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Low-dimensional Topology and Number Theory

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Abstract. The workshop brought together topologists and number theorists with the intent of exploring the many tantalizing connections between these areas.

Mathematics Subject Classification (2010): 57xx, 11xx.

Introduction by the Organisers

The workshop Low-Dimensional Topology and Number Theory, organised by Paul E. Gunnels (Amherst), Walter Neumann (New York), Don Zagier (Bonn) and Adam S. Sikora (New York) was held August 20th – August 26th, 2017. This meeting was a part of a long-standing tradition of collaboration of researchers in these areas. The preceding meeting under the same name took place in Oberwolfach three years ago. At the moment the topic of most active interaction between topologists and number theorists are quantum invariants of 3-manifolds and their asymptotics. This year's meeting showed significant progress in the field.

The workshop was attended by many researchers from around the world, at different stages of their careers - from graduate students to some of the most established scientific leaders in their areas. The participants represented diverse backgrounds. There were 26 talks ranging from 30 to 60 minutes intertwined with informal discussions.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, “US Junior Oberwolfach Fellows”. Moreover, the MFO and the workshop organizers would like to thank the Simons Foundation for supporting Stavros Garoufalidis in the “Simons Visiting Professors” program at the MFO.

Workshop: Low-dimensional Topology and Number Theory

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Abstracts

The Pink-Zilber Conjecture and the generalized Cosmetic Surgery Conjecture

BoGwang Jeon

Dehn filling is one of the most fundamental topological operations in the field of low dimensional topology. More than 50 years ago, W. Lickorish and A. Wallace showed that any closed connected orientable 3-manifold can be obtained by a Dehn filling on a link complement. In the late 1970's, W. Thurston, as a part of his revolutionary work, showed Dehn filling behaves very nicely under hyperbolic structure by proving that if the original 3-manifold is hyperbolic, then almost all of its Dehn fillings are also hyperbolic. Since then, understanding hyperbolic Dehn filling has become a central topic in the study of 3-dimensional geometry and topology.

However many quantitative questions regarding Dehn filling are still unanswered even for simple cases. For instance the following conjecture, which was proposed by C. Gordon in 1990 [1] (see also Kirby's problem list [5]), is one of the basic questions in the topic, but the complete answer is unknown:

Conjecture 1 (Cosmetic Surgery Conjecture (Hyperbolic Case))**.** *Let $\mathcal{M}$ be a 1-cusped hyperbolic 3-manifold. Let $\mathcal{M}(p/q)$ and $\mathcal{M}(p'/q')$ be the p/q and p'/q'-Dehn filled manifolds of it (respectively) which are also hyperbolic. If*

$$p/q \neq p'/q',$$

then there is no orientation preserving isometry between $\mathcal{M}(p/q)$ and $\mathcal{M}(p'/q')$.

The study of unlikely intersections was first initiated by E. Bombieri, D. Masser, and U. Zannier in the 1990's and it has grown and become an active research area in number theory nowadays [8]. Recently it turned out that results in this field could provide powerful tools to understand algebraic invariants of hyperbolic Dehn fillings. For example, using P. Habegger's work, the author proved the following theorem in [3] and [4]:

Theorem 1. *Let $\mathcal{M}$ be an k-cusped hyperbolic 3-manifold. Then the height of a Dehn filling point of any hyperbolic Dehn filling of $\mathcal{M}$ is uniformly bounded.*

This leads to the following two corollaries:

Corollary 1. *Let $\mathcal{M}$ be an k-cusped hyperbolic 3-manifold. For $D > 0$, there are only a finite number of hyperbolic Dehn fillings of $\mathcal{M}$ whose trace field degrees are bounded by D.*

Corollary 2. *There are only a finite number of hyperbolic 3 -manifolds of bounded volume and trace field degree.*

Basically the reason that unlikely intersection theory is applicable to the study of hyperbolic Dehn filling is we can interpret this geometric and topological phenomenon as an algebro-geometric one. More precisely, we can view a hyperbolic k-cusped manifold as an k-dimensional algebraic variety, and Dehn filling on it as the intersection between the corresponding variety and an algebraic subgroup whose index is given by the Dehn filling coefficient. Thus unlikely intersection theory provides a natural framework to understand algebraic invariants of hyperbolic Dehn fillings.

In this talk, following along the same lines, we explain another application of unlikely intersection theory to a problem of hyperbolic Dehn filling. First, we resolve the Cosmetic Surgery Conjecture for a hyperbolic 1-cusped manifold whose cusp shape is not quadratic except for finitely many exceptions:

Theorem 2. *Let $\mathcal{M}$ be a 1-cusped hyperbolic 3-manifold whose cusp shape is non-quadratic. Then, for sufficiently large $|p|+|q|$ and $|p'|+|q'|$,*

$$\mathcal{M}(p/q) \cong \mathcal{M}(p'/q')$$

if and only if

$$p/q = p'/q'$$

where $\cong$ repsents an orientation preserving isometry.

An analogous extension of the above theorem to a more cusped manifold is clearly false. For example, if $\mathcal{M}$ is the Whitehead link complement, then, since it allows the symmetry between two given cusps, $\mathcal{M}(p_1/q_1, p_2/q_2)$ is equal to $\mathcal{M}(p_2/q_2, p_1/q_1)$ for any p_1/q_1 and p_2/q_2.

Let τ_1 and τ_2 be two cusp shapes of a 2-cusped hyperbolic 3-manifold $\mathcal{M}$. If there is a symmetry between two cusps, then it allows the following relation between τ_1 and τ_2:

$$\tau_1 = \frac{a\tau_2 + b}{c\tau_2 + d}$$

where $a, b, c, d \in \mathbb{Z}$ and $ad - bc = \pm 1$. Thus having no symmetry between two cusps can be rephrased algebraically as follows:

$$1, \tau_1, \tau_2, \tau_1\tau_2 \quad \text{are linearly independent over} \quad \mathbb{Q}. \tag{1}$$

When τ_1, τ_2 satisfies the condition (1), we say $\mathcal{M}$ has rationally independent cusp shapes. Under this hypothesis, we can extend the result of Theorem 2 to 2-cusped manifolds as the following theorem shows:

Theorem 3. *Let $\mathcal{M}$ be a 2-cusped hyperbolic 3-manifold having non-quadratic, rationally independent cusp shapes. Then, for sufficiently large $|p_i|+|q_i|$ and $|p_i'|+|q_i'|$ $(1 \leq i \leq 2)$,*

$$\mathcal{M}(p_1/q_1, p_2/q_2) \cong \mathcal{M}(p_1'/q_1', p_2'/q_2')$$

if and only if

$$(p_1/q_1, p_2/q_2) = (p_1'/q_1', p_2'/q_2').$$

For an n-cusped hyperbolic 3-manifold, we generalise (1) as follows:

Definition 1. Let $\mathcal{M}$ be a n-cusped manifold and $\tau_1, \ldots, \tau_n$ be its cusp shapes. We say $\mathcal{M}$ has rationally independent cusp shapes if the elements of the set $\{\tau_{i_1} \cdots \tau_{i_l} \mid 1 \leq i_1 < \cdots < i_l \leq n\}$ is linearly independent over $\mathbb{Q}$.

Having this definition, we extend Theorem 2 to the general case as follows:

Theorem 4. *Let $\mathcal{M}$ be a n-cusped ($n \geq 3$) hyperbolic 3-manifold having non-quadratic and pairwise rationally independent cusp shapes. If the Pink-Zilber conjecture is true, then, for sufficiently large $|p_i| + |q_i|$ and $|p'_i| + |q'_i|$ ($1 \leq i \leq n$),*

$$\mathcal{M}\big(p_1/q_1, \ldots, p_n/q_n\big) \cong \mathcal{M}\big(p'_1/q'_1, \ldots, p'_n/q'_n\big)$$

if and only if

$$\big(p_1/q_1, \ldots, p_n/q_n\big) = \big(p'_1/q'_1, \ldots, p'_2/q'_2\big).$$

References

[1] C. Gordon, *Dehn surgery on knots*, Proceedings of the ICM, Vol. I, II (Kyoto, 1990) Math. Soc. Japan, Tokyo (1991), 631-642.

[2] P. Habegger, *On the bounded height conjecture*, IMRN (2009), 860-886.

[3] B. Jeon, *Hyperbolic 3-manifolds of bounded volume and trace field degree*, arxiv.org/abs/1305.0611.

[4] B. Jeon, *Hyperbolic 3-manifolds of bounded volume and trace field degree II*, arxiv.org/abs/1409.2069.

[5] R. Kirby, *Problems in low-dimensional topology*, AMS/IP Studies in Advanced Mathematics, Vol 2, part 2, Amer. Math. Soc. (1997).

[6] W. Neumann, D. Zagier, *Volumes of hyperbolic three-manifolds*, Topology (1985), 307-332.

[7] W. Thurston, *The Geometry and Topology of 3-manifolds*, Princeton University Mimeographed Notes (1979).

[8] U. Zannier, *Some problems of unlikely intersections in Arithmetic and Geometry*, Princeton University Press (2012).

Graph complexity and Mahler measure

Daniel S. Silver

(joint work with Susan G. Williams)

In 1933, D.H. Lehmer discovered the remarkable polynomial

$$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$$

with Mahler measure equal to 1.17628 . . .. This remains the smallest known Mahler measure greater than 1. Lehmer's paper implicitly raises the following question.

Lehmer's Question: Given $\epsilon > 0$, does there exist an integral polynomial $f(x)$ with $1 < M(f) < 1 + \epsilon$?

Although Lehmer's Question remains open, notable partial results have been obtained. One of them is a theorem of C. Smyth that tells us that when considering Lehmer's Question it suffices to consider only polynomials $f(x)$ with the property that $f(x^{-1}) = f(x)$. We will call such polynomials *palindromic*.

In previous work [4] the authors reformulated Lehmer's Question both in terms of Alexander polynomials of knots in lens spaces as well as pseudo-Anosov surface automorphisms. Our present purpose is to reformulate Lehmer's Question in terms of graph complexity.

Consider a graph G as above in the punctured plane $\mathbb{R}^2 \setminus \{0\}$. Lifting G to the universal cover of the punctured plane produces a graph $\tilde{G}$ with an action by the infinite cyclic group $\mathbb{Z} = \langle x \rangle$. We call $\tilde{G}$ a 1-*periodic graph*, and we denote the intermediate r-fold covering graph, for $r > 0$, by G_r.

Although $\tilde{G}$ is not a finite graph, the $\mathbb{Z}$-action enables us to define a Laplacian matrix $L_{\tilde{G}}$ for $\tilde{G}$. It is of the same size as L_G but has entries in $\mathbb{Z}[x, x^{-1}]$ rather than $\mathbb{Z}$. For this, we choose a section of V_G; that is, a lift to $\tilde{G}$ of each vertex of G. All other vertices of $\tilde{G}$ are then described via the action of $\mathbb{Z}$. The matrix $L_{\tilde{G}}$ is described just as L_G.

We call the determinant of $L\tilde{G}$ the *Laplacian polynomial* of $\tilde{G}$. We denote it by $\Delta_{\tilde{G}}$. It is well defined, independent of our choice of section. It is easy to see that it is a palindromic polynomial.

Example 1. If G has a single vertex and edge labeled $\epsilon = \pm 1$ and wrapping s times around the origin of the plane, then $\Delta_{\tilde{G}} = \epsilon(-x^s + 2 - x^{-s})$.

Proposition 1. *A polynomial $f(x)$ is the Laplacian polynomial of a 1-periodic graph if and only if it has the form $(x - 2 + x^{-1})g(x)$, for some palindromic g.*

Example 2. Recall that Lehmer's polynomial $x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1$ has the smallest known Mahler measure greater than 1. Multiplying by $x^{-5}(x-2+x^{-1})$ produces the palindromic polynomial $x^6-x^5-x^4+x^2+x^{-2}-x^{-4}-x^{-5}+x^{-6}$ with the same Mahler measure. We can pair terms of the latter and express it as $-(-x^6 + 2 - x^6) + (-x^5 + 2 - x^{-5}) + (-x^4 + 2 - x^{-4}) - (-x^2 + 2 - x^{-2})$. It is now easy to construct a 1-periodic graph $\tilde{G}$ with this Laplacian polynomial. The graph has a single vertex and four edges. Two edges labeled $+1$ wind four and five times, respectively, around the origin. Two edges labeled -1 run two and six times around.

The following theorem follows from a deep result in algebraic dynamics of D. Lind, K. Schmidt and T. Ward (see [2] or Theorem 21.1 of [3]). However, a direct, elementary proof is possible.

Theorem 1. *For any 1-periodic graph $\tilde{G}$,*

$$M(\Delta_{\tilde{G}}) = \lim_r \kappa_{G_r}^{1/r}$$

Using Theorem 1 and the observation that the graphs constructed in the proof of Proposition 1 have nonzero tree complexity, we obtain:

Corollary 1. *Lehmer's Question is equivalent to: Given $\epsilon > 0$, does there exist a graph G in the punctured plane such that*

$$1 < \lim_r \tau_{G_r}^{1/r} < 1 + \epsilon?$$

Remark 1. The graphs G of Corollary 1 need not be embedded. (Indeed the graph constructed in Example 2 is easily shown to be nonplanar.) We do not know whether the corollary remains true if we require that G be embedded. If true, then Lehmer's Question would be a question about determinant density of knots and links. See [1] for possible geometric consequences.

References

[1] A. Champanerkar, I. Kofman and J.S. Purcell, *Geometrically and diagrammatically maximal knots*, J. London Mathematical Society, **94** (2016), 883–908.

[2] D.A. Lind, K. Schmidt and T. Ward, *Mahler measure and entropy for commuting automorphisms of compact groups*, Inventiones Math. **101** (1990), 593–629.

[3] K. Schmidt, *Dynamical Systems of Algebraic Origin*, Birkhäuser, Basel, 1995.

[4] D.S. Silver and S.G. Williams, *Lehmer's question, knots and surface dynamics*, Math. Proc. Camb. Phil. Soc., **143** (2007), 649–661.

[5] D.S. Silver and S.G. Williams, *Graph complexity and Mahler measure*, preprint (2017), arXiv:1701.06097

The colored Jones polynomial and surfaces in 3-manifolds.

Christine Ruey Shan Lee

The Strong Slope Conjecture made by Garoufalidis [1] and extended by Kalfagianni and Tran [2] predicts an interesting relationship between essential surfaces in the knot complement and a quantum knot invariant, the colored Jones polynomial. The conjecture predicts that the asymptotics of the degree of the polynomial determine the boundary slopes and other topological information such as the Euler characteristic and number of sheets of essential surfaces, called Jones surfaces, realizing the conjecture.

In this talk I will give an overview of recent advances on the conjecture, with specific focus on other potential relationships between the colored Jones polynomial and the geometry of the knot complement observed from those results in my recent work.

With Efstratia Kalfagianni, we developed a normal surface algorithm to verify the Strong Slope Conjecture and indicated a possible relationship between the number of sheets of a Jones surface and the Jones period of the knot [3]. Building on the work of Futer, Kalfagianni, and Purcell, I have introduced the class of near-alternating links approximating the class of adequate links [4]. For this class, I proved the Strong Slope Conjecture and derived two-sided volume bounds from the stable coefficients of the colored Jones polynomial, extending many of the previous results known for adequate knots. Lastly, I will discuss my current project with Roland van der Veen and Stavros Garoufalidis [5] on the Strong Slope Conjecture for 3-tangle Montesinos knots, which, in contrast to the aforementioned examples, yields many rational Jones slopes. This indicates a different Jones surface for which the relationship to the complement of the knot has yet to be clarified.

References

[1] S. Garoufalidis, *The Jones slopes of a knot*, Quantum Topology **2** (2011), 43–69.

[2] E. Kalfagianni and A. Tran, *Knot cabling and the degree of the colored Jones polynomial*, New York Journal of Mathematics **21** (2015), 905–941.

[3] E. Kalfagianni and C. Lee, *Normal and Jones surfaces of knots*, https://arxiv.org/abs/1702.06466.
[4] C. Lee, *Jones slope and coarse volume for near-alternating links*, https://arxiv.org/abs/1708.04900.
[5] S. Garoufalidis, C. Lee, and R. van der Veen, *Tangle replacements and Jones slopes for Montesinos knots*, in preparation.

An application of TQFT to modular representation theory

Gregor Masbaum
(joint work with P. Gilmer)

Let p be an odd prime, and K be an algebraically closed field of characteristic p. For $g \geq 1$ an integer, we consider the symplectic group $\mathrm{Sp}(2g, K)$, thought of as an algebraic group of rank g. It is well-known that the classification (due to Chevalley) of rational simple $\mathrm{Sp}(2g, K)$-modules up to isomorphism is independent of the characteristic. For a dominant weight λ, let $L_p(\lambda)$ denote the simple $\mathrm{Sp}(2g, K)$-module with highest weight λ. We recall that λ is dominant iff it is a linear combination of the fundamental weights ω_i $(i = 1, \ldots, g)$ with nonnegative integer coefficients.

In characteristic zero, the dimension and the formal character of a simple $\mathrm{Sp}(2g, \mathbb{C})$-module can be computed from the Weyl character formula. But explicit dimension formulae for the modules $L_p(\lambda)$ for $p > 0$ are quite rare, except in rather special situations. In particular, for weights outside the fundamental alcove, no general dimension formula is known. A conjectural formula by Lusztig for primes in a certain range was shown to hold for $p >> 0$ by Andersen-Jantzen-Soergel [1] but was recently shown not to hold for all p in the hoped-for range by Williamson [7].

In our joint work [5] we show that Topological Quantum Field Theory (TQFT) can give new information about the dimensions of some of these simple modules. Specifically, we show that for every prime $p \geq 5$ and in every rank $g \geq 3$, there is a family of $p-1$ dominant weights λ, lying outside of the fundamental alcove except for one weight in rank $g = 3$, for which we can express the dimension of $L_p(\lambda)$ by formulae similar to the Verlinde formula in TQFT. We found this family as a byproduct of Integral SO(3)-TQFT [2], an integral refinement of the Witten-Reshetikhin-Turaev TQFT associated to SO(3). More precisely, we use Integral SO(3)-TQFT in what we call the 'equal characteristic case' which we studied in [4]. The family of weights λ we found together with our formulae for $\dim L_p(\lambda)$ is given in the following Theorem. We can also compute the weight space decomposition of $L_p(\lambda)$ for these weights λ; see [5, Theorem 1.9].

Theorem. *Let $p \geq 5$ be prime and put $d = (p-1)/2$. For rank $g \geq 3$, consider the following $p-1$ dominant weights for the symplectic group* $\mathrm{Sp}(2g, K)$ *:*

$$\lambda = \begin{cases} (d-1)\omega_g & (\textit{Case}\ \mathrm{I}) \\ (d-c-1)\omega_g + c\,\omega_{g-1} \quad \textit{for}\ 1 \leq c \leq d-1 & (\textit{Case}\ \mathrm{II}) \\ (d-c-1)\omega_g + (c-1)\omega_{g-1} + \omega_{g-2} \quad \textit{for}\ 1 \leq c \leq d-1 & (\textit{Case}\ \mathrm{III}) \\ (d-2)\omega_g + \omega_{g-3} & (\textit{Case}\ \mathrm{IV}) \end{cases}$$

Put $\varepsilon = 0$ in Case I *and* II *and $\varepsilon = 1$ in Case* III *and* IV*. Then*

$$\dim L_p(\lambda) = \frac{1}{2}\left(D_g^{(2c)}(p) + (-1)^{\varepsilon}\delta_g^{(2c)}(p)\right) \quad \textit{where} \tag{1}$$

$$D_g^{(2c)}(p) = \left(\frac{p}{4}\right)^{g-1} \sum_{j=1}^{d} \left(\sin \frac{\pi j(2c+1)}{p}\right)\left(\sin \frac{\pi j}{p}\right)^{1-2g} \tag{2}$$

$$\delta_g^{(2c)}(p) = (-1)^c \frac{4^{1-g}}{p} \sum_{j=1}^{d} \left(\sin \frac{\pi j(2c+1)}{p}\right)\left(\sin \frac{\pi j}{p}\right)\left(\cos \frac{\pi j}{p}\right)^{-2g}, \tag{3}$$

and c is the same c used in the definition of λ, except in Case I *and* IV*, where we put $c = 0$. In Case* IV *in rank $g = 3$, $\omega_{g-3} = \omega_0$ should be interpreted as zero.*

Formula (2) is an instance of the famous Verlinde formula in TQFT. Formula (3) appeared first in [4]. Note that the difference between the two formulae is that certain sines in (2) have become cosines in (3), and the overall prefactor is different. For fixed g, both $D_g^{(2c)}(p)$ and $\delta_g^{(2c)}(p)$ can be expressed as polynomials in p and c.

When $p = 5$, the list above produces (in order) the fundamental weights $\omega_g, \omega_{g-1}, \omega_{g-2}, \omega_{g-3}$. These are exactly the weights considered by Gow [6], who gave an explicit construction of $L_p(\omega_i)$ for the last $p-1$ fundamental weights (that is: ω_i where $i \geq g-p+1$). But for $p > 5$, our weights are different from those of Gow. It is intriguing that both Gow's and our family of weights have $p-1$ elements.

Question. Can one find similar Verlinde-like dimension formulae for other families of dominant weights?

In [3], we answered this question affirmatively for the $p-1$ fundamental weights considered by Gow. On the other hand, SO(3)-TQFT is just one of the simplest TQFTs within the family of Witten-Reshetikhin-Turaev TQFTs, and it is conceivable that other Integral TQFTs might produce more families of weights λ where the methods of the present paper could be applied.

Another interesting question is whether our character formulae coincide with those given by Lusztig's conjecture. As already mentioned, Lusztig's conjecture is known to be false in general by work of Geordie Williamson. But some experimental evidence seems to indicate that it may hold true for our highest weights.

References

[1] H. H. Andersen, J. C. Jantzen, W. Soergel, *Representations of quantum groups at a p-th root of unity and of semisimple groups in characteristic p: independence of p.* Astérisque **220** (1994)
[2] P. Gilmer, G. Masbaum, *Integral lattices in TQFT*, Ann. Scient. Ec. Norm. Sup. **40**, (2007), 815–844
[3] P. Gilmer, G. Masbaum, *Dimension formulas for some modular representations of the symplectic group in the natural characteristic*, J. Pure Appl. Algebra **217** (2013) 82-86.
[4] P. Gilmer, G. Masbaum, *Irreducible factors of modular representations of mapping class groups arising in Integral TQFT*, Quantum Topology **5** (2014), 225–258
[5] P. Gilmer, G. Masbaum, *An application of TQFT to modular representation theory*, Invent. Math. (to appear) arXiv:1606.09532
[6] R. Gow. *Construction of $p-1$ irreducible modules with fundamental highest weight for the symplectic group in characteristic p*, J. Lond. Math. Soc. **58** (1998) 619–632.
[7] G. Williamson. *Schubert calculus and torsion explosion.* Journal of the AMS (to appear) arXiv:1309.5055

One step beyond the Alexander polynomial

Roland van der Veen
(joint work with Dror Bar-Natan)

We present a simple, strong knot invariant that is closely related to the Alexander polynomial and seems to share many of its good properties. For example, unlike the commonly used quantum invariants such as the Jones polynomial, our invariant is computable in polynomial time.

Consider a (long) knot K presented as a proper smooth embedding of $[0,1]$ into the closed unit ball such that the projection on the third coordinate is a generic immersion γ in the plane, see for example Figure 1. More specifically, assume that there is an $n \in \mathbb{N}$ such that γ has the following properties. The points $\gamma(\frac{k}{n+1})$ where $k \in \{1, \dots, n\}$ are the union of all double points and all points where γ' is parallel to the positive x-axis. The double points are known as crossings and the latter as cuaps. Close to any crossing we assume γ' has positive y-coordinate. The sign of a crossing is the sign of the x-coordinate of γ' at the overpass. A crossing is denoted $X^{\sigma}_{i,j}$ where σ is the sign and i, j are the labels of the over and under strand. The sign of a cuap is the sign of the y-direction of γ''. A cuap is denoted u^{σ}_i where σ is the sign and i is its label.

Let E^i_j be the elementary matrix with a single non-zero entry 1 at the (i, j)-th place. Define the matrices

$$Q = \sum_{X^{\sigma}_{i,j}} \sigma t^{\frac{\sigma}{2}} (E^j_j - E^i_j) \quad W = \sum_{i<j} E^i_j \quad c = \prod_{X^{\sigma}_{i,j}, u^{\sigma}_i} t^{-\sigma}$$

$$B = I - (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) WQ \quad G = Q\mathrm{adj}(B) \quad H = \mathrm{adj}(B)W$$

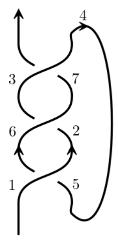
$$Z_G = (t - t^{-1}) \sum_{j=2}^{n} \sum_{a,b<j} G^j_a \left(\frac{1}{2} G^j_b + \sum_{c>j} G^c_b \right)$$

$$Z_H = \sum_{X^\sigma_{i,j}} \frac{\sigma}{2}((1-t^\sigma)H^i_j)^2 - \frac{\sigma}{2}((1+t^\sigma)H^j_j)^2 + \sigma t^\sigma(H^j_i H^i_j + H^i_i H^j_j)$$

$$+t^\sigma(1-t)H^j_i((1+\sigma)H^j_j + (1-\sigma)H^i_i) + \det(B)\sum_{u^\sigma_i} \sigma H^i_i$$

Theorem 1. *(Bar-Natan, van der Veen, 2017)* [BV17]
$c^{\frac{1}{2}}\det(B)$ *is the Alexander polynomial* Δ *and* $Z_1 = c(Z_G + Z_H)$ *is a new knot invariant. Both are elements of* $\mathbb{Z}[t, t^{-1}]$ *computable in polynomial time.*

When normalized as $\rho_1 = -\frac{t}{(1-t)^2}(Z_1 - t\Delta\frac{d}{dt}\Delta)$ our invariant appears to be closely related to Rozansky's expansion of the colored Jones polynomial [Ro98][Ov13].
Expanding in $h = q-1$ they $J_\alpha(q) = \sum_{n\geq 0} h^n(\sum_{0\leq m\leq n} D_{m,n}(\alpha h)^{2m})$. Now there exist Laurent polynomials $P^{(n)}$ such that $\sum_{m\geq 0} D_{m,n+2m}(\alpha h)^{2m} = \frac{P^{(n)}(q^\alpha)}{\Delta^{2n+1}(q^{\frac{\alpha}{2}} - q^{-\frac{\alpha}{2}})}$.
We conjecture $\rho_1(t^2) = \frac{t^2}{(1-t^2)^2}P^{(1)}(t)$.



$$W = \begin{pmatrix} 0&1&1&1&1&1&1\\ 0&0&1&1&1&1&1\\ 0&0&0&1&1&1&1\\ 0&0&0&0&1&1&1\\ 0&0&0&0&0&1&1\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0 \end{pmatrix} \quad Q = \begin{pmatrix} 0&0&0&0&-\sqrt{t}&0&0\\ 0&\sqrt{t}&0&0&0&0&0\\ 0&0&0&0&0&0&-\sqrt{t}\\ 0&0&0&0&0&0&0\\ 0&0&0&0&\sqrt{t}&0&0\\ 0&-\sqrt{t}&0&0&0&0&0\\ 0&0&0&0&0&0&\sqrt{t} \end{pmatrix} \quad c = t^{-2}$$

Figure 1. A diagram for the trefoil knot 3_1. The double points at the crossings and the right-pointing cuaps are enumerated in order of appearance. The matrices W, Q and the number c necessary for computation of the invariant Z_1 are listed next to it.

Example: Trefoil. We illustrate the computation of Z for the trefoil knot 3_1 shown in Figure 1. Notice this is the mirror image of the one used in the knot tables.

$$B = \begin{pmatrix} 1&0&0&0&1-t&0&0\\ 0&t&0&0&1-t&0&0\\ 0&t-1&1&0&1-t&0&1-t\\ 0&t-1&0&1&1-t&0&1-t\\ 0&t-1&0&0&1&0&1-t\\ 0&0&0&0&0&1&1-t\\ 0&0&0&0&0&0&1 \end{pmatrix} \qquad \Delta_{3_1}(t) = c^{\frac{1}{2}}\det(B) = t - 1 + t^{-1}$$

$$G = \begin{pmatrix} 0 & t^{\frac{3}{2}} - t^{\frac{1}{2}} & 0 & 0 & -t^{\frac{3}{2}} & 0 & t^{\frac{3}{2}} - t^{\frac{5}{2}} \\ 0 & t^{\frac{1}{2}} & 0 & 0 & t^{\frac{3}{2}} - t^{\frac{1}{2}} & 0 & t^{\frac{5}{2}} - 2t^{\frac{3}{2}} + t^{\frac{1}{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & -t^{\frac{5}{2}} + t^{\frac{3}{2}} - t^{\frac{1}{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t^{\frac{1}{2}} - t^{\frac{3}{2}} & 0 & 0 & t^{\frac{3}{2}} & 0 & t^{\frac{5}{2}} - t^{\frac{3}{2}} \\ 0 & -t^{\frac{1}{2}} & 0 & 0 & t^{\frac{1}{2}} - t^{\frac{3}{2}} & 0 & -t^{\frac{5}{2}} + 2t^{\frac{3}{2}} - t^{\frac{1}{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & t^{\frac{5}{2}} - t^{\frac{3}{2}} + t^{\frac{1}{2}} \end{pmatrix} \qquad Z_G = t^4 - \frac{3t^2}{2} + \frac{1}{2}$$

$$H = \begin{pmatrix} 0 & t^2 - t + 1 & t & t & t & t^2 & t^2 \\ 0 & 0 & 1 & 1 & 1 & t & t \\ 0 & 0 & t - t^2 & 1 & 1 & t & t \\ 0 & 0 & t - t^2 & t - t^2 & 1 & t & t \\ 0 & 0 & 1 - t & 1 - t & 1 - t & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & t^2 - t + 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$Z_H = t^4 - 3t^3 + \frac{7t^2}{2} - t - \frac{1}{2}$$

It follows that $Z_1(3_1) = c(Z_G + Z_H) = 2 - t^{-1} - 3t + 2t^2$ and its normalization is $\rho_1(t) = -t - t^{-1}$. Comparing to the value in the table in the next section we notice the minus sign caused by taking mirror image.

Table and conjectures. In the table below we list the Alexander polynomial $\Delta(t)$ of each knot together with the normalized version of our invariant $\rho_1 = -\frac{t}{(1-t)^2}(Z_1 - t\Delta\frac{d}{dt}\Delta)$. As both Laurent polynomials appear to be symmetric with respect to $t \mapsto t^{-1}$ only non-negative coefficients are listed.

Knot	Alexander	ρ_1 normalization of Z
 3_1	$t-1$	t
 4_1	$3-t$	0
 5_1	t^2-t+1	$2t^3+3t$
 5_2	$2t-3$	$5t-4$
 6_1	$5-2t$	$t-4$
 6_2	$-t^2+3t-3$	t^3-4t^2+4t-4
 6_3	t^2-3t+5	0
 7_1	t^3-t^2+t-1	$3t^5+5t^3+6t$
 7_2	$3t-5$	$14t-16$
 7_3	$2t^2-3t+3$	$-9t^3+8t^2-16t+12$
 7_4	$4t-7$	$32-24t$
 7_5	$2t^2-4t+5$	$9t^3-16t^2+29t-28$
 7_6	$-t^2+5t-7$	$t^3-8t^2+19t-20$
 7_7	t^2-5t+9	$8-3t$

All prime knots up to 10 crossings are distinguished by the pair (Δ, ρ_1). This is better than the pair (Khovanov, HOMFLY) which fails to distinguish the knots $(8_{16}, 10_{156})$.

A longer table and the computer program that produced it are available at `http://www.rolandvdv.nl/MLA/`

We conjecture that ρ_1 satisfies $\rho_1(-K) = -\rho_1(K)$ and the highest power of t is less than or equal to 2 genus$(K) - 1$. For example for knot $K = 12_{n313}$ the genus is 2 and the highest power of t in $\rho_1(K)$ is 2 and $\Delta_K = 1$.

References

[BV17] D. Bar-Natan and R. van der Veen. *A polynomial time knot polynomial*, 2017. preprint Arxiv 1708.040853.

[Ov13] A. Overbay. *Perturbative expansion of the colored Jones polynomial*, 2013. University of North Carolina PhD thesis.

[Ro98] L. Rozansky, *The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial*, Adv. math. **134-1** (1998), 1–31.

Distribution of Betti numbers of manifolds

Fernando Rodriguez-Villegas
(joint work with Tamás Hausel)

We consider the question of how Betti numbers are distributed for random algebraic varieties. For smooth, projective varieties the even/odd Betti numbers are symmetric and unimodal by Poincaré duality and the Hard Lefschetz theorem respectively. It is easy to see that for the Grassmanian $\mathrm{Gr}(r, n + r)$ as $n \to \infty$ for fixed r the Betti numbers, appropriately scaled, converge to the B-spline distribution $\chi^{(1)} * \cdots * \chi^{(1)}$ where $\chi^{(1)} = \delta_{[-\frac{1}{2},\frac{1}{2}]}$. Hence as $r \to \infty$ too the distribution tends to a Gaussian.

It is less clear what to expect for say smooth affine varieties. We consider the case of generic Nakajima quiver varieties. Because these are semiprojective there is a form of the Hard Lefschetz theorem that holds. In particular the say even Betti numbers must be increasing in an initial segment. How far?

We expect that even Betti numbers of these Nakajima quiver varieties to tend to the Airy distribution as the dimension vector tends to infinity (all entries going to infinity independently). We prove this for a related family corresponding to the complete graphs. It involves the asymptotic expansion of the series

$$\sum_{n\geq 0} q^{\binom{n}{2}} \frac{T^n}{n!}.$$

as $q \to 1$. This in turn relies on the asymptotic growth of the number of connected graphs on n vertices with fixed Betti number b_1.

Arithmetic Chern-Simons invariants

Ted Chinburg

(joint work with F. M. Bleher, R. Greenberg, M. Kakde, G. Pappas, and M. J. Taylor)

This note is a report on some recent work by several groups of authors concerning an arithmetic version of Chern Simons theory proposed by M. Kim in [6]. Kim's work is based on the approach to 2 + 1 dimensional topological quantum field theory by Dijkgraaf and Witten [4] and by Freed and Quinn [5].

Kim's invariants may be defined in the following way in the simplest case. Suppose F is a number field containing the multiplicative group $\tilde{\mu}_n$ generated by a primitive n^{th} root of unity. Define O_F to be the ring of integers of F, and let $X = \mathrm{Spec}(O_F)$. Let $\pi_1(X, \eta)$ be the étale fundamental group of X relative to a fixed base point η. Then $\pi_1(X, \eta)$ is the Galois group of a maximal everywhere unramified extension of F. Let $f : \pi_1(X, \eta) \to G$ be a fixed homomorphism to an abstract finite group G. Let μ_n be the sheaf of n^{th} roots of unity in the étale topology on X. We let G act trivially on $\tilde{\mu}_n = \mu_n(X)$. Pick a class $c \in H^3(G, \tilde{\mu}_n)$. Then $f^*c \in H^3(\pi_1(X, \eta), \tilde{\mu}_n)$ defines via Cech cohomology a class $f_X^* c \in H^3(X, \mu_n)$. By the global duality theorem of Artin and Verdier [7, p. 538], there is a canonical isomorphism $\mathrm{inv}_n : H^3(X, \mu_n) \to \mathbb{Z}/n\mathbb{Z}$. Kim's invariant [6] associated to c and f is the class

$$S(f, c) = \mathrm{inv}_n(f_X^* c) \in \mathbb{Z}/n\mathbb{Z}. \tag{1}$$

The topological counterpart of this case is that of Chern Simons invariants for a finite gauge group G and a compact three manifold; see [5].

In [2], H. Chung, D. Kim, M. Kim, J. Park and H. Yoo proved that Kim's invariant can be non-trivial even when the finite group in question is cyclic of order two. Their approach is to compare local and global trivializations of Galois three cocycles. Using this method they construct infinitely many examples in which the invariant is non-trivial and the finite group involved is either $\mathbb{Z}/2$, $\mathbb{Z}/2 \times \mathbb{Z}/2$ or the symmetric group S_4.

In [1], F. Bleher, T. Chinburg, R. Greenberg, M. Kakde, G. Pappas and M. Taylor used a different approach to prove a formula for Kim's invariant in (1) in terms of Artin maps when $G = \mathbb{Z}/n$. One consequence is that for all $n > 1$, there are infinitely many number fields F over which there are both trivial and non-trivial Kim invariants associated to cyclic G of order n. The construction also shows that Kim's invariant in the cyclic case is a specialization of a bilinear pairing in Galois cohomology which resembles, but is different from, one considered by McCallum and Sharifi in [8]. It was shown in [1] that by replacing $H^3(G, \tilde{\mu}_n)$ by $\mathrm{Ext}^3_{\mathbb{Z}[G]}(\mathbb{Z}/n, \mu_n)$, one arrives at an invariant which combines the McCallum-Sharifi pairing and Kim's invariant.

In [3], H. Chung, D. Kim, M. Kim, J. Park, G. Pappas and H. Yoo then defined an arithmetic counterpart of the linking pairing of two curves in a three manifold to give another way of defining Kim's invariant when G is a finite cyclic group. They showed that in this case, the invariant is related to the class invariant homomorphism of Galois module structure theory. They also gave a different proof of the Artin map formula of [1].

In his original paper [6], Kim also considered the arithmetic counterpart of Chern Simons invariants for finite gauge groups and three manifolds whose boundary is a union of surfaces. In the above contructions, one replaces $X = \mathrm{Spec}(O_F)$ by $\mathrm{Spec}(O_{F,S})$ when S is a finite set of finite places of F. Following the approach of topological quantum field theory, Kim's invariant in this case is an element of a torsor for $\mathbb{Z}/n$ which is push out of $\mathbb{Z}/n$ torsors for each place in S. More recently, Bleher, Chinburg, Greenberg, Kakde, and Taylor have generalized this construction by using $\mathrm{Ext}^3_{\mathbb{Z}[G]}(\mathbb{Z}/n, \tilde{\mu}_n)$ rather than $H^3(G, \tilde{\mu}_n)$. The resulting invariant is an element in a torsor for a larger group than $\mathbb{Z}/n$, namely the cokernel of the natural map from a global Ext^2 group to product local Ext^2 groups associated to the places in S.

References

[1] F. M. Bleher, T. Chinburg, R. Greenberg, M. Kakde, G. Pappas, and M. J. Taylor, M. J., *Unramified arithmetic Chern-Simons invariants*, arXiv:1705.07110.

[2] H. J. Chung, D. Kim, M. Kim, J. Park, and H. Yoo, H., *Arithmetic Chern-Simons theory II*, arXiv:1609.03012.

[3] H. J. Chung, D. Kim, M. Kim, G. Pappas, J. Park, and H. Yoo, H.,Chung, *Abelian arithmetic Chern-Simons theory and arithmetic linking numbers*, arXiv:1706.03336v1.

[4] R. Dijkgraaf and E. Witten, E., *Topological gauge theories and group cohomology*, Comm. Math. Phys. **129** (1990), no. 2, 393–429.

[5] D. S. Freed and F. Quinn, *Chern-Simons theory with finite gauge group*, Comm. Math. Phys. **156** (1993), no. 3, 435–472.

[6] M. Kim, M., *Arithmetic Chern Simons Theory I*, arXiv:1510.05818.

[7] B. Mazur, *Notes on étale cohomology of number fields*, Ann. Sci. École Norm. Sup. (4) **6** (1973), 521–552 (1974).
[8] W. McCallum, and R. Sharifi, *A cup product in the Galois cohomology of number fields*, Duke Math. J., 120, no. 2, (2003), 269–310.

Some remarks on Mahler measure for arbitrary tori

Matilde N. Lalín

Definition 1. The Mahler measure of a non-zero rational function $P \in \mathbb{C}(x_1, \dots, x_n)$ is defined by

$$\mathrm{m}(P) := \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n},$$

where $\mathbb{T}^n = \{(x_1, \dots, x_n) \in \mathbb{C}^n : |x_1| = \cdots = |x_n| = 1\}$.

Mahler measure raises very interesting problems in connection to the distribution of values, such as Lehmer's question. Here, we are interested in exact formulas for multivariable polynomials.

Example 1. Cassaigne and Maillot [Mai00] proved the following formula. Let a, b, c be nonzero complex numbers. Then

$$\pi \mathrm{m}(ax+by+c) = \begin{cases} \alpha \log |a| + \beta \log |b| + \gamma \log |c| + D\left(\left|\frac{a}{b}\right| e^{i\gamma}\right) & \triangle, \\ \pi \log \max\{|a|, |b|, |c|\} & \text{not } \triangle, \end{cases} \tag{1}$$

where $\triangle$ stands for the statement that $|a|$, $|b|$, and $|c|$ are the lengths of the sides of a triangle; and α, β, and γ are the angles opposite to the sides of lengths $|a|$, $|b|$ and $|c|$ respectively.The function D is the Bloch–Wigner dilogarithm given by

$$D(x) = \mathrm{Im}(\mathrm{Li}_2(x)) + \arg(1-x) \log |x|, \tag{2}$$

where $\mathrm{Li}_2(x)$ is the classical dilogarithm, and the corresponding term codifies the volume of an ideal hyperbolic tetrahedron in $\mathbb{H}^3 \cong \mathbb{C} \times \mathbb{R}_{\geq 0}$ with basis the triangle whose sides are $|a|$, $|b|$, and $|c|$ and fourth vertex infinity.

Notice that the case $a = b = c = 1$ was proven earlier by Smyth [Smy81].

Example 2. Boyd [Boy98] systematically computed numerical examples for several families of polynomials including

$$R_\alpha(x, y) := (1+x)(1+y)(x+y) - \alpha xy,$$
$$S_{k,\beta}(X, Y) = Y^2 + kXY - X^3 - \beta X,$$

where α, k, b are integral parameters. He found numerical formulas of the form

$$\mathrm{m}(R_\alpha(x, y)) \stackrel{?}{=} r_\alpha L'(E_{N(\alpha)}, 0),$$
$$\mathrm{m}(S_{k,\beta}(X, Y)) \stackrel{?}{=} \frac{1}{4} \log |\beta| + s_{k,b} L'(E_{N(k,\beta)}, 0),$$

where r_α, $s_{k,\beta}$ are rational numbers, the L-functions are attached to elliptic curves that are defined by $R_\alpha(x, y) = 0$, and $S_{k,\beta}(X, Y) = 0$ respectively, and the question mark stands for a numerical formula that is true for at least 20 decimal places. N denotes the conductor of the corresponding elliptic curve. In addition, Boyd noticed that the square-free part of β must divide k for such a formula to hold.

Some of Boyd's conjectures have been proven but most remain conjectural. The following table gives a complete list (to our knowledge) of the identities that have been proven for $R_\alpha(x, y)$ for α integral.

α	r_α	N	Proven by
-4	2	36 (CM)	Rodriguez-Villegas [RV99]
2	1/2	36 (CM)	Rodriguez-Villegas [RV99]
-8	10	14	Mellit [Mel12]
1	1	14	Mellit [Mel12]
7	6	14	Mellit [Mel12]
-2	3	20	Rogers–Zudilin [RZ12]
4	2	20	Rogers–Zudilin [RZ12]

The following table has a few conjectural numerical formulas and information about the two proven results for the family $S_{k,\beta}(X, Y)$.

β	k	$s_{k,\beta}$	N	Proven by
1	3	7/2	17	Zudilin [Zud14] + L.–Ramamonjisoa [LR17]
1	4	1/4	192	
-1	2	2	20	Touafek [Tou08a, Tou08b]+Bertin [Ber15]
-1	3	2	145	
2	4	1/4	256	
2	6	1/40	2336	
-2	4	1/20	768	
-2	6	1/48	2848	
3	6	1/16	828	
-3	6	1/72	4464	

Deninger [Den97] related the Mahler measure to the regulator and used this connection to predict the appearance of L-functions of elliptic curves. This was further explored by Rodriguez-Villegas [RV99], who found that the polynomial must be tempered, which means that the Mahler measures of the sides of its Newton polygon are zero. (For example, the family $S_{k,\beta}(X, Y)$ yields a tempered polynomial iff $|\beta| = 1$.)

In recent collaboration with Mittal [LM17] we propose the following extension of Mahler measure.

Definition 2. Let $a_1, \ldots, a_n \in \mathbb{R}_{>0}$. The $(a_1, \ldots, a_n)$-Mahler measure of a non-zero rational function $P \in \mathbb{C}(x_1, \ldots, x_n)$ is defined by

$$\mathrm{m}_{a_1,\ldots,a_n}(P) := \frac{1}{(2\pi i)^n} \int_{\mathbb{T}_{a_1} \times \cdots \times \mathbb{T}_{a_n}} \log |P(x_1, \ldots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n},$$

where $\mathbb{T}_a = \{x \in \mathbb{C} : |x| = a\}$.

The idea of considering arbitrary tori in the integration was initially proposed to us by Rodriguez-Villegas a long time ago. With this definition, Cassaigne and Maillot's formula (1) can be interpreted as $\mathrm{m}_{a,b,c}(x+y+z)$.

We have proved the following results for generalized tori.

Theorem 1. *[Lalín and Mittal* [LM17]*]*

$$\mathrm{m}_{a,a}(Y^2+2XY-X^3+X) = \begin{cases} 2\log a + 2L'(E_{20},0) & \frac{\sqrt{5}-1}{2} \le a \le \frac{1+\sqrt{5}}{2}, \\ 3\log a & a \ge \frac{3+\sqrt{13}}{2}, \\ \log a & 0 < a \le \frac{-3+\sqrt{13}}{2}, \end{cases}$$

$$\mathrm{m}_{a^2,a}((1+x)(1+y)(x+y)+2xy) = \begin{cases} 4\log a + 3L'(E_{20},0) & 1 \le a \le A_+, \\ 2\log a + 3L'(E_{20},0) & A_- \le a \le 1, \end{cases}$$

where $A_\pm = \sqrt{\frac{1+\sqrt{5}\pm\sqrt{2\sqrt{5}+2}}{2}}$.

These results rest on the work of Rogers and Zudilin for $R_{-2}(x,y)$ and of Touafek and Bertin for $S_{2,-1}(X,Y)$. The main difficulty that arises when trying to extend to the general case of arbitrary $a,b \in \mathbb{R}_{>0}$ is the study of certain integration path resulting from applying Jensen's formula to the initial integral. In particular, we do not know how to give a formula when the path is not closed.

Finally, we remark that by applying some basic changes of variables, the formulas above can also be expressed in terms of classical Mahler measures of non-tempered polynomials, which is interesting in its own right.

References

[Ber15] Marie-José Bertin, *Mahler measure, regulators, and modular units*, Workshop lecture at *"The Geometry, Algebra and Analysis of Algebraic Numbers"*, Banff International Research Station, Banff, Canada, 2015.

[Boy98] David W. Boyd, *Mahler's measure and special values of L-functions*, Experiment. Math. **7** (1998), no. 1, 37–82. MR 1618282

[Den97] Christopher Deninger, *Deligne periods of mixed motives, K-theory and the entropy of certain* $\mathbf{Z}^n$*-actions*, J. Amer. Math. Soc. **10** (1997), no. 2, 259–281. MR 1415320

[LM17] Matilde N. Lalín and Tushant Mittal, *The Mahler measure of arbitrary tori*, ArXiv e-prints (2017).

[LR17] Matilde N. Lalín and Frank Ramamonjisoa, *The Mahler measure of a Weierstrass form*, Int. J. Number Theory **13** (2017), no. 8, 2195–2214. MR 3681698

[Mai00] Vincent Maillot, *Géométrie d'Arakelov des variétés toriques et fibrés en droites intégrables*, no. 80, 2000. MR 1775582

[Mel12] A. Mellit, *Elliptic dilogarithms and parallel lines*, ArXiv e-prints (2012).

[RV99] F. Rodriguez-Villegas, *Modular Mahler measures. I*, Topics in number theory (University Park, PA, 1997), Math. Appl., vol. 467, Kluwer Acad. Publ., Dordrecht, 1999, pp. 17–48. MR 1691309

[RZ12] Mathew Rogers and Wadim Zudilin, *From L-series of elliptic curves to Mahler measures*, Compos. Math. **148** (2012), no. 2, 385–414. MR 2904192

[Smy81] C. J. Smyth, *On measures of polynomials in several variables*, Bull. Austral. Math. Soc. **23** (1981), no. 1, 49–63. MR 615132

[Tou08a] Nouressadat Touafek, *From the elliptic regulator to exotic relations*, An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat. **16** (2008), no. 2, 117–125. MR 2473805
[Tou08b] ———, *Thèse de doctorat*, 2008, Université de Constantine, Algeria.
[Zud14] Wadim Zudilin, *Regulator of modular units and Mahler measures*, Math. Proc. Cambridge Philos. Soc. **156** (2014), no. 2, 313–326. MR 3177872

A meromorphic extension of the 3D-index

Rinat Kashaev
(joint work with Stavros Garoufalidis)

1. Introduction

Let X be an ideal triangulation of an oriented 3-manifold M with one cusp and a fixed peripheral structure, i.e. a fixed basis of the first integral homology of the torus boundary obtained after cutting out an open neighbourhood of the cusp. The 3D-index of X introduced by Dimofte–Gaiotto–Gukov in [3] is a map $I_X\colon \mathbb{Z}^2 \to \mathbb{Z}[\![q]\!]$ which is well defined for X admitting a strict angle structure. In general, the 3D-index is not a well defined quantity in the case of arbitrary ideal triangulations of M. Nonetheless, it is expected to be a topological invariant, in particular, the coefficients of the corresponding q-series are expected to have interesting geometrical interpretation in terms of generalised normal surfaces [4]. The building block in the construction of the 3D-index is the tetrahedral index $I_\Delta\colon \mathbb{Z}^2 \to \mathbb{Z}[\![q]\!]$ defined as

$$I_\Delta(m,e) = \sum_{n=\max(0,-e)}^{\infty} (-1)^n \frac{q^{n(n+1)-(2n+e)m}}{(q^2;q^2)_n (q^2;q^2)_{n+e}}.$$

Theorem 1 ([5])**.** *Let $q \in \mathbb{R}_{\neq 0}$ be such that $|q| < 1$, X an ideal triangulation of an oriented 3-manifold M with 1 cusp and a fixed peripheral structure. Then there exists a meromorphic function $J_X\colon \left(\mathbb{C}_{\neq 0}\right)^2 \to \mathbb{C}$ such that*

1. *$J_X = J_{X'}$ if X and X' are related by a shaped $2-3$ or $3-2$ Pachner move;*
2. *if X admits a strict angle structure, then J_X admits a convergent Laurent series expansion of the form*

$$J_X(s,t) = \sum_{m,e\in\mathbb{Z}} s^m t^e I_X(m,e).$$

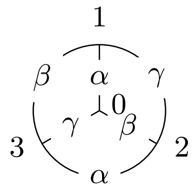
Remark 1. The construction of the function J_X can be extended to a generalised TQFT on shaped triangulations of all oriented pseudo 3-manifolds. In particular, one obtains invariants of hyperbolic $\mathbb{R}^3$ with conical singularities along string links, namely with the components having conical angles $\alpha_1, \dots, \alpha_n$ satisfying the condition

$$\sum_{i=1}^{n} \alpha_i = (n-1)2\pi.$$

2. Construction of the partition function

In what follows, for two sets A and B, we denote by B^A the set of all maps from A to B. We also denote by X_i the set of i-dimensional cells in a CW-complex X. A *triangulation* for us will mean a CW-complex X, where each n-cell is given by a characteristic map of the form $\alpha\colon \Delta^n \to X$, where Δ^n is the standard n-dimensional simplex, and for each face map $f_i\colon \Delta^{n-1} \to \Delta^n$, the composition $\alpha \circ f_i$ is the characteristic map of an $(n-1)$-cell.

We say a triangulation X of an oriented pseudo 3-manifold M is *shaped* if each tetrahedron of X carries the structure of an ideal hyperbolic tetrahedron. Les us fix a *state* of X given by a map $x\colon X_1 \to \mathbb{T}$, where $\mathbb{T}$ is the complex unit circle. For each tetrahedron $T \in X_3$ with the shape structure given by dihedral angles α, β, γ arranged according to this picture



$$\alpha + \beta + \gamma = \pi$$

we associate the weight function

$$B(T, x) = c(q) G_q\left((-q)^{\alpha/\pi} x_3/x_2\right) G_q\left((-q)^{\beta/\pi} x_1/x_3\right) G_q\left((-q)^{\gamma/\pi} x_2/x_1\right)$$

where $x_i := x(v_0 v_i) x(v_j v_k)$ for $\{i, j, k\} = \{1, 2, 3\}$, $v_i v_j$ denoting the edge of T connecting the vertices v_i and v_j, and we use the functions

$$G_q(z) := \frac{(-q/z; q)_\infty}{(z; q)_\infty}, \quad c(q) := \frac{(q; q)_\infty^2}{(q^2; q^2)_\infty}.$$

The *partition function* of X is defined by the formula

$$Z(X) := \int_{\mathbb{T}^{X_1}} \mathrm{d}\mu(x) \prod_{T \in X_3} B(T, x), \quad \mathrm{d}\mu(x) := \prod_{e \in X_1} \frac{\mathrm{d}x(e)}{x(e) 2\pi i}.$$

An edge of X is called *balanced* if the sum of dihedral angles around it is equal to 2π. The partition function $Z(X)$ is invariant under the $2-3$ or $3-2$ Pachner moves provided the relevant edge is balanced. As $Z(X)$ appears to be a meromorphic function in complexified angle variables, the balancing conditions are achieved by analytic continuation from positive angle variables, by balancing first all three-valent edges (i.e. the edges shared by exactly three distinct tetrahedra) and only then balancing all other edges. Additionally, $Z(X)$ is gauge invariant with respect to the Neumann–Zagier hamiltonian flows generated by total dihedral angles around the edges of X.

When X is an ideal triangulation of a 1-cusped oriented 3-manifold M with a fixed peripheral structure, the meromorphic index $J_X(s, t)$ is obtained from $Z(X)$ by balancing all edges and relating the remaining degrees of freedom to s and t by using the peripheral structure, see [5] for details.

3. Tetrahedral index and a quantum dilogarithm over $\mathbb{T} \times \mathbb{Z}$

The tetrahedral weight function can be written in the form

$$B(T, x) = \varphi_q \left((-q)^{\beta/\pi} x_1/x_3, (-q)^{\alpha/\pi} x_2/x_3 \right)$$

where

$$\varphi_q(x, y) = \sum_{m,e \in \mathbb{Z}} \tilde{I}_\Delta(m, e) \frac{x^e}{y^m}, \quad \tilde{I}_\Delta(m, e) := (-q)^e I_\Delta(m, e),$$

and the topological invariance is based on the five term integral identity

$$\varphi_q(x, y)\varphi_q(u, v) = \int_{\mathbb{T}} \varphi_q \left(uy, \frac{v}{z} \right) \varphi_q \left(\frac{xyuv}{z}, z \right) \varphi_q \left(xv, \frac{y}{z} \right) \frac{\mathrm{d}z}{z 2\pi i}$$

We can also write

$$\varphi_q(x, y) = \sum_{m \in \mathbb{Z}} \phi_q(1/x, m) y^{-m},$$

where

$$\phi_q(x, m) := \frac{(-q^{1-m} x; q^2)_\infty}{(-q^{1-m}/x; q^2)_\infty}$$

is the quantum dilogarithm over the Pontryagin self-dual locally compact abelian group $\mathbb{T} \times \mathbb{Z}$ with the gaussian exponential $\langle (x, m) \rangle = x^m$, see [2], which was first found in [6]. This fact puts the 3D-index into the general framework underlying the Teichmüller TQFT of [1].

References

[1] Jørgen Ellegaard Andersen and Rinat Kashaev, *A TQFT from Quantum Teichmüller theory*, Comm. Math. Phys. **330** (2014), no. 3, 887–934.

[2] Jørgen E. Andersen and Rinat M. Kashaev, *Complex Quantum Chern-Simons*, arXiv:1409.1208, Preprint 2014.

[3] Tudor Dimofte, Davide Gaiotto, and Sergei Gukov, *3-manifolds and 3d indices*, Adv. Theor. Math. Phys. **17** (2013), no. 5, 975–1076.

[4] Stavros Garoufalidis, Craig D. Hodgson, Neil R. Hoffman, and J. Hyam Rubinstein, *The 3D-index and normal surfaces*, Illinois J. Math. (2017).

[5] Stavros Garoufalidis, Rinat Kashaev, *A meromorphic extension of the 3D Index*, arXiv:1706.08132, Preprint 2017.

[6] S. L. Woronowicz, *Operator equalities related to the quantum $E(2)$ group*, Comm. Math. Phys. **144** (1992), no. 2, 417–428.

Skein algebras of surfaces and quantum $SL_2(\mathbb{C})$

Thang Le

Let Σ be an oriented compact surface and $\mathcal{R}$ be a commutative ring with unit 1 and a distinguished invertible element $q^{1/2} \in \mathcal{R}$. A 1-dimensional submanifold α of $\Sigma \times (0,1)$ is *framed* if it is equipped with a *framing*, i.e. a continuous choice of a vector transverse to α at each point of α. The *Kauffman bracket skein algebra* $\mathcal{S}(\Sigma)$, introduced by Przytycki and Turaev, is defined as the $\mathcal{R}$-module spanned by isotopy classes of framed unoriented links in $\Sigma \times (0,1)$ modulo the skein relation and the trivial loop relation of Figure 1

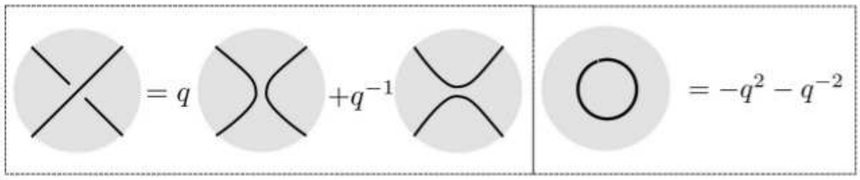


Figure 1. The skein relation (left) and the trivial loop relation (right).

As usual, links are presented by diagrams on Σ and framing is vertical, i.e. the framing at a point is parallel to the $(0,1)$ factor and points in the direction of 1.

The skein algebra is closely related to classical objects such as the $SL_2(\mathbb{C})$-character variety, the Teichmüller spaces, and quantum objects such as the Jones polynomial; it plays an important role as it can serve as a bridge between quantum topology and classical topology.

The Teichmüller spaces can be studied through an ideal triangulation of the surface, by work of Thurston, Bonahon, Penner. We want also to study the skein algebra through triangulations of the surface. For this we need to extend the definition of $\mathcal{S}(\Sigma)$ to involve the boundary $\partial\Sigma$ of Σ.

A surface Σ is called a punctured bordered surface if $\Sigma = \overline{\Sigma} \setminus \mathcal{P}$, where $\overline{\Sigma}$ is a compact oriented surface and $\mathcal{P}$ is a finite set such that every boundary component of $\overline{\Sigma}$ has at least one point in $\mathcal{P}$. A connected component of $\partial\Sigma := \partial\overline{\Sigma} \setminus \mathcal{P}$ is called a *boundary edge* of Σ. Note that any boundary edge is diffeomorphic to the open interval $(0,1)$.

A $\partial\Sigma$*-tangle* is an unoriented, framed, compact, properly embedded 1-dimensional submanifold $\alpha \subset \Sigma \times (0,1)$ such that:

- at every point of $\partial\alpha = \alpha \cap (\partial\Sigma \times (0,1))$ the framing is *vertical*, and
- for any boundary edge b, the points of $\partial_b(\alpha) := \partial\alpha \cap (b \times (0,1))$ have distinct heights.

Two $\partial\Sigma$-tangles are *isotopic* if they are isotopic in the class of $\partial\Sigma$-tangles. The emptyset, by convention, is a $\partial\Sigma$-tangle which is isotopic only to itself.

For a $\partial\Sigma$-tangle α define a partial order on $\partial(\alpha)$ by: $x > y$ if x and y are in the same boundary edge and x has greater height. If $x > y$ and there is no z such that $x > z > y$, then we say x and y are *consecutive*.

A *stated $\partial\Sigma$-tangle* α is a $\partial\Sigma$-tangle α equipped with a *state*, which is a function $s : \partial\alpha \to \{+,-\}$.

The *(Kauffman bracket) stated skein module* $\mathcal{S}_{\mathrm{s}}(\Sigma)$ is the $\mathcal{R}$-module freely spanned by isotopy classes of stated $\partial\Sigma$-tangles modulo the above skein relation and the trivial loop relation (see Figure 1), and the boundary relations in Figure 2.

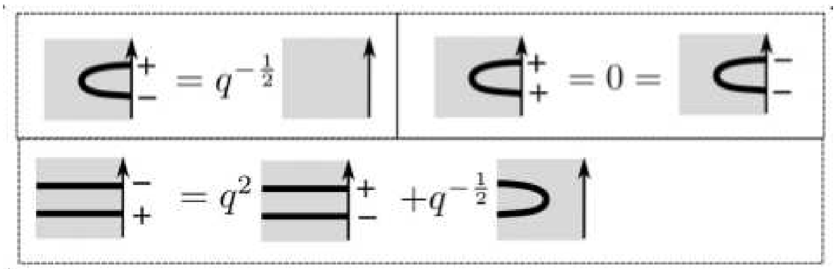


FIGURE 2. The boundary relations.

Here in these identities, each shaded part is a part of Σ, with a stated $\partial\Sigma$-tangle diagram on it. Each arrowed line is part of a boundary edge, and the order on that part is indicated by the arrow and the points on that part are consecutive in the height order. The order of other end points away from the picture can be arbitrary and are not determined by the arrows of the picture. On the right hand side of the first identity of Figure 2, the arrow does not play any role; it is there only because the left hand side has an arrow.

Suppose a and b are distinct boundary edges of Σ. Let $\Sigma' = \Sigma/(a = b)$, which is obtained from Σ by gluing a and b together. The canonical projection $\mathrm{pr} : \Sigma \to \Sigma'$ induces a projection $\widetilde{\mathrm{pr}} : \Sigma \times (0,1) \to \Sigma' \times (0,1)$. Let $c = \mathrm{pr}(a) = \mathrm{pr}(b)$.

A $\partial\Sigma'$-tangle $\alpha \subset (\Sigma' \times (0,1))$, is said to be *vertically transverse to* c if

- α is transverse to $c \times (0,1)$,
- the points in $\partial_c\, \alpha := \alpha \cap (c \times (0,1))$ have distinct heights, and have vertical framing.

Suppose α is a $\partial\Sigma'$-tangle vertically transverse to c. Then $\tilde{\alpha} := \widetilde{\mathrm{pr}}^{-1}(\alpha)$ is a $\partial\Sigma$-tangle. Suppose in addition α is stated, with state $s : \partial\alpha \to \{\pm\}$. For any $\boldsymbol{\varepsilon} : \alpha \cap (c \times (0,1)) \to \{\pm\}$ define $\tilde{\alpha}(\boldsymbol{\varepsilon})$ to be $\tilde{\alpha}$ equipped with state $\tilde{s}$ defined by $\tilde{s}(x) = s(\mathrm{pr}(x))$ if $\mathrm{pr}(x) \in \partial\alpha$ and $\tilde{s}(x) = \boldsymbol{\varepsilon}(\mathrm{pr}(x))$ if $\mathrm{pr}(x) \in c$. We call $\tilde{\alpha}(\boldsymbol{\varepsilon})$ *a lift* of α. If $|\alpha \cap (c \times (0,1))| = k$, then α has 2^k lifts.

Theorem 1. *Suppose a and b are two distinct boundary edges of a punctured bordered surface Σ. Let $\Sigma' = \Sigma/(a = b)$, and c be the image of a (or b) in Σ'.*

(a) There is a unique $\mathcal{R}$-algebra homomorphism $\rho : \mathcal{S}_{\mathrm{s}}(\Sigma') \to \mathcal{S}_{\mathrm{s}}(\Sigma)$ such that if α is a stated $\partial\Sigma'$-tangle vertically transverse to c, then $\rho(\alpha) = \sum_\beta [\beta]$, where the sum is over all lifts β of α, and $[\beta]$ is the element in $\mathcal{S}_{\mathrm{s}}(\Sigma)$ represented by β.

(b) In addition, ρ is injective.

(c) For four distinct boundary edges a_1, a_2, b_1, b_2 of Σ, the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{S}_{\rm s}(\Sigma/(a_1 = b_1, a_2 = b_2)) & \xrightarrow{\rho} & \mathcal{S}_{\rm s}(\Sigma/(a_1 = b_1)) \\ \rho\downarrow & & \downarrow\rho \\ \mathcal{S}_{\rm s}((\Sigma/(a_2 = b_2) & \xrightarrow{\rho} & \mathcal{S}_{\rm s}(\Sigma). \end{array} \tag{1}$$

Suppose an ideal triangulation of Σ is given. This means there is a finite collection $\tilde{\mathcal{F}}$ of disjoint ideal triangles and a finite collection of disjoint pairs of elements in $\tilde{\mathcal{E}}$, the set of all edges of ideal triangles in $\tilde{\mathcal{F}}$, such that Σ is obtained from $\bigsqcup_{\mathfrak{T}\in\tilde{\mathcal{F}}} \mathfrak{T}$ by gluing the two edges in each pair. It may happen that two edges of one triangle are glued together.

From Theorem 1 we have an injective algebra homomorphism

$$\rho : \mathcal{S}_{\rm s}(\Sigma) \to \bigotimes_{\mathfrak{T}\in\tilde{\mathcal{F}}} \mathcal{S}_{\rm s}(\mathfrak{T}). \tag{2}$$

The map ρ is described explicitly by Theorem 1. The algebra $\mathcal{S}_{\rm s}(\mathfrak{T})$ has a simple presentation and is not difficult to investigate using algebraic method. In particular, by going to a quotient of $\mathcal{S}_{\rm s}(\mathfrak{T})$ one can recover from (2) the quantum trace map of Bonahon and Wong.

When Σ is a bigon, the stated skein algebra $\mathcal{S}_{\rm s}(\Sigma)$ is naturally isomorphic to the quantum matrix algebra $SL_2(q)$, and many algebraic facts concerning $SL_2(q)$ have nice pictorial descriptions in terms of stated skein algebras (joint work with F. Costantino).

References

[1] T. Lê, *Triangular decomposition of skein algebras*, preprint arXiv:1609.04987, 2016. Quantum Topology, to appear.

Automorphic forms for $g \geq 1$

Werner Nahm

(joint work with Marianne Leitner)

If $(f_1, \ldots, f_k)$ is a vector valued modular function, then $\sum_{i=1}^{k} |f_i|^2$ can be conceived as a map from the moduli space of flat metric tori to $\mathbb{R}_+$. The f_i can be recovered by acting with $\partial, \bar{\partial}$ on this real analytic function. In conformal field theory (CFT) one extends this map to tori with arbitrary metric g by

$$\delta \log Z(g) = \frac{c}{48\pi} \int_{\rm torus} \delta(\log\rho)\, R\, d\,{\rm vol}, \tag{*}$$

where $g = \rho g_0$ varies in a Weyl class of metrics with fixed conformal structure given by g_0, $R\,d\,$vol is a Riemann curvature $\times$ volume form for g, $c \in \mathbb{R}$ a fixed constant called the central charge. Z is called the partition function. One has

$$Z(g) = \sum_{i=1}^{k} |f_i(\tau)|^2 \quad (\tau \text{ given by } g) \text{ for flat } g.$$

Eq. (*) can be integrated.

$$\log \frac{Z(\rho g)}{Z(g)} = \frac{c}{96\pi} \int (\log \rho)\, ((R\, d\,\mathrm{vol})(\rho g) + (R\, d\,\mathrm{vol})(g))\,.$$

(The + sign comes from $x^2 - y^2 = (x-y)(x+y)$.)

A special case is the $(2,5)$ minimal model with $c = -\frac{22}{5}$ and

$$f_1 = q^{-\frac{1}{60}} \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n}, \quad f_2 = q^{\frac{11}{60}} \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n}$$

the Rogers-Ramanujan functions ($(q)_n = (1-q)\cdots(1-q^n)$).

Following Einstein, one introduces the energy-momentum field as a functional derivative w.r.t. g. In local coordinates x^1, x^2, $g = g_{\mu\nu} dx^\mu dx^\nu$ $(\mu, \nu = 1, 2)$

$$T_{\mu\nu}(x) = \frac{1}{2} \frac{\delta}{\delta g^{\mu\nu}(x)}$$

$$\text{(e.g.} \quad Z(g + \varepsilon h) - Z(g) = \varepsilon \int h^{\mu\nu} \frac{\delta Z}{\delta g^{\mu\nu}}\, d\,\mathrm{vol} + O(\varepsilon^2) \quad).$$

n-fold derivatives of Z are called n-point functions of T, with the notation

$$(**) \qquad \big\langle T_{\mu_1\nu_1}(x_1) \cdots T_{\mu_n\nu_n}(x_n) \big\rangle = 2^{-n} \frac{\delta}{\delta g^{\mu_1\nu_1}(x_1)} \cdots \frac{\delta}{\delta g^{\mu_n\nu_n}(x_n)} Z.$$

The x_i have to be pairwise different to avoid singularities. Invariance under diffeomorphisms yields

$$\mathcal{D}_\mu T^{\mu\nu} = 0 \quad (\mathcal{D}_\mu\text{: covariant derivative}),$$

where for fields ϕ one defines $\phi = 0 \Leftrightarrow \big\langle \phi \cdots \big\rangle = 0$, $\big\langle \partial_\mu \phi \cdots \big\rangle = \partial_\mu \big\langle \phi \cdots \big\rangle$, $\big\langle (a_1\phi_1 + a_2\phi_2) \cdots \big\rangle = a_1 \big\langle \phi_1 \cdots \big\rangle + a_2 \big\langle \phi_2 \cdots \big\rangle$ with a_1, a_2 functions on the torus.

The prescribed behaviour under Weyl transformation yields

$$g_{\mu\nu} T^{\mu\nu} = -\frac{c}{24\pi} R. \quad (\text{with } \big\langle 1 \cdots \big\rangle = \langle \cdots \rangle)$$

Using complex coordinates z with $g = 2\rho\, dz d\bar{z}$

$$T_{\mu\nu} dx^\mu dx^\nu = T_{zz} dz^2 + 2T_{z\bar{z}} dz d\bar{z} + T_{\bar{z}\bar{z}} d\bar{z}^2$$

and defining $T(z)$ by

$$\frac{T(z)}{2\pi} = T_{zz} + \frac{c}{24} \left(\partial_z^2 \log \rho - \frac{1}{2} (\partial_z \log \rho)^2 \right)$$

(analogously for $\bar{T}$) one finds

- $T(z)$ is invariant under Weyl rescalings $g \to \rho g$
- $\partial_{\bar{z}} T = 0$

- Under holomorphic maps $u \to z(u)$, T transforms as

$$T(z)dz^2 = T(u)du^2 - \frac{c}{12}S(z,u)du^2 \quad \text{where} \quad S(z,u) = \frac{\partial_u^3 z}{\partial_u z} - \frac{3}{2}\left(\frac{\partial_u^2 z}{\partial_u z}\right)^2$$

- Virasoro OPE $T(z_1)T(z_2) = \frac{c/2}{(z_1-z_2)^4} + \frac{T(z_1)+T(z_2)}{(z_1-z_2)^2} + R(z_1,z_2)$, where $R(z_1,z_2)$ is regular. Taylor expansion of R defines new fields Φ_n:

$$R(z_1,z_2) = \sum_{n=0}^{\infty}(z_1-z_2)^n \Phi_n(z_2).$$

(By $R(z_1,z_2) = R(z_2,z_1)$ the Φ_{2n+1} are determined by the Φ_{2n}.)

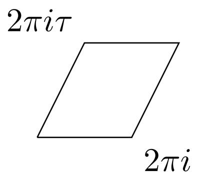
In general the Φ_n are not linearly expressible in terms of T. As can be checked by considering the behaviour of the Virasoro OPE under holomorphic transformation there is only one case where $\Phi_0 \sim T''$ is possible, namely $c = -22/5$ and

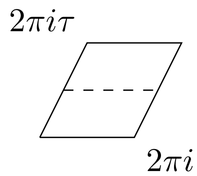
$$T(z_1)T(z_2) = \frac{-11/5}{(z_1-z_2)^4} + \frac{T(z_1)+T(z_2)}{(z_1-z_2)^2} - \frac{1}{10}(T''(z_1)+T''(z_2)) + O((z_1-z_2)^2).$$

This defines the $(2,5)$ minimal model. In this case, Z can be determined as follows:

- $\langle T(z)\rangle$ is a holomorphic doubly periodic function of z. Thus it only depends on τ. Notation: $\langle T(z)\rangle = \langle T\rangle$. In particular, $\langle T''(z)\rangle = 0$.
- For the torus given by the lattice $\langle 2\pi i, 2\pi i\tau\rangle$ and the corresponding Weierstrass function $\mathcal{P}$

$$\langle T(z_1)T(z_2)\rangle = -\frac{11}{5}\left(\mathcal{P}''(z_1-z_2) - \frac{E_4}{120}\right)Z + 2\mathcal{P}(z_1-z_2)\langle T\rangle.$$

Changing the metric of the torus 

along a closed line 

so that $\tau \to \tau + d\tau$, the equations (**) specialize to the ODE

$$\frac{1}{2\pi i}\frac{d}{d\tau}Z = \langle T\rangle$$

$$\frac{1}{2\pi i}\frac{d}{d\tau}\langle T\rangle = \oint \langle T(z_1)T(z)\rangle dz_1 = \frac{11}{3600}E_4 Z + \frac{1}{6}E_2\langle T\rangle,$$

with solution

$$Z = \tilde{f}_1 f_1 + \tilde{f}_2 f_2, \quad \partial_\tau \tilde{f}_i = 0 \text{ for } i = 1,2.$$

Complex conjugation yields Z up to conventional normalization.

All of this can be generalized not only to arbitrary CFTs with a finite number of characters f_i but also to surfaces of higher genus g[1]. The constant curvature metric is out of reach, thus it is convenient to use flat surfaces, with curvature

[1]This g is different from g the metric

concentrated on vertices in multiples of 2π. More precisely one must regularize by putting flat circles around the vertices. One obtains

$$Z(g) = \prod r_i^{\nu_i} Z_{\text{reg}},$$

where the r_i are the radii of the circles and the ν_i are integral multiples of $c/24$.

For hyperelliptic curves it suffices to use curve $\to \mathbb{P}^1$ with flat metric on the Gauss plane, sufficiently large radius for the circle(s) around ∞, sufficiently small radius for the circles around (other) ramification points. We made an explicit computation of the $(2,5)$ minimal model and genus 2 along the lines described above. It suffices to consider eqs. (**) for $n = 0, 1, 2$. We put one ramification point at ∞, the others ar $X_1, \ldots, X_5$. W.r.t. each $\frac{d}{dX_i}$ one obtains a system of five linear ODEs. Compatibility is implied by the existence of the CFT (or to be checked by tedium).

The equations w.r.t. $\frac{d}{dX_i}$ involve the functions $f, \vartheta_i, \dot{\vartheta}_i, \ddot{\vartheta}_i, \varphi_i$. Let $p(t) = \prod_{i=1}^5 (t - X_i)$, $\dot{p} = \frac{dp}{dt}$ etc., $c = -22/5$

$$\mathcal{D}_i = \frac{d}{dX_i} - \frac{c}{8} \sum_{j \neq i} \frac{1}{X_i - X_j}. \quad \text{Then}$$

$$\mathcal{D}_i f = \frac{2}{\dot{p}(X_i)} \vartheta_i$$

$$\mathcal{D}_i \vartheta_i = \left(\frac{9}{10} \frac{\ddot{p}}{\dot{p}} \vartheta_i + \frac{3}{10} \dot{\vartheta}_i + \frac{7c}{320} \frac{\ddot{p}^2}{\dot{p}} f - \frac{7c}{480} \dddot{p} f \right) \Big|_{t=X_i}$$

$$\mathcal{D}_i \dot{\vartheta}_i = \left(\frac{11}{30} \frac{\dddot{p}}{\dot{p}} \vartheta_i + \frac{7}{10} \frac{\ddot{p}}{\dot{p}} \dot{\vartheta}_i + \frac{7}{10} \ddot{\vartheta}_i + \frac{7c}{480} \frac{\ddot{p}\dddot{p}}{\dot{p}} f - \frac{c}{480} p^{IV} f \right) \Big|_{t=X_i}$$

$$\mathcal{D}_i \ddot{\vartheta}_i = \frac{2}{\dot{p}(X_i)} \beta_i$$

$$\mathcal{D}_i \beta_i = \left(\frac{9}{10} \frac{\dddot{p}}{\dot{p}} \beta_i + \frac{1601}{6000} p^V \vartheta_i + \frac{p^{IV}}{20} \dot{\vartheta}_i + \frac{143}{1200} \dddot{p} \ddot{\vartheta}_i + \frac{293c}{48000} \ddot{p} p^V f + \frac{c}{480} \dddot{p} p^{IV} f \right) \Big|_{t=X_i}$$

(Frobenius indices $\frac{7}{10}$ (3 times), $\frac{11}{10}$ (twice), no logarithms). The systems are connected by

$$\vartheta_j = \vartheta_i + (X_j - X_i) \dot{\vartheta}_i + \frac{1}{2} (X_j - X_i)^2 \ddot{\vartheta}_i - \frac{c}{160} (X_j - X_i)^3 \dddot{p}(X_i) f,$$

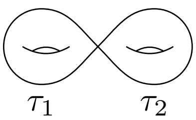
$$\dot{\vartheta}_j = \dot{\vartheta}_i + (X_j - X_i) \ddot{\vartheta}_i - \frac{3c}{160} (X_j - X_i)^2 \dddot{p}(X_i) f,$$

$$\ddot{\vartheta}_j = \ddot{\vartheta}_i - \frac{3c}{80} (X_j - X_i) \dddot{p}(X_i) f,$$

$$\beta_i = B$$

$$+ \left(\frac{101}{600} p^{IV} \theta_i - \frac{1}{600} \dddot{p} \dot{\vartheta}_i + \frac{37}{210} \ddot{p} \ddot{\vartheta}_i + \left(\frac{49c}{72000} \dddot{p}^2 + \frac{67c}{12000} \ddot{p} p^{IV} - \frac{13c}{960} \dot{p} p^V \right) f \right) \Big|_{t=X_i}$$

(B independent of i).

Note that the only denominator is $\dot{p}(X_i) = \prod_{j \neq i} (X_i - X_j)$, so that the system has only regular singularities. When 3 ramification points approach each other, the curve splits as 

and four solutions of the system[2] reduce to products $f_i(\tau_1)f_j(\tau_2)$, $i,j=1,2$ in the guise

$$f_1(\tau) = \left(\frac{X(1-X)}{16}\right)^{-\frac{1}{30}} F_{21}\left(\frac{3}{10}, -\frac{1}{10}, \frac{3}{5}, X\right)$$

$$f_2(\tau) = \left(\frac{X(1-X)}{16}\right)^{\frac{11}{30}} F_{21}\left(\frac{7}{10}, \frac{11}{10}, \frac{7}{5}, X\right)$$

$$X = \frac{\vartheta_2^4(\tau)}{\vartheta_3^4(\tau)}, \qquad F_{21}\text{: Gauss hypergeometric function,}$$

$$\vartheta_2(\tau) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} q^{n^2/2}, \quad \vartheta_3(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2/2}.$$

The order of the corresponding systems for higher genus increases exponentially,

$$5^{g/2}F_{g-2} \quad \text{for } g \text{ even,} \quad 5^{(g-1)/2}(F_{g-1}+F_{g-3}) \quad \text{for } g \text{ odd,}$$

$$F_0, F_1, F_2, \ldots = 1, 1, 2, \ldots \quad \text{the Fibonacci numbers.}$$

For hyperelliptic surfaces one gets smaller systems of order F_{2g}. In the first case one gets automorphic forms for the mapping class group, in the second case for the braid group of the ramification points.

Bloch groups, units, modularity and Nahm sums

Don Zagier
(joint work with Stavros Garoufalidis)

The Volume Conjecture (by Kashaev in [1]) and the Arithmeticity Conjecture (by Garoufalidis in [2], and Dimofte, Gukov, Lenells and myself in [3]) predict that the N-th Kashaev invariant $\langle K\rangle_N$ of a hyperbolic knot K has an asymptotic expansion

$$(*) \qquad \langle K\rangle_N \sim N^{3/2} e^{CN} \lambda_0 \left(a_0 + a_1 \frac{2\pi i}{N} + a_2 \left(\frac{2\pi i}{N}\right)^2 + \cdots\right)$$

as $N \to \infty$, where C is essentially the volume of $S^3 \setminus K$ and λ_0^2 and a_j $(j \geq 0)$ belong to the trace field F of the knot.

A refinement of this is given by the Modularity Conjecture, which in particular predicts the existence of a similar expansion for $\mathcal{J}^K(q)$ as q tends to any root of unity, where $\mathcal{J}^K$ is a suitable extension of the Kashaev invariant $\langle K\rangle_N = \mathcal{J}^K(e^{2\pi i/N})$ to all roots of unity. This conjecture was originally formulated by me for the case of the Figure 8 knot [4], and has now been proved in that case and verified numerically for many other knots by Garoufalidis and myself. In all cases the expansion of $\mathcal{J}^K(q)$ near a fixed primitive n-th root of unity ζ has the

[2] The fifth solution is $\sim \eta^{-2/5}(\tau_1)\eta^{-2/5}(\tau_2)$ and leads to a new field (the second conformal block).

same form as (*) except that the coefficients a_j now lie in the cyclotomic extension $F_n = F(\zeta)$ and that λ_0 must be replaced by $\lambda_0 \varepsilon^{1/n}$ for some unit ε of F_n.

We guessed that this mysterious unit ε should depend on the knot only through its class $[K] \in B(F)$, the Bloch group of F (whose definition was recalled in the talk). This suggested the existence of a canonical map

$$B(F)/nB(F) \to U_n/U_n^n \quad (U_n = \text{units of } F_n).$$

In joint work with Frank Calegari [5], we proved the existence if such a map for any number field F and n prime to some number depending only on F, as well as a corresponding map from $K_3(F)/nK_3(F)$ to U_n/U_n^n, where K_3 is the third algebraic K-group of F. Both maps are injective, with (the same) known image. The map is given explicitly by associating to an element $\sum n_i[X_i]$ ($n_i \in \mathbb{Z}$, $X_i \in F$, $\sum n_i(X_i) \wedge (1 - X_i) = 0$) the element $u = \prod D_\zeta(x_i)^{n_i}$ in the Kummer extension $H = F(x_1, x_2, \dots)$ of F_n, where $x_i = \sqrt[n]{X_i}$ and $D_\zeta(x) := \prod_{k=1}^{n-1}(1 - \zeta^k x)^k$, and then showing that $u \in \varepsilon\,\{H^\times\}^n$ for a unique element ε of U_n/U_n^n.

As an unexpected corollary of this result, we also obtained a proof of Nahm's conjecture relating the modularity of certain special q-hypergeometric series ("Nahm sums") to the vanishing of a certain class in the Bloch group of $\overline{\mathbb{Q}}$. The proof depends on showing that the asymptotics of any Nahm sum as q tends to a root of unity ζ_n is given by a formula containing the above unit [6], together with the observation that the limiting values at cusps of a modular function can only take on finitely many distinct values.

References

[1] R. M. Kashaev, *The hyperbolic volume of knots from the quantum dilogarithm*, Lett. Math. Phys. **39** (1997), 269–275.

[2] Stavros Garoufalidis and Thang T. Q. Lê, *The colored Jones function is q-holonomic*, Geom. Topol. **9** (2005), 1253–1293. MR 2174266

[3] Tudor Dimofte, Sergei Gukov, Jonatan Lenells, and Don Zagier, *Exact results for perturbative Chern-Simons theory with complex gauge group*, Commun. Number Theory Phys. **3** (2009), no. 2, 363–443.

[4] Don Zagier, *Quantum modular forms*, Quanta of maths, Clay Math. Proc., vol. 11, Amer. Math. Soc., Providence, RI, 2010, pp. 659–675.

[5] Frank Calegari, Stavros Garoufalidis, and Don Zagier, *Bloch groups, algebraic K-theory, units, and Nahm's conjecture*, Preprint, 2017.

[6] Stavros Garoufalidis and Don Zagier, *Asymptotics of Nahm sums at roots of unity*, Preprint, 2017.

Milnor invariants in Ihara theory and triple power residue symbols

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1. Based on the analogies between knots and primes ([11]), mod 2 Milnor invariants $\mu_2(j_1 \cdots j_n)$ of certain rational primes $p_1, \dots, p_r$ were introduced in [10], as arithmetic analogues of Milnor invariants of a link ([9]). For example, $(-1)^{\mu_2(12)}$ coincides with the Legendre symbol (p_1/p_2). Assuming $\mu_2(ij) = 0$ $(1 \le i, j \le 3)$, $(-1)^{\mu_2(123)}$ is proved to equal the Rédei symbol $[p_1, p_2, p_3]$, which describes the decomposition of p_3 in a dihedral extension R, determined by p_1 and p_2, of degree 8 over $\mathbb{Q}$

$$R = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{\alpha}),$$

where $\alpha = x + y\sqrt{p_1}$ and x, y are certain integers satisfying $x^2 - p_1y^2 - p_2z^2 = 0$ with some non-zero integer z (cf. [14]). Recently, mod 3 Milnor invariants $\mu_3(ij)$ and $\mu_3(123)$ were introduced for certain primes $\mathfrak{p}_i = (\pi_i)$ $(1 \le i \le 3)$ of the Eisenstein field $\mathbb{Q}(\zeta_3)$, $\zeta_3 := \exp(2\pi\sqrt{-1}/3)$ ([1]). As in the mod 2 case, $\zeta_3^{\mu_3(12)}$ coincides with the cubic residue symbol $(\pi_1/\pi_2)_3$ and, assuming $\mu_3(ij) = 0$ for $1 \le i, j \le 3$, $[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_3 := \zeta_3^{\mu_3(123)}$ describes the decomposition of $\mathfrak{p}_3$ in a mod 3 Heisenberg extension K, determined by $\mathfrak{p}_1$ and $\mathfrak{p}_2$, of degree 27 over $\mathbb{Q}(\zeta_3)$

$$K = \mathbb{Q}(\zeta_3)(\sqrt[3]{\pi_1}, \sqrt[3]{\pi_2}, \sqrt[3]{\theta}),$$

where $\theta = x + y\sqrt[3]{\pi_1} + z(\sqrt[3]{\pi_1})^2$ and x, y, z are certain algebraic integers in $\mathbb{Z}[\zeta_3]$ satisfying $x^3 + \pi_1y^3 + \pi_1^2z^3 - 3\pi_1xyz - \pi_2^3w^3 = 0$ with some $w \in \mathbb{Z}[\zeta_3]$ ([ibid]). We note that a key ingredient to define well these mod l Milnor invariants of primes is the theory of pro-l extensions of number fields with restricted ramification due to Koch et al. (cf. [6]).

2. Ihara considered an arithmetic analogue of Artin's representation of a braid group ([4], [5]). Let l be a prime number and let k be a finite algebraic number field. He considered a certain continuous representation of the absolute Galois group $\mathrm{Gal}_k := \mathrm{Gal}(\overline{k}/k)$ on the free pro-l group $F_r^{(l)}$ on $x_1, \ldots, x_r$

$$\mathrm{Ih} : \ \mathrm{Gal}_k \longrightarrow \mathrm{Aut}(F_r^{(l)})$$

so that for $g \in \mathrm{Gal}_k$ and $1 \le j \le r$,

$$\mathrm{Ih}(g)(x_j) = y_j(g)x_j^{\chi_l(g)}y_j(g)^{-1}.$$

Here $\chi_l : \mathrm{Gal}_k \to \mathbb{Z}_l^\times$ is the l-cyclotomic character and $y_j(g) \in F_r^{(l)}$ is the unique pro-l word such that $y_j(g) \equiv \prod_{i \ne j} x_i^{e_i} \bmod [F_r^{(l)}, F_r^{(l)}]$ for some $e_i \in \mathbb{Z}_l$. In fact, the representation Ih is obtained from the natural action of Gal_k on the pro-l étale fundamental group $\pi_1^{(l)}(\mathbb{P}^1_{\overline{k}} \setminus \{a_0, a_1, \ldots, a_r\}; v_0)$, where a_i's are k-rational points with $a_0 = \infty$ and v_0 is a k-rational tangential base point at ∞. Following [5] and [15], the pro-l word $y_j(g)$ can be described in the following geometric way: For $1 \le i \le r$, choose a k-rational tangential point v_i at each a_i and let γ_i be a path from v_0 to v_i. Let x_i' be a small circle starting from v_i in the opposite clockwise direction. We set $x_i := \gamma_i^{-1} \cdot x_i' \cdot \gamma_i$, where paths are composed from the right. Then $\pi_1^{(l)}(\mathbb{P}^1_{\overline{k}} \setminus \{a_0, a_1, \ldots, a_r\}; v_0)$ is generated by $x_0, x_1, \ldots, x_r$, subject to the relation $x_r \cdots x_1x_0 = 1$. By [15], we can show that $y_j(g) = g(\gamma_j)^{-1} \cdot \gamma_j$ for $1 \le j \le r$.

Let $M^{(l)} : F_r^{(l)} \to \mathbb{Z}_l\langle\langle X_1, \ldots, X_r\rangle\rangle$ be the pro-l Magnus homomorphism defined by $M^{(l)}(x_i) = 1 + X_i$ for $1 \le i \le r$. The *l-adic Milnor number* $\mu^{(l)}(g; i_1 \cdots i_n j) \in \mathbb{Z}_l$ is defined by the coefficient of $X_{i_1} \cdots X_{i_n}$ in the pro-l Magnus expansion of $y_j(g)$: $M^{(l)}(y_j(g)) = 1 + \sum_{1 \le i_1, \ldots, i_n \le r} \mu^{(l)}(g; i_1 \cdots i_n j) X_{i_1} \cdots X_{i_n}$. We set $\mu^{(l)}(g; i) := 0$ $(1 \le i \le r)$. The *l-adic Milnor invariant* $\overline{\mu}^{(l)}(g; I)$ for a multi-index I with length

$|I| \geq 2$ is defined by taking modulo a certain indeterminacy $\Delta^{(l)}(g;I)$ (ideal of $\mathbb{Z}_l$):

$$\overline{\mu}^{(l)}(g;I) := \mu^{(l)}(g;I) \bmod \Delta^{(l)}(g;I).$$

We note that $\Delta^{(l)}(g;I) = 0$ if $\mu^{(l)}(g;J) = 0$ for any J with $|J| < |I|$ and $g \in \mathrm{Gal}_{k(\zeta_{l^\infty})}$, where $k(\zeta_{l^\infty})$ is the field obtained by adjoining all l-powerth roots of unity. The l-adic Milnor invariants of Galois elements satisfy some properties similar to the braid case (cf. [8]).

Now, as we mentioned in Section 1, mod l Milnor invariants $\overline{\mu}_l(I)$ of primes for $l = 2$ and for $l = 3$ and $|I| \leq 3$, which may be regarded as arithmetic analogues of Milnor invariants of links. Since Milnor invariants of a braid b coincide with those of the link obtained by closing b ([7], [12]), the analogy with topology suggests to ask the following

Question. Is there any relation between $\mu_l(I)$ and $\mu^{(l)}(g;I)$ for $l = 2, 3$?

3. In this section, we give an answer to the above Question for the case that $l = 2, 3$ and $|I| = 3$.

Let p_1, p_2 be distinct prime numbers satisfying $p_1 \equiv p_2 \equiv 1 \bmod 4$ and $\left(\frac{p_1}{p_2}\right) = 1$. Then there are integers x, y and z satisfying $x^2 - p_1 y^2 - p_2 z^2 = 0, (x, y, z) = 1, y \equiv 0 \bmod 2$ and $x - y \equiv 1 \bmod 4$. Then Rédei's field $R := \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{\alpha})$, $\alpha = x + y\sqrt{p_1}$, is a dihedral extension over $\mathbb{Q}$ of degree 8 unramified outside p_1, p_2 and ∞ ([14]).

In the setting of Section 1, for simplicity, we consider the case that $a_0 = \infty, a_1 = 0, a_2 = 1, a_3 = p_1(y/x)^2$ $(r = 3)$. Let S be the finite set of primes which divide $p_1 y^2$ or $x^2 - p_1 y^2 = p_2 z^2$ or l, and let Ω be the maximal Galois extension of $\mathbb{Q}$ unramified outside $S \cup \{\infty\}$. By [2], Ih factors through $\mathrm{Gal}(\Omega/\mathbb{Q})$. For a prime $p \notin S$, $\sigma_p \in \mathrm{Gal}(\Omega/\mathbb{Q})$ denote a Frobenius automorphism over p. Note that $R \subset \Omega$. Let p_3 be a prime number satisfying $p_3 \notin S$, $p_3 \equiv 1 \bmod 4$ and $\left(\frac{p_i}{p_3}\right) = \left(\frac{x}{p_3}\right) = 1$ $(i = 1, 2)$. Recall that the Rédei triple symbol $[p_1, p_2, p_3]$ is given by $\sigma_{p_3}(\sqrt{\alpha})/\sqrt{\alpha}$.

Let t be the coordinate on $\mathbb{P}^1$. Consider the algebraic function $f(t)$ and the function field $\mathfrak{R}$ defined by

$$f(t) := \sqrt{1 + \sqrt{t}}, \quad \mathfrak{R} := \mathbb{Q}(t)(\sqrt{t}, \sqrt{1-t}, f(t)).$$

Then $\mathfrak{R}/\mathbb{Q}(t)$ is a dihedral extension of degree 8 unramified outside $t = 0, 1, \infty$. Note that $\mathfrak{R}$ coincides with R when t is specialized to $p_1(y/x)^2$. By the condition on p_3 and similar computations as in [13], we have

$$y_3(\sigma_{p_3}) \equiv [x_1, x_2]^{\mu^{(2)}(\sigma_{p_3};123)} \bmod (F_3^{(2)})^2[F_3^{(2)}, [F_3^{(2)}, F_3^{(2)}]].$$

By [16], the monodromy transformations of $f(t)$ along 2 paths $\sigma_{p_3}(\gamma_3)^{-1} \cdot \gamma_3$ and $[x_1, x_2]^{\mu^{(2)}(\sigma_{p_3};123)}$ are $f(t) \mapsto \sigma_{p_3}(\sqrt{\alpha})/\sqrt{\alpha} f(t)$ and $f(t) \mapsto (-1)^{\mu^{(2)}(\sigma_{p_3};123)} f(t)$, respectively. Therefore we obtain the following

Theorem. *We have*

$$[p_1, p_2, p_3] = (-1)^{\mu^{(2)}(\sigma_{p_3};123)}$$

and hence $\mu_2(123) = \mu^{(2)}(\sigma_{\mathfrak{p}_3}; 123) \bmod 2$.

Similarly, for the case that $l = 3$ and $k = \mathbb{Q}(\zeta_3)$, we obtain $[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_3 = \zeta_3^{\mu^{(3)}(\sigma_{\mathfrak{p}_3};123)}$ and $\mu_3(123) = \mu^{(3)}(\sigma_{\mathfrak{p}_3}; 123) \bmod 3$ for certain primes $\mathfrak{p}_i$'s of $\mathbb{Q}(\zeta_3)$.

References

[1] F. Amano, Y. Mizusawa, M. Morishita, *On mod 3 triple Milnor invariants and triple cubic residue symbols in the Eisenstein number field*, arXiv:1412.6894.

[2] G. Anderson, Y. Ihara, *Pro-l branched coverings of $\mathbb{P}^1$ and higher circular l-units*, Ann. of Math. (2) **128** (1988), no. 2, 271–293.

[3] Y. Ihara, *Profinite braid groups, Galois representations and complex multiplications*, Ann. of Math. (2) **123** (1986), no. 1, 43–106.

[4] Y. Ihara, *Arithmetic analogues of braid groups and Galois representations*, Braids (Santa Cruz, CA, 1986), 245–257, Contemp. Math., **78**, Amer. Math. Soc., Providence, RI, 1988.

[5] Y. Ihara, Braids, Galois groups, and some arithmetic functions, Proc. of the ICM, (Kyoto, 1990), 99–120, Math. Soc. Japan, Tokyo, 1991.

[6] H. Koch, *Galoissche Theorie der p-Erweiterungen*, Springer-Verlag, Berlin-New York; VEB Deutscher Verlag der Wissenschaften, Berlin, 1970.

[7] H. Kodani, *Arithmetic topology on braid and absolute Galois groups*, Doctoral Thesis, Kyushu University, 2017.

[8] H. Kodani, M. Morishita, Y. Terashima, *Arithmetic topology in Ihara theory*, to appear in Publ. Res. Inst. Math. Sci., Kyoto Univ.

[9] J. Milnor, *Isotopy of links, in Algebraic Geometry and Topology*, A symposium in honor of S. Lefschetz (edited by R.H. Fox, D.C. Spencer and A.W. Tucker), 280–306 Princeton University Press, Princeton, N.J., 1957.

[10] M. Morishita, *On certain analogies between knots and primes*, J. Reine Angew. Math. **550** (2002), 141–167.

[11] M. Morishita, *Knots and Primes – An Introduction to Arithmetic Topology*, Universitext, Springer, 2011.

[12] K. Murasugi, B.I. Kurpita, *A study of braids, Mathematics and its Applications*, **484**, Kluwer Academic Publishers, Dordrecht, 1999.

[13] H. Nakamura, Z. Wojtkowiak, *On explicit formulae for l-adic polylogarithms*, In: Arithmetic fundamental groups and noncommutative algebra (Berkeley, CA, 1999), 285–294, Proc. Sympos. Pure Math., **70**, Amer. Math. Soc., Providence, RI, 2002.

[14] L. Rédei, *Ein neues zahlentheoretisches Symbol mit Anwendungen auf die Theorie der quadratischen Zahlkörper I*, J. Reine Angew. Math., **180** (1939), 1-43.

[15] Z. Wojtkowiak, *On l-adic iterated integrals. I. Analog of Zagier conjecture*, Nagoya Math. J. **176** (2004), 113–158.

[16] Z. Wojtkowiak, *A remark on nilpotent polylogarithmic extensions of the field of rational functions of one variable over* $\mathbb{C}$, Tokyo J. Math. **30** (2007), no. 2, 373–382.

Higher depth quantum modular forms

Kathrin Bringmann
(joint work with Antun Milas, Jonas Kaszian)

1. Quantum modular forms

To better understand the functions in the title, let us first recall quantum modular forms, introduced by Zagier (see [4] for many examples).

Roughly speaking, *quantum modular forms* are functions $f : \mathcal{Q} \to \mathbb{C}$ ($\mathcal{Q} \subseteq \mathbb{Q}$), for which the error of modularity ($M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$)

$$f(\tau) - (c\tau + d)^{-k} f(M\tau) \tag{1}$$

is "nice". The definition is intentionally vague to include many examples; the examples of interest for us in particular require (1) to be extendable to open subsets of $\mathbb{R}$ and to be real-analytic. Note that for classical modular forms (which live on the complex upper half plane instead of $\mathbb{Q}$) (1) is zero.

A famous motivating example comes from Kontsevich's "strange function" which was investigated by Zagier [3]. This function is defined as

$$KZ(q) := \sum_{m\geq 0} (q;q)_m,$$

where for $m \in \mathbb{N}_0 \cup \{\infty\}$, $(a;q)_m := \prod_{j=0}^{m-1}(1 - aq^j)$ denotes the usual *q-Pochhammer symbol*. This function does not converge on any open subset of $\mathbb{C}$, but converges for q any root of unity and is actually a finite sum. Zagier's study of KZ depends on the identity ($q := e^{2\pi i\tau}$ throughout)

$$\sum_{m\geq 0} \left(\eta(\tau) - q^{\frac{1}{24}} (q;q)_m\right) = \eta(\tau) D(\tau) + \frac{1}{2}\widetilde{\eta}(\tau). \tag{2}$$

Here $\eta(\tau) := q^{\frac{1}{24}} \prod_{n\geq 1}(1 - q^n) = \sum_{m\geq 1}\left(\frac{12}{m}\right) q^{\frac{m^2}{24}}$ is Dedekind's η-function, $\left(\frac{\cdot}{\cdot}\right)$ the Kronecker symbol, $D(\tau) := -\frac{1}{2} + \sum_{m\geq 1} \frac{q^m}{1-q^m}$ and $\widetilde{\eta}(\tau) := \sum_{m\geq 1}\left(\frac{12}{m}\right) m q^{\frac{m^2}{24}}$ the formal Eichler integral of η. The key observation of Zagier is that in (2), the functions $\eta(\tau)$ and $\eta(\tau)D(\tau)$ vanish of infinite order as $\tau \to \frac{h}{k} \in \mathbb{Q}$. So at a root of unity ζ, $KZ(\zeta)$ is essentially the limiting value of the Eichler integral of η. Zagier then related this function asymptotically to

$$\int_{-\overline{\tau}}^{i\infty} \frac{\eta(w)}{(-i(w+\tau))^{\frac{3}{2}}} dw. \tag{3}$$

The error of modularity of this function can easily be determined and is real-analytic. Note that integrals of the shape (3) also occur in the setting of mock modular forms. These generalize Ramanujan's mock theta functions which he introduced in his last letter to Hardy. Mock modular forms are not quite modular, but can be "completed" by adding a non-holomorphic piece (see [5] for details).

Konsevitch's strange functions occur also in knot theory. It belongs to the Habiro ring of analytic functions of root of unity.

Returning to quantum modular forms, further examples of quantum modular forms were investigated in the setup of characters of vertex algebra modules. I do not want to give background here but just say that they take the particularly nice shape ($s \in \mathbb{N}, s < p$)

$$\frac{F_{p-s,p}(p\tau)}{\eta(\tau)},$$

where

$$F_{j,p}(\tau) := \sum_{m\in\mathbb{Z}} \operatorname{sgn}\left(m + \frac{j}{2p}\right) q^{\left(m+\frac{j}{2p}\right)^2}$$

is a *false theta function.* It is called "false theta" since getting rid of the sgn-factor yields the theta function $\sum_{m\in\mathbb{Z}} q^{\left(m+\frac{j}{2p}\right)^2}$, a modular form of weight $\frac{1}{2}$. Quantum modularity of $F_{j,p}$ is now given by relating it to a non-holomorphic Eichler integral, as in (3) (see [2] for details). Of course this example is not as interesting as Kontsevich's strange function as it exists as q-series.

2. Higher depth quantum modular forms

Our motivating example is a certain q-series appearing in representation theory of vertex algebras and W-algebras. They are sometimes called "higher rank false theta functions". They appear from extracting the constant term of certain multivariable Jacobi forms. The constant term can be interpreted as the character of the zero weight space of the corresponding Lie algebra representation. In the case of the simple Lie algebra $\mathfrak{sl}_3$, the false theta function takes the following shape ($p \in \mathbb{N}$, $p \geq 2$):

$$F(q) := \sum_{\substack{m_1,m_2\geq 1\\ m_1\equiv m_2 \pmod 3}} \min(m_1,m_2) q^{p\left(m_1^2+m_2^2+m_1m_2\right)-m_1-m_2+\frac{1}{p}} \times \left(1-q^{m_1}\right)\left(1-q^{m_2}\right)\left(1-q^{m_1+m_2}\right).$$

In [1], we decomposed this function as $F(q) = F_1(q) + F_2(q)$, where

$$F_1(q) := \sum_{\alpha\in\mathscr{S}} \varepsilon_1(\alpha) \sum_{n\in\alpha+\mathbb{N}_0^2} q^{Q(n)} + \frac{1}{2}\sum_{m\in\mathbb{Z}} \operatorname{sgn}\left(m+\frac{1}{p}\right) q^{\left(m+\frac{1}{p}\right)^2},$$

$$F_2(q) := \sum_{\alpha\in\mathscr{S}} \varepsilon_2(\alpha) \sum_{n\in\alpha+\mathbb{N}_0^2} n_2 q^{Q(n)} - \frac{1}{2}\sum_{m\in\mathbb{Z}} \left|m+\frac{1}{p}\right| q^{\left(m+\frac{1}{p}\right)^2},$$

with quadratic form $Q(x,y) := 3x^2+3xy+y^2$, $\mathscr{S} \subset \mathbb{Q}^2$ a finite set, and $\varepsilon_1, \varepsilon_2$ functions on that set. The function F_1 and F_2 turn out to have generalized quantum modular properties. This connection goes via an analogue of (3). For instance, we showed that F_1 asymptotically agrees with an integral of the shape

$$\int_{-\overline{\tau}}^{i\infty}\int_{w_1}^{i\infty} \frac{f(w_1,w_2)}{\sqrt{-i(w_1+\tau)}\sqrt{-i(w_2+\tau)}} dw_1 dw_2,$$

where f has weight $\frac{3}{2}$ in both variables. Modular properties follow from the modularity of f which in turn gives quantum modular properties of F_1. That motivates the following definition: We call the resulting functions higher depth quantum modular forms. Roughly speaking, *depth two quantum modular forms* satisfy, in the simplest case, the modular transformation property $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$

$$f(\tau) - (c\tau + d)^{-k} f(M\tau) \in \mathcal{Q}_\kappa(\Gamma)\mathcal{O}(R) + \mathcal{O}(R),$$

where $\mathcal{Q}_\kappa(\Gamma)$ is the space of quantum modular forms of weight κ and $\mathcal{O}(R)$ the space of real analytic functions on $R \subset \mathbb{R}$.

Clearly, we can construct examples of depth two simply by multiplying two (depth one) quantum modular forms. Non-trivial examples arise from F.

Theorem 1. *For $p \geq 3$, the higher rank false theta function F can be written as the sum of two depth two quantum modular forms of weight one and two.*

3. Open question

The above construction is just the starting point. There are many things left which one may investigates. Of particular interest to me is the construction of a "strange function" and building a theory of such functions. It feels to me that interesting examples could for example come from knot theory.

References

[1] K. Bringmann, J. Kaszian, and A. Milas, *Higher depth quantum modular forms, multiple Eichler integrals, and $\mathfrak{sl}_3$ false theta functions*, Quanta of Math. **11** (2010), 659–675.

[2] K. Bringmann and A. Milas, *W-algebras, higher rank false theta functions, and quantum dimensions*, Selecta Mathematica (2017), 1–30.

[3] D. Zagier, *Vassilev invariants and a strange indentity related to the Dedekind eta-functions*, Topology **40** (2001), 945–960.

[4] D. Zagier, *Quantum modular forms*, Quanta of Math. **11** (2010), 659–675.

[5] S. Zwegers, *Mock Theta functions*, Ph.D. Thesis (2002).

Volume & determinant densities, and biperiodic alternating links

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(joint work with Ilya Kofman, Jessica Purcell)

For a hyperbolic link K, let $\mathrm{vol}(K)$ denote the hyperbolic volume of $S^3 - K$, $\det(K)$ denote its determinant and $c(K)$ denote its crossing number. We define the *volume density* of K as $D_{\mathrm{vol}}(K) = \mathrm{vol}(K)/c(K)$, and its *determinant density* as $D_{\det}(K) = 2\pi \log \det(K)/c(K)$. It is known that $D_{\mathrm{vol}}(K) \in (0, v_{\mathrm{oct}})$, and conjectured that $D_{\det}(K) \in (0, v_{\mathrm{oct}})$, where v_{oct} is the hyperbolic volume of a regular ideal octahedron. In [3] we proposed the following conjecture relating the volume and determinant densities of hyperbolic links, and proved that the constant 2π in the conjecture is sharp:

Conjecture 1 (Vol-Det Conjecture [3])**.** *For any alternating hyperbolic link K,*

$$\mathrm{vol}(K) < 2\pi \log \det(K).$$

A biperiodic alternating link $\mathcal{L}$ is an alternating link in the plane which can be isotoped to be invariant under translations by a two-dimensional lattice Λ. Let $L = \mathcal{L}/\Lambda$ be the alternating quotient link, which can be realized as a link in the thickened torus $T^2 \times (-1, 1)$.

In this paper, we prove the Vol-Det conjecture for infinite families of knots and links which arise from studying biperiodic alternating links. We briefly explain our method of constructing such families.

Følner convergence of link diagrams. Given a biperiodic link $\mathcal{L}$, we study families of links $\{K_n\}$ which diagrammatically converge to $\mathcal{L}$ in a controlled manner. This idea is made precise as follows.

Definition 1 (Følner convergence of links [2, 3])**.** For a subgraph $H \subset G$, let ∂H denote the set of vertices of H that share an edge with a vertex not in H, and let $|\cdot|$ denote the number of vertices in a graph. For a link K, let $G(K)$ denote the projection graph of K. We will say that a sequence of alternating links $\{K_n\}$ *Følner converges almost everywhere* to the biperiodic alternating link $\mathcal{L}$, denoted by $K_n \stackrel{\mathrm{F}}{\to} \mathcal{L}$, if the respective projections graphs $\{G(K_n)\}$ and $G(\mathcal{L})$ satisfy the following conditions:

- (i) there are subgraphs $G_n \subset G(K_n)$ which form an exhaustive nested sequence of connected subgraphs i.e. $\{G_n \subset G \mid \; G_n \subset G_{n+1}, \; \bigcup_n G_n = G\}$ such that $\displaystyle\lim_{n\to\infty} \frac{|\partial G_n|}{|G_n|} = 0$,
- (ii) $G_n \subset G(\mathcal{L}) \cap (n\Lambda)$, and
- (iii) $\displaystyle\lim_{n\to\infty} |G_n|/c(K_n) = 1$.

Determinant density convergence. The determinant of a knot is one of the oldest knot invariants that can be directly computed from a knot diagram. For any knot or link K,

$$\det(K) = |\det(M + M^T)| = |H_1(\Sigma_2(K); \mathbb{Z})| = |\Delta_K(-1)| = |V_K(-1)|,$$

where M is any Seifert matrix of K, $\Sigma_2(K)$ is the 2–fold branched cover of K, $\Delta_K(t)$ is the Alexander polynomial and $V_K(t)$ is the Jones polynomial of K (see, e.g., [7]).

For a biperiodic alternating link $\mathcal{L}$, the two Tait graphs $G_{\mathcal{L}}$ and $G^*_{\mathcal{L}}$ are planar duals and are both biperiodic. We form the overlaid bipartite biperiodic graph $G^b_{\mathcal{L}} = G_{\mathcal{L}} \cup G^*_{\mathcal{L}}$, whose black vertices are the vertices of $G_{\mathcal{L}}$ and of $G^*_{\mathcal{L}}$, and white vertices are points of intersection of their edges. We study the toroidal dimer model on $G^b_{\mathcal{L}}$ and obtain the characteristic polynomial $p(z, w)$ of the toroidal dimer model [6]. In [2] we prove

Theorem 1 ([2])**.** *Let $\mathcal{L}$ be any biperiodic alternating link, with toroidally alternating Λ–quotient link L and crossing number $c(L)$. Let $p(z, w)$ be the characteristic* *polynomial of the toroidal dimer model on* $G^b_{\mathcal{L}}$*. Then*

$$K_n \xrightarrow{\text{F}} \mathcal{L} \quad \Longrightarrow \quad \lim_{n\to\infty} \frac{\log\det(K_n)}{c(K_n)} = \frac{m(p(z,w))}{c(L)},$$

where $m(p(z,w))$ *is the Mahler measure of* $p(z,w)$*.*

Semi-regular biperiodic links. A biperiodic alternating link $\mathcal{L}$ is called *semi-regular* if the link projection is isomorphic, as plane graphs, to a biperiodic edge-to-edge Euclidean tiling with convex regular polygons, such that all vertices are 4–valent. Such Euclidean tilings are k–uniform tilings, where k is the number of orbits of vertices (see [5]). There exist infinitely many such tilings. In [4] we study the hyperbolic geometry of semi-regular biperiodic alternating links in detail. One of the consequences of this is the following:

Theorem 2 ([4])**.** *Let* $\mathcal{L}$ *be a semi-regular biperiodic alternating link, with toroidally alternating* Λ*–quotient link* L *and crossing number* $c(L)$*. Let* K_n *be any sequence of alternating links such that* $K_n \xrightarrow{\text{F}} \mathcal{L}$*. Then for almost all* n*,*

$$\frac{\text{vol}(K_n)}{c(K_n)} \leq \frac{\text{vol}(T^2 \times I - L)}{c(L)}$$

Mahler measure and hyperbolic volume. Mahler measure of two-variable polynomials has been related to hyperbolic volume, most notably in the work of Smyth [8], and Boyd-Rodriguez-Villegas [1], by finding families of polynomials whose Mahler measure equals sum of hyperbolic volumes. However for this paper we conjecture the following inequality:

Conjecture 2. *Let* $\mathcal{L}$ *be a semi-regular biperiodic alternating link, with toroidally alternating* Λ*–quotient link* L *and crossing number* $c(L)$*. Let* $p(z,w)$ *be the characteristic polynomial of the toroidal dimer model on* $G^b_{\mathcal{L}}$*. Then*

$$\text{vol}(T^2 \times (-1,1) - L) \leq 2\pi\, m(p(z,w)).$$

Rhombitrihexagonal link. A simple example which verifies Conjecture 2 is the Rhombitrihexagonal link as shown in Figure 1. In this case we get a strict inequality.

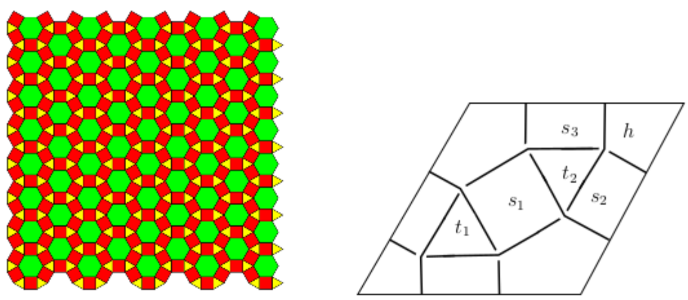


FIGURE 1. Rhombitrihexagonal link and its quotient in $T^2 \times I$.

Using the strict inequality in Conjecture 2 and the volume bound for semi-regular links in Theorem 2 lets us prove the Vol-Det conjecture for an infinite family of links.

Theorem 3. *Let $\mathcal{L}$ be the Rhombitrihexagonal link and let K_n be any sequence of alternating links such that $K_n \xrightarrow{\text{F}} \mathcal{L}$. Then for almost all n,*

$$\operatorname{vol}(K_n) < 2\pi \log \det(K_n).$$

References

[1] D. Boyd and F. Rodriguez-Villegas, *Mahler's measure and the dilogarithm. I*, Canad. J. Math. **54** (2002), no. 3, 468–492.

[2] Abhijit Champanerkar and Ilya Kofman, *Determinant density and biperiodic alternating links*, New York J. Math. **22** (2016), 891–906. MR 3548129

[3] Abhijit Champanerkar, Ilya Kofman, and Jessica S. Purcell, *Geometrically and diagrammatically maximal knots*, J. Lond. Math. Soc. (2) **94** (2016), no. 3, 883–908. MR 3614933

[4] Abhijit Champanerkar, Ilya Kofman, and Jessica S. Purcell, *Geometry of biperiodic alternating links*, preprint.

[5] Branko Grünbaum and Geoffrey C. Shephard, *Tilings by regular polygons. Patterns in the plane from Kepler to the present, including recent results and unsolved problems*, Math. Mag. **50** (1977), no. 5, 227–247. MR 0500451

[6] R. Kenyon, A. Okounkov, and S. Sheffield, *Dimers and amoebae*, Ann. of Math. (2) **163** (2006), no. 3, 1019–1056.

[7] W. B. Raymond Lickorish, *An introduction to knot theory*, Graduate Texts in Mathematics, vol. 175, Springer-Verlag, New York, 1997. MR 1472978

[8] C. J. Smyth, *On measures of polynomials in several variables*, Bull. Austral. Math. Soc. **23** (1981), no. 1, 49–63. MR 615132

Simplicial schemes and triangulated spaces

Christian Zickert

We formulate some results of Fock and Goncharov [1], and Garoufalidis, Thurston and Zickert [2] using the language of simplicial schemes. A simplicial scheme associates (tautologically) a scheme to each triangulated space. The simplicial scheme of configurations of points in $\mathrm{SL}(n)/N$ associates the Fock–Goncharov $\mathcal{A}$-space to a triangulated surface, and the Ptolemy variety of Garoufalidis–Thurston–Zickert to a triangulated 3-manifold. If one instead considers the simplicial scheme of configurations of points in a fixed vector space, one obtains schemes with very interesting structures. This scheme has explicit coordinates similar to the Plucker coordinates on the Grassmannian, and it appears that the schemes associated to even dimensional spaces have canonical 2-forms (possibly symplectic), and that the schemes associated to odd dimensional manifolds have polylogarithm invariants, and invariants in the (higher) Bloch groups. These invariants can be explicitly expressed in terms of the Plucker coordinates.

References

[1] Vladimir Fock and Alexander Goncharov. Moduli spaces of local systems and higher Teichmüller theory. *Publications mathématiques de l'IHÉS*, 103(1):1–211, 2006.
[2] Stavros Garoufalidis, Dylan P Thurston, Christian K Zickert, et al. The complex volume of $SL(n, \mathbb{C})$-representations of 3-manifolds. *Duke Mathematical Journal*, 164(11):2099–2160, 2015.

An overlooked problem: What are the Betti numbers of closed manifolds?

Matthias Kreck
(joint work with Don Zagier)

Question. Given natural numbers $(b_0 = 1, b_1, b_2, \ldots, b_n)$. Is there a closed smooth n-manifold M with Betti number $b_i(M) = b_i$?

Two cases. Case 1: $b_n = 0$ or M non-orientable. Then we prove

Theorem 1. *$b_0 = 1, b_1, \ldots, b_{n-1}, b_n = 0$ can be realized as Betti numbers of a closed manifold $M \Leftrightarrow \sum(-1)^i b_i = 0$ for n odd.*

Case 2: $b_n = 1$ or M orientable.
Necessary condition from Poincaré duality:

$$b_i = b_{n-i} \quad \text{and for } n = 4k+2: \quad b_{2k+1} \text{ even.}$$

Observation. If these conditions are fulfilled and in addition for $n = 4k$ we have b_{2k} even, then b_i can be realized as Betti numbers of connected sum of products of spheres.

Thus the **open problem** is, which $b_0 = 1, b_1, \ldots, b_{4k} = 1$ with b_{2k} odd are Betti numbers of closed $4k$-manifolds?

We will concentrate from now on the case where $b_1 = \cdots = b_{2k-1} = 0$. To formulate our results we have to formulate the Hirzebruch Signature Theorem for a manifold M^{4k} with these Betti numbers:

$$\operatorname{sign} M = \langle L_k(p_i(M)), [M] \rangle$$
$$= \begin{cases} \langle s_{2k} p_{2k}(M), [M] \rangle & \text{if } k \text{ is odd} \\ \langle s_{2k} p_{2k}(M), [M] \rangle + \langle \frac{1}{2}(s_k^2 - s_{2k}) p_k^2(M), [M] \rangle & \text{if } k \text{ is even,} \end{cases}$$

where $s_k = 2^{2k}(2^{2k-1} - 1)\frac{|B_{2k}|}{(2k)!}$.

Theorem 2. *The only dimensions $n \equiv 0 \mod 4$, for which there is a M^n with $b_1 = \cdots = b_{n/2-1} = 0$ and $b_{n/2}$ an odd number are $n = 4$ and $n = 8k$ with*

$$k = 2^a + 2^{a'}.$$

Theorem 3. *For $k = 2^a + 2^{a'}$ and a given odd natural number b there is a closed oriented manifold M^{8k} with $b_1(M) = \cdots = b_{4k-1}(M) = 0$ and $b_{4k}(M) = b \Leftrightarrow$ there is an odd number s with $1 \leq s \leq b$ and integers u and v such that*

(1) $s = s_{2k}u + \frac{1}{2}(s_k^2 - s_{2k})v$
(2) $\left(\frac{(-1)^{k+1}s_k}{(2k-1)!} + \frac{1}{2(4k-1)}\right)v - \frac{u}{(4k-1)!} \in \mathbb{Z}[\frac{1}{2}]$
(3) $\frac{v}{(2k-1)^2} \in \mathbb{Z}[\frac{1}{2}]$,

where $v \geq 0$ *for* $s = b$*;* $v > 0$ *and* $v \neq 7 \mod 8$ *for* $s = b = 3$*;* $v = r^2$ *for* $s = b = 1$.

Remark. The case $s = b = 1$ was first proved by Kennard and Su.

Theorem 4. *There exists an n-dimensional manifold with total Betti number 3 (i.e., with* $b_0 = b_{n/2} = b_n = 1$ *and no other non-vanishing Betti numbers) for* $n = 4$*, 8, 16, 32,* 128 *and* 256 *and for no other n less than* 10^5*, with 13 possible exceptions* $n = 544$*, 4160, 8224, 16448, 32776, 32832, 33280, 40960, 49152, 65536, 65600, 65792, and* 66560.

The six values of n for which such a "rational projective plane" exists were found independently by Kennard and Su. The number-theoretical analysis suggests strongly that rational projective planes exist in only finitely many dimensions, and that there are almost certainly at most one or two further examples.

New q-series invariants of 3-manifolds

Sergei Gukov

(joint work with Marcos Marino, Du Pei, Pavel Putrov, Cumrun Vafa)

For knots and links in $M_3 = \mathbb{R}^3$, the suitably normalized WRT invariants computed by the 3d Chern-Simons TQFT with coupling constant / "level" k turn out to be polynomial in the variable $q = e^{2\pi i/k}$ with integer coefficients. This curious fact, observed in the past century, found a new life in the 21st century: the integer coefficients of knot invariants are Euler characteristics of the Hilbert spaces $\mathcal{H}(\mathbb{R}^3, K)$ assigned to a knot K by a 4d TQFT with surface operators / foams:

$$\text{WRT}(\mathbb{R}^3, K) = \sum_{i,j} (-1)^i q^j \dim \mathcal{H}^{i,j}(\mathbb{R}^3, K) \tag{1}$$

The physical interpretation of the space $\mathcal{H}$ as a certain Q-cohomology (or, equivalently, as a space of particular BPS states) proposed in [1] quickly led to many new predictions and connections between various areas, which include knot contact homology [2], gauge theory [3, 4], algebras of interfaces [5, 6], and many more [7].

A typical example of a simple knot invariant for which the lift (1) to a space $\mathcal{H}$ was not known prior to [1] is the $sl(4)$ quantum invariant of the figure-8 knot:

$$1 + \frac{1}{q^4} - \frac{1}{q} - q + q^4. \tag{2}$$

The physics computation, on the other hand, led to a doubly-graded space (of BPS states) with the Poincaré polynomial

$$1 + \frac{1}{q^4t^2} + \frac{1}{qt} + qt + q^4t^2. \tag{3}$$

which was later verified by purely mathematical techniques, as soon as they became sufficiently developed. In a similar way, the proposed physical setup led to concrete predictions for $sl(N)$ homologies of knots and links, including colors (decoration by representations of $sl(N)$) and other root systems [8, 9].

Although knots and 3-manifolds often appear on the same footing in Chern-Simons theory and in low-dimensional topology — *e.g.* every 3-manifold can be obtained from a 3-sphere by a sequence of surgeries on some knots — a much desired generalization of (1) to arbitrary 3-manifolds faces an immediate challenge. No normalization and change of variables can turn the Chern-Simons (WRT) invariants of M_3 into polynomials or power series with integer powers and coefficients.

This challenge was recently overcome in [10, 11, 12], where a set of new invariants $\widehat{Z}_a(M_3)$ was proposed, such that

$$\mathrm{WRT}(M_3) \;=\; \sum_a e^{ik\mathrm{CS}(a)} \left(\lim_{q \to e^{2\pi i/k}} \sum_b S_{ab}\, \widehat{Z}_b(q) \right) \tag{4}$$

Here, the labels a and b run over[1] Abelian flat connections on M_3, and S_{ab} are simple numerical coefficients known explicitly (in particular, they do not depend on q or k). More importantly, the new invariant $\widehat{Z}_a(M_3, q)$ is a power series in q with integer coefficients and, as in (1), has a lift to a vector space $\mathcal{H}(M_3)$ which in the physical framework is the space of BPS states, a.k.a. Q-cohomology of the fivebrane theory with respect to a topological supercharge Q. This machinery has been applied to some concrete examples which so far include Lens spaces, circle bundles over Riemann surfaces, Seifert spaces, and general plumbing 3-manifolds [12, 10]. The next obvious step is to extend it to calculation of homology groups for other 3-manifolds and to understand the structural properties of the resulting spaces of BPS states. Surprisingly, they appear to exhibit much of the same structure as $sl(N)$ knot homologies, *e.g.* action of the differentials d_N, *etc.*

References

[1] S. Gukov, A. S. Schwarz and C. Vafa, *Khovanov-Rozansky homology and topological strings*, Lett. Math. Phys. **74** (2005) 53–74, hep-th/0412243.

[2] M. Aganagic, T. Ekholm, L. Ng and C. Vafa, *Topological Strings, D-Model, and Knot Contact Homology*, Adv. Theor. Math. Phys. **18** (2014) 827–956, 1304.5778.

[3] S. Gukov, *Gauge theory and knot homologies*, Fortsch. Phys. **55** (2007) 473–490, 0706.2369.

[4] E. Witten, *Fivebranes and Knots*, 1101.3216.

[5] N. Carqueville and D. Murfet, *Computing Khovanov-Rozansky homology and defect fusion*, Algebr. Geom. Topol. **14** (2014) 489–537, 1108.1081.

[1] More precisely, the labels a and b run over connected components of the moduli space of Abelian flat connections on M_3.

[6] S. Chun, S. Gukov and D. Roggenkamp, *Junctions of surface operators and categorification of quantum groups*, arXiv:1507.06318 [hep-th].
[7] S. Gukov, M. Khovanov and J. Walcher, *Physics and Mathematics of Link Homology*, Contemp. Math. **680** (2016).
[8] S. Gukov, S. Nawata, I. Saberi, M. Stosic and P. Sulkowski, *Sequencing BPS Spectra*, JHEP **1603** (2016) 004, arXiv:1512.07883 [hep-th].
[9] R. Elliot and S. Gukov, *Exceptional knot homology*, J. of Knot Theory and Its Ramifications **25**, no. 03, 1640003 (2016), arXiv:1505.01635 [math.QA].
[10] S. Gukov, P. Putrov and C. Vafa, *Fivebranes and 3-manifold homology*, JHEP **1707** (2017) 071, arXiv:1602.05302 [hep-th].
[11] S. Gukov, M. Marino and P. Putrov, *Resurgence in complex Chern-Simons theory*, arXiv:1605.07615 [hep-th].
[12] S. Gukov, D. Pei, P. Putrov and C. Vafa, *BPS spectra and 3-manifold invariants*, arXiv:1701.06567 [hep-th].

Special subgroups of Bianchi groups

Michelle Chu

Recent progress in 3-manifold theory has determined many virtual properties relating to surfaces. In particular, the resolution of the virtual Haken and the virtual fibering conjectures for finite volume hyperbolic 3-manifolds relies heavily on geometric group theory techniques; indeed, these results follow from a stronger theorem which states that 3-manifolds are virtually special [2, 7]. We say M is *special* in the sense of [5] if its fundamental group $\pi_1(M)$ embeds in a right-angled-Coxeter group (RACG). A special 3-manifold inherits many nice properties from the embedding in the RACG, in particular, it is Haken and virtually fibered by RFRS [1]. Prior to Agol's resolution, virtually special was known for Bianchi groups [3] and more generally for arithmetic lattices of simplest type [4].

There has been recent interest in determining special finite index subgroups of the fundamental groups of hyperbolic 3-manifolds. In [6], Spreer and Tillmann found a special subgroup of index 60 in the fundamental group of the Seifert-Weber dodecahedral space, which is arithmetic of simplest type. The aim of this talk is to determine special finite index subgroups of Bianchi groups. Let d be a square-free positive integer and $\mathcal{O}_d$ the ring of integers of the quadratic field $\mathbb{Q}(\sqrt{-d})$.

Theorem 1. *The Bianchi group* $\mathrm{PSL}_2(\mathcal{O}_d)$ *contains a subgroup* Δ_d *which embeds geometrically in a RACG and has index*

$$[\mathrm{PSL}_2(\mathcal{O}_d) : \Delta_d] = \begin{cases} 48 & \textit{if } d \equiv 1, 2 \mod 4 \\ 120 & \textit{if } d \equiv 3 \mod 8 \\ 72 & \textit{if } d \equiv 7 \mod 8. \end{cases}$$

Furthermore, Δ_d *is as congruence subgroup of level 2 whenever* $d \equiv 1, 2 \mod 4$ *and otherwise a congruence subgroup of level 4.*

Arithmetic Kleinian groups associated to rational quadratic form will have have congruence subgroups which are special (but perhaps of high level). It would be interesting to see whether special subgroups of any arithmetic hyperbolic lattices can always be achieved by congruence subgroups.

References

[1] Ian Agol. Criteria for virtual fibering. *J. Topol.*, 1(2):269–284, 2008.

[2] I. Agol. The virtual Haken conjecture. *Doc. Math.*, 18:1045–1087, 2013. With an appendix by Agol, Daniel Groves, and Jason Manning

[3] I. Agol, D. D. Long, and A. W. Reid. The Bianchi groups are separable on geometrically finite subgroups. *Ann. of Math.* (2), 153(3):599–621, 2001.

[4] N. Bergeron, F. Haglund, and D. T. Wise. Hyperplane sections in arithmetic hyperbolic manifolds. *J. Lond. Math. Soc.* (2), 83(2):431–448, 2011.

[5] F. Haglund and D. T. Wise. Special cube complexes. *Geom. Funct. Anal.*, 17(5):1551–1620, 2008.

[6] J. Spreer and S. Tillmann. Unravelling the Dodecahedral Spaces. arXiv:1702.08080, 2017.

[7] D. T. Wise. *The structure of groups with quasiconvex hierarchy.* 2011. Unpublished manuscript. 187 pages.

Analogies connecting asymptotic problems [from number fields to 3-manifolds]

Farshid Hajir

Introduction. Analogies have a long history of driving discovery in mathematics. The goal in this short expository talk is to use an example from algebraic number theory to motivate a question about volumes of cusped 3-manifolds refining a theorem of Adams.

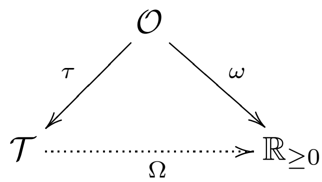
We first give a quick sketch of the formalization of the concept of asymptotically good families, closely following [2], with the difference that we keep track of "good" objects as those of "low cost" as opposed to "high quality." To begin, we require a *context* $\mathscr{C} = (\mathcal{O}, \mathcal{T}, \tau, \omega)$, where $\mathcal{O}, \mathcal{T}$ are sets and τ, ω are maps $\tau : \mathcal{O} \to \mathcal{T}$ and $\omega : \mathcal{O} \to \mathbb{R}_{\geq 0}$. Here, $\mathcal{O}$ is the set of objects of interest, $\mathcal{T}$ is a paramater space of *types* of the objects, and ω is the *critical invariant* measuring the "cost" of the object (for some contexts not discussed here it is more natural to consider $\alpha = \omega^{-1}$ as the "quality" of the object). Often ω is expressed as a ratio of two other invariants and hence measures a "unit cost" for the object at hand. The parameter space is usually a familiar and countable set; for convenience, we will assume that τ is surjective. It goes without saying that our normalization is such that "good" objects are those of low cost.

A *family* $\mathcal{F}$ in $\mathcal{O}$ is a sequence $F_1, F_2, \cdots$ of **pairwise distinct** elements of $\mathcal{O}$. We say that $\mathcal{F} = (F_i)$ is *isotypic of type* t if every member of $\mathcal{F}$ has type t, i.e. $\tau(F_i) = t$ for all i. We extend ω to families by putting $\omega(\mathcal{F}) = \limsup_{i\to\infty} \omega(F_i)$, for $\mathcal{F} = (F_1, F_2, \cdots)$ and say that $\mathcal{F}$ is *asymptotically good* if $\omega(\mathcal{F}) < \infty$. In the contexts we have in mind, it is typically difficult to construct asymptotically good families, or at least to do so explicitly.

The main object of interest attached to a context $\mathscr{C} = (\mathcal{O}, \mathcal{T}, \tau, \omega)$, namely the *asymptotic envelope* function $\Omega : \mathcal{T} \to \mathbb{R}_{\geq 0}$ is given by

$$\Omega(t) := \inf_{\mathcal{F} \text{ of type } t} \omega(\mathcal{F}),$$

where the infimum is taken over all isotypic families of type t. Thus, the map Ω is induces by τ and α as in the following diagram.



It is clear that the asymptotic envelope function is a measure not of the cost of individual objects, but rather of the existence of infinite non-repeating strings of those of a fixed type having bounded cost. In many contexts, one is able to estimate $\Omega(t)$ by upper and lower bounds but of course would like to have an explicit formula for $\Omega(t)$ itself. Typically, the theory provides a natural and "decent" lower bound, meaning one that is believed to be sharp. Interestingly, the source of this lower bound is usually a zeta function known or at least suspected to satisfy an appropriate Riemann hypothesis. Obtaining upper bounds involves the creation of examples with extremal properties, usually from objects carrying inordinately many symmetries – it is not surprising that automorphic forms are a typical source.

Number Fields. For the context of number fields $\mathscr{C}_{\mathrm{nf}}$, the set of objects $\mathcal{O}_{\mathrm{nf}}$ consists of fields K of finite degree $n(K)$ over $\mathbb{Q}$. The type of a number field is defined to be $\tau(K) = r_1(K)/n(K)$; it is the proportion of the embeddings of K into $\mathbb{C}$ with image contained in $\mathbb{R}$. The space of possible types in this context is $\mathcal{T} = [0,1] \cap \mathbb{Q}$. As the critical invariant, we choose the *logarithmic root discriminant*:

$$\omega_{\mathrm{nf}}(K) := \frac{\log|\operatorname{disc}(K)|}{n(K)},$$

where $\operatorname{disc}(K)$ is the absolute discriminant of K and $n(K) = [K : \mathbb{Q}]$ is its absolute degree; for the field $\mathbb{Q}$, we have $\omega(\mathbb{Q}) = 0$. The resulting asymptotic envelope function $\Omega_{\mathrm{nf}}(t)$ is known as the Martinet function [4]. Under the Generalized Riemann Hypothesis (GRH), we have a bound due to Stark, Odlyzko and Serre, namely

$$\Omega_{\mathrm{nf}}(t) \geq \log(8\pi) + \gamma + \frac{\pi}{2} t.$$

There is an unconditional lower bound as well. Such lower bounds (but with worst constants) were first derived by Minkowski using his Geometry of Numbers. As for upper bounds, the only source of good families in $\mathscr{C}_{\mathrm{nf}}$ we currently know are nested fields $K_0 \subsetneq K_1 \subsetneq \cdots$ which are ramified at finitely many places and shallowly ramified (they exist by a theorem of Golod and Shafarevich). As a result we do not have an explicit upper bound , though by [3], we have $\Omega_{\mathrm{nf}}(0) \leq \log(83)$ and $\Omega_{\mathrm{nf}}(1) \leq 1/\log(955)$.

3-manifolds. We now consider the context $\mathscr{C}_{\mathrm{cm}}$ where the objects are cusped hyperbolic 3-manifolds of finite volume. For an n-cusped 3-manifold M with r_1 Klein bottle cusps and $n - r_1 = r_2$ torus cusps, we define the *orientation type* and *normalized volume* of M to be, respectively

$$\tau(M) := \frac{r_1}{n}, \qquad \omega_{\mathrm{cm}}(M) := \frac{\mathrm{vol}(M)}{n}.$$

If we consider the cusps of a 3-manifold to be analogous to the places at infinity for a number field, we are led to the question: does the volume of an n-cusped hyperbolic 3-manifold grow linearly with n? The answer is yes. Indeed, we have the following theorem of Adams [1]: *If M is an n-cusped hyperbolic 3-manifold, then* $\mathrm{vol}(M) \geq v_3 n$ *where v_3 is the volume of the regular ideal tetrahedron.*

We note that Adams' proof relies on Minkowski's geometry of numbers. Even without this fact as a provocation, it is natural for a number-theorist to wonder whether Adams' theorem can similarly be refined for contributions from torus cusps and Klein-bottle cusps. We venture that $\omega_{\mathrm{cm}}(M)$ is a reasonable analogue of the logarithmic root discriminant for number fields and therefore ask what can be said about $\Omega_{\mathrm{cm}}(t)$ for $t \in [0,1] \cap \mathbb{Q}$ beyond what comes from Adams' theorem, namely $\Omega_{\mathrm{cm}}(t) \geq v_3$. Can one prove an explicit upper bound for $\Omega_{\mathrm{cm}}(t)$? It would be very interesting, for instance, if it can be established that $\Omega_{\mathrm{cm}}(t)$ is a linear function of t, or that it meets a fixed linear upper bound for many values of t. The analogue of such bounds for number fields appear to be beyond reach at the moment.

References

[1] C. Adams, *Volumes of N-cusped hyperbolic 3-manifolds*, J. London Math. Soc. (2) **38** (1988), no. 3, 555–565.

[2] F. Hajir, *Asymptotically Good Families*, Actes de la Conférence "Fonctions L et Arithmétique," 121–128, Publ. Math. Besancon, Algébre Théorie Nr., Lab. Math Besancon, 2010.

[3] F. Hajir and C. Maire, *Tamely ramified towers and discriminant bounds for number fields II*, *J. Symb Computation* **33** (2002) no. 4, 415-423.

[4] J. Martinet, *Tours de corps de classes et estimations de discriminants*, Invent. Math. **44** (1978), no. 1, 65–73.

Knots and their related q-series

Stavros Garoufalidis

(joint work with Don Zagier)

We discuss empirical asymptotic properties of q-series and their relation of the Kashaev invariant of knots. This is a tale of several independent discoveries, in many parts, and with a yet unfinished ending.

In this paper we want to tell a story about q-series and quantum invariants of knots that seems to us very interesting. We will tell it in detail for the 4_1 (or figure 8) knot, the simplest hyperbolic knot, and will discuss a few other cases in the final section. The story relates

- the colored Jones polynomials and Kashaev invariant of a knot and the conjectural modularity properties of the latter,
- certain special q-hypergeometric series arising in connection with quantum spin networks, with the Dimofte-Gaiotto-Gukov index of triangulations, and certain invariants of knots defined by Dimofte and the first author, and
- state integrals (integrals of the Faddeev quantum dilogarithm) of the type studied by Hikami, Kashaev and others.

The statements, most of which are only empirical, mostly concern the asymptotic properties of these functions and their interrelations.

1. The Kashaev invariant and the function $g(q)$

To any knot K is associated a sequence of Laurent polynomials $J_n^K(q) \in \mathbb{Z}[q, q^{-1}]$ called the colored Jones polynomials of K. We do not repeat the general definition, which can be found in many places and is not relevant for us, but simply give the formula for the 4_1 knot:

$$J_n^{4_1}(q) \;=\; \sum_{m\geq 0} q^{-mn} \prod_{j=1}^{m} \bigl(1-q^{n-j}\bigr)\bigl(1-q^{n+j}\bigr) \qquad (n=1,2,3,\dots)\,.$$

(The sum terminates at $m = n-1$.) If we specialize q to be a root of unity, say $q^N = 1$, then it is clear from this expression that $J_n^{4_1}(q)$ is periodic in n of period N, so we can extrapolate backwards and define $J_n^{4_1}(q)$ also for $n \leq 0$. In particular, this gives us the new function

$$J_0^{4_1}(q) \;=\; \sum_{m=0}^{\infty} \prod_{j=1}^{m} \bigl(1-q^{-j}\bigr)\bigl(1-q^{j}\bigr) \qquad (q \text{ a root of unity})\,. \tag{1}$$

It is known by the work of Murakami and Murakami [11] that the (similarly defined) invariant $J_0^K(e^{2\pi i/N})$ for any knot K is equal to the knot invariant $\langle K\rangle_N$ defined by Kashaev [10]. The famous volume conjecture of Kashaev states that for any hyperbolic knot K the logarithm of $\langle K\rangle_N$ is asymptotically equal to CN as N tends to infinity, where C equals the (complexified) hyperbolic volume of the knot divided by 2π. There are very few cases for which the volume conjecture has been rigorously proved, but for the 4_1 knot it is quite easy using the Euler-Maclaurin formula and standard asymptotic techniques, because all of the terms in (1) are positive, and one finds the much more precise formula

$$J_0^{4_1}(N) \;\sim\; N^{3/2}\, \widehat{\Phi}(1/N) \tag{2}$$

with $\widehat{\Phi}(x)$ defined by

$$\widehat{\Phi}(x) \;=\; e^{C/x}\, \Phi(x)\,, \tag{3}$$

where C is the real number

$$(4) \qquad C \;=\; \frac{1}{2\pi}\,\mathrm{Vol}\big(S^3 \smallsetminus 4_1\big) \;=\; \frac{1}{\pi}\,\mathrm{Li}_2(e^{\pi i/3}) \;=\; 0.3230659\cdots$$

and where $\Phi(x)$ is the formal power series with real coefficients having the form

$$(5) \qquad \Phi(x) \;=\; \sum_{j=0}^{\infty} a_j x^j\,, \qquad a_j \;=\; \frac{1}{\sqrt[4]{3}}\Big(\frac{\pi}{36\sqrt{3}}\Big)^j \frac{A_j}{j!}$$

with $A_j \in \mathbb{Q}$, the first values being given by

j	0	1	2	3	4	5	6	7
A_j	1	11	697	$\frac{724351}{5}$	$\frac{278392949}{5}$	$\frac{244284791741}{7}$	$\frac{1140363907117019}{35}$	$\frac{212114205337147471}{5}$

A proof of (2) is given in the appendix, and an extension to all roots of unity is given in [8]. The weaker asymptotic formula with $\Phi(x)$ replaced by its constant term a_0 was proved by Andersen and Hansen [1].

The surprising discovery that we made, completely by accident, is that there is a close connection between the asymptotic expression occurring here and the radial asymptotics of the function in the unit disk defined by

$$(6) \quad g(q) \;=\; (q)_\infty \sum_{n=0}^{\infty}(-1)^n \frac{q^{n(3n+1)/2}}{(q)_n^3} \;=\; 1 - q - 2q^2 - 2q^3 - 2q^4 + q^6 + \cdots\,.$$

(Here $(q)_n = (1-q)(1-q^2)\cdots(1-q^n)$ is the usual q-Pochhammer symbol.) The infinite sum in (6) occurred in the work of the first author on the stability of the coefficients of the evaluation of the regular quantum spin network [6, Sec.7], and in the course of a numerical investigation of its asymptotics as $q \to 1$ we discovered empirically the following:

Conjecture We have

$$(7) \qquad g\big(e^{2\pi i z}\big) \;\sim\; \sqrt{z}\,\widehat{\Phi}(z) \;+\; \sqrt{-z}\,\widehat{\Phi}(-z)$$

to all orders in z as z tends to 0 in the upper half-plane.

(It was to achieve this simple statement that we included the factor $(q)_\infty$ in (6).)

The authors now a proof of this conjecture. We mention two other formulas for g, noticed by us and verified by Sander Zwegers,

$$(8) \qquad g(q) \;=\; \frac{1}{(q)_\infty} \sum_{n,m=0}^{\infty} (-1)^{n+m} \frac{q^{(n+m)(n+m+1)/2}}{(q)_n\,(q)_m} \;=\; \sum_{n=0}^{\infty}(-1)^n \frac{q^{n(n+1)/2}}{(q)_n^2}\,.$$

These expressions are of interest because, unlike the original series in (6) whose origin had no obvious connection with the knot, these series are related to it: the first one, which was shown to us by Tudor Dimofte, is typical of the series occurring in his work with Gaiotto and Gukov [2, 3, 5] on the 3D index of a triangulation, while the second one is typical of those occurring in the work of Dimofte and the first author on q-series associated to ideal triangulations of cusped 3-manifolds [4].

Equation (7) turns out to be only part of a bigger story. On the one hand, the power series $\Phi(x)$ is only a special case at $\alpha = 0$ of the more general asymptotic series $\Phi_\alpha(x)$ ($\alpha \in \mathbb{Q}$) occurring in the modularity conjecure for $J_0^{4_1}(q)$ made by the second author in [12], and these turned out to be were related in exactly the same way to the asymptotics of $g(q)$ for $q = e^{2\pi i\alpha - \epsilon}$ as $\epsilon \searrow 0$. On the other hand, the q-series $g(q)$ and the asymptotic formula (7) are related to the Dimofte-Gaiotto-Gukov index and to the Hikami-Kashaev state integral [7]. This is explained in the talk and in the paper under preparation [9].

References

[1] J. Andersen and S.K. Hansen, *Asymptotics of the quantum invariants for surgeries on the figure 8 knot*, Journal of Knot Theory and its Ramifications, **15** (2006) 479–548.
[2] T. Dimofte, D. Gaiotto and S. Gukov, *Gauge theories labelled by three-manifolds*, Communications in Mathematical Physics, **325** (2014) 367–419.
[3] T. Dimofte, D. Gaiotto and S. Gukov, *3-manifolds and 3d indices*, Advances in Theoretical and Mathematical Physics, **17** (2013) 975–1076.
[4] T. Dimofte and S. Garoufalidis, *The quantum content of the gluing equations*, Geometry & Topology, **17** (2013) 1253–1315.
[5] S. Garoufalidis, *The 3D-index of an ideal triangulation and angle structures*, Ramanujan J., **40** (2016) 573–604.
[6] S. Garoufalidis, *Quantum Knot Invariants*, Mathematische Arbeitstagung 2012.
[7] S. Garoufalidis and R. Kashaev *From state integrals to q-series*, Math. Research Letters, **24** (2017) 781–801.
[8] S. Garoufalidis and D. Zagier, *Quantum Modularity of the Kashaev invariant*, in preparation.
[9] S. Garoufalidis and D. Zagier, *Knots and their related q-series*, in preparation.
[10] R. Kashaev, *A link invariant from quantum dilogarithm*, Modern Phys. Lett. A **10** (1995), 1409–1418.
[11] H. Murakami and J. Murakami, *The colored Jones polynomials and the simplicial volume of a knot*, Acta Mathematica **186** (2001) 85–104.
[12] D. Zagier, *Quantum modular forms*, Quanta of maths, Clay Math. Proc., **11** 659–675.

Azumaya algebras and hyperbolic knots

Matthew Stover
(joint work with Ted Chinburg and Alan W. Reid)

Broadly, the purpose of this work [1] is to introduce methods from arithmetic geometry to study the $\mathrm{SL}_2(\mathbb{C})$ character variety of a (hyperbolic) knot group. Let K be a knot in the 3-sphere S^3 and set $\Gamma = \pi_1(S^3 \smallsetminus K)$. Fix a meridian/longitude pair $\mu, \lambda \in \Gamma$. We assume that K is *hyperbolic*, meaning that there is a discrete and faithful representation

$$\rho_\infty : \Gamma \to \mathrm{SL}_2(\mathbb{C})$$

such that $S^3 \smallsetminus K$ is homeomorphic to the quotient $\mathbf{H}^3/\rho_\infty(\Gamma)$ of hyperbolic 3-space $\mathbf{H}^3$ by the action of $\rho_\infty(\Gamma)$ by isometries.

For a pair of coprime integers $\alpha = (p, q)$, let M_α denote the closed manifold that results from performing α-Dehn surgery on $S^3 \smallsetminus K$. A famous result of Thurston says that, for all but finitely many α, M_α is a closed hyperbolic 3-manifold. More precisely, there is a representation

$$\Gamma \twoheadrightarrow \pi_1(M_\alpha) = \Gamma/\langle\!\langle \mu^p \lambda^q \rangle\!\rangle \hookrightarrow \mathrm{SL}_2(\mathbb{C}), \qquad \rho_\alpha : \Gamma \to \mathrm{SL}_2(\mathbb{C})$$

with discrete cocompact image and M_α homeomorphic to $\mathbf{H}^3/\rho_\alpha(\Gamma)$.

Recall that the character of an irreducible $\mathrm{SL}_2(\mathbb{C})$ representation is uniquely determined by the conjugacy class of the representation. In this language, Thurston proved that the character χ_∞ of ρ_∞ defines a smooth point of an affine curve component $C(K)$ of the character variety

$$X(\Gamma) = \mathrm{Hom}(\Gamma, \mathrm{SL}_2(\mathbb{C}))/\!/\, \mathrm{SL}_2(\mathbb{C}),$$

and the points χ_α associated with the characters of the representations ρ_α algebraically converge to χ_∞ as $|p| + |q| \to \infty$. The curve $C(K)$ is typically called the *canonical component* of the character variety.

One goal of this project is to use the arithmetic geometry of the canonical component to prove theorems about arithmetic invariants of the hyperbolic 3-manifolds M_α. The *trace field* of M_α is the field

$$k_\alpha = \mathbb{Q}(\chi_\alpha(\gamma) \ : \ \gamma \in \Gamma),$$

and Weil rigidity implies that k_α is a number field. There is also the associated quaternion algebra

$$A_\alpha = \left\{ \sum_{i=1}^{n} x_i \rho_\alpha(\gamma_i) \ : \ \gamma_i \in \Gamma \, , \, x_i \in k_\alpha \, , \, n \in \mathbb{N} \right\} \subseteq \mathrm{M}_2(\mathbb{C}).$$

The field k_α is well-known to be closely related to the lengths of closed geodesics on M_α, and the algebra A_α, which carries information about the eigenvalues of hyperbolic elements of $\pi_1(M_\alpha)$, has been of great use in studying geometric problems like isospectrality. See [7] for more about these important topological invariants.

Associated with A_α are its *invariants* $\mathrm{Inv}(A_\alpha)$, which is a finite set consisting of (1) real embeddings of k_α and (2) prime ideals of the integer ring $\mathcal{O}_\alpha$ of k_α. To simplify the discussion, let $\mathrm{inv}(A_\alpha)$ denote the (finite) set $\{p_i\}$ of rational primes such that $\mathfrak{p}_i$ divides $p_i\mathcal{O}_\alpha$ for some prime ideal $\mathfrak{p}_i \in \mathrm{Inv}(A_\alpha)$. In other words, p_i is the characteristic of the finite field $\mathcal{O}_\alpha/\mathfrak{p}_i$.

Consider the set

$$\mathrm{inv}(K) = \{p \ : \ p \in \mathrm{inv}(A_\alpha) \, , \, M_\alpha \text{ a hyperbolic Dehn surgery on } S^3 \smallsetminus K\}.$$

Using the program Snap [2], one can compute $\mathrm{inv}(A_\alpha)$ for small Dehn surgeries on $S^3 \smallsetminus K$, and an interesting dichotomy appears: for some knots, the set $\mathrm{inv}(K)$ appears quite rigid over all slopes α, whereas for other knots $\mathrm{inv}(K)$ seems more random and possibly infinite. See Table 1 for some data.

Using the theory of quaternion *Azumaya algebras*, we are able to give a theoretical explanation for this dichotomy. Specifically, the problem ends up intimately tied to the question as to whether the assignment $\chi_\alpha \mapsto A_\alpha$ extends to define

Knot	Primes appearing in various $\mathrm{inv}(A_\alpha)$
Figure-8 knot	2
5_2	3, 5, 13, 181
$(-2,3,7)$ pretzel knot	3, 5, 13, 149, 211
6_1	$\emptyset$

Table 1. Primes appearing for small Dehn surgeries.

an Azumaya algebra over the entire smooth projective model $\widetilde{C}$ of the canonical component $C(K)$. Heuristically, an Azumaya algebra over $\widetilde{C}$ is an assignment of a quaternion algebra to each point; see [8] for a more precise definition. The reason we care about extending over the smooth projective model is precisely the following theorem of Harari [4]: An Azumaya algebra defined over a Zariski open subset $\mathcal{U}$ of a geometrically integral projective smooth curve $\widetilde{C}$ extends over all of $\widetilde{C}$ if and only if there exists a finite set of rational primes S such that, if A_z is the algebra associated with the point $z \in \mathcal{U}$, then $\mathrm{inv}(A_z) \subseteq S$. If no such finite set S exists, then in fact

$$\{p \ : \ p \in \mathrm{inv}(A_z) \text{ for some } z \in \widetilde{C}\}$$

has positive Dirichlet density.

Analogous to the above definition of A_α, one can associate a quaternion algebra A_ρ to the point on $\widetilde{C}$ determined by the character of any (absolutely) irreducible representation ρ, and in [1] we show that this defines an Azumaya algebra over the Zariski-open subset $\mathcal{U} \subset \widetilde{C}$ associated with the characters of irreducible representations. Moreover, analogous to Culler and Shalen's analysis of ideal points [3] (in particular, irreducibility of the associated tree action), we show that one can also include the set of ideal points of $\widetilde{C}$ in $\mathcal{U}$. Then $\widetilde{C} \smallsetminus \mathcal{U}$ consists of the points associated with characters of reducible representations.

Using the close connection between nonabelian reducible representations of knot groups and their Alexander polynomials, we completely characterize when the above defines an Azumaya algebra over $\widetilde{C}$. The main result of [1] is the following:

Theorem 1 ([1])**.** *Let K be a hyperbolic knot with $\Gamma = \pi_1(S^3 \smallsetminus K)$, and suppose that its Alexander polynomial $\Delta_K(t)$ satisfies:*

($\star$) For any root z of $\Delta_K(t)$ in an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and w a square root of z, we have an equality of fields $\mathbb{Q}(w) = \mathbb{Q}(w + w^{-1})$.

Then the above construction extends to give an Azumaya algebra over the entire smooth projective model $\widetilde{C}$ of the canonical component of the $\mathrm{SL}_2(\mathbb{C})$ character variety of K.

Conversely, assuming that the point on $\widetilde{C}$ determined by a nonabelian reducible representation is smooth (e.g., if the associated root of the Alexander polynomial is a simple root [5]), then the converse holds: if ($\star$) fails, then the Azumaya algebra cannot extend. One easily checks that ($\star$) holds for the figure-8 knot and for 6_1, but fails for 5_2 and the $(-2,3,7)$ pretzel knot. Moreover, using the theory of minimal regular integral models, we prove that when ($\star$) holds the set S in Harari's theorem is precisely the (finite) set of primes p for which ($\star$) fails modulo p. In particular, we can confirm:

Theorem 2 ([1]). *Let M_α be a hyperbolic Dehn surgery on the figure-8 knot. Then* $\mathrm{inv}(A_\alpha) \subseteq \{2\}$.

In [1] we also study connections with the existence of SU(2) representations of knot groups. For example, we can prove under certain technical assumptions on the arithmetic properties of the canonical component that it contains a circle of characters of irreducible SU(2) representations. Irreducible SU(2) representations are known to exist by work of Kronheimer–Mrowka [6], though it is not known that these representations necessarily determine characters on the canonical component. Our results indicate that one can also study this problem via arithmetic geometry of the character variety.

References

[1] T. Chinburg, A. W. Reid, and M. Stover, *Azumaya algebras and canonical components*, arXiv:1706.00952.

[2] D. Coulson, O. A. Goodman, C. D. Hodgson, and W. D. Neumann, Computing arithmetic invariants of 3-manifolds, *Experiment. Math.*, **9** (2000), no. 1, 127–152.

[3] M. Culler and P. B. Shalen, Varieties of group representations and splittings of 3-manifolds, *Ann. of Math. (2)*, **117** (1983), no. 1, 109–146.

[4] D. Harari, Méthode des fibrations et obstruction de Manin, *Duke Math. J.*, **75** (1994), no. 1, 221–260.

[5] M. Heusener, J. Porti, and E. Suárez Peiró, Deformations of reducible representations of 3-manifold groups into $\mathrm{SL}_2(\mathbb{C})$, *J. Reine Angew. Math.*, **530** (2001), 191–227.

[6] P. B. Kronheimer and T. S. Mrowka, Dehn surgery, the fundamental group and SU(2), *Math. Res. Lett.*, **11** (2004), no. 5-6, 741–754.

[7] C. Maclachlan and A. W. Reid, *The Arithmetic of Hyperbolic 3-Manifolds*, Graduate Texts in Mathematics **219**, Springer (2003).

[8] J. S. Milne, *Étale Cohomology*, Princeton Mathematical Series **33**, Princeton University Press (1980).

On the volume conjecture of quantum knot invariants

Jun Murakami

The relation between a quantum invariant and the hyperbolic volume of the complement of a knot was discovered by R. Kahsaev [2], and then his invariant turned out to be a specialization of the colored Jones invariant [4]. The colored Jones invariant is a complex valued invariant, and now we have the following conjecture.

Volume Conjecture. For a hyperbolic knot K in S^3,

$$\lim_{N\to\infty} \frac{2\pi \log J_N(K)}{N} = \mathrm{Vol}(K) + \sqrt{-1}\,\mathrm{CS}(K), \tag{1}$$

where $J_N(K)$ is the colored Jones invariant corresponding to the N dimensional irreducible representation of sl_2 whose parameter q is specialized to the N-th primitive root of unity $e^{\frac{2\pi\sqrt{-1}}{N}}$, $\mathrm{Vol}(K)$ and $\mathrm{CS}(K)$ are the hyperbolic volume and the Chern-Simons invariant of the knot complement $S^3 \setminus K$. The righthand side of (1) is called the *complex volume* of $S^3 \setminus K$.

The proof of this conjecture is not difficult for the figure-eight knot and the Borromean rings, but it is difficult for other cases, and now proved for hyperbolic knots up to 7 crossings in [5], [7] and [6]. The strategy to prove the conjecture is the following.

Step 1: Replace quantum factorials in the quantum invariants by quantum dilogarithm functions. The quantum factorial is defined for integer values, while quantum dilogarithm function is defined for continuous parameters, and whose value at an integer coincides with the quantum factorial. Moreover, the main asymptotics of the quantum dialogarithm function is given by the (classical) dilogarithm function. The function obtained from the quantum invariant by replacing the quantum factorials by dilogarithm functions is called the *volume potential function.*

Step 2: By using Poisson summation formula, replace sums by integrals.
Poissson summation formula:

$$\sum_{m\in\mathbf{Z}} f(m) = \sum_{m\in\mathbf{Z}} \widehat{f}(m) \quad \text{where} \quad \widehat{f}(x) = \int_{\mathbf{R}} e^{-2\pi\sqrt{-1}xt}\, f(t)\, dt.$$

Step 3: Apply the saddle point method to the integral of the volume potential function, then the value at certain saddle point of the volume potential function coincide with the complex volume of the knot complement. Yokota [9] pointed out that the saddle point equation corresponds to the glueing equation of the ideal tetrahedral decomposition of the knot complement.

The difficult part is to apply the saddle point method in Step 3. To apply the saddle point method, we have to check that the integral domain satisfy certain geometric condition. In general, we need multivariable integral and the condition for multivariable case is not so easy to check. At this moment, it is done by case-by-case check for each knot.

Even though the proof is not given, the volume potential function constructed from the quantum invariant contains various geometric information of knots. For example, Yokota [8] shows that a good deformation of the volume potential function coincides with the Neumann-Zagier function of the knot complement.

Almost 20 years after Kashav's discovery, Chen-Yang [1] found that the volume conjecture holds not only for knots but also for closed 3 manifolds if the parameter q is replaced by the second N-th root of unity $e^{\frac{4\pi\sqrt{-1}}{N}}$ instead of $e^{\frac{2\pi\sqrt{-1}}{N}}$, and they generalized the volume conjecture for the Turaev-Viro invariant and the Witten-Reshetikhin-Turaev invariant of 3-manifolds. The Turaev-Viro invariant is obtained from a tetrahedral decomposition of a 3-manifold by assigning the quantum $6j$ symbol to each tetrahedron. By applying the above 3 steps to the Turaev-Viro invariant, a solution of saddle point equation gives the edge lengths of the geometric tetrahedral decomposition of the 3-manifold, and the quantum $6j$ symbol corresponds to the volume.

Chen-Yang's version of the volume conjecture is also valid for the quantum invariant of knotted graphs in S^3 introduced by Kirillov-Reshetikhin [3]. The face model construction of this invariant also uses the quantum $6j$ symbols. In this case, the solution of the saddle point equation consists of complex numbers. If the parameter is a real or pure-imaginal number, then it corresponds to the length or angle at the edge, but if tis is a complex number, then it is a problem to understand the geometric object corresponds to the quantum $6j$ symbol. Quantum invariants are often formulated by quantum R-matrices, but the formulation by the quantum $6j$ symbol is also useful and universal.

References

[1] Q. Chen and T. Yang, *A volume conjecture for a family of Turaev-Viro type invariants of 3-manifolds with boundary*, preprint, arXiv:1503.02547.

[2] R. M. Kashaev, *The hyperbolic volume of knots from the quantum dilogarithm*, Lett. Math. Phys. **39** (1997), 269–275.

[3] A. Kirillov and N. Yu. Reshetikhin, *Representations of the algebra $U_q(sl(2))$, q-orthogonal polynomials and invariants of links*, Infinite-dimensional Lie algebras and groups (Luminy-Marseille, 1988), 285–339, Adv. Ser. Math. Phys., **7**, World Sci. Publ., Teaneck, NJ, 1989.

[4] H. Murakami and J. Murakami, *The colored Jones polynomials and the simplicial volume of a knot*, Acta Math. **186** (2001), 85–104.

[5] T. Ohtsuki, *On the asymptotic expansion of the Kashaev invariant of the* 5_2 *knot*, Quantum Topol. **7** (2016), 669–735.

[6] T. Ohtsuki, *On the asymptotic expansions of the Kashaev invariant of the knots with 7 crossings*, preprint.

[7] T. Ohtsuki and Y. Yokota, *On the asymptotic expansions of the Kashaev invariant of the knots with 6 crossings*, Math. Proc. Camb. Phil. Soc. (to appear).

[8] Y. Yokota, *On the potential functions for the hyperbolic structures of a knot complement*, Invariants of knots and 3-manifolds (Kyoto, 2001), 303–311, Geom. Topol. Monogr., **4**, Geom. Topol. Publ., Coventry, 2002.

[9] Y. Yokota, *On the complex volume of hyperbolic knots*, J. Knot Theory Ramifications **20** (2011), 955–976.

Theta Series for Indefinite Quadratic Forms

Sander Zwegers

(joint work with Don Zagier)

Let $Q : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a non-degenerate quadratic form of signature (r, s). We denote the associated symmetric matrix by A (so $Q(x) = \frac{1}{2}x^{\mathrm{T}}Ax$, $A^{\mathrm{T}} = A$) and assume that A has entries in $\mathbb{Z}$. Further let $B : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ be the associated bilinear form: $B(x, y) := x^{\mathrm{T}}Ay = Q(x + y) - Q(x) - Q(y)$.

For the case that A is positive definite ($s = 0$), it's a classical result that the corresponding theta series

$$\Theta_A(\tau) := \sum_{\ell\in\mathbb{Z}^n} q^{Q(\ell)} \qquad (q = e^{2\pi i\tau},\ \mathrm{Im}\,\tau > 0)$$

is a (holomorphic) modular form of weight $n/2$ (on some subgroup of $\mathrm{SL}_2(\mathbb{Z})$, with some character), see [6] for the case that A is even.

If for example we take $A = 2I_4$, then the theta function Θ_A is modular of weight 2 on $\Gamma_0(4)$. Writing this Θ_A as a linear combination of Eisenstein series we obtain Jacobi's four-square theorem, which gives a formula for the number of ways that a given positive integer m can be represented as the sum of four squares:

$$|\{\ell \in \mathbb{Z}^4 \mid \ell_1^2 + \ell_2^2 + \ell_3^2 + \ell_4^2 = m\}| = 8 \sum_{d \mid m,\, 4 \nmid d} d$$

For $s > 0$ the situation is a bit more complicated, because the sum $\sum_{\ell \in \mathbb{Z}^n} q^{Q(\ell)}$ doesn't converge. However, we can remedy this by restricting the sum over the full lattice to the sum over a cone. For $s = 1$ such a construction can be found in [3] and [9]: Let

$$\Theta_m^{c_1,c_2}(\tau) := \sum_{k \in \mathbb{Z}^n} \big\{\operatorname{sgn} B(c_1, k) - \operatorname{sgn} B(c_2, k)\big\}\, m(k)\, q^{Q(k)},$$

where $c_1, c_2 \in \mathbb{R}^n$ are such that $Q(c_1) = Q(c_2) = -1$, $B(c_1, c_2) < 0$ (c_1, c_2 belong to the same component of $\{c \in \mathbb{R}^n \mid Q(c) = -1\}$) and m is a periodic function on $\mathbb{Z}^n$. This function $\Theta_m^{c_1,c_2}$ is holomorphic, but in general not modular. Now consider the function $\widehat{\Theta}_m^{c_1,c_2}$ given by

$$\widehat{\Theta}_m^{c_1,c_2}(\tau) := \sum_{k \in \mathbb{Z}^n} \big\{E\big(B(c_1, k) y^{\frac{1}{2}}\big) - E\big(B(c_2, k) y^{\frac{1}{2}}\big)\big\}\, m(k)\, q^{Q(k)} \qquad (y = \operatorname{Im} \tau),$$

where

$$E(z) := 2 \int_0^z e^{-\pi u^2} du = \operatorname{sgn}(z) - \operatorname{sgn}(z) \int_{z^2}^{\infty} u^{-\frac{1}{2}} e^{-\pi u} du$$

(see [9]). This function $\widehat{\Theta}_m^{c_1,c_2}$ is modular (which can easily be shown using a result of Vignéras [7]), but in general not holomorphic. We can view $\widehat{\Theta}_m^{c_1,c_2}$ as the modular "completion" of $\Theta_m^{c_1,c_2}$. As an application one can use these functions to study the modular behaviour of Ramanujan's mock theta functions (see [9]). Further, in certain special cases one has $\Theta_m^{c_1,c_2} = \widehat{\Theta}_m^{c_1,c_2}$, so then the function is both holomorphic *and* modular.

Recently, analogous constructions have been found for arbitrary $s \geq 0$, see [1], [4], [8], [5] and [2] (in chronological order). In this talk we discuss the construction of holomophic theta functions for simplicial cones and we give a very simple and elegant formula for their modular "completions".

References

[1] S. Alexandrov, S. Banerjee, J. Manschot, and B. Pioline, *Indefinite theta series and generalized error functions*, preprint (2016), arXiv:1606.05495.

[2] J. Funke and S. Kudla, *Theta integrals and generalized error functions II*, preprint (2017), arXiv:1708.02969.

[3] L. Göttsche and D. Zagier, *Jacobi forms and the structure of Donaldson invariants for 4-manifolds with $b_+ = 1$*, Sel. math., New ser. 4 (1998), 69–115

[4] S. Kudla, *Theta integrals and generalized error functions*, preprint (2016), arXiv:1608.03534.

[5] C. Nazaroglu, *r-tuple error functions and indefinite theta series of higher-depth*, preprint (2016), arXiv:1609.01224.
[6] B. Schoeneberg, *Das Verhalten von mehrfachen Thetareihen bei Modulsubstitutionen*, Math. Ann. 116 (1939), 511–523.
[7] M.-F. Vignéras, *Séries thêta des formes quadratiques indéfinies*, In: Modular functions of one variable VI, Springer Lecture Notes 627 (1977), 227–239.
[8] M. Westerholt-Raum, *Indefinite theta series on cones*, preprint (2016), arXiv:1608.08874.
[9] S. Zwegers, *Mock Theta Functions*, PhD dissertation (2002), Utrecht.

Homology of torus knots

Anton Mellit

We begin by asking some motivational questions.

Question 1. For which knots does the HOMFLY-PT polynomial have positive coefficients in q and $-a$?

Our results below imply that all torus knots have this property. Probably all algebraic knots also have this property. Since HOMFLY-PT polynomial is the equivariant Euler characteristic of the *triply graded Kohvanov-Rozansky* homology HHH, the condition that the triply graded homology is concentrated in even degrees implies HOMFLY-positivity. So one may also ask

Question 2. For which knots is HHH concentrated only in even degrees?

Furthermore, we expect that the homological invariant can be upgraded to an invariant whose values are algebraic varieties. In this case, maybe the evenness of homology can be explained by geometry of the variety.

Then we proceed to formulate our result for torus knots. The connection between homologies of torus knots and Macdonald polynomials was conjectured by Aganagic and Shakirov [2]. A more precise combinatorial formulation was given by Gorsky and Negut [3]. The algebraic combinatorics of Macdonald polynomials is related to combinatorics of Dyck paths via *shuffle conjectures*, solved in [4]. Using the technique of Elias and Hogancamp [5] and our interpretation of Dyck paths as "1-dimensional movies" we prove the conjecture about homology of torus knots:

Theorem 1. *The Poincaré polynomial of* HHH *of the* (m, n) *torus knot, up to a monomial factor, is given by the following sum over* (m, n)*-Dyck paths*

$$\mathcal{P}_{m,n} \sim \sum_{\pi} t^{\mathrm{area}(\pi)} q^{h_+(\pi)} \prod_{v \in v^*(\pi)} (1 - aq^{-k(v)}),$$

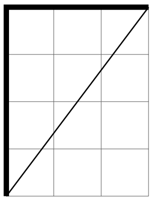
where for each (m, n)*-Dyck path* π *we use the following notations:*

- $\mathrm{area}(\pi)$ *is the number of* 1×1 *lattice squares contained between the path and the diagonal,*
- $v^*(\pi)$ *is the set of outer vertices of* π *without the vertex most distant from the diagonal,*

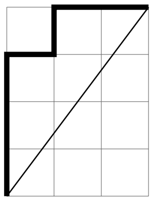
- *$k(v)$ for each $v \in v^*(\pi)$ is the number of vertical steps (=number of horizontal steps) of π intersected by the line parallel to the diagonal passing through v.*
- *$h_+(\pi)$ is the number of pairs (s_1, s_2) where s_1 resp. s_2 is the horizontal resp. vertical step of π, s_1 is to the left of s_2 and there exists a line parallel to the diagonal intersecting both s_1 and s_2.*

In particular we see, that the coefficients of $\mathcal{P}_{m,n}$ expanded in q, t and $-a$ are positive, which implies that HHH is concentrated in even degrees. Setting $t = q^{-1}$ one obtains the HOMFLY polynomial, so it also has positive coefficients.

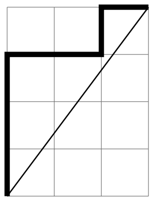
Example 1. For $(m, n) = (3, 4)$ we have 5 Dyck paths, which are displayed below together with their contributions:



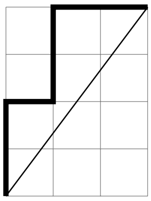
t^3q^6



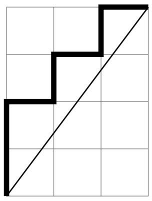
$t^2q^7(1 - aq^{-1})$



$tq^7(1 - aq^{-1})$



$tq^8(1 - aq^{-1})$



$q^9(1 - aq^{-1})(1 - aq^{-2})$

So we obtain

$$\mathcal{P}_{3,4} \sim t^3 + t^2q + tq^2 + q^3 + qt - a(t^2 + qt + q^2 + q + t) + a^2.$$

The HOMFLY polynomial is obtained by setting $t = q^{-1}$:

$$\begin{aligned}\mathrm{HOMFLY}_{3,4} &\sim q^{-3} + q^{-1} + 1 + q + q^3 - a(q^{-2} + q^{-1} + 1 + q + q^2) + a^2 \\ &= z^6 + 6z^4 + 10z^2 + 5 - a(z^4 + 5z^2 + 5) + a^2,\end{aligned}$$

where $z = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$. The last expression is provided so that one can match our formula with The Knot Atlas [6].

References

[1] Mikhail Khovanov. Triply-graded link homology and Hochschild homology of Soergel bimodules. *International Journal of Mathematics* **18**(2007), 869–885.

[2] M. Aganagic and S. Shakirov. Knot Homology from Refined Chern-Simons Theory. arXiv:1105.5117 (2011).

[3] Eugene Gorsky and Andrei Neguţ. Refined knot invariants and Hilbert schemes. *Journal de Mathématiques Pures et Appliquées* **104**(2015), 403–435.

[4] A. Mellit. Toric braids and (m, n)-parking functions. arXiv:1604.07456 (2016).

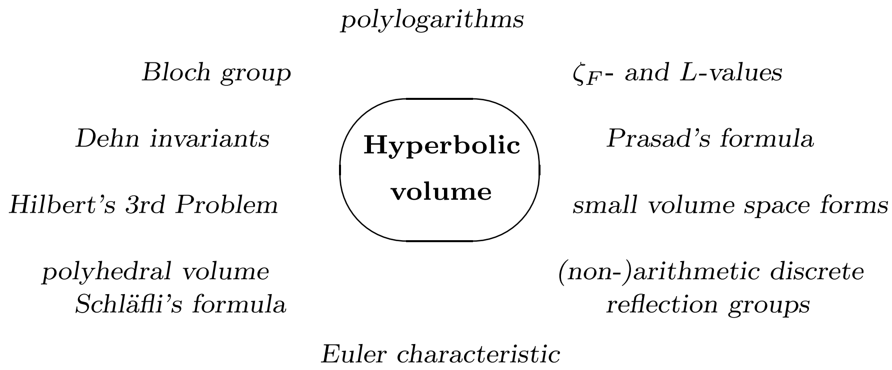
[5] B. Elias and M. Hogancamp. On the computation of torus link homology. arXiv:1603.00407 (2016).

[6] *The Knot Atlas.* http://katlas.org/wiki/T(4,3).

Algebraic aspects of hyperbolic volume

Ruth Kellerhals

When considering volumes of hyperbolic orbifolds and manifolds $Q = \mathbb{H}^n/\Gamma$ as given by the volumes of fundamental polyhedra for the discontinuous action of the group $\Gamma \subset \mathrm{Isom}\mathbb{H}^n$, several algebraic aspects (beside the obvious analytical ones) appear. A rough picture is provided by the following chart.



In even dimensions, by the (generalised) Theorem of Gauss-Bonnet, the volume of Q is proportional to the Euler characteristic $\chi(Q)$ which can be computed in different combinatorial and homological ways. When $n = 2$, there is a very satisfactory picture about the area spectrum of hyperbolic orbifolds. For $n \geq 4$ and if Q is an orientable and arithmetically defined n-orbifold, one has Prasad's important volume formula. For even $n \geq 4$, Belolipetsky in 2004, 2006 exploited this formula and deduced explicit values for the minimal volume orbifolds (for a survey with more details, see [4], for example). In particular, Belolipetsky showed that the quotient of $\mathbb{H}^4$ by the rotation subgroup of the arithmetic discrete reflection group with fundamental Coxeter simplex S given by the Coxeter diagram $[5, 3, 3, 3]$ is the unique compact oriented arithmetic orbifold of minimal volume. Other optimality results in dimension 4 are known when looking at arbitrary orbifolds and manifolds with cusps (see [3] and [6]). Furthermore, the identification of minimal volume cusped orientable arithmetic hyperbolic n-orbifolds of even dimension up to $n = 18$ has recently been established in [5]. However, in the compact smooth case, the smallest known 4-manifold has Euler characteristic equal to 8, only, and was constructed by Conder and Maclachlan [1]. In fact, by means of the computer package MAGMA, they found a suitable torsionfree subgroup of the Coxeter simplex group given by $[5, 3, 3, 3]$ above.

In odd dimensions, there is a structural difference in view of the volume spectra of hyperbolic orbifolds and manifolds for $n = 3$ and $n > 3$, respectively, and much more is known in the low dimensional case $n = 3$. By results of Jørgensen, Thurston and Gromov, the volume spectrum for hyperbolic 3-manifolds is non-discrete (well-ordered of order type ω^ω and with limits points given by cusped manifolds) while, by a result of Wang, the volume spectrum for hyperbolic n-manifolds is discrete if $n \neq 3$. Furthermore, minimality results are known for the small part of all the diverse restricted volume spectra in dimension 3 (see also [4]). For example, the Gieseking manifold built from the ideal regular tetrahedron is the unique cusped 3-manifold of minimal volume. However, an analogous optimality result in the non-compact case for $n = 5$ is not known.

In [2, Theorem 1.3], Goncharov proved a substantial generalisation to $n = 5$ of a result of Dupont, Sah, Neumann and Thurston for $n = 3$ which can be stated in terms of the Bloch group $B_2(F)$ of a field F as follows. For each finite volume hyperbolic 5-manifold M there are finitely many algebraic numbers z_i, $i \in I$, satisfying an algebraic identity of the form

$$\sum_{i\in I} \{z_i\}_2 \otimes z_i = 0 \quad \text{in} \quad B_2(\overline{\mathbb{Q}}) \otimes \overline{\mathbb{Q}}^* \,, \tag{1}$$

in such a way that

$$\mathrm{vol}_5(M) = \sum_{i\in I} \mathcal{L}_3(z_i) \,,$$

where $\mathcal{L}_3(z)$ denotes a certain generalised trilogarithm function.
However, for Goncharov's result, in particular in the compact setting, we do not dispose of any non-trivial example in view of (1) (meaning $z_1 \neq 1$ and $|I| > 1$).

There is a construction due to Ratcliffe and Tschantz [7], [8] of a cusped hyperbolic 5-manifold M_1 (in-)directly related to the ideal right-angled polyhedron P^5 of Vinberg. They computed the volume of the manifold M_1 and obtained $7\zeta(3)/4$ (notice that $\mathcal{L}_3(1) = \zeta(3)$). In the ongoing work together with Conder, we constructed a cusped hyperbolic 5-manifold M_0 with a (conjectural) big degree of symmetry whose volume should be comparatively small.

References

[1] M. Conder, C. Maclachlan, *Compact hyperbolic 4-manifolds of small volume*, Proc. Amer. Math. Soc. **133** (2005), 2469–2476.

[2] A. Goncharov, *Volumes of hyperbolic manifolds and mixed Tate motives*, J. Amer. Math. Soc., **12** (1999), 569–618.

[3] T. Hild, R. Kellerhals, The FCC lattice and the cusped hyperbolic 4-orbifold of minimal volume, J. Lond. Math. Soc. *75* (2007), 677–689.

[4] R. Kellerhals, *Hyperbolic orbifolds of minimal volume*, Comput. Methods Funct. Theory **14** (2014), 465–481.

[5] R. Kellerhals, *On minimal covolume hyperbolic lattices*, Mathematics **5** (2017), available online at http://www.mdpi.com/2227-7390/5/3/43.

[6] J. Ratcliffe, S. Tschantz, *The volume spectrum of hyperbolic 4-manifolds*, Experiment. Math. **9** (2000), 101–125.

[7] J. Ratcliffe, S. Tschantz, *Volumes of integral congruence hyperbolic manifolds*, J. Reine Angew. Math. **488** (1997), 55–78.

[8] J. Ratcliffe, S. Tschantz, *Integral congruence two hyperbolic 5-manifolds*, Geom. Dedicata **107** (2004), 187–209.

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