Arbeitsgemeinschaft mit aktuellem Thema:

Additive combinatorics, entropy and fractal geometry

Mathematisches Forschungsinstitut Oberwolfach

October 8-13, 2017

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Objectives

This series of twenty-one talks aims to cover many of the recent developments in the dimension theory of self-similar measures and their projections and intersections, especially on Furstenberg's conjectures on sets defined by digit restrictions [11, 24, 17, 31, 40], the overlaps conjecture for self-similar sets and measures [16], the Bernoulli convolutions problem [29, 36, 4, 5]; as well as recent improvements to classical projection theorems and their relatives [2, 3, 22]. These works have relied upon, and in many cases introduced, techniques from ergodic theory, additive combinatorics, Fourier analysis and algebraic number theory. We hope to cover many of these during the course of the Arbeitsgemeinschaft.

Given the breadth of topics and results, and the technical nature of some of the proofs, talks are intended to provide proof sketches more often than complete proofs. Speakers should attempt to convey the main ideas rather than give full details.

Background material

Participants should review basic definitions and results of Hausdorff dimension of sets and measures and of ergodic theory, including:

Dimension theory Hausdorff and box (Minkowski) dimension and the relation between them, upper and lower Hausdorff and pointwise dimension of a probability measure on $\mathbb{R}^d$, the mass distribution principle and Frostman's lemma (relating dimension of a closed set and the measures on it). Possible sources are [7, 20].

Ergodic theory Measure preserving systems, ergodicity and ergodic decomposition, mean and pointwise ergodic theorems. Alternatively a comfortable relationship with Markov processes and martingales is a close substitute. Possible sources are [6, 38].

Introductory lectures

1. **Self-similar sets and measures**

   This talk outlines properties of self-similar sets and measures and some of the central conjectures (in some cases, now theorems) about them. Define iterated function systems and their attractors, self-similar sets and measures, strong separation and open set condition, similarity dimension of sets and measures (sometimes called Lyapunov dimension), calculate dimension under strong separation/OSC, equality of Hausdorff and box dimension (if time permits, state Falconer's implicit method). Exact overlaps conjecture, Marstrand's projection and slice theorems (in the plane), Furstenberg's slice conjecture and its context. Sources: [1, 33, 20, 11].

2. **Introduction to entropy**

   Entropy will be one of the main technical tools in this series for computing dimension. Here we survey the definition and basic properties of entropy, and its connections with dimension: Shannon entropy and conditional entropy, basic identities and inequalities, convexity, continuity in weak topology and total variation, dyadic partitions, interpretation of chain rule in terms of conditional measures on dyadic cells. Entropy dimension, relation with pointwise dimension and box dimension. Existence of entropy dimension for self-similar measures. Sources: Notes will be provided for entropy basics, additionally [38] (for general properties of entropy), [25] (for the existence of entropy dimension for self-similar measures), [9] (for the relation between entropy and pointwise dimensions).

Ergodic techniques

3. **CP-Processes**

   CP-processes, introduced by Furstenberg in [12], provide a way to represent fine-scale structure of measures in a suitable dynamical system. This talk will present the definition and basic properties of CP-processes: Preliminary material on Markov chains, CP-processes, dimension of CP-processes, constructing CP-processes via averaging and as pointwise averages. CP-processes associated to self-similar measures and products of deleted digit measures. Sources: the first part of the talk should follow [15, Chapter 6], see also [12, 28]. Further notes will be provided.

4. **Furstenberg's dimension conservation theorem and local entropy averages**

   Furstenberg proved that, for certain fractal measures, a dimension conservation phenomenon holds, in which the dimension of the image measure under linear projection is compensated for in the conditional measures on the fibers of the map. In this talk we prove Furstenberg's dimension conservation theorem for CP-processes [12, Section 3] and note its implications for self-similar measures defined by homotheties, and their projections, namely, exact dimensionality of self-similar measures with overlaps. Then, formulate and prove the local entropy averages lemma (including projection version), [17, Section 4]. Sources: [12, 17], see also [10].

5. **Peres-Shmerkin and Hochman-Shmerkin projection theorems**

   For certain dynamically-defined measures, linear images are "as large as they can be". In this talk we prove the Peres-Shmerkin and Hochman-Shmerkin theorems on projections of self-similar measures and their products: For this we prove local entropy averages bounds for the projection of measures generating a CP-process, and semicontinuity of projection dimension for CP-processes [17, Section 8]. Then, combining Marstrand's theorem with additional symmetries of the CP-processes in question, prove the Peres-Shmerkin/Hochman-Shmerkin theorem about projections of self-similar sets with dense rotations and for products of product measures in non-commensurable bases, and sketch the analogous result for products of $\times m$- and $\times n$-invariant sets [17, Section 9 and 10]. See also [24, 15].

6. **Wu's proof of Furstenberg's intersection conjecture**

   Wu [40] gave a dynamical proof of Furstenberg's intersections conjecture that is strongly based on CP-processes. In this talk we describe the needed background and prove the theorem: reduction to the case of deleted digits sets, Sinai's theorem, dimension of conditional measures, disjointness, radial-Fubini argument. Source: [40].

Entropy, algebra and additive combinatorics

7. **Some additive combinatorics**

   Additive combinatorics is a collection of tools for understanding sum-sets in abelian (and sometimes other) groups. We survey classical results for finite sets and for measures (via entropy). State Freiman's theorem [35, Theorem 5.32] and Tao's entropy variant [34, Theorem 1.11]. Then present Petridis's elegant proof of the Plünnecke-Rusza inequality [26, Theorem 1.1 and 1.2] and the Kaimanovich-Vershik entropy variant of Plünnecke-Rusza [16, Lemma 4.7].

8. **Hochman's inverse theorem for entropy**

   Hochman's inverse theorem describes the statistical structure of measures whose convolutions does not increase entropy. First, define components of a measure, concentration and uniform components. Then sketch the proof of entropy growth of repeated convolutions via the central limit theorem and multiscale analysis. Finally, derive the inverse theorem. Sources: [16, Section 4].

9. **Hochman's theorem on self-similar measures with overlaps**

   The dimension of self-similar measures can be computed classically under the open set condition. Hochman's theorem provides a weaker condition which lets the dimension be computed in many cases where there are more substantial overlaps. First sketch the proof that self-similar measures have uniform entropy dimension [16, Section 5.1]. Then prove Hochman's theorem that dimension drop implies super-exponential concentration of cylinders, and a variant using random walk entropy [16, Theorem 1.1 and 1.3], see also [4, Section 3.4]. State implications in the algebraic case and state the consequences for the 1-dimensional Sierpinski gasket (state Garsia's separation lemma [13, Lemma 1.51], it will be proved in a later lecture. Note: the lemma is also proved in [16, Lemma 5.10] but the statement and proof omit the requirement on the heights of the polynomials must be bounded). Discuss implications for parametric families.

10. **Background on Bernoulli convolutions**

    Bernoulli convolutions (BC) have been studied since the 1930s. In this talk we touch on some of the main classical and modern results. Prove Erdős's result on singularity of BC for inverse-Pisot numbers, and establish the Erdős-Kahane lemma on generic decay of the Fourier transform. Derive generic smoothness near 1. Survey Garsia numbers, explain idea of Solomyak's transversality results. This talk is based on [23].

11. **Algebraic aspects of Bernoulli convolutions**

    This talk provides additional algebraic background material and connections with Bernoulli convolutions. Define the Mahler measure of an algebraic number. Define Pisot and Salem numbers and state the Lehmer conjecture. Prove that BC with Salem parameters do not have power Fourier decay ([23, Sec. 5]). State Garsia's separation lemma [13, Lemma 1.51] and explain the proof of Garsia's theorem that the entropy is strictly less than $\log \lambda$ if λ^{-1} is Pisot. State (sketch proof if time permits) Mahler's theorem on separation of roots [5, Lemma 21]. Describe implications of Hochman's theorem and the problem on separation of roots of $-1, 0, 1$ polynomials [16, Theorem 1.9 and Question 1.10]. See also [27].

12. **Shmerkin's theorem on smoothness of BC and projections**

    In this talk we prove Shmerkin's theorem that BC are smooth outside a zero-dimensional set of parameters. State the relation between dimension and Fourier transform [29, Sec. 2.1]. Prove Shmerkin's smoothing lemma [29, Lemma 2.1], and derive Shmerkin's theorem on smoothness of BC [29, Thm 1.1]. Describe variations for projections of self-similar and product sets and other generalizations [29, 32]. Also show the Nazarov-Peres-Shmerkin example of projections with singular measures [21, Section 4].

13. **Varjú's theorem on smoothness of BC for algebraic parameters**

    State Varjú's main theorem ([36, Theorem 1]). State Garsia's entropy condition for AC ([14, Theorem I.5]). Define the average entropy $H(X; r)$ at a scale r; state and prove some of its properties. State Varjú's inverse theorems for the entropy of convolutions ([36, Theorem 2,3]. Derive Varjú's main theorem on absolute continuity from the inverse theorems. If time permits outline the proof of Theorems 2 or 3. Main sources: [36, 37].

14. **Breuillard-Varjú's inequality between entropy and Mahler measure**

    The goal of this talk is to prove [4, Theorem 5], which relates the entropy of a Bernoulli convolution with algebraic parameter λ to the Mahler measure of λ. State the main result and its corollary (full dimension for algebraic parameters near 1 assuming the Lehmer conjecture). Start with a proof of the upper bound (Lemma 16). Define differential entropy and sketch proof of Madiman's submodularity inequality [4, Sec 2.4]. Define the Gaussian-averaged entropy $H(X; A)$ at scale A, state its basic properties (Lemma 9) and explain how they follow from submodularity of entropy. Prove the lower bound in Theorem 5. Main sources: [4, 37].

15. **Breuillard-Varjú's results on the dimension of Bernoulli convolutions with transcendental parameters**

    This talk builds on Talk 13, but is (logically) independent of Talk 14. State the main theorem [5, Theorem 1] and its corollaries (Corollary 3,4). Compare with Hochman's theorem (Theorem 7). Show how the effective nullstellensatz implies an initial entropy lower bound for $\mu_\lambda^{(n)}$ at scale

$n^{-O(n)}$ unless λ has an excellent algebraic approximation (Theorem 17). Recall Varjú's inverse theorem from Talk 12 (i.e. [5, Theorem 8]). Explain how this can be used to bootstrap to full dimension and outline the proof of Theorem 1. If time permits prove Lemma 13. Main Sources: [5, 37].

Other additive combinatorics methods

16. **Orponen's distance set theorem**

    Falconer's distance problem asks whether a set in the plan of dimension > 1 must have a positive Lebesgue measure of distances represented by it pairs of points. This talk covers Orponen's results, in which entropy features prominently. First survey background material on the distance set conjecture and the theorems of Wolff and Bourgain. Then define Ahlfors-David regular sets, and prove Orponen's bound on the box dimension of distance sets of AD-regular sets. If time allows, discuss some of the extensions of Shmerkin [30] and how CP-processes enter the picture. Sources: [8, 39, 2, 22, 30].

17. **Balog-Szemerédi-Gowers Theorem**. The goal of this talk is to introduce one of the most powerful tools arising from additive combinatorics, the theorem of Balog-Szemerédi-Gowers. Introduce the concepts of additive energy and partial sumsets. State the Balog-Szemerédi-Gowers Theorem (emphasizing both the energy and partial sum versions), indicate why it is sharp up to the values of several constants, and give as much details of its proof as time allows. State the asymmetric version of Balog-Szemerédi-Gowers due to Tao-Vu [35, Theorem 2.35]. The material is contained in [35, Sections 2.3, 2.5, 2.6 and 6.4]. Note however that the proof of B-S-G in [35, Section 6.4] contains several misprints, see errata in Tao's webpage or, alternatively, see [19].

18. **Bourgain's sum-product and projection theorems. Part I**. In this talk we start discussing Bourgain's discretized sum-product and projection theorems, a collection of interconnected results at the intersection of geometric measure theory and additive combinatorics, which have found several applications in ergodic theory. State the main results (Theorems 1-4) from [3], giving heuristics about the connections between the different statements. Sketch the additive part of the proof, [3, Sections 2 and 3], leading up to the statement of [31, Theorem 3.1] (may also mention [31, Corollary 3.10]). Emphasize in particular the amplification argument that uses the Plünnecke-Ruzsa inequalities. See [2] in addition to [3].

19. **Bourgain's sum-product and projection theorems. Part II**. Building on the material from the previous talk, explain some of the ideas involved in the proofs of the main results from [3]. In particular, emphasize how the Balog-Szemerédi-Gowers is applied to deduce the projection theorem from the sum-product theorem.

20. **Shmerkin's theorem on L^q dimensions, and applications. Part I**. Introduce the concept of L^q dimension and explain its relationship to Frostman exponents and the size of fibers [31, Section 1.3]. Explain how one can deduce Furstenberg's slice conjecture from a statement about L^q dimensions of convolutions of self-similar measures. State [31, Theorem 1.11] in the case X is a singleton (in which case the statement deals with classical self-similar measures). State [31, Theorems 1.3 and 1.5 and Corollary 6.3] (with a discussion of their history/related results), and explain how they follow from [31, Theorem 1.11] and the L^p smoothing lemma from [32, Theorem 4.4].

21. **Shmerkin's theorem on L^q dimensions, and applications. Part II**. State the inverse theorem for L^q norms of convolutions, [31, Theorem 2.1]. Do not prove it, but sketch how it follows from the asymmetric version of the Balog-Szemerédi-Gowers Theorem and the additive part of Bourgain's sum-product Theorem. Outline the rest of the proof of [31, Theorem 1.11] in the case in which X is a singleton (the outline in [31, Section 1.6] may be useful). In particular, indicate the role of the differentiability of the L^q spectrum. For this part, see [18] in addition to [31]. Emphasize the similarities and differences with Hochman's results from Talks 8 and 9.

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