma/C^*(\widetilde{M})^\Gamma)) \simeq K_*(M)\,, \quad K_*(C^*(\widetilde{M})^\Gamma) \simeq K_*(C^*_r\Gamma)$$

and how, using it, it is possible to define:

- a fundamental class $[\not{D}_g] \in K_{*+1}(D^*(\widetilde{M})^\Gamma/C^*(\widetilde{M})^\Gamma) = K_*(M)$, which is independent of g
- an index class $\mathrm{Ind}(\widetilde{\not{D}}_g) = \delta[\not{D}_g] \in K_*(C^*(\widetilde{M})^\Gamma)$ which vanishes if g is of PSC but that it is also independent of g
- a rho class $\rho(\widetilde{\not{D}}_g) \in K_{*+1}(D^*(\widetilde{M})^\Gamma)$, a lift of $[\not{D}_g]$, whenever g is of PSC.

These are the K-theoretic invariants we wanted to define. We briefly explained how the index class gives a more general obstruction than the numeric index to the existence of metrics of PSC and how the rho class can be used in order to sharpen the result of Piazza and Schick (work of Xie and Yu, [8]). We also briefly recalled how these invariants and a suitable Atiyah-Patodi-Singer index class can be used in order to map the Stolz surgery sequence to the Higson-Roe analytic surgery sequence (work of Piazza and Schick, [6]). Crucial to all this is the delocalized Atiyah-Patodi-Singer index theorem of Piazza and Schick, relating
(i) the Atiyah-Patodi-Singer index class of a spin Dirac operator of a manifold with boundary with PSC on the boundary
(ii) the rho class of the spin Dirac operator of the boundary.
We ended this long introduction by explaining Zenobi's approach [9] to these K-theoretic invariants via the adiabatic groupoid associated to the groupoid $G := \widetilde{M} \times_\Gamma \widetilde{M}$ (with units M and with obvious range and source maps); we stressed the fact that this approach holds for any Lie groupoid G with Lie algebroid $\mathfrak{A}$; in particular, there is a delocalized Atiyah-Patodi-Singer index theorem in this generality.

After this long introduction on the closed smooth case we passed to Thom-Mather stratified spaces and to the possible metrics that we can consider on their regular part. We concentrated for simplicity on a Thom-Mather stratified pseudomanifold ${}^{\mathrm{S}}X$ of depth 1; the singularities can be non-isolated. We introduced the following four types of metrics: (i) incomplete edge; (ii) complete edge; (iii) fibered cusp; (iv) fibered boundary, and explained the corresponding pseudodifferential calculi for the complete edge and the fibered boundary metrics: these are pseudodifferential calculi (due respectively to Mazzeo and Mazzeo-Melrose) on the resolution of ${}^{\mathrm{S}}X$, denoted X, a manifold with fibered boundary. For incomplete edge metrics and for the numeric invariants we considered so far, we stated results of Albin and Gell-Redman [1] regarding the numeric index and of Piazza and Vertman [7] regarding the Cheeger-Gromov rho invariant. We also stated recent results of Piazza and Albin regarding the three K-theoretic invariants. All these results (numeric and K-theoretic) give sufficient conditions for the existence of these invariants and then provide fundamental properties of them along the usual lines; in this incomplete case they have been established in the depth 1 case only.

In the last part of this talk, based on joint work with Zenobi, we passed to a Thom-Mather stratified pseudomanifold ${}^{\mathrm{S}}X$ of arbitrary depth. The resolution X of ${}^{\mathrm{S}}X$ is a manifold with corners with an iterated boundary fibration structure on the boundary. We endow X^{reg} with an *iterated* fibered boundary metric g. Thanks to work of Debord-Lescure-Rochon [3], based in turn on work of Mazzeo-Melrose [4], there is a fibered boundary pseudodifferential calculus on X and the Dirac operator $\not{D}_g$ is an element of it. We show that if the links of ${}^{\mathrm{S}}X$ inherit PSC metrics, then $\not{D}_g$ is fully elliptic. According to work of Debord-Lescure-Rochon there is then a well defined fundamental class $[\not{D}_g] \in K_*({}^{\mathrm{S}}X)$. The class $[\not{D}_g]$ is defined à la Kasparov, using a parametrix for the fully elliptic operator defined by $\not{D}_g$.

There is a different description of the fundamental class $[\not{D}_g] \in K_*({}^{\mathrm{S}}X)$, also due to Debord-Lescure-Rochon, that employs a specific groupoid and that brings us back, via Zenobi's approach, to our main goal of defining the three fundamental K-theoretic invariants on stratified manifolds. We explained this in the depth 1 case. Let S be the singular locus of ${}^{\mathrm{S}}X$; let X be the resolution of ${}^{\mathrm{S}}X$; we know that X is a manifold with boundary ∂X equal to H, a fibration $H \xrightarrow{\pi} S$ with base S and fiber Z; the interior of X is X^{reg}.

The groupoid G_π considered in the work of Debord-Lescure-Rochon is a groupoid over X; it is explicitly given by $X^{\mathrm{reg}} \times X^{\mathrm{reg}}$ over X^{reg} and $H \underset{S}{\times} TS \underset{S}{\times} H \times \mathbb{R}$ over H; its algebroid is ${}^{\mathrm{fb}}TX$, the fibered-boundary-tangent bundle, with its natural anchor map.

There is also a Γ-equivariant version of it, G_π^Γ, again a groupoid over $X \times [0,1]$. Let us denote by $(G_\pi^\Gamma)^0_{ad}$ the restriction of the adiabatic deformation of G_π^Γ to $X \times [0,1)$. The C*-algebra of this groupoid fits into the following exact sequence

$$0 \to C^*_r(\widetilde{X}^{reg}_\Gamma \underset{\Gamma}{\times} X^{reg}_\Gamma \times (0,1)) \to C^*_r((G_\pi^\Gamma)^0_{ad}) \to C^*_r(T^{\mathrm{NC}}X) \to 0.$$

with $T^{\mathrm{NC}}X$ a groupoid over $X^{reg} \times \{0\} \cup H \times [0,1)$ given explicitly by the disjoint union ${}^{\mathrm{fb}}TX \cup (H \underset{S}{\times} TS \underset{S}{\times} H \times \mathbb{R}) \times (0,1)$. Using the groupoid pseudodifferential calculus for G_π and the hypothesis that the links have PSC one can show that the Dirac operator $\not{D}_g$ defines a class $\sigma_{\mathrm{nc}}(\not{D}_g) \in K_*(C^*_r(T^{\mathrm{NC}}X))$ and thus a class $\delta(\sigma_{\mathrm{nc}}(\not{D}_g))$ in $K_*(C^*_r(X^{reg}_\Gamma \underset{\Gamma}{\times} X^{reg}_\Gamma)) = K_*(C^*_r\Gamma)$. Now, a theorem ultimately due to Debord, Lescure and Rochon states that $C({}^{\mathrm{S}}X)$, the algebra of continuous functions on the stratified pseudomanifold, is K-dual to the C*-algebra $C^*_r(T^{\mathrm{NC}}X)$: thus $K_*({}^{\mathrm{S}}X) \cong K_*(C^*_r(T^{\mathrm{NC}}X))$. Moreover, under this isomorphism the class $\sigma_{\mathrm{nc}}(\not{D}_g)$

correspond to $[\not{D}_g] \in K_*({}^{\mathrm{S}}X)$. Finally, one can show that $\delta(\sigma_{\mathrm{nc}}(\not{D}_g))$ is the index class defined via a parametrix construction by the Γ-equivariant operator associated to $\not{D}_g$. Summarizing: using Poincaré duality, we have defined two out of the three K-theoretic invariants we wanted to define, namely the fundamental class and the index class. Assume now that g has positive scalar curvature everywhere on X; then the index class vanishes and we can define a class $\rho(g_\pi) \in K_*(C^*_r((G^\Gamma_\pi)^0_{ad}))$ as a specific lift of the class $[\sigma_{nc}(\not{D}_g)]$. This is our rho-class. We ended this talk by showing that thanks to the delocalized APS index theorem for groupoids, due to Zenobi, this rho class gives a well-defined map from $\mathrm{Conc}_{fb}(X)$, the set of concordance classes of fibered boundary metrics on X, to $K_*(C^*_r((G^\Gamma_\pi)^0_{ad}))$.

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Edge-cone Einstein metrics and the Yamabe invariant

Kazuo Akutagawa

(joint work with Ilaria Mondello)

We study edge-cone Einstein metrics on smooth closed manifolds and an application to the Yamabe invariant. This is a joint work with Ilaria Mondello (Créteil, FR) [4].

Definition 1. Let M be a closed n-manifold ($n \geq 3$) and $\Sigma = \sqcup_{j=1}^{\ell} \Sigma_j$ a closed smooth submanifold of codimension 2. (For simplicity, we assume here that Σ is connected, that is, $\ell = 1$.) For any point $x_0 \in \Sigma$, one can find local coordinates $\{U, (x^1, x^2, x^3, \cdots, x^n)\}$ satisfying $\Sigma \cap U = \{x^1 = x^2 = 0\}$. We then introduce an associated *transversal polar coordinate* system $(\rho, \theta, x^3, \cdots, x^n)$ by setting $x^1 = \rho \cos\theta,\ x^2 = \rho \sin\theta$. Now fix a positive constant $\beta > 0$. An *edge-cone* *metric g of cone angle $2\pi\beta$ on (M, Σ)* is a smooth Riemannian metric on $M - \Sigma$ which takes the form (cf. [6,12])

$$g = \overline{g} + \rho^{1+\kappa}\mathcal{E}, \qquad \overline{g} = d\rho^2 + \beta^2\rho^2 d\theta^2 + (\Phi^* p)_{ij} \quad (3 \le i, j \le n),$$

where $\kappa > 0$ is a positive constant, p is a smooth Riemannian metric on Σ, and $\Phi : U \to \Sigma$ is the natural projection. Here, $\mathcal{E} = (\mathcal{E}_{AB})$ $(1 \le A, B \le n)$ is a symmetric tensor field on M which is *infinite conormal regular* along Σ. An *edge-cone Einstein metric* h is an edge-cone metric which is also an Einstein metric on $M - \Sigma$.

Our first result is the following:

Theorem A. Let h be an edge-cone Einstein metric of cone angle $2\pi\beta$ $(0 < \beta < 1)$ on (M, Σ). Then, there exists a family of $C^{2,\alpha}$-metrics $\{g_\delta\}_{0<\delta<\delta_0}$ on M which satisfies the following:

$$(1) \quad \mathrm{Ric}_{g_\delta} \ge (1 - C \cdot \delta)\mathrm{Ric}_h, \qquad (2) \quad |V_{g_\delta} - V_h| \le C \cdot \delta,$$

where $C > 0$ is a constant independent of $\delta > 0$, Ric_h and V_h denote respectively the Ricci curvature tensor of h and the volume of (M, h).

Remark 1. When h is a Kähler-Einstein metric with cone singularities on a Fano manifold X along an anti-canonical divisor Σ, Chen-Donaldson-Sun [8] have proved the existence of smooth Kähler metrics satisfying the above conditions (1), (2). However, since their method heavily depends on the Kählerness of X, it cannot apply to the real case. On the proof of Theorem A, our main tools are Donaldson's Schauder type estimates on edge-cone manifolds in [9] and Kobayashi's family of sophisticated cutoff functions in [11].

Corollary. Under the same setting as that in Theorem A, we have

$$Y(M) \ge \liminf_{\delta \searrow 0} Y(M, [g_\delta]) \ge R_h \cdot V_h^{2/n} = Y(M, [h]),$$

where $Y(M), Y(M, [g_\delta])$ and R_h denote respectively the Yamabe invariant of M, the Yamabe constant of $(M, [g_\delta])$ and the scalar curvature of h.

Remark 2. The first inequality follows from the definition of the Yamabe invariant of M. The second inequality follows from the result of Ilias [10]. The third equality follows from the computation of Yamabe constants of edge-cone Einstein metrics by Mondello [13]. Corollary implies that the Yamabe invariant $Y(M)$ can be estimated from below in terms of the Yamabe constants of a sort of *singular* Einstein metrics, that is, edge-cone Einstein metrics.

Definition 2. (cf. [6]) The standard round metric $g_S = g_{S^n}$ of constant curvature 1 on the n-sphere S^n can be written as a doubly warped product

$$g_S = dr^2 + \sin^2 r d\theta^2 + \cos^2 r \cdot g_{S^{n-2}} =: h_1$$

on $S^n - (S^{n-2} \sqcup S^1) = (0, \frac{\pi}{2}) \times S^1 \times S^{n-2} \ni (r, \theta, x)$. For each $\beta > 0$ ($\beta \neq 1$), we define the *standard edge-cone metric* h_β *of cone angle* $2\pi\beta$ *on* (S^n, S^{n-2}) by

$$h_\beta := dr^2 + \beta^2 \sin^2 r d\theta^2 + \cos^2 r \cdot g_{S^{n-2}}.$$

Since each h_β is locally isomorphic to $h_1 = g_S$, h_β is of constant curvature 1, and hence it is an edge-cone Einstein metric.

Our second result is the following:

Theorem B. For any $\beta \geq 2$, there is no (edge-cone) Yamabe metric on $(S^n, [h_\beta])$.

Remark 3. (1) Viaclovsky [14] has proved that some compact 4-orbifold do not admit orbifold Yamabe metrics. His proof depends on an generalization of Obata's Theorem for Einstein metrics. Our proof depends on a branched covering version of Aubin's Lemma for finite coverings [7], [5]. We also note that the dimension of the singularities of Viaclovsky's example is 0. On the other hand, the one of our example is $n-2$.
(2) We conjecture that, for any β ($1 < \beta < 2$), the same conclusion as that in Theorem B still holds.

Outline of Proof of Theorem B. Let $P : (S^n, h_\beta) \to (S^n, h_{\beta/2})$ be the natural branched double covering along S^{n-2}. One can check that the Aubin type Lemma still holds for the branched covering P, that is,

$$Y(S^n, [h_\beta]) > Y(S^n, [h_{\beta/2}])$$

provided that
(1) the existence of an edge-cone Yamabe metric $u^{4/(n-2)} \cdot h_\beta$ ($u > 0$) on $(S^n, [h_\beta])$,
(2) the following equality holds:

$$\int_{S^n} u \, \Delta_{h_\beta} u \, d\mu_{h_\beta} = - \int_{S^n} |\nabla u|^2 d\mu_{h_\beta}.$$

Suppose that the assertion (1) holds. Set $r(p) := \mathrm{dist}_{h_\beta}(p, S^{n-2})$ for $p \in S^n$. By the regularity result in [2,3], on each ε-open neighborhood $U_\varepsilon := U_\varepsilon(S^{n-2})$

$$0 < c \leq u \leq C, \qquad \partial_r u(r, \theta, x) = O(r^{\frac{1}{\beta}-1}) \quad \text{as} \quad r \searrow 0.$$

Hence,

$$\int_{S^n - U_\varepsilon} u \, \Delta_{h_\beta} u \, d\mu_{h_\beta} + \int_{S^n - U_\varepsilon} |\nabla u|^2 d\mu_{h_\beta} = \int_{\partial U_\varepsilon} u \, \partial_r u \, d\sigma_{h_\beta} = O(\varepsilon^{\frac{1}{\beta}}) \searrow 0$$

as $\varepsilon \searrow 0$. Then, the assertion (2) holds. On the other hand, if $\beta/2 \geq 1$, Mondello's result [13] (cf. [1]) implies that

$$Y(S^n, [h_\beta]) = Y(S^n, [g_S]) = Y(S^n, [h_{\beta/2}]).$$

This contradicts the previous strict inequality. $\square$

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From scalar curvature rigidity phenomena to min-max geodesics

Alessandro Carlotto
(joint work with Camillo De Lellis)

In recent years we have witnessed significant progress in the study of the large-scale structure of asymptotically flat Riemannian manifolds, which naturally arise in general relativity as models for isolated gravitating systems. An important result, in this direction, is the following:

Rigidity Theorem A ([6])**.** Let (M^3, g) be an asymptotically flat Riemannian manifold of non-negative scalar curvature and assume it contains a non-compact area-minimizing boundary. Then (M, g) is isometric to the Euclidean space.

We shall just mention here that this assertion has significant implications for the behaviour of large isoperimetric domains on the one hand and for the zero set of static potentials on the other, see [6] and references therein for further context and applications. At a purely geometric level, a natural question arising from the theorem above is whether asymptotically flat manifolds may/should in fact contain asymptotically planar minimal surfaces with Morse index equal to one. In this respect, R. Schoen described to the author of the present report a possible approach to such problem via min-max techniques for the area functional. The scope of this lecture is to outline such approach in a simpler case, with the goal of constructing properly embedded geodesics on asymptotically conical surfaces, a problem which can in fact be regarded as the lower-dimensional analogue of the one described above. Before getting there, let us remark that there are ample classes of highly anisotropic asymptotically flat manifolds where non-compact minimal surfaces exist in abundance: indeed, as a special case of the general localization scheme presented in [8] one can in fact construct asymptotically flat, scalar flat, metrics on $\mathbb{R}^3$ of positive mass which are exactly Euclidean on a half-space. As a result, the problem above is more interesting for the smaller class of asymptotically Schwarzschildean data, where an even stronger obstruction is known to hold:

Rigidity Theorem B ([5])**.** Let (M^3, g) be an asymptotically Schwarzschildean Riemannian manifold of non-negative scalar curvature and assume it contains a non-compact, properly embedded, stable minimal surface. Then (M, g) is isometric to the Euclidean space.

In fact, to this date we do not know even a single example of single-ended asymptotically Schwarzschildean manifold containing a non-compact minimal surface that is not totally geodesic. Heuristically, this is rather odd as in the *ground state* represented by Euclidean $\mathbb{R}^3$ every plane through the origin is (trivially) minimal and it would be natural to expect persistence of minimal surfaces at least for sufficiently small perturbations; yet this conclusion seems remarkably delicate to be established.

The Hopf-Rinow theorem asserts that a Riemannian manifold (M^n, g) is complete if and only if for every basepoint $p \in M$ the exponential map $\exp_p(\cdot)$ is defined on the whole tangent space T_pM. So one could assert that through every point of a complete Riemannian manifold there pass infinitely many geodesics defined over the entire real line. Nevertheless, in general any such geodesic will neither be embedded nor be proper. Based on this remark, S. Cohn-Vossen posed in 1936 the question whether any complete Riemannian plane $(\mathbb{R}^2, g)$ does in fact contain a properly embedded geodesic defined over $\mathbb{R}$. After a series of partial results, this question was finally settled in the affirmative by V. Bangert in 1981 (see [1] as well as the related works [2,3]). In the same article, he asked whether it is in fact the case that always *infinitely many* such geodesics actually exist. This problem is still completely open for a general complete metric on $\mathbb{R}^2$, but one should mention here remarkable contributions by Shioya [13], Bonk-Lang [4], Fernandez-Melian [11] and Shioya-Shiohama-Tanaka [12] among others. Even in relatively simple cases, like for instance that of compactly supported perturbations of radially symmetric complete metrics on $\mathbb{R}^2$ we are actually very far from obtaining a final answer to the question above: in that setting we have learnt from Bangert himself that always at least two distinct geodesics should exist, but even such mild assertion is still to be proven. In striking contrast to that situation, there are broad classes of complete, open, two-dimensional Riemannian manifolds for which the problem posed by Bangert does have a fully satisfactory answer.

Theorem. [7] An asymptotically conical surface of non-negative scalar curvature contains infinitely many, geometrically distinct, properly embedded geodesics defined over $\mathbb{R}$, each of Morse index less or equal than one. If the scalar curvature is assumed to be positive equality holds.

Asymptotically conical surfaces can be regarded as models for isolated systems in 2+1 gravity, thus they are the counterpart of asymptotically Euclidean Riemannian manifolds both from a physical and a geometric viewpoint. Correspondingly, the requirement that the scalar curvature be non-negative descends from the so-called dominant energy condition at the level of time-symmetric initial data sets.

We shall further remark that none of the geodesics mentioned in the theorem can in fact be length-minimizing unless in the trivial case of Euclidean $\mathbb{R}^2$. Indeed, if that were the case the surface in question would split as a result of the Cheeger-Gromoll theorem. This suggests that any minimizing scheme for the problem above is inevitably doomed to fail and thus min-max methods come into play.

The proof of this theorem is articulated in three steps, with some ancillary results of independent interest and potentially broad applicability. The first step, which may be referred to as *finite-scale construction* relies on the following assertion:

Proposition. Let (N, g) be a complete Riemannian manifold of dimension two, without boundary, and for given distinct points p, q assume that there exist two embedded geodesics $\gamma_1, \gamma_2 : [0,1] \to N$ bounding a disk-type region, and such that the mountain-pass condition

$$\Lambda > \max\{E(\gamma_1), E(\gamma_2)\}$$

holds. Here

$$X := \{\gamma \in W^{1,2}([0,1], N) \mid \gamma(0) = p, \gamma(1) = q\}$$
$$\Sigma := \{H \in C([0,1], X) \mid H(0) = \gamma_1, H(1) = \gamma_2\}$$

and we have set

$$\Lambda := \min_{H \in \Sigma} \max_{s \in [0,1]} E(H(s))$$

where for an element $\gamma \in X$

$$E(\gamma) = \int_0^1 g(\dot\gamma(t), \dot\gamma(t))\, dt.$$

Then there exists a parametrized embedded geodesic $\gamma_3 : [0,1] \to N$, whose support does not coincide with that of γ_1 or γ_2 and whose energy equals the value Λ. Furthermore, γ_3 has Morse index less or equal than one (as a critical point of the energy functional).

This is essentially a geometric mountain-pass theorem, and the key point here is that we require the third geodesic to be embedded, which is proven by using the recently obtained *resolution of singularities procedure* in order to effectively convert homotopies into isotopies (see [9]). Then we apply this general proposition to prove that any couple of far out antipodal points on an asymptotically conical surface can be joined by three distinct geodesics, two stable segments and a min-max one inbetween. Also, by suitably estimating the min-max value Λ both from above and from below, we can obtain a full asymptotic characterization of the three geodesic segments above, which shall converge (as we blow-down to unit scale) to the three geodesics connecting two-antipodal points on a flat wedge under pointwise identification of the two sides.

That conclusion being gained, one needs to make sure that the sequence of min-max geodesic segments one obtains does not drift off to infinity together with the corresponding endpoints. This follows from a suitable application of the Gauss-Bonnet theorem together with the aforementioned blow-down analysis. Hence, modulo some extra technical work, one can take a suitable limit of these min-max geodesic segments, thereby obtaining a properly embedded geodesic $\gamma : \mathbb{R} \to S$ where (S, g) is the asymptotically conical surface in question.

As a third and final step, one has to check that the construction above, when performed for sequences of antipodal points diverging along rays defined by different angles are indeed distinct, or in other words that one can actually obtain a bijection between $\mathbb{RP}^1$ and the space of properly embedded geodesics that are produced via this procedure. This step relies on a non-twisting lemma, which in turn exploits the fact that, in the setting of the theorem, a geodesic ray must asymptote to a coordinate ray at a certain specific rate. Once again, the pathologic twisting behaviour is then ruled out by applying the Gauss-Bonnet theorem to suitably constructed, possibly multiply-connected, domains in the asymptotic region.

Apart from the specific application to the construction of inextendible min-max geodesics, we expect this sort of approach to be useful in various other problems in Geometry and Analysis.

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Construction of stationary blackhole solution to the 4 + 1 vacuum Einstein equation with non-spherical horizons

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(joint work with Marcus Khuri, Gilbert Weinstein)

Spacetime is a Lorentzian n-manifold $(\mathcal{N}^{n+1}, \mathbf{g})$ satisfying the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R_{\mathbf{g}}\, \mathbf{g}_{\mu\nu} = T_{\mu\nu}$$

where $R_{\mu\nu}$ is the Ricci curvature of the metric $\mathbf{g}$, $R_{\mathbf{g}}$ the scalar curvature of the Lorentzian metric $\mathbf{g}$, and $T_{\mu\nu}$ is the energy-momentum-stress tensor of the matter fields. When vacuum; $T_{\mu\nu} = 0$, taking the trace of the Einstein vacuum equations we get $R_{\mathbf{g}} = 0$, and hence the VEE is equivalent to $R_{\mu\nu} = 0$..

Let (M, g) and (N, h) be Riemannian manifolds, and let $\varphi\colon M \to N$ be a continuously differentiable map. The Dirichlet energy density of φ is

$$|d\varphi|^2 = h_{ij}\, g^{\mu\nu}\, \partial_\mu \varphi^i\, \partial_\nu \varphi^j.$$

If $\Omega \subset\subset M$ then the *energy* of φ over Ω is $E_\Omega(\varphi) = \int_\Omega |d\varphi|^2\, dV_g$.

A map φ is *harmonic* if it is critical with respect to E_Ω for every $\Omega \subset\subset M$. The harmonic map equations are the vanishing of the tension field:

$$\tau^i = g^{\mu\nu} \left(\partial_\mu \partial_\nu \varphi^i - {}^M\Gamma^\sigma_{\mu\nu} \partial_\sigma \varphi^i + {}^N\Gamma^i_{jk} \partial_\mu \varphi^j \partial_\nu \varphi^k \right) = 0.$$

which is a cross section $\tau(\varphi) = \tau^i \partial_i \colon M \to TN$ of the pull back bundle $\varphi^{-1}TN$.

We introduce the so-called Ernst reduction scheme in 3 + 1 dimension. Let $(\mathcal{N}^{3+1}, \mathbf{g})$ be asymptotically flat, stationary and axially symmetric, and let G be the symmetry group ($\cong \mathbb{R} \times U(1)$). We further define X: the (Lorentzian) norm (> 0) of the Killing field generator $\xi = \partial/\partial\phi$, and Y: the potential of the *twist* $*(\xi \wedge d\xi)$. Then the two quantities $\varphi = (X, Y)\colon \mathcal{M}/G \to \mathbb{H}^2$ form a (weighted) harmonic map, i.e. a critical point of the energy

$$\int \left(\frac{|\nabla X|^2 + |\nabla Y|^2}{X^2} \right) \rho\, d\mu_g,$$

where g is the quotient metric on $\mathcal{M}/G$ and ρ is the area element of the orbits of symmetry. Then there are the following observations.

Fact 0: ρ is harmonic, and $\nabla\rho \neq 0$. Let z be a conjugate harmonic function.
Fact 1: $g = e^{2\nu}(d\rho^2 + dz^2)$ on $\mathcal{M}/G =$ the ρz-half plane ($\rho > 0$)
Fact 2: $X \approx \rho^2$ near $\Gamma \subset \{z - \text{axis}\}$, $\{X = 0\} \sim$ the physical *axis*.

The map φ is identified with an θ-axisymmetric harmonic map (B. Carter)

$$\varphi : \mathbb{R}^3 \setminus \Gamma \to \mathbb{H}^2 \ni (X, Y).$$

with $\varphi(\rho, \theta, z) = (X, Y)$. (Note $d\mu_{\mathbb{R}^3} = [\rho d\rho dz]d\theta$ with θ is a dummy variable.)) Furthermore, Y is constant on any component of Γ, and the difference between constants is the angular momentum induced by the existence of the horizon between the two components. The *data* consists of Γ together with a constant for each component of Γ.

Two maps $\varphi_1, \varphi_2 \colon M \to N$ are *asymptotic* if $\mathrm{dist}_N(\varphi_1, \varphi_2)$ is bounded. $\varphi_0 \colon \mathbb{R}^3 \setminus \Gamma \to \mathbb{H}^2$ is a *model map* for a given data if $\tau(\varphi_0)$ is $O(r^{-2-\epsilon})$ at infinity, $|\tau(\varphi_0)|$ is bounded and φ_0 assumes the given constants on Γ. The asymptotics of a harmonic map uniquely determine the data, and conversely given the data, there exists a unique harmonic map (Weinstein [9]). When reconstructing the spacetime metric $\mathbf{g}$, conical singularities can appear on (bounded components of) Γ. (Bach-Weyl 1921, Li-Tian 1991, Weinstein [9])

We now seek stationary solutions in the higher dimension, namely find spacetimes $(\mathcal{N}^{4+1}, \mathbf{g})$ of the Einstein vacuum equations with three mutually commuting Killing fields $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial \phi_1}$, $\frac{\partial}{\partial \phi_2}$, the generators of an isometry group $\mathbb{R} \times U(1)^2$ acting on N.

If $(\mathcal{N}^{4+1}, \mathbf{g})$ has an event horizon, its connected components are diffeomorphic to 3-manifolds Σ^3 of positive Yamabe type, i.e. each component admits a metric of positive scalar curvature (Galloway-Schoen [2]). Under the additional symmetry condition above, the list of topological types (Hollands-Yazadjiev [3]) is further restricted to S^3, $S^1 \times S^2$, $\mathbb{R}P^3$, $L(n, m)$. Examples of solutions include

- No horizon: Minkowski space: $\mathbb{R}^{4,1}$ with its metric
- S^3: Myers-Perry ([6])— 4+1 analog of Kerr solution, vacuum.
- $S^2 \times S^1$: Emperan-Reall ([1]) Pomeransky-Senkov([7])— *Black Ring*, vacuum.
- $L(2, 1)$: Kunduri-Lucietti ([4]) — *Black Lens*, minimal supergravity.
- $L(n, 1)$: Nozawa-Tomizawa ([8]) — minimal supergravity.

Following the $3 + 1$-dimensional cases, solutions $\mathbf{g}$ on $(\mathcal{N}^{4+1} \setminus \{\text{axis}\})$ can be modeled by the Weyl-Papapetrou Coordinates

$$\left(\mathbb{R} \times U(1)^2\right) \times \{(\rho, z) \colon z \in \mathbb{R}, \quad \rho > 0\}$$

with

$$\mathbf{g} = G_{ij}\, dx^i\, dx^j + e^{2\nu}(d\rho^2 + dz^2) \quad (y^1 = \phi^1, y^2 = \phi^2, y^3 = t)$$

and

$$\rho = \sqrt{|\det G_{ij}|}, \quad \nu = \nu(\rho, z), \quad G_{ij} = G_{ij}(\rho, z).$$

Note: ρ is harmonic, and thus a coordinate function, i.e. $\nabla\rho \neq 0$, and Let z be its conjugate harmonic function.

The Einstein vacuum equations now reduce to a *harmonic map* ([5])

$$\varphi\colon \mathbb{R}^3 \setminus A \to SL(3,\mathbb{R})/SO(3) := X, \text{ with}$$

$$G_{ij}dx^i dx^j = -\frac{\rho^2}{f}dt^2 + \sum_{1\le i,j,\le 2}^{2} f_{ij}(d\phi^i + \omega^i dt)(d\phi^j + \omega^j dt),$$

where $f = \det(f_{ij})$, define a positive definite symmetric matrix with det $= 1$:

$$\Phi = \frac{1}{f}\begin{pmatrix} 1 & -v_1 & -v_2 \\ -v_1 & f\,f_{11} + (v_1)^2 & f\,f_{12} + v_1 v_2 \\ -v_2 & f\,f_{21} + v_2 v_1 & f\,f_{22} + (v_2)^2 \end{pmatrix}$$

representing a point $\Psi \in SL(3,\mathbb{R})/SO(3)$ ($\Phi = \Psi^T\Psi$). Here v_i is the Ernst potential for the Killing field $\frac{\partial}{\partial \phi^i}$.

On the z-axis $\{\rho = \sqrt{|\det G|} = 0\}$, we have $\dim\ker(G(0,z)) = 1$ except at isolated values $\{p_i\}_{i=1}^N$ on the z-axis (corners).

Let $(0,z)$ be a non-corner point, and let $0 \neq V \in \ker G$. There are two cases:

- V *null* $\Rightarrow$ the rod $p_i \le z \le p_{i+1}$ corresponds to *a horizon rod.*
- V *spacelike* $\Rightarrow$ the rod $p_j \le z \le p_{j+1}$ corresponds to *an axis rod.*

In the second case, one can scale V so that $V = k\frac{\partial}{\partial t} + n\frac{\partial}{\partial \phi^1} + m\frac{\partial}{\partial \phi^2}$ and $n, m \in \mathbb{Z}$ with $gcd(n,m) = 1$.

Definition. The *rod structure* consists of the set of axis rods $\{\Gamma_i\}_{i=1}^N$ together with the integers (n_i, m_i) for each axis rod, plus the values of Ernst potentials v_1, v_2 for each axis rod Γ_i.

The rod structure encodes the topological types of the horizons, as the three manifolds on our lists have singular torus foliations whose singular fibers are S^1's, appearing at one of the axis rods.

We present our main theorem. Denote by Γ the set of axis rods.

Theorem 1 (Khuri-Weinstein-Yamada 2017)**.** *Given any rod structure* $\Gamma = \cup\Gamma_i$, *and any set of axis rod constants* $v_1 = a_i, v_2 = b_i$ *for* Γ_i, *there exists* a unique harmonic map $\varphi\colon \mathbb{R}^3 \setminus A \to \Gamma$ *with singularities on* Γ *whose asymptotic behavior near* Γ *and spacetime infinity corresponds to the correct rod structure and to asymptotic flatness respectively.*

When reconstructing the spacetime, there may be *conical singularities* along the (bounded) axes.

Open problem: Can these be removed by adjusting the parameters? (Clearly such adjustments are possible in some cases; Myers-Perry, Emperan-Reall.)

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Positive scalar curvature metrics on manifolds with controlled geometry at infinity

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(joint work with Elmar Schrohe)

We report on the recent progress concerning the index theory of certain non-compact manifolds and potential applications to the study of positive scalar curvature (PSC) metrics on such manifolds [2].

1. Lie manifolds

The Lie manifolds were introduced by Ammann, Lauter and Nistor, cf. [1]. The definition of a Lie manifold entails an axiomatization of compactifications of certain complete Riemannian manifolds (open manifolds with controlled singular geometry at infinity). Denote by M a compact manifold with corners, ∂M the boundary and $M_0 := M \setminus \partial M$ the interior of M. We denote by $\mathcal{V} \subset \Gamma(TM)$ *a Lie algebra of vector fields* and $\mathcal{V}_b$ the Lie algebra of vector fields *tangent to the boundary* ∂M. By a *Lie manifold* we mean a tuple $(M, \mathcal{V})$ such that M is a compact manifold with corners and $\mathcal{V}$ is a projective $C^\infty(M)$-module, finitely generated, closed under Lie bracket, $\mathcal{V} \subset \mathcal{V}_b$ and such that a local basis of vector fields $\{X_1, \cdots, X_n\}$ around $x \in M_0$ also gives a local basis of TM_0. Given a Lie manifold $(M, \mathcal{V})$ we obtain further structures. By the Serre-Swan theorem there is a Lie algebroid

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\pi"] \arrow[d, "\varrho"'] & M \\ TM \arrow[ur, "p"] & \end{tikzcd}$$

with anchor ϱ such that $\mathcal{A}_{|M_0} \cong TM_0$ and $\Gamma(\mathcal{A}) \cong \mathcal{V}$. By [4] given the Lie algebroid from above, there is an s-connected Lie groupoid $\mathcal{G} \rightrightarrows M$ such that $\mathcal{A}(\mathcal{G}) \cong \mathcal{A}$ and $\mathcal{G}_{|M_0} \cong M_0 \times M_0$. *Briefly:* Lie algebroids coming from Lie manifolds are *integrable*.

2. Index formula

Let $(M, \mathcal{V})$ be a Lie manifold with Lie algebroid $\mathcal{A} \to M$ and integrating Lie groupoid $\mathcal{G} \rightrightarrows M$. We fix a *compatible* metric $g = g_{\mathcal{A}}$, i.e. a positive definite symmetric bilinear form (a euclidean structure) on $\mathcal{A}$. Denote by $\mathrm{Cl}(\mathcal{A}) \to M$ the bundle of Clifford algebras $\bigcup_{x \in M} \mathrm{Cl}(\mathcal{A}_x, g(x))$ and fix a complex $\mathrm{Cl}(\mathcal{A})$-module $W \to M$. We also assume that W is $\mathbb{Z}_2$-graded, $W = W^+ \oplus W^-$ and the grading is compatible with the Clifford action, i.e.

$$c(\mathrm{Cl}(\mathcal{A})^+)W^{\pm} \subseteq W^{\pm} \quad \text{and} \quad c(\mathrm{Cl}(\mathcal{A})^-)W^{\pm} \subseteq W^{\mp}.$$

We denote by ∇^W a first order operator in the $\mathcal{V}$-differential operators which is a Clifford $\mathcal{A}$-connection, i.e.

$$\nabla_X^W(c(Y)\xi) = c(\nabla_X Y)\xi + c(Y)\nabla_X^W(\xi), \ \xi \in \Gamma(W), \ X, Y \in \Gamma(\mathcal{A}). \tag{1}$$

The geometric Dirac operator D is defined as the composition $D = c \circ (\mathrm{id} \otimes \sharp) \circ \nabla^W$, acting on $\Gamma(W)$,

$$\Gamma(W) \xrightarrow{\nabla^W} \Gamma(W \otimes \mathcal{A}^*) \xrightarrow{\mathrm{id}\otimes\sharp} \Gamma(W \otimes \mathcal{A}) \xrightarrow{c} \Gamma(W),$$

where c denotes Clifford multiplication and $\sharp$ is the conjugate-linear isomorphism $\mathcal{A} \cong \mathcal{A}^*$ induced by the metric g. Note that c is a $\mathcal{V}$-operator of order 0 and ∇^W is a $\mathcal{V}$-operator of order 1, hence D is in $\mathrm{Diff}_{\mathcal{V}}^1(M; W)$. The principal symbol of D satisfies $\sigma_1(D)\xi = ic(\xi) \in \mathrm{End}(W)$, hence it is invertible for $\xi \neq 0$, and D is elliptic.

According to the representation theory of Ammann-Lauter-Nistor, cf. [1], the geometric Dirac operator $D = D^W$ has a $\mathcal{G}$-*equivariant lifting* to a family of Dirac operators $\not{D} = (\not{D}_x)_{x \in M}$ on the source-fibers $\mathcal{G}_x = s^{-1}(x)$ of the groupoid $\mathcal{G}$. There is a $\mathcal{G}$-equivariant family of metrics $(g_x)_{x \in M}$ such that $(\mathcal{G}_x, g_x)$ are complete Riemannian manifolds of bounded geometry for $x \in M$ and $(g_x)_{x \in M}$ descends to the restriction of g to M_0, by right invariance of the action of $\mathcal{G}$. We assume that $\mathcal{G}$ is Hausdorff and fulfills the Nistor Fredholm conditions, i.e. $\not{D} \colon H^1(\mathcal{G}) \to L^2(\mathcal{G})$ is Fredholm if and only if the indicial symbol $\mathcal{R}(\not{D}) = (\not{D}_x)_{x \in \partial M}$ is pointwise invertible.

Theorem 1. *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid such that $\mathcal{A}(\mathcal{G}) \cong \mathcal{A}$ and assume that $\mathcal{G}$ fulfills the Nistor condition. Then if $D = D^W$ is a Fredholm geometric Dirac operator over $(M, \mathcal{V})$, the Fredholm index*

$$\mathrm{ind}(D) = {}^{\mathcal{V}}\!\!\!\!\!\!\;\;\;\;\fint \hat{A}(\nabla) \wedge \exp F^{W/S} + {}^{\mathcal{V}}\eta(D) \tag{2}$$

where $F^{W/S}$ is the twisting curvature and $\hat{A}(\nabla)$, for the curvature tensor R obtained from the compatible metric, denotes the form given by the formal power series

$$h(R) = \left(-\frac{i}{2\pi}\right)^{\frac{n}{2}} \det\left(\frac{\frac{1}{2}R}{\sinh(\frac{1}{2}R)}\right)^{\frac{1}{2}}.$$

The function ${}^{\mathcal{V}}\eta$ *is the renormalized* η*-invariant which is given by the integrated trace defect*

$$ {}^{\mathcal{V}}\eta(D) := \frac{1}{2}\int_0^\infty {}^{\mathcal{V}}\mathrm{Tr}_s([D, De^{-tD^2}])\,dt. $$

3. A secondary invariant

The index formula yields obstructions to compatible PSC metrics by an application of the Lichnerowicz formula for Lie manifolds:

$$ D^2 = \Delta^W + c(F^{W/S}) + \frac{\kappa}{4}, \tag{3} $$

where $\kappa = \kappa_g$ denotes the scalar curvature associated to the fixed compatible metric g. The twisting curvature $F^{S/W} \in \Lambda^2(\mathrm{End}_{\mathrm{Cl}(\mathcal{A})}W)$ is part of the decomposition of the curvature $(\nabla^W)^2 = c(R) + F^{W/S}$ obtained via the isomorphism $\mathrm{End}(W) \cong \mathrm{Cl}(\mathcal{A}) \otimes \mathrm{End}_{\mathrm{Cl}(\mathcal{A})}W$. The operator $c(F^{S/W}) \in \Psi^0_{\mathcal{V}}(M;W)$ is a 0-order operator contained in the Lie calculus of pseudodifferential operators[1]. Assuming that $\|c(F^{S/W})\| < \frac{\min_{x\in M}\kappa(x)}{4}$ in $\mathcal{L}(L^2_{\mathcal{V}}(M;W))$ operator norm, then if $\kappa > 0$ everywhere, we obtain via (3) that the index of $D = D^W$ must vanish. One obtains the usual obstruction to compatible PSC metrics on Lie manifolds.

The *deformation groupoid* describing the Fredholm index is studied e.g. in recent work of Carrillo-Rouse, Lescure and Monthubert [3]. These deformation groupoids are used by Zenobi [5] to define a secondary ρ-invariant. We consider the groupoid integrating the Lie algebroid ${}^{\mathrm{ad}}\mathcal{A} = \mathcal{A} \times [0,1]$ with anchor ${}^{\mathrm{ad}}\varrho\colon {}^{\mathrm{ad}}\mathcal{A} \to TM \times T[0,1]$, ${}^{\mathrm{ad}}\mathcal{A} \ni (x,v,t) \mapsto (x, tv, t, 0) \in TM \times T[0,1]$, the *adiabatic groupoid* $\mathcal{G}^{ad} = \mathcal{A}(\mathcal{G}) \times \{0\} \cup \mathcal{G} \times (0,1] \rightrightarrows M_{ad} := M \times [0,1]$ associated to $\mathcal{G} \rightrightarrows M$. The *Fredholm groupoid* $\mathcal{G}_{\mathcal{F}} = \mathcal{A}(\mathcal{G}) \times \{0\} \cup \mathcal{G}_{|\partial M} \times (0,1) \cup M_0 \times M_0 \times (0,1] \rightrightarrows M_{\mathcal{F}} := M_{ad} \setminus (\partial M \times \{1\})$ is obtained by removing $\partial M \times \partial M \times \{1\}$ from $\mathcal{G}^{ad}$. The *non-commutative tangent bundle* groupoid $\mathcal{T} = \mathcal{A}(\mathcal{G}) \times \{0\} \cup \mathcal{G}_{|\partial M} \times (0,1) \rightrightarrows M_{\partial} := M_{\mathcal{F}} \setminus (M_0 \times (0,1])$ is obtained by removing the saturated sub-groupoid $M_0 \times M_0 \times (0,1]$ from $\mathcal{G}_{\mathcal{F}}$. Setting $\mathring{\mathcal{G}}_{\mathcal{F}} := \mathcal{G}_{\mathcal{F}|[0,1)}$, by the Nistor property of the groupoid, we obtain a short exact sequence with completely positive section:

$$ C^*_r(M_0 \times M_0) \otimes C_0(0,1) \rightarrowtail C^*_r(\mathring{\mathcal{G}}_{\mathcal{F}}) \xrightarrow{e_0} C^*_r(\mathcal{T}) $$

By restricting (via e_0) to the sub-groupoid $\mathcal{T}$. Note that $C^*_r(M_0 \times M_0) \cong \mathcal{K}$, the compact operators on the interior. The Fredholm index is the connecting map $K_0(C^*_r(\mathcal{T})) \to K_1(\mathcal{K} \otimes C_0(0,1)) \cong K_0(\mathcal{K}) \cong \mathbb{Z}$ in the corresponding six-term exact sequence. Given a compatible PSC metric g, the index of the associated Spin-Dirac operator D vanishes. The *ρ-invariant* $\rho(D,g) \in K_0(C^*_r(\mathring{\mathcal{G}}_{\mathcal{F}}))$ is the lifting of the full symbol class of the Spin-Dirac operator D. An ongoing project concerns the study of the space of compatible PSC metrics on Lie manifolds via the secondary ρ-invariant, e.g. to relate the concordance classes of metrics to Spin-bordism classes of metrics. While the notion of bordism can be defined also

[1] In fact all geometric operators considered here are elements of this calculus, cf. [1]

in the category of Lie manifolds, it is an open problem to define an appropriate composition of concordance classes of metrics, relative to a fixed compatible PSC metric. This step would involve a suitable generalization of the surgery theorem of Gromov-Lawson and Schoen-Yau to our setting.

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Cauchy problems for Lorentzian special holonomy and generalised imaginary Killing spinors

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(joint work with Helga Baum, Andree Lischewski)

We consider the following problems: for a Riemannian manifold (M, g), find a Lorentzian manifold $(\overline{M}, \overline{g})$, containing (M, g) as a Cauchy hypersurface, and with

(A) a parallel null vector field V, or,

(B) a parallel null spinor field ϕ.

Since the Dirac current V_ϕ associated to a parallel null spinor field ϕ is a parallel null vector field, problem (A) is more general. The constraint conditions that are imposed on (M, g) by the conditions in (A) arise from decomposing V into $V = uT - U$, with a unit normal T to M and U tangent to M, as

$$u^2 = g(U, U) \neq 0, \qquad \nabla^g U + u\, W = 0, \tag{1}$$

with $W = -\nabla^{\overline{g}} T$ the Weingarten operator of $M \subset \overline{M}$. This condition implies that $U^\flat$ is a closed one-form. Constraint conditions for (B) are obtained from spinor calculus for hypersurfaces: the spinor ϕ induces a spinor φ on M with

$$U_\varphi \cdot \varphi = i\|\phi\|\varphi, \qquad \nabla^g_X \varphi = \frac{i}{2} W(X) \cdot \varphi, \qquad \text{for all } X \in TM. \tag{2}$$

Spinor fields that satisfy the differential equation in (2) are called *generalised imaginary Killing spinors*. In [8] we solve problem (A):

Theorem. *Let (M, g, U) be a smooth Riemannian manifold with a smooth vector field U satisfying the constraints (1), and define the metric $\tilde{g} = -dt^2 + g$ on $\mathbb{R} \times M$. Then there is a unique Lorentz metric $\overline{g}$ on an open neighbourhood $\overline{M}$ of M in*

$\mathbb{R} \times M$ such that $(\overline{M}, \overline{g})$ admits a unique parallel null vector field V, contains M as Cauchy hypersurface, and satisfies the initial conditions $V|_M = u\partial_t - U$,

$$\overline{g}|_M = \tilde{g}|_M, \quad \partial_t \overline{g}|_{TM \times TM} = -uW, \tag{3}$$

as well as the gauge fixing condition $\overline{g}_{\mu\nu}\overline{g}^{\alpha\beta}C^{\nu}{}_{\alpha\beta} = 0$, where $C = \nabla^{\overline{g}} - \nabla^{\tilde{g}}$.

For the proof we apply the theory of quasilinear symmetric hyperbolic PDE to

$$(d + \delta^{\overline{g}})\omega = 0, \qquad Ric(\overline{g}) = Q \circ \mathrm{pr}_{TM}^{\omega_1^\sharp}, \qquad \nabla^{\overline{g}}_{\omega_1^\sharp} Q = 0,$$

a system for a differential form $\omega = \omega_0 + \omega_1 + \omega_2 + \dots$, a bilinear form Q and a metric $\overline{g}$, and where the projection onto TM defined by the vector $\omega_1^\sharp$ is used.

By parallel transporting a spinor with conditions (2) with respect to $\overline{g}$ along the flow of V, we also solve problem B:

Corollary. *Let (M, g, φ) be a smooth Riemannian manifold with a smooth spinor field satisfying conditions (2). Then there is a unique Lorentz metric $\overline{g}$ on an open neighbourhood $\overline{M}$ of M in $\mathbb{R} \times M$ such that $(\overline{M}, \overline{g})$ has a unique parallel null spinor field ϕ, contains M as Cauchy hypersurface, and satisfies the initial conditions (3) and $\phi|_M = \varphi$, as well as the above gauge fixing condition.*

This result was obtained in [9] directly by reducing the system

$$D^{\overline{g}}\phi = 0, \qquad Ric(\overline{g}) = f\,(V_\phi^\flat)^2, \qquad df(V_\phi) = 0,$$

for spinor ϕ, a metric $\overline{g}$ and a function f, and in which $D^{\overline{g}}$ is the Dirac operator of $\overline{g}$, to a quasilinear symmetric hyperbolic PDE system.

In the analytic category these results were already obtained in [2] in a slightly stronger version: in addition to an analytic Riemannian manifold obeying the constraints we can prescribe an analytic function λ on $\mathbb{R} \times M$ and use the Cauchy-Kowalevski Theorem to obtain a unique family g_t of Riemannian metrics such that $\overline{g} = -\lambda^2 dt^2 + g_t$. A similar result about extending generalised *real* Killing spinors to parallel spinors on a Riemannian manifold was obtained in [1].

These results are motivated by the construction of globally hyperbolic Lorentzian manifolds with special holonomy. A semi-Riemannian manifold has *special holonomy* if the holonomy algebra is reduced from the full orthogonal algebra $\mathfrak{so}(p,q)$ but does not admit a non-degenerate invariant subspace. For Riemannian manifolds special holonomy algebras are irreducible in $\mathfrak{so}(m)$ and they were classified by Berger [4]. The construction of Riemannian manifolds with special holonomy lead to several landmark results in differential geometry. In contrast to the Riemannian case, there are no irreducible subalgebras in $\mathfrak{so}(1, m-1)$. Hence, special Lorentzian holonomy algebras are indecomposable subalgebras in the stabiliser of a null line, i.e., in $\mathfrak{p} = (\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathbb{R}^n$. The classification of special Lorentzian holonomy algebras follows from results in [3] and in [7], where it was shown that the projection of the holonomy algebra to $\mathfrak{so}(n)$ in $\mathfrak{p}$ is a Riemannian holonomy algebra. By the general holonomy principle, Lorentzian manifolds with special holonomy admit a line bundle in the tangent bundle that is invariant under parallel transport. Local metrics realising all possible holonomy algebras were constructed in [5]. A special case of this is when the Lorentzian manifold admits a parallel null vector field as in problem A. The theorem together with the following result from [8] allows to construct globally hyperbolic manifolds with special holonomy that have have a complete Cauchy hypersurface.

Proposition. *A Riemannian manifold (M,g) admits a vector field U with $u = g(U,U) \neq 0$ and $dU^\flat = 0$ if and only if it is locally isometric to*

$$\big((a,b)\times N, \hat{g} = \tfrac{1}{u^2}ds^2 + h_s\big), \quad \text{with a family } h_s \text{ of Riemannian metrics on } N.$$

If the vector field $\frac{1}{u^2}U$ is complete, then the universal cover of (M,g) is globally isometric to $\hat{g}$ on $\mathbb{R}\times N$ with a simply connected manifold N.

If N is compact and u is bounded, then for any family of Riemannian metrics h_s, the Riemannian manifold $(\mathbb{R}\times N,\hat{g})$ is complete and $U = u^2\frac{\partial}{\partial s}$ satisfies $dU^\flat = 0$.

In [6] it was shown that the projection $\mathfrak{g}$ to $\mathfrak{p}$ of an indecomposable Lorentzian holonomy algebra is also the holonomy of the connection $\nabla^{\mathbb{S}}$ that is induced on the *screen bundle* $\mathbb{S} = V^\perp/\mathbb{R}V$ from $\nabla^{\overline{g}}$. For Lorentzian manifolds $(\overline{M},\overline{g})$ arising from the above Cauchy problem, $\mathbb{S}$ can be identified with the subbundle $V^\perp \cap T^\perp$ in $T\overline{M}$. In [8] we establish the following one-to-one correspondence between

(i) sections $\hat{\omega}$ of the bundle $\otimes^{a,b}\mathbb{S} \to \overline{M}$ with $\nabla^{\mathbb{S}}\omega = 0$,
(ii) sections ω of the bundle $\otimes^{a,b}U^\perp \to M$ with $\nabla^\perp\omega = 0$, and
(iii) families $(\omega_s)_{s\in I}$ of sections of $\otimes^{a,b}TN \to N$ with $\nabla^{g_s}\,\omega_s = 0$ and

$$\partial_s\omega_s = \tfrac{1}{2}(\partial_s g)\bullet\omega_s, \tag{4}$$

in which $\otimes^{a,b}E$ denotes the a-fold tensor product of a vector bundle E with the b-fold tensor product of E^*, $\nabla^\perp$ is the connection induced from ∇ on $U^\perp$, and $\bullet$ denotes the natural action of endomorphisms on tensors. Hence, holonomy reductions from $\mathfrak{p}$ to $\mathfrak{g}\ltimes\mathbb{R}^n$, where $\mathfrak{g}$ is the stabiliser of some tensor or spinor, are reflected in the constrained geometry of (M,g) as follows:

$\mathfrak{g}$		h_s is family of
$\mathfrak{so}(p)\oplus\mathfrak{so}(q)$	$\iff$	product metrics
$\mathfrak{u}(n/2)$	$\iff$	Kähler metrics with (4),
$\mathfrak{su}(n/2)$	$\iff$	Ricci-flat Kähler metrics with (4) and $\mathrm{div}^{h_s}(\dot{h}_s) = 0$
$\mathfrak{sp}(n/4)$	$\iff$	hyper-Kähler metrics with (4)
$\mathfrak{g}_2$ / $\mathfrak{spin}(7)$	$\iff$	of G_2 / $Spin(7)$-metrics with (4).

As a consequence we obtain a local classification of Riemannian manifolds that admit a generalised imaginary Killing spinor. We leave as an *open problem* the status of the flow equation (4) for families of Kähler-forms or, in the case of G_2 or $Spin(7)$ geometries, for families of parallel stable 3-forms or generic 4-forms.

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Sphere Theorems, Space Forms, and positive scalar curvature

Wilderich Tuschmann

(joint work with Michael Wiemeler)

The well-known constructions of torpedo (or also: Pinocchio) metrics show that there cannot be upper diameter bounds in terms of just a lower positive bound on the scalar curvature function of a Riemannian manifold. However, already back in 1963, Leon Green (compare [2]) proved in particular the following remarkable rigidity result:

A closed Riemannian n-manifold with scalar curvature $\mathrm{scal}_M \geq n(n-1)$ *and injectivity radius equal to* π *is isometric to the* n*-dimensional unit sphere* $S^n(1)$.

Thus, when replacing diameter by injectivity radius, this result can be viewed as a scalar curvature analogue to the well-known maximal diameter sphere theorems obtained by Toponogov and Cheng for manifolds of positive sectional curvature ≥ 1 and, more generally, positive Ricci curvature $\geq n-1$.

It is therefore natural to ask which conditions on the injectivity radius, or, more generally, conjugate radius, of a closed Riemannian n-manifold M with positive *scalar* curvature will guarantee stability of Green's above-mentioned results in the sense that M can still be recognized as being homeomorphic, or even diffeomorphic, to the standard n-sphere or, respectively, to an n-dimensional spherical space form.

Of course, stability results which actually imply diffeomorphism are of much more significance in this context than merely topological ones, because exotic spheres with positive scalar curvature are known to abound.

Our main result and its corollaries provide positive answers to whether Green's rigidity results are differentiably stable as follows:

Theorem 1. *For all $n \in \mathbb{N}$, $C, \lambda_0, \lambda_1, i_0 > 0$ and $0 \leq \beta < 1$ there exists $\epsilon = \epsilon(n, C, \lambda_0, \lambda_1, i_0, \beta) > 0$ such that every closed n-dimensional Riemannian manifold M with*

$$\int_M \operatorname{scal}_M \operatorname{dvol}_M \geq n(n-1) \operatorname{Vol} M \qquad \operatorname{conj} M \geq \pi - \epsilon$$

$$\operatorname{Ric}_M \geq -\lambda_0 \qquad \|\nabla \operatorname{Ric}_M\| \leq \lambda_1$$

$$\operatorname{Vol} M \leq \frac{C}{(\pi - \operatorname{conj} M)^\beta} \qquad \operatorname{inj} M \geq i_0$$

is diffeomorphic to a spherical space form.

This result yields various new recognition and stability theorems for closed manifolds with positive curvatures and, in particular, new sphere theorems.

First, we have the following new sphere theorem for manifolds with positive mean scalar curvature:

Corollary 2. *For all $n \in \mathbb{N}$ and $\lambda_1, d > 0$ there exists $\epsilon = \epsilon(n, \lambda_1, d) > 0$ such that every closed n-dimensional Riemannian manifold M with*

$$\int_M \operatorname{scal}_M \operatorname{dvol}_M \geq n(n-1) \operatorname{Vol} M \qquad \operatorname{inj} M \geq \pi - \epsilon$$

$$\operatorname{diam} M \leq d \qquad \|\nabla \operatorname{Ric}_M\| \leq \lambda_1$$

is diffeomorphic to the standard n-sphere.

We do believe that the bound on the first covariant derivative of the Ricci tensor is not necessary for the corollary to remain true. Notice that also the following question is so far completely open:

Does there exist $\epsilon = \epsilon(n) > 0$ such that every closed n-dimensional Riemannian manifold M with $\operatorname{scal}_M \geq n(n-1)$ and injectivity radius $\geq \pi - \epsilon(n)$ is diffeomorphic to the standard n-sphere?

Second, the main result above has also applications to Einstein manifolds. Since Berger's initial studies in the 1960s much work in Riemannian geometry has also been devoted to the problem to classify all Einstein manifolds which satisfy further curvature conditions or other suitable metric or topological constraints. For Einstein manifolds with positive Einstein constant and large conjugate radius, we obtain in particular the following differentiable stability result, which does not require any further curvature bounds:

Corollary 3. *For all $n \in \mathbb{N}$ there exists $\epsilon = \epsilon(n) > 0$ such that every closed simply connected n-dimensional Einstein manifold M with Einstein constant $n-1$ and conjugate radius $\operatorname{conj} M \geq \pi - \epsilon$ is diffeomorphic to the standard n-sphere.*

Notice in this context that in [1] families of inequivalent Einstein metrics with positive Einstein constant were constructed on the $(4m+1)$-dimensional Kervaire spheres, where $m \geq 2$. The Kervaire spheres are known to be exotic in dimensions $n \neq 2^k - 3$, where $k \geq 3$.

Our proofs of the above results rely heavily on $C^{k,\alpha}$ convergence techniques and precompactness results by Anderson and Anderson-Cheeger, as well as on the following important

Lemma 4. *Let $\lambda_1, i_0 > 0$ and $n \in \mathbb{N}$. Then there is a constant $\lambda_0(\lambda_1, i_0, n)$ such that for every n-dimensional complete Riemannian manifold M with $\|\nabla \operatorname{Ric}_M\| \leq \lambda_1$ and $\operatorname{conj} M \geq i_0$ one has $\operatorname{Ric}_M \leq \lambda_0$.*

For the details of the proofs as well as more results and open questions we refer the reader to our work [3]. We thank the organizers for an inspiring conference and the MFO and its staff for their hospitality and friendliness.

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Ricci positive metrics on spheres

David J. Wraith
(joint work with Boris Botvinnik, Mark Walsh)

Given a manifold M, let $\mathcal{R}(M)$ denote the space of all Riemannian metrics on M with the smooth topology, let $\mathcal{R}^{Ric>0}(M)$ be the space of Ricci positive metrics on M, and so on. As the diffeomorphism group of M acts by pull-back on these spaces, we can form the corresponding moduli spaces $\mathcal{M}(M) = \mathcal{R}(M)/\mathrm{Diff}(M)$, $\mathcal{M}^{Ric>0}(M) = \mathcal{R}^{Ric>0}(M)/\mathrm{Diff}(M)$ etc.

The focus of this note will be the so-called *observer moduli space.* This is $\mathcal{M}_{x_0}(M) := \mathcal{R}(M)/\mathrm{Diff}_{x_0}(M)$ where $\mathrm{Diff}_{x_0}(M)$ is the *observer diffeomorphism group*: the group of diffeomorphisms of M which fix both a basepoint $x_0 \in M$ and $T_{x_0}M$. It turns out that the observer moduli space offers certain technical advantages over the full moduli space, and as a consequence it has been the focus of much recent attention.

Our basic motivating question is to understand what we can say about the topology of $\mathcal{R}^{Ric>0}(M)$ and its moduli spaces. There are currently very few results in this direction, but we wish to draw attention to the following:

- (Kreck-Stolz [5]) There exist manifolds M^7 with $Ric > 0$ (specifically certain S^1-bundles over $\mathbb{C}P^2 \times \mathbb{C}P^1$) such that $|\pi_0 \mathcal{M}^{Ric>0}(M)| = \infty$.
- (Wraith [8]) For all $n \geq 2$ there are homotopy spheres Σ^{4n-1} such that $|\pi_0 \mathcal{M}^{Ric>0}(\Sigma)| = \infty$. (This family includes all S^{4n-1} and infinitely many exotic spheres. The result can be extended to large families of highly-connected manifolds in these dimensions.)
- (Crowley-Schick-Steimle [2]) If M^m is spin with $m \geq 6$ and $n \equiv 0, 1 \mod 8$, $n \geq m$, then $\pi_{n-m} \mathcal{R}^{Ric>0}(M)$ contains order 2 elements.

Our main result is the following:

Theorem 1. *Given $k \in \mathbb{N}$, there exists $N = N(k) \in \mathbb{N}$ such that for all $n > N$, n odd, $\pi_{4k} \mathcal{M}^{Ric>0}_{x_0}(S^n)$ contains infinite order elements.*

It should be noted that the analogous result for positive scalar curvature is established in [1]. The topological foundations that we use are borrowed from this paper, however the geometric arguments we propose are very different.

The key advantage of using the observer moduli space is that the observer diffeomorphism group acts freely on $\mathcal{R}(M)$. This leads to a fibration

$$\mathrm{Diff}_{x_0}(M) \to \mathcal{R}(M) \to \mathcal{M}_{x_0}(M).$$

As $\mathcal{R}(M) \simeq *$, the above is a universal $\mathrm{Diff}_{x_0}(M)$-bundle, and we observe that

$$\mathcal{R}(M) = \mathrm{EDiff}_{x_0}(M), \quad \mathcal{M}_{x_0}(M) = \mathrm{BDiff}_{x_0}(M).$$

We obtain an associated bundle

$$M \to \mathcal{R}(M) \times_{\mathrm{Diff}_{x_0}(M)} M \to \mathcal{M}_{x_0}(M) = \mathrm{BDiff}_{x_0}(M), \tag{$\dagger$}$$

which is universal among bundles with fibre M and structural group $\mathrm{Diff}_{x_0}(M)$.

Given a continuous map $f : S^i \to \mathrm{BDiff}_{x_0}(M)$, pulling back ($\dagger$) gives an M-bundle over S^i with structural group $\mathrm{Diff}_{x_0}(M)$. It turns out that there exists a natural fibrewise metric (determined by f) on this pull-back bundle. For more details see page 61 of [7], however the basic observation underlying this fact is that we can view $\mathcal{R}(M)$ in two ways: as fibered by copies of $\mathrm{Diff}_{x_0}(M)$, or as fibered by families of metrics (which differ by elements of Diff_{x_0}).

Note that 'fibrewise' means on the vertical tangent bundle, that is, on the fibres only. Asking for a fibrewise Ricci positive metric on a bundle is a much weaker requirement than asking for a Ricci positive metric on the total space. Given a metric on the total space of a bundle such that the intrinsic curvature of each of the fibres is Ricci positive, there is no need for the Ricci curvature of the bundle as a whole to be positive. As an illustration of this, note that we can scale fibres using a arbitrary smooth positive function on the base and still retain fibrewise Ricci positivity, however it is clear that such a rescaling can affect the global Ricci curvature in complicated ways.

The main theorem follows from:

Theorem 2. *Given $k \in \mathbb{N}$, there exists $N = N(k) \in \mathbb{N}$ such that for all $n > N$, n odd, the map*

$$\iota_* \otimes \mathbb{Q} : \pi_{4k}(\mathcal{M}^{Ric>0}_{x_0}(S^n)) \otimes \mathbb{Q} \to \pi_{4k}(\mathcal{M}_{x_0}(S^n)) \otimes \mathbb{Q}$$

is a surjection, where ι is inclusion.

This result produces Theorem 1 when combined with the following:

Theorem 3 (Farrell-Hsiang, [3]). *Given $\ell \in \mathbb{N}$, there exists $N = N(\ell)$ such that for all $n > N$ we have that $\pi_\ell(BDiff_{x_0}(S^n)) \otimes \mathbb{Q}$ is isomorphic to $\mathbb{Q}$ if n is odd and ℓ is a multiple of 4, and zero otherwise.*

Our proof strategy is as follows: an element on the right-hand side of $\iota_* \otimes \mathbb{Q}$ can be represented by an S^n-bundle over S^{4k} equipped with a fibrewise Riemannian metric, and it suffices to show that this bundle admits a fibrewise Ricci positive metric. It turns out that there is a generating set for $\pi_{4k}(\mathcal{R}(S^n)/\mathrm{Diff}_{x_0}(S^n)) \otimes \mathbb{Q}$ for which the elements can be represented by 'Hatcher bundles', which are S^n-bundles over S^{4k} which are topologically, but not smoothly trivial.

Hatcher bundles decompose into a union of two identical disc bundles, which we will call 'Hatcher disc bundles'. We therefore need to

(1) understand how to equip Hatcher disc bundles with fibrewise Ricci positive metrics, and
(2) understand how to glue two such disc bundles within fibrewise Ricci positivity.

In brief, Hatcher disc bundles can be constructed by splitting the base S^{4k} into hemispheres $D^{4k}_\pm$, and taking disc bundles over these together with a suitable gluing map along the boundaries. The disc bundle over D^{4k}_- is taken to be a product, whereas the bundle over D^{4k}_+ is non-trivial when viewed as a bundle with given boundary trivialisation. This latter bundle is constructed as the union of a disc bundle and an annulus bundle, glued together in a highly non-trivial manner. We refer the reader to [4] for more details.

We next note that the fibres of the disc bundle over D^{4k}_+ can be equipped with Ricci positive metrics in a natural way, exhibiting certain rotational symmetries. The same is trivially true for the bundle over D^{4k}_-. It turns out that the gluing map which produces the Hatcher disc bundle over S^{4k} is a fibrewise isometry with respect to these fibrewise metrics, as a consequence of the rotational symmetry. This produces a fibrewise Ricci positive metric on the Hatcher disc bundle after gluing, as required.

In order to glue the two Hatcher disc bundles together within fibrewise Ricci positivity we note the following result:

Theorem 4 (Perelman, [6]). *Given manifolds M_1, M_2 equipped with Ricci positive metrics, suppose the boundaries are non-empty and isometric. Then if the normal curvatures at the boundary of M_1 are greater than the negatives of the corresponding normal curvatures at the boundary of M_2, with all normal curvatures computed with respect to the outward normal, then the metric on $M_1 \cup M_2$ can be smoothed in a neighbourhood of the join to give a global Ricci positive metric.*

As we need to simultaneously glue all corresponding pairs of Hatcher disc bundle fibres, we prove a family version of the Perelman result, which is a consequence of an explicit proof argument we provide for the Perelman construction.

Finally, it remains to argue that the Perelman normal curvature condition is satisfied in our case for all fibres, but this condition can be controlled by a careful choice of the disc fibres.

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Boundary value problems for the Dirac operator on Riemannian and Lorentzian manifolds

Christian Bär

(joint work with W. Ballmann, S. Hannes, A. Strohmaier)

We discuss boundary value problems for the Dirac operator on compact manifolds with boundary, both in Riemannian and in Lorentzian signature. These two cases are analytically very different in nature.

1. Riemannian manifolds and elliptic operators

Let M be a compact Riemannian manifold with boundary ∂M. Suppose M carries a spin structure so that we can form the spinor bundle $SM \to M$. We assume that the dimension n of M is even so that the spinor bundle splits into chirality subbundles, $SM = S_R M \oplus S_L M$. Finally, we give ourselves a Hermitian vector bundle $E \to M$ with compatible connection and form the twisted spinor bundles $V_{R/L} = S_{R/L}M \otimes E$. Then the twisted Dirac operator $D : C^\infty(M, V_R) \to C^\infty(M, V_L)$ is defined. It is an elliptic first-order linear differential operator.

Let A_0 be the Dirac operator on the boundary ∂M and $P_+ = \chi_{[0,\infty)}(A_0)$ the spectral projector onto the sum of eigenspaces to nonnegative eigenvalues. Then the Atiyah-Patodi-Singer (APS) boundary conditions for M are given by

$$P_+(f|_{\partial M}) = 0.$$

Theorem (Atiyah-Patodi-Singer [1]). *Under APS boundary conditions the Dirac operator is Fredholm and its index is given by*

$$\mathsf{ind}(D_{\mathrm{APS}}) = \int_M \hat{\mathsf{A}}(M) \wedge \mathsf{ch}(E) + \int_{\partial M} T(\hat{\mathsf{A}}(M) \wedge \mathsf{ch}(E)) - \frac{h(A_0) + \eta(A_0)}{2}.$$

Here h denotes the dimension of the kernel and η the η-invariant.

Which other boundary conditions besides APS will work?

The first thing one might be tempted to try does not work. Namely, if we replace P_+ by the complementary projector $P_- = \mathsf{id} - P_+$ then the anti-Atiyah-Patodi-Singer (aAPS) boundary conditions

$$P_-(f|_{\partial M}) = 0$$

do not give a Fredholm operator.

In [2,3] we describe a class of "elliptic" boundary conditions which do give a Fredholm operator and also good analytic properties such as elliptic boundary regularity up to the boundary. These boundary conditions are plenty enough to contain generalized APS conditions, classical local elliptic conditions in the sense of Lopatinsky-Shapiro and also transmission conditions which allow for cut-and-paste arguments for the index. In particular, one obtains a very simple and natural proof the relative index theorem by Gromov and Lawson [7].

2. Lorentzian manifolds and hyperbolic operators

Now we let M be a globally hyperbolic Lorentzian manifold with boundary which consists of two smooth spacelike Cauchy hypersurfaces. The Dirac operator on M is still defined but is now hyperbolic rather than elliptic. Since the boundary is Riemannian APS conditions still make sense.

Theorem (Bär-Strohmaier [5]). *Under APS boundary conditions the Dirac operator is Fredholm and the kernel consists of smooth spinors. Its index is given by formally the same formula as in the Riemannian case.*

The regularity of the spinors in the kernel is surprising since solutions to the Dirac equation on a Lorentzian manifold can be very irregular in general. Unlike elliptic regularity which is a local phenomenon, the regularity is in this case of global nature and uses the boundary conditions in a crucial manner.

In [3] the Lorentzian index theorem is used to compute the *chiral anomaly* in quantum field theory on curved spacetimes.

In contrast to the Riemannian case, aAPS boundary conditions do yield a Fredholm operator. On the other hand, the elliptic boundary conditions in [2,3] give a Fredholm operator only under additional technical assumptions [4]. Examples show that these assumptions cannot be dropped.

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The Yamabe problem on stratified spaces

Ilaria Mondello

The aim of this talk is to present some recent results about a familiar problem in geometric analysis, the Yamabe problem, in an unfamiliar singular setting, given by stratified spaces. There are many different reasons to study the geometry of singular metric spaces: singularities naturally arise from quotients of smooth manifolds (for example orbifold singularities); also, if we consider a sequence of smooth Riemannian manifolds, its Gromov-Hausdorff limit, when it exists, is not necessarily smooth. Moreover, when studying "canonical metrics", which for example minimize a functional, it can be wiser to keep in account singular metrics as well. This is an approach that has been proven to be successful, for example in proving the existence of a Kähler-Einstein metric on a Fano manifold.

One of the interests of stratified spaces is that they can appear as limits or quotients: the simplest example of a stratified space is the American football, a surface with two isolated conical singularities obtained as the quotient of the sphere $\mathbb{S}^2$ by the group generated by rotation around an axis. Moreover, stratified spaces generalize the notion of isolated conical singularities in the following sense: a stratified space can be decomposed in a regular dense set, which is an open smooth manifold of dimension n, and in a singular set, that is the union of a finite number of *singular strata* Σ^j. Each Σ^j is a manifold of dimension j strictly smaller than $(n-1)$; for every point in Σ^j there exists a neighbourhood homeomorphic to the product of a ball in $\mathbb{R}^j$ and a cone $C(Z_j)$, where Z_j is the *link* of the stratum, it can be a stratified space as well, and determines how complicated the local geometry is next to a singularity. When a stratum has codimension 2, the link is a circle and the neighbourhood of a singular point is homeomorphic to $\mathbb{B}^{n-2}(\varepsilon) \times C(\mathbb{S}^1)$ (see in the figure, a manifold of dimension 3 with a conical singularity along a curve). We can then associate an angle to the stratum of codimension 2, which is the angle of the two-dimensional cone $C(\mathbb{S}^1)$. Note that the angle can be smaller or larger than 2π, this corresponding respectively to positive or negative Alexandrov curvature.

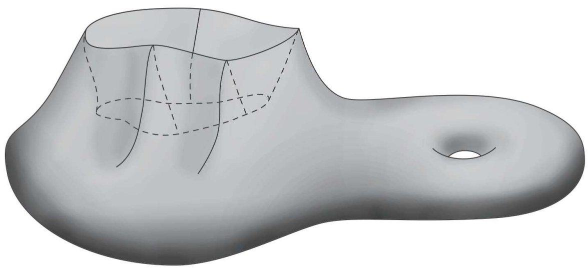

FIGURE 1. An example of a 3-dimensional stratified space.

Stratified spaces have been studied in topology (starting form H. Whitney, R. Thom...); there exists a wide literature about spectral analysis on manifolds with isolated conical singularities or simple edges, mainly using the tools of microlocal analysis (J. Cheeger, R. Melrose, R. Mazzeo...). Our goal is to study stratified spaces from the point of view of classical Riemannian geometry. In particular we are interested in a classical question: which is the "best" metric on a stratified space? There are many ways to choose what "best" means, and one possible good feature for a metric is to have constant curvature (in some sense). A first step towards this direction is to look for a metric of constant scalar curvature in the conformal class $[g]$ of a given metric g. This is known to be the Yamabe problem, and it has been solved on compact smooth manifolds (M^n, g), $n \geq 3$, thanks to the work of H. Yamabe, N. Trudinger, T. Aubin and R. Schoen: there always exists a metric conformal to g of constant scalar curvature. The proof is based on a variational approach and strongly depends on the relationship between a conformal invariant of the manifold, the Yamabe constant $Y(M^n, [g])$, and a reference constant, the Yamabe invariant of the round sphere, $Y(S^n, [g_0])$.

The situation is similar on a compact stratified space. Under an appropriate assumption on the integrability of the scalar curvature, one can define the Yamabe constant of (X, g):

$$Y(X, [g]) = \inf_{\tilde{g} \in [g]} \frac{\displaystyle\int_X \mathrm{Scal}_{\tilde{g}} dv_{\tilde{g}}}{\mathrm{vol}_{\tilde{g}}(X)^{\frac{n}{2}}}$$

A metric attaining the Yamabe constant has constant scalar curvature and is called a Yamabe metric. On a compact smooth manifold, the Yamabe invariant of the sphere can be roughly seen as the Yamabe constant of a very small ball on the manifold, when considering conformal factors supported on the ball; in the case of a stratified space, small balls can be singular and, as a consequence, we need to introduce a different reference constant to keep in account the local geometry around singular points. This is the *local Yamabe constant*:

$$Y_\ell(X) = \inf_{p \in X} \lim_{r \to 0} Y(B(p, r)) = \inf_{p \in \Sigma^j} Y(\mathbb{R}^j \times C(Z_j)).$$

An existence result by K. Akutagawa, G. Carron and R. Mazzeo [2] states that if the Yamabe constant $Y(X,[g])$ is strictly less then the local Yamabe constant of the stratified space, then there exists a Yamabe metric. Still, there are some significant differences with the smooth case. First, in [4] J. Viaclovsky gave an example of a four-dimensional orbifold with one isolated singularity, for which the Yamabe constant coincides with the local one and a Yamabe metric does not exist. Besides, the only case in which the explicit value of the local Yamabe constant was known consisted of orbifolds with isolated conical singularities, thanks to a result in [1].

Observe that the local Yamabe invariant depends on the local geometry of the singularities, and in particular on the links. In [3], we showed that it is possible to compute the local Yamabe constant whenever the links are endowed with an Einstein metric:

Theorem. *Let (X^n, g) be a stratified space with a singular stratum Σ^j and associated link Z. Let k denote the metric of Z. If $Ric_k = (\dim(Z)-1)k$ on the regular set of Z, then the local Yamabe constant is given by:*

$$Y_\ell(X) = \left(\frac{\mathrm{vol}_k(Z)}{\mathrm{vol}(\mathbb{S}^{n-j-1})}\right)^{\frac{2}{n}} Y(\mathbb{S}^n,[g_0]).$$

In particular, this includes strata of codimension 2 and orbifolds with non-isolated conical singularities. In these cases the previous result reads as follows:

- When the cone angle α along the stratum Σ^{n-2} is less than 2π, we have
$$Y_\ell(X) = \left(\frac{\alpha}{2\pi}\right)^{\frac{2}{n}} Y(\mathbb{S}^n,[g_0]).$$
- When the cone angle along the stratum Σ^{n-2} is larger than 2π, then
$$Y_\ell(X) = Y(\mathbb{S}^n,[g_0]).$$
- In presence of an orbifold singularity with finite group $\Gamma \subset O(n)$, we get
$$Y_\ell(X) = \frac{Y(\mathbb{S}^n,[g_0])}{|\Gamma|^{\frac{2}{n}}}.$$

In order to prove the above theorem, we distinguish two situations: when the stratum of codimension 2 has angle larger or smaller than 2π. In the second situation, if the Ricci curvature is positive on the regular set, we proved a lower bound for the spectrum of the Laplacian and therefore a Sobolev inequality with explicit constants, only depending on the volume and the dimension of the space. This leads to a lower bound for the global Yamabe constant, which is attained in the Einstein case. In particular, when angles are less than 2π, an Einstein metric is a Yamabe metric, which is not the case for angles larger than 2π. In order to get the result about the *local* Yamabe constant, it is not difficult to show that, when the link Z is endowed with an Einstein metric, the product $\mathbb{R}^j \times C(Z)$ is conformally equivalent to a compact Einstein stratified space, for which we can compute the Yamabe constant. When the angle along the stratum of codimension

2 is larger than 2π, we compute the Yamabe constant of $\mathbb{R}^{n-2} \times C(\mathbb{S}^1)$, by using isoperimetric profiles and proving that it is equal to the one of the round sphere.

Obtaining the value of the local Yamabe constant allows one to further study the question of existence, and non-existence, of a Yamabe metric on stratified spaces. In particular, in the case of a codimension 2 singularity with angle larger than 2π, we can apply a local argument by Aubin, and show that if the dimension is larger than or equal to 6, and the metric is not locally conformally flat, then the Yamabe constant is strictly less than the local one. Therefore, a Yamabe metric exists. Nevertheless, in a joint work with K. Akutagawa, we show that on the sphere with codimension 2 singularity of angle larger than 4π a Yamabe metric does not exists; we conjecture the same result for angle larger than 2π.

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Highly connected 7-manifolds and non-negative curvature

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(joint work with Martin Kerin, Krishnan Shankar)

We construct a six-parameter family of highly connected 7-manifolds with cyclic π_3 that admit metrics of non-negative sectional curvature. Among these manifolds are all exotic 7-spheres and also some manifolds that are not homotopy equivalent to S^3-bundles over S^4.

We start with cohomogeneity-1-manifolds $P^{10}_{\underline{a},\underline{b}}$ with group diagram

$$\begin{array}{ccccc} & & G = S^3 \times S^3 \times S^3 & & \\ & \nearrow & & \nwarrow & \\ K_- = \mathrm{Pin}\,(2)_{\underline{a}} & & & & \mathrm{Pjn}\,(2)_{\underline{b}} = K_+ \\ & \nwarrow & & \nearrow & \\ & & Q & & \end{array}$$

The positions of the subgroups $\mathrm{Pin}(2)_{\underline{a}} \cong \mathrm{Pin}(2) \cong \mathrm{Pjn}(2)_{\underline{b}}$ in G are determined by the numbers a_1, a_2, a_3 and b_1, b_2, $b_3 \in 4\mathbb{Z}+1$, respectively. We demand that $\gcd(a_1, a_2, a_3) = 1 = \gcd(b_1, b_2, b_3)$. Finally, $Q = K_- \cap K_+ \cong \{\pm 1, \pm i, \pm j, \pm k\} \subset S^3 \subset \mathbb{H}$. By a classical result by Grove and Ziller [3], the manifolds $P^{10}_{\underline{a},\underline{b}}$ admit metrics with sectional curvature $\sec \geq 0$.

If in addition $\gcd(a_1, a_2 \pm a_3) = 1 = \gcd(b_1, b_2 \pm b_3)$, then the subgroup $1 \times \Delta S^3 \subset G$ acts freely on $P^{10}_{\underline{a},\underline{b}}$. By the O'Neill-Gray formula, the quotient $M^7_{\underline{a},\underline{b}}$

also admits a metric with sec ≥ 0. It can be understood as cylinder with cross-section a biquotient $S^3 \times S^3 /\!/ Q$, whose ends are identified with biquotients of the form $S^3 \times S^3 /\!/ \mathrm{Pin}(2)$.

The manifolds $M^7_{\underline{a},\underline{b}}$ are 2-connected with $\pi_3(M^7_{\underline{a},\underline{b}}) \cong \mathbb{Z}/n \cong H^4(M)$, where

$$n = \frac{1}{8}\left(a_1^2\,(b_3^2 - b_2^2) - b_1^2\,(a_3^2 - a_2^2)\right) \in \mathbb{Z}\,.$$

If $n = 1$, the manifold is a homotopy sphere, and it suffices to compute its Eells-Kuiper invariant $\mu(M)$ to determine its diffeomorphism type. If $n > 1$ and the linking form on $H^4(M)$ is non-standard, then M is not homotopy equivalent to an S^3-bundle over S^4.

Classically, the Eells-Kuiper invariant is defined using a 0-bordism of M. Here, we use Donnelly's formula instead, who expresses $\mu(M)$ in terms of η-invariants and a curvature correction term intrinsically on M. We remark that the quotient $B^4_{a_1,b_1}$ of $P^{10}_{\underline{a},\underline{b}}$ by the subgroup $1 \times S^3 \times S^3 \subset G$ is an orbifold. The singular locus consists of two copies of $\mathbb{R}P^2$. The natural projection $M^7_{\underline{a},\underline{b}} \to B^4_{a_1,b_1}$ is a generalised Seifert fibration with generic fibre S^3. The fibres over the singular $\mathbb{R}P^2$s are lens spaces. Using the adiabatic limit formula for η-invariants of Seifert fibrations [1], we get

$$\begin{aligned}
\mu\big(M^7_{\underline{a},\underline{b}}\big) &= \frac{|n| - a_1^2 b_1^2 m^2}{2^5 \cdot 7 \cdot n} + D(a_1; 4, a_3 + a_2, a_3 - a_2) \\
&\qquad\qquad - D(b_1; 4, b_3 + b_2, b_3 - b_2) \in \mathbb{Q}/\mathbb{Z}\,,\\
\text{where} \qquad m &= \frac{1}{8a_1^2 b_1^2}\left(a_1^2(b_2^2 + b_3^2 + 8) - b_1^2(a_2^2 + a_3^2 + 8)\right)\\
\text{and} \quad D(q; p_1, p_2, p_3) &= \sum_{\ell=1}^{\frac{|q|-1}{2}} \sum_{\substack{(i,j,k)=\\ \circlearrowleft(1,2,3)}} \frac{p_i}{2^5 \cdot 7 \cdot q^2} \cdot \frac{14 \cos\frac{p_i \pi \ell}{q} + \cos\frac{p_j \pi \ell}{q} \cos\frac{p_k \pi \ell}{q}}{\sin^2\frac{p_i \pi \ell}{q} \sin\frac{p_j \pi \ell}{q} \sin\frac{p_k \pi \ell}{q}}\,.
\end{aligned}$$

The Milnor spheres carry metrics of sec ≥ 0, again by [3]. We obtain sec ≥ 0-metrics on the remaining exotic 7-spheres by considering $M_{(-3,-3,1),(1,4r+1,4r+1)}$ for $r = -3$, -1, 1, 2, 4, 8, 11, and 15, see [2, Cor D].

Finally, let p be a prime such that $p \equiv 5 \bmod 8$. Then we can show that the linking form $\mathrm{lk}\colon H^4(M) \times H^4(M) \to \mathbb{Z}$ of $M = M_{(p,1,-3),(p,-3,5)}$ is non-standard. In other words, there is no generator of the cyclic group $H^4(M) \cong \mathbb{Z}/n\mathbb{Z}$ such that $\mathrm{lk}(k,\ell) = \pm k\ell/n \in \mathbb{Q}/\mathbb{Z}$ for k, $\ell \in \mathbb{Z}/n\mathbb{Z}$. Because all S^3-bundles over S^4 have standard linking form, this shows that the family $M^7_{\underline{a},\underline{b}}$ contains members that are not even homotopy equivalent to S^3-bundles over S^4.

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