Mathematisches Forschungsinstitut Oberwolfach

Report No. 6/2018

DOI: 10.4171/OWR/2018/6

Interplay of Analysis and Probability in Applied Mathematics

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11 February – 17 February 2018

Abstract. This workshop continued to foster the collaboration between researchers working in analysis and probability, respectively. Some core areas, in which this happens with high success, belong to the objectives of this meeting: stochastic homogenization of various quantities in random media and random operators, metastability in several particle models with stochastic input that are triggered by physics reasonings, emergence of macroscopic effects in large random structures like graphs or permutations. A main feature present was the exploration of the benefit of a high-level combination of methods from both fields: analysis and probability.

Mathematics Subject Classification (2010): 35B27, 35R60, 47A75, 47B80, 60C05, 60F05, 60F10, 60F17, 60G50, 60H25, 60J10, 60J25, 60J27, 60K37, 80A30.

Introduction by the Organisers

Analysis and probability are fundamental parts of mathematics and have been in contact since ever. In the 20th century, in both areas more and more specialised methods and concepts have been developed in great depth. It was not easy for either of the two to keep track with the developments in the other, or even to see how they could influence and help each other and even push forward the other's evolution. Since a few decades, there is a desire to again increase the contact between the two areas on the highest level. This workshop stands in a series of several MFO workshops by now that do precisely that: it brings together people from both fields, including those that cannot be clearly assigned to any of the two

areas. The goal is to put together modern methods and ideas from both and use them to create knowledge that is out of reach for each area alone. We think that the preceding workshops have been increasingly instrumental in this sense, and the current one was even more successful.

For achieving this high goal, one has to identify suitable research subjects. One of the most fruitful ones is the area of *stochastic homogenisation* in the broadest sense. Indeed, this is about random spatial structures (random media, random equations, random operators, random processes in random media, ...) on a large scale and the identification of their macroscopic behaviour in terms of a few characteristic quantities. One of the most popular models is the random conductance model and here random processes in this kind of medium, for which a number of refined results were presented, like a quantification of the velocity in the central limit theorem (Andres), local limit theorems in time-dependent variants of the random conductance model (Slowik), Harnack inequalities and variants of the central limit theorem (e.g., superdiffusive Lévy flight-type processes Bianchi) and anomalies for less standard medium distributions (Deuschel, Kumagai, Boukhadra), and the dichotomy between localised and homogenised behaviour (Flegel). Furthermore, instead of a random proces in the random conductance model (respectively in a space-continuous variant of it), also other fundamental objects were considered like the Poisson equation (Flegel, Piatnitski), or Laplace-operator-driven energies in such medium (Neukamm), or random perturbations of the ergodic theorem in random Hamilton-Jacobi equations (Cardaliaguet). But also different approaches and aspects of geometric fine properties of random media and also different types of random media have been studied, like fluctuations of various scales in the Gaussian behaviour of more general observables in large boxes for a general elliptic random field (Otto), or a random grid defined by Voronoi tesselations of a random point cloud (Heida).

Another very fruitful research area that falls into our goals is the area of *stochastic particle systems* that are inspired by physics reasoning. Here one of the main tasks is to identify and to prove macroscopic effects emerging from microscopic random input quantities and their interaction in various ways. Particular interest receive models in which a positive percentage of the particles exhibit a very unusual behaviour like condensation or clumping (in static models), or a discrepancy between long and short time-stretches in which drastically different behaviours occur (in dynamic models). The first type falls into phenomena like *phase transitions*, and the second may be summarized by the term *metastability*. Here we had a few inspiring presentations, for example by Dereich on condensation in certain types of random graphs, and by Mörters on a kind of metastability in a version of the contact process on a particular type of a random time-dependent graph. Grosskinsky presented a macroscopic description of particle systems with mutual repellence and attraction in terms of mean-field equations that exhibit interesting phase transitions of condensation type. More classical, but very deep and sophisticated to prove, metastability phenomena in particle dynamics of Langevin-type were derived by Lelièvre, whose expansions employ a deep saddle

point analysis. Giuliani reported on emergence of striped ground states in Ising models with long range interactions, presenting work in which for the first time the emergence of mesoscopic structures (stripes) in those models is proved rigorously. Macroscopic shape results for a dynamic microscopic model, the Hastings-Levitov model, were presented by Norris. Another dynamic model, the linear Boltzmann equation, was analysed using gradient flows by Bertini.

A third area where probability and analysis naturally come together are models coming from *continuum and quantum physics*. Often, the focus here is similar to the one of the previous paragraph, namely to connect microscopic fundamental models with macroscopic, observable phenomena. Sometimes, however, it is also very useful and fruitful to analyse the macroscopic models directly. In quantum mechanical models, a currently active research area is the stochastic representation of quantum many body systems at positive temperature by probabilistic models. Lees gave an introduction into several different loop models and presented some results on phase transitions on trees. Adams presented the bosonic loop measure, which is intimately connected to Bose-Einstein condensation. In many of these measures, a prominent role is played by random permutations, whose probability weight is a complicated function of the cycle structure and the geometry. While treatment of such models is currently out of reach, some simpler models can be analysed and yield interesting phenomena. Zeindler presented such a model of permutations without macroscopic cycles. Mailler studied another combinatorial model, namely measure-valued Polya urns, and presented limit laws for the colour distributions in such urns. Direct investigation of macroscopic models was carried out by Souganidis on the long time behaviour of rough Hamilton-Jacobi equations, by Schlichting who investigated a non-local equation related to coarsening and nucleation, by Faggionato, who presented studies on periodically driven Markov processes inspired by molecular motors in cells, and by Patterson, who studied large deviations for reaction fluxes.

The workshop was visited by a well-picked mixture of about 50 probabilists and analysts of various academic ages, from those finishing soon their PhD, via more advanced postdocs, till well-known long-standing experts. The organisers chose to give many of the participants the opportunity to present their results, and most of the talks were for 45 minutes. In this way, a lot of current streamings could be communicated, discussed and compared.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, "US Junior Oberwolfach Fellows".

Workshop: Interplay of Analysis and Probability in Applied Mathematics

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Abstracts

Loop measures for space-time random walks

Stefan Adams
(joint work with Quirin Vogel)

Recently ([1, 7]) Markovian Loop measures have become an active field in probability theory with its origin going back to Symanzik [9]. In this work [2] we are going to investigate a novel setting of these loop measures, namely we consider first loop soups with varying intensity $\mu \leq 0$ (chemical potential in physics terms) and secondly we study Markovian loop measures on graphs with an additional "time" dimension, that is space-time random walks, their loop measures, and Poisson loop processes. Interesting phenomena appear when the additional coordinate of the space-time process is on a discrete torus with non-symmetric jumping rates. The projection of these space-time random walk loop measures onto the spacial dimensions are loop measures on the underlying graph, and in the limiting regime of unbounded torus size these loop measures converge to the so-called Bosonic loop measures. Via this space-time setting we provide a natural probabilistic definition of Bosonic loop measures. These novel loop measures have similarities with the Markovian loop measures only that they give weights to loops of certain lengths, namely any length which is multiple of a given time horizon $\beta > 0$ which serves as an additional parameter. The Bosonic loop measures not only have the probabilistic motivation as outlined, a second major interest in these objects stems from the fact that the total weight of the Bosonic loop measure for a finite graph is exactly the logarithm of the grand-canonical partition function of a non-interacting Bose gase on the finite graph in thermodynamic equilibrium at inverse temperature $\beta > 0$ and chemical potential $\mu \leq 0$. The study on Markovian loop measures has been outlined in the lecture notes [7] and [8] with more recent developments in [6].

Setting We define loop measures for random walks on finite graphs. Let $\Lambda \subset \mathbb{Z}^d$ be a finite set. We consider a random walk on Λ with killing, and we denote $Q = (q(x,y))_{x,y\in\Lambda}$ the generator matrix given by $Q = \lambda(P - I)$, where $P = (p(x,y))_{x,y\in\Lambda}$ is the transition matrix defined by $p(x,y) = \frac{w(x,y)}{\lambda(x)}$ with $(w(x,y))_{x,y\in\Lambda}$ being an irreducible matrix with non-negative symmetric entries and normalising $\lambda(x) = \kappa(x) + \sum_{y\sim x} w(x,y), x \in \Lambda$, with killing vector $(\kappa_x)_{x\in\Lambda}$. The space of loops is defined as the uncountable union

$$\Gamma := \bigcup_{t>0} \Gamma_t \,.$$

For each $\omega \in \Gamma$ we denote the time horizon or the length of the loop ω by $\ell(\omega)$ being the unique $t > 0$ such that $\omega \in \Gamma_t$. We study the following new loop measures which are related to the well-known Markovian loop measures (see e.g. [7] for an overview on Markovian loop measures).

Let $C \subset \Gamma$ be measurable, $\mu \leq 0$, and $\beta > 0$. The *Markovian loop measure* $M_{\Lambda,\mu}$ on Λ with chemical potential μ is defined as

$$M_{\Lambda,\mu}[C] = \sum_{x\in\Lambda} \int_0^\infty \frac{\mathrm{e}^{t\mu}}{t} \mathbb{P}^t_{x,x}(C)\,\mathrm{d}t = \sum_{x\in\Lambda} \int_0^\infty \frac{\mathrm{e}^{t\beta\mu}}{t} \mathbb{P}^{\beta t}_{x,x}(C)\,\mathrm{d}t, \tag{1}$$

where $\mathbb{P}^t_{x,y}$ denotes the random walk bridge measure going from x to y with time horizon $[0,t]$. The *Bosonic loop measure* $M^B_{\Lambda,\mu,\beta}$ on Λ with chemical potential μ and time horizon β is defined as

$$M^B_{\Lambda,\mu,\beta}[C] = \sum_{x\in\Lambda} \sum_{j=1}^\infty \frac{\mathrm{e}^{\beta j\mu}}{j} \mathbb{P}^{j\beta}_{x,x}(C)\ . \tag{2}$$

The main focus in this work is the Bosonic loop measure $M^B_{\Lambda,\mu,\beta}$ which is defined in similar way to the Markovian loop measure ([7]) but differs in having support only on loops with time horizons being a multiples of the given parameter β. The Bosonic loop measures for simple random walks are linked to equilibrium quantum statistical mechanics. The main idea and novelty of our study is to view these measures as natural Markovian loop measures in a space-time structure where we add an additional "time" torus to the given graph, i.e., the graph is $\Lambda \times \mathbb{T}_N$ with a discrete torus $\mathbb{T}_N$ of size N. The torus process is considered to be independent from the spatial process on Λ, that is, the weights are

$$w_N(x,\tau;y,\sigma) = \begin{cases} N\beta^{-1} & \text{if } \sigma = \tau + 1 \mod N,\ x = y\,, \\ w(x,y) & \text{if } \tau = \sigma, \\ 0 & \text{otherwise,} \end{cases}$$

$x, y \in \Lambda, \tau, \sigma \in \mathbb{T}_N$. We first show that in the torus limit $N \to \infty$ the spatial projection of the space-time Markovian loop measure converges to the Bosonic loop measure. Our main results then concern isomorphism theorems and generalised Symanzik's type formulae for moments of the occupation time fields. The major challenges are the missing symmetry of the space-time random walk ruling out real Gaussian discription of the occupation field. Taking the so-called torus limit, the limiting procedure allows to represent quantum correlation functions as space-time Green functions of the corresponding complex Gaussian measure. Thus we deliver a purely probabilistic derivation of Bosonic loop measures and quantum correlation functions. Interestingly, the occupation time distributions are given by complex Gaussian measures which we apply only to functionals of the moduli of any field component, henceforth not considering the inherent phase factors. However, we think to analyse these complex measures in greater detail in the future to study Bose-Einstein condensation phenomena and their possible connection to random interlacements ([8]) and to general permanental processes ([6]). The Bosonic loop measure and its Poisson loop process is a natural extension of random walk loop soups, and it seems feasible to define Bosonic loop measures and soups also over the continuum space $\mathbb{R}^d$. In [2] we briefly outline which of our results would hold in the so-called thermodynamic limit $\Lambda \uparrow \mathbb{Z}^d$.

References

[1] S. Adams and D. Brydges. *Loop measures and isomorphism theorems for the Helffer-Sjostrand random walk*, in preparation (2018).

[2] S. Adams and Q. Vogel, *Space-time random walk loop measures*, 32 pages, arXiv:1711.02766, November (2017).

[3] S. Adams, A. Collevecchio W. König, *The free energy of non-dilute many-particle systems*, Annals of Probability Vol. **39**, No. 2, (2011) 683–728.

[4] S. Adams and T. Dorlas, *Asymptotic Feynman-Kac formulae for large symmetrised systems of random walks*, Annals de l'Institut Henri Poincaré (B) Probabilités et Statistique **44**, (2008), 837-875.

[5] D. Brydges, J. Fröhlich, and T. Spencer, *The random walk representation of classical spin systems and correlations inequalities*, Comm. Math. Phys. **83**, (1982) 123–190.

[6] P.J. Fitzsimmons and J. Rosen, *Markovian loop soups: permanental processes and isomorphism theorems*, Electron. J. Probab. Volume **19**, no. 60, (2014) 1–30.

[7] Y. Le Jan, *Markov Paths, Loops and Fields*, Lecture Notes in Mathematics Volume **2026**, Springer (2011).

[8] A. Sznitman, *Topics in Occupation Times and Gaussian Free Fields*, Notes of the course "Special topics in probability", Spring 2012, Zürich Lectures in Advanced Mathematics, EMS, Zürich (2012).

[9] K. Symanzik, *Euclidean quantum field theory*, in Scuola internazionale di Fisica 'Enrico Fermi', XLV Coso, Academic Press, (1969) 152–223.

Berry-Esseen Theorem for the Random Conductance Model

Sebastian Andres

(joint work with Stefan Neukamm)

Stochastic homogenization of elliptic equations in divergence form with random coefficients started from the pioneering works of Kozlov [10] and Papanicolaou-Varadhan [13]. They established a *qualitative* homogenization result, which (adjusted to a discrete setting) can be rephrased as follows. The unique bounded solution u_ε to the elliptic finite difference equation

$$\nabla^* \omega \nabla u_\varepsilon = \varepsilon^2 f(\varepsilon\,\cdot) \qquad \text{on } \mathbb{Z}^d \tag{1}$$

with ω describing stationary and ergodic, uniformly elliptic, random coefficients, and f an appropriate right-hand side, e.g. $f \in C_c(\mathbb{R}^d)$ with zero mean, converges after a rescaling to the solution u_0 of the deterministic, elliptic equation

$$-\nabla \cdot \omega_{\mathrm{hom}} \nabla u_0 = f \qquad \text{on } \mathbb{R}^d,$$

where ω_{hom} denotes a deterministic coefficient matrix, the so-called *homogenized coefficients*. Quantitative stochastic homogenization is concerned with finding the rate of convergence of u_ε towards u_0. Recently, in [6, 7, 5, 3] optimal error bounds have been obtained in the uniformly elliptic case under strong mixing assumptions.

In probability theory, the model for the random walk in random environment generated by the operator in (1) is known as the *random conductance model*. More precisely, consider the Euclidean lattice $\mathbb{Z}^d$ with $d \geq 2$ and let E_d be the set of non oriented nearest neighbour bonds, i.e. $E_d = \{e = \{x, y\} : x, y \in \mathbb{Z}^d, |x - y| = 1\}$. The random environment is given by non-negative, stationary ergodic random

variables $(\omega_e, e \in E_d)$, defined on $(\Omega, \mathbb{P})$. We write $\omega_{xy} = \omega_{\{x,y\}} = \omega_{yx}$. Let $(X_t, t \geq 0, P_x^\omega, x \in \mathbb{Z}^d)$ be the continuous time random walk on $\mathbb{Z}^d$, which jumps according to the transitions $P(x,y) = \omega_{xy}/\sum_y \omega_{xy}$ associated with the generator

$$\mathcal{L}^\omega f(x) = \sum_{y \sim x} \omega_{xy}(f(y) - f(x)) = -\nabla^* \omega \nabla f(x).$$

A key feature of this random walk is its reversibility with respect to the counting measure. Since the law of the waiting times does depend on the location, X is also called the *variable speed random walk* (VSRW). In the study of the random conductance model the question whether an *invariance principle* holds has been object of very active research, see the surveys [4, 11] and references therein. One recent result for general ergodic environments is the following.

Theorem 1 (Quenched invariance principle [1])**.** *Suppose $d \geq 2$. Let $(\omega_e)_{e \in E_d}$ be stationary ergodic and $p, q \in (1, \infty]$ be such that $1/p + 1/q < 2/d$ and assume that $\mathbb{E}\big[(\omega_e)^p\big] < \infty$ and $\mathbb{E}\big[(\omega_e)^{-q}\big] < \infty$ for any $e \in E_d$. Then, for $\mathbb{P}$-a.e. ω, the rescaled process $X_t^{(n)} := \frac{1}{n} X_{n^2 t}$ converges (under P_0^ω) in law to a Brownian motion on $\mathbb{R}^d$ with a deterministic non-degenerate covariance matrix Σ^2.*

The invariance principle for X is closely related to homogenization of the associated generator $\mathcal{L}^\omega$; in particular, the covariance matrix of the limiting process and the homogenized coefficients are related by the identity $\Sigma^2 = 2\omega_{\mathrm{hom}}$. In view of the quantified results in stochastic homogenization mentioned in the beginning, our goal is to established a quantified version of the invariance principle in form of a *Berry Esseen theorem*. For this purpose, following [6, 7, 5], we assume that $\mathbb{P}$ satisfies a certain spectral gap estimate.

Assumption (Spectral Gap)**.** *Suppose $\mathbb{P}$ is stationary, and assume that there exists $\rho > 0$ such that*

$$\mathbb{E}\big[(u - \mathbb{E}[u])^2\big] \leq \frac{1}{\rho} \sum_{e \in E_d} \mathbb{E}\Big[\big(\partial_e u\big)^2\Big], \tag{SG}$$

for any random field $u \in L^2(\Omega)$. Here, the vertical derivative *$\partial_e u$ is defined as*

$$\partial_e u(\omega) := \limsup_{h \to 0} \frac{u(\omega + h\delta_e) - u(\omega)}{h},$$

where $\delta_e : E^d \to \{0, 1\}$ stands for the Dirac function satisfying $\delta_e(e) = 1$ and $\delta_e(e') = 0$ if $e' \neq e$.

Any stationary environment satisfying (SG) is ergodic. In a sense (SG) can be interpreted as a quantified version of ergodicity.

Let now $\xi \in \mathbb{R}^d$ be fixed and set $\sigma_\xi^2 := \xi \cdot \Sigma^2 \xi$. Then, the invariance principle in Theorem 1 yields for $\mathbb{P}$-a.e. ω,

$$\lim_{t \to \infty} P_0^\omega \left[\xi \cdot X_t \leq \sigma_\xi x \sqrt{t} \, \right] = \Phi(x), \tag{2}$$

where $\Phi(x) := (2\pi)^{-1/2} \int_{-\infty}^{x} e^{-u^2/2}\, du$ denotes the distribution function of the standard normal distribution. In our main result we quantify the speed of convergence in (2). We write $\mathbb{P}_0[\cdot] = \int_\Omega P_0^\omega[\cdot]\, d\mathbb{P}(\omega)$ for the annealed measure.

Theorem 2 (Berry-Esseen theorem [2])**.** *Let $d \geq 3$ and suppose that* (SG) *holds. For any $\varepsilon > 0$ there exist exponents $p, q \in (1, \infty)$ such that, if $\mathbb{E}\big[(\omega_e)^p\big] < \infty$ and $\mathbb{E}\big[(\omega_e)^{-q}\big] < \infty$ for any $e \in E_d$, the following hold.*

(i) There exists a constant $c > 0$ such that for all $t \geq 0$,

$$\sup_{x\in\mathbb{R}} \Big| \mathbb{P}_0\big[\xi \cdot X_t \leq \sigma_\xi x \sqrt{t}\ \big] - \Phi(x) \Big| \ \leq \ \begin{cases} c\, t^{-\frac{1}{10}+\varepsilon} & \text{if } d = 3, \\ c\, t^{-\frac{1}{5}+\varepsilon} & \text{if } d \geq 4. \end{cases}$$

(ii) There exists a random variable $\mathcal{X} \in L^1(\mathbb{P})$ such that if $d = 3$ for $\mathbb{P}$-a.e. ω,

$$\int_0^\infty \Big(\sup_{x\in\mathbb{R}} \Big| P_0^\omega\big[\xi \cdot X_t \leq \sigma_\xi x \sqrt{t}\ \big] - \Phi(x) \Big| \Big)^5 (t+1)^{-\frac{1}{2}-\varepsilon}\, dt \ \leq \ \mathcal{X}(\omega) \ < \ \infty$$

and if $d \geq 4$ for $\mathbb{P}$-a.e. ω,

$$\int_0^\infty \Big(\sup_{x\in\mathbb{R}} \Big| P_0^\omega\big[\xi \cdot X_t \leq \sigma_\xi x \sqrt{t}\ \big] - \Phi(x) \Big| \Big)^5 (t+1)^{-\varepsilon}\, dt \ \leq \ \mathcal{X}(\omega) \ < \ \infty.$$

In the case of uniformly elliptic i.i.d. conductances an annealed Berry-Esseen theorem as in (i) has been proven in [12] for arbitrary dimension $d \geq 1$ with rate $t^{-1/5}$ in $d \geq 3$. Theorem 2 extends this result to unbounded and correlated random conductances. To our knowledge (ii) is the first quenched Berry-Essen-type result for the random conductance model.

The proof is based on the classical corrector approach by Kipnis-Varadhan, i.e. the random walk is decomposed into a martingale part and a remainder term, and we need to quantify both, the speed of convergence of the martingale part and the smallness of the remainder. For the martingale part we use a general Berry-Esseen bound for martingales established in [8] (cf. also [9]). This requires a result on the speed of convergence of the opérateur carré du champ associated with $\mathcal{L}^\omega$, for which we need to extend a variance decay estimate for the semigroup of the process of the environment as seen from the particle into our degenerate setting. Such a variance estimate, which plays a central role in quantitative stochastic homogenization, has been established in [5] for uniformly elliptic conductances satisfying (SG).

References

[1] S. Andres, J.-D. Deuschel, and M. Slowik. Invariance principle for the random conductance model in a degenerate ergodic environment. *Ann. Probab.* **43**, no. 4, 1866–1891 (2015).

[2] S. Andres, S. Neukamm. Berry-Esseen theorem and quantitative homogenization for the random conductance model with degenerate conductances. Preprint, available at arXiv:1706.09493.

[3] S. Armstrong, T. Kuusi, and J.-C. Mourrat. The additive structure of elliptic homogenization. *Invent. Math.* **208**, no. 3, 999–1154 (2017).

[4] M. Biskup. Recent progress on the random conductance model. *Probab. Surv.* **8**, 294–373 (2011).

[5] A. Gloria, S. Neukamm, and F. Otto. Quantification of ergodicity in stochastic homogenization: optimal bounds via spectral gap on Glauber dynamics. *Invent. Math.*, **199**, no. 2, 455–515 (2015).
[6] A. Gloria and F. Otto. An optimal variance estimate in stochastic homogenization of discrete elliptic equations. *Ann. Probab.* **39**, no. 3, 779–856 (2011).
[7] A. Gloria and F. Otto. An optimal error estimate in stochastic homogenization of discrete elliptic equations. *Ann. Appl. Probab.* **22**, no. 1, 1–28 (2012).
[8] E. Haeusler. On the rate of convergence in the central limit theorem for martingales with discrete and continuous time. *Ann. Probab.* **16**, no. 1, 275–299 (1988).
[9] C. C. Heyde and B. M. Brown. On the departure from normality of a certain class of martingales. *Ann. Math. Statist.* **41**, 2161–2165 (1970).
[10] S. M. Kozlov. The averaging of random operators. *Mat. Sb. (N.S.)* **109** (151), no. 2, 188–202, 327, (1979).
[11] T. Kumagai. *Random walks on disordered media and their scaling limits*, volume 2101 of *Lecture Notes in Mathematics*. Springer, Cham, 2014.
[12] J.-C. Mourrat. A quantitative central limit theorem for the random walk among random conductances. *Electron. J. Probab.*, 17:no. 97, 17, 2012.
[13] G. C. Papanicolaou and S. R. S. Varadhan. Boundary value problems with rapidly oscillating random coefficients. In *Random fields, Vol. I, II (Esztergom, 1979)*, volume 27 of *Colloq. Math. Soc. János Bolyai*, pages 835–873. North-Holland, Amsterdam-New York, 1981.

A gradient flow approach to linear Boltzmann equations

Lorenzo Bertini
(joint work with Giada Basile, Dario Benedetto)

We consider linear Boltzmann equations of the form

$$(\partial_t + b(v)\cdot\nabla_x) f(t,x,v) = \int \pi(dv')\sigma(v,v')\big[f(t,x,v') - f(t,x,v)\big] \tag{1}$$

where $x \in \mathbb{T}^d$, the d-dimensional torus, $\pi(dv)$ is a reference probability measure on the velocity space $\mathcal{V}$, $b\colon \mathcal{V} \to \mathbb{R}^d$ is the drift, $\sigma(v,v')\pi(dv')$ is the scattering kernel and f is the density of the one-particle distribution with respect to $dx\,\pi(dv)$. We assume the detailed balance condition, i.e., $\sigma(v,v') = \sigma(v',v)$. Examples of linear Boltzmann equations of this form are the Lorentz gas [5], the evolution of a tagged particle in a Newtonian system in thermal equilibrium [6], and the propagation of lattice vibrations in insulating crystals [2].

Using the shorthand notation $f = f(t,x,v)$, $f' = f(t,x,v')$, we set

$$\eta^f = \eta^f(t,x,v,v') := \sigma(f - f') = \sigma(v,v')\big[f(t,x,v) - f(t,x,v')\big]$$

and rewrite the linear Boltzmann equation (1) in the form

$$\begin{cases} \big(\partial_t + b(v)\cdot\nabla_x\big) f(t,x,v) + \int \pi(dv')\,\eta(t,x,v,v') = 0 \\ \eta = \eta^f \end{cases}$$

where $\eta\colon [0,T]\times\mathbb{T}^d\times\mathcal{V}\times\mathcal{V}\to\mathbb{R}$ is antisymmetric with respect to the exchange of velocities.

Referring to [1] for the details, given a time interval $[0,T]$, we rewrite the identity $\eta = \eta^f$ as the following inequality that expresses the decay of the entropy along the solutions to (1),

$$\mathcal{H}(f(T)) + \int_0^T dt\, \mathcal{E}(f(t)) + \mathcal{R}(f,\eta) \le \mathcal{H}(f(0)). \tag{2}$$

Here $\mathcal{H}$ is the relative entropy with respect to $dx\,\pi(dv)$, i,e.,

$$\mathcal{H}(f) := \iint dx\,\pi(dv)\, f \log f,$$

$\mathcal{E}$ is the Dirichlet form of the square root of f, i.e.,

$$\mathcal{E}(f) := \int dx \iint \pi(dv)\pi(dv')\,\sigma(v,v')\big[\sqrt{f'} - \sqrt{f}\big]^2,$$

and the *kinematic term* $\mathcal{R}$ is defined by

$$\mathcal{R}(f,\eta) := \int_0^T dt \int dx \iint \pi(dv)\pi(dv')\,\Psi_\sigma(f,f';\eta)$$

in which $\sigma = \sigma(v,v')$ and

$$\Psi_\kappa(p,q;\xi) = \xi \mathrm{arcsinh}\frac{\xi}{2\kappa\sqrt{pq}} - \Big[\sqrt{\xi^2 + 4\kappa^2 pq} - 2\kappa\sqrt{pq}\Big].$$

Both $\mathcal{E}$ and $\mathcal{R}$ can be expressed by variational formulae that imply their lower semi-continuity and convexity on the set of density f satisfying the entropy bound $\sup_{t\in[0,T]} \mathcal{H}(f(t)) \le \ell$, $\ell > 0$. It is then straightforward to prove existence and stability of the formulation (2). Uniqueness follows from the argument in [4].

The entropy dissipation formulation (2) of (1) allows to discuss the diffusive limit of linear Boltzmann equation, see e.g., [3], in the framework of the gradient flow formulation of the heat equation; in particular by assuming only equiboundedness of the entropy at the initial time.

Let $\epsilon > 0$ be the scaling parameter and denote by $(f^\epsilon, \eta^\epsilon)$ the diffusively rescaled solution of the linear Boltzmann equation. According to the gradient flow formulation, the pair $(f^\epsilon, \eta^\epsilon)$ satisfies

$$\partial_t f^\epsilon(t,x,v) + \frac{1}{\epsilon} b(v)\cdot\nabla_x f^\epsilon(t,x,v) + \frac{1}{\epsilon^2}\int \pi(dv')\eta^\epsilon(t,x,v,v') = 0 \tag{3}$$

$$\mathcal{H}(f^\epsilon(T)) + \frac{1}{\epsilon^2}\int_0^T dt\,\mathcal{E}(f^\epsilon(t)) + \frac{1}{\epsilon^2}\mathcal{R}(f^\epsilon,\eta^\epsilon) \le \mathcal{H}(f^\epsilon(0)). \tag{4}$$

We set

$$\rho^\epsilon(t,x) := \int \pi(dv) f^\epsilon(t,x,v)$$

$$j^\epsilon(t,x) := \frac{1}{\epsilon}\int \pi(dv) f^\epsilon(t,x,v) b(v).$$

Since $\eta^\epsilon(t,x,v,v')$ is antisymmetric with respect to the exchange of v and v', by integrating (3) with respect to $\pi(dv)$ we deduce the continuity equation

$$\partial_t \rho^\epsilon + \nabla \cdot j^\epsilon = 0. \tag{5}$$

Let $H(\rho) := \int dx\, \rho \log \rho$ the entropy of the probability density ρ. Assuming $\rho^\epsilon(0) \to \rho(0)$ and $\mathcal{H}(f^\epsilon(0)) \to H(\rho(0))$ we would like to take the inferior limit in the inequality (4) deducing

$$H(\rho(T)) + \int_0^T dt\, E(\rho(t)) + R(\rho, j) \le H(\rho(0)), \tag{6}$$

that corresponds to the gradient flow formulation of the heat equation for the pair (ρ, j) satisfying the continuity equation. Here E is the Fisher information, i.e.,

$$E(\rho) = 2 \int dx\, \nabla\sqrt{\rho} \cdot D \nabla \sqrt{\rho}$$

and

$$R(\rho, j) = \frac{1}{2} \int_0^T dt \int dx\, \frac{1}{\rho(t)} j(t) \cdot D^{-1} j(t),$$

where the positive definite $d \times d$ matrix D is diffusion coefficient.

This step is accomplished in [1] under suitable conditions on the scattering kernel σ and the drift b implying homogenization of the velocity on the diffusive scale.

References

[1] Basile G., Benedetto D., Bertini L.; *A gradient flow approach to linear Boltzmann equations*, Preprint `arXiv:1707.09204`

[2] Basile G., Olla S., Spohn H.; *Energy transport in stochastically perturbed lattice dynamics*, Arch. Ration. Mech. Anal. 195, 1, 171-203, 2010.

[3] Bensoussan A., Lions J. L., Papanicolaou G.; *Boundary layers and homogenization of transport processes*, Publ. RIMS, Kyoto Univ. 15, 53-157, 1979.

[4] Gigli N.; *On the Heat flow on metric measure spaces: existence, uniqueness and stability*, Calc. Var. Part. Diff. Eq. 39, 101120, 2010.

[5] Lorentz H.A.; *The motion of electrons in metallic bodies*, Proc. Acad. Amst. 7, 438453, 1905.

[6] Spohn, H.; *Kinetic equations from Hamiltonian dynamics: Markovian limits*, Rev. Mod. Phys., 52, 3, 569-615, 1980.

Random walk in a non-integrable random scenery time

Alessandra Bianchi

(joint work with Marco Lenci, Françoise Pène)

Anomalous diffusions are stochastic processes $X(t)$, $t \in \mathbb{R}^+$, having an asymptotic variance which does not grow linearly in time, that is $\mathbb{E}(X^2(t)) \sim t^\delta$ with $\delta \neq 1$. This phenomenon is quite common in applications and it is especially related to the transport in inhomogeneous material, e.g., the motion of a light particle in an optical lattice [6, 7]. The basic mathematical models for anomalous diffusions

are *Lévy flights*, which are random walks with step length provided by an i.i.d. sequence of Lévy α-stable random variables with $\alpha \in (0, 2)$ (see [10, 5]). In this simple case, the motion is indeed provided by an asymptotic super-diffusive behavior with $\delta = 2$, for $\alpha \in (0, 1]$, and $\delta = 3 - \alpha$, for $\alpha \in (1, 2)$ (Lévy scheme). To model the motion in inhomogeneous material, one would like to take also into account that steps are mutually correlated by their positions, which we may identify with the presence of scatterers in the media. To this aim, in [4] the so-called *Lévy-Lorentz gas* were introduced. This is linear interpolation of a one-dimensional random walk in a Lévy-type random environment, where scatterers are placed as a renewal point process with inter-distances having a Lévy-type distribution, and jump probabilities depend on whether the position of the walker is on a scatterer or not.

We are then interested in providing a characterization of this process under the quenched and annealed laws (LLN, scaling limits, large deviation of the empirical speed), and in determining whether (and under which law) the asymptotic behavior is super-diffusive. The theory of random walks in random environments have been intensively studied in the last forty years and many important results have been achieved, especially for one-dimensional systems that are generally quite well understood. Even so, classical results do not apply to this setting, mainly because of the non-ellipticity of the environment, and a different analysis is required.

The range of $\alpha \in (1, 2)$, when inter-distances between scatterers having finite mean but infinite variance, was first studied in [1, 8] in the annealed setting, and then extended in the quenched setting in a recent work in collaboration with Cristadoro, Lenci and Ligabò (see [3]), where we proved that the quenched law of the process satisfies a classical CLT and has moments converging to the moments of a diffusion. While the annealed CLT follows trivially from these results, there are not sharp results on the asymptotical behavior of the annealed second moment which is then still under debate, as the results in [1, 8] are not completely in agreement and may lead to different conclusions.

In the present work we investigate the annealed behavior of the process for $\alpha \in (0, 1)$, when inter-distances between scatterers having infinite mean. Under this hypothesis, some previous works where suggesting the super-diffusivity of the process, and in particular the results in [4] and in [1, 8] where some annealed quantities related to the second moment were estimated and numerically simulated. Here we confirm and extend these predictions, proving, for the first time to our knowledge, that *Lévy-Lorentz gas* is super-diffusive for $\alpha \in (0, 1)$. In particular we establish the convergence of the finite-dimensional distributions of the process under a super-diffusive scaling with exponent $1/1 + \alpha > 1/2$, and we characterize the scaling limit. This is explicitely given by the composition of three processes: The α stable process obtained as the scaling limit of the Lévy environment, the Brownian motion obtained as the scaling limit of an underlying random walk, and the inverse of the Kesten-Spitzer process. This last process, that was introduced in [9] as the scaling limit of *random walks in random scenery*, appears in this context as the scaling limit of the sequence of time-lengths between to consecutive

collisions, or in other words, the sequence of jump-lengths of the random walk on the environment that is coupled to the continuous time process. This is indeed the key observation and tool in the proof of our main result.

The open problems we are interested in, are the following:

1. Study of the regime of $\alpha \in (0,1)$: tightness under a suitable topology, moments convergence w.r.t. the quenched and annealed law.
2. Construction and characterization of an analogous two-dimensional model, also following the physical analysis of *Lévy glasses* given in [6, 2].

References

[1] C. W. J. Beenakker, C. W. Groth, and A. R. Akhmerov. *Nonalgebraic length dependence of transmission through a chain of barriers with a Lévy spacing distribution* Phys. Rev. B **79** (2009).

[2] C. W. Groth, A. R. Akhmerov, and C. W. J. Beenakker. *Transmission probability through a Lévy glass and comparison with a Lévy walk*. Phys. Rev. E **85** (2012).

[3] A. Bianchi, G. Cristadoro, M. Lenci and M. Ligabò. *Random walks in a one-dimensional Lévy random environment*. J. Stat. Phys. **16** 1 (2016).

[4] E. Barkai, V. Fleurov and J. Klafter. *One-dimensional Lévy–Lorentz Gas*. Phys. Rev. E **61** (2000).

[5] E. Barkai, J. Klafter. *Anomalous diffusion in the strong scattering limit: a Lévy walk approach*. Chaos, kinetics and nonlinear dynamics in fluids and plasmas (Carry-Le Rouet, 1997), 373393, *Lecture Notes in Physics*, **511**. Springer, Berlin, 1998.

[6] P. Barthelemy, J. Bertolotti, D.S. Wiersma. *A Lévy flight for light*. Nature **453** (2008), 495.

[7] J. Bertolotti, K. Vynck, D.S. Wiersma. *Multiple scattering of light in superdiffusive media*. Phys. Rev. Lett. **105** (2010), 163902.

[8] R.Burioni, L. Caniparoli, A. Vezzani. *Lévy Walks and scaling in quenched disordered media*. Phys. Rev. E **81** (2010).

[9] H. Kesten, F. Spitzer *A limit theorem related to a new class of self-similar processes*. Z. Wahrsch. Verw. Gebiete **50** 1 (1979), 5–25.

[10] *Lévy flights and related topics in physics*. Proceedings of the International Workshop held in Nice, June 2730, 1994. Edited by Michael F. Shlesinger, George M. Zaslavsky and Uriel Frisch. *Lecture Notes in Physics*, **450**. Springer-Verlag, Berlin, 1995.

Anomalies in the random conductance model

Omar Boukhadra

This talk concerns the anomalies of the heat-kernel in the random conductance model (RCM). In the first place, we give ourselves a family of non negative random variables associated with the non oriented edges of the grid $\mathbb{Z}^d$: $(\omega_e)_e = (\omega_{xy}), e \in E_d = \{e = \{x,y\} : x,y \in \mathbb{Z}^d, |x-y| = 1\}$. These random variables are called random conductances. The realization of these gives a random environment ω in which one defines a (quenched) discrete-time random walk with transition probabilities

$$\mathsf{P}_\omega(x,y) = \frac{\omega_{xy}}{\pi(x)}, \qquad \pi(x) = \sum_{y:|x-y|=1} \omega_{xy} \tag{1}$$

The n-th power of the transition kernel P_ω is

$$\mathsf{P}^n_\omega(x,y) = P^x_\omega(X_n = y). \tag{2}$$

Let $o \in \mathbb{Z}^d$ denote the origin and define the conditional measure

$$\mathbb{P}_o(\cdot) := \mathbb{P}(\cdot \mid o \in \mathscr{C}). \tag{3}$$

where $\mathscr{C}$ is the unique infinite connected cluster along edges with positive conductances, which exists a.s. when we suppose that $\mathbb{P}(\omega_e > 0)$ is larger than the critical threshold $p_c(d)$ for bond percolation on $\mathbb{Z}^d$.

The result available on arXiv that I would like to talk about is the following which basically reconsiders and improves the result and techniques in [2].

Theorem 1. *Let $d \geq 2$ and $\alpha \in (0, 1/2)$. Suppose that the environment is governed by bounded i.i.d. random conductances such that $\mathbb{P}(\omega_e \in [0,1]) = 1$ and*

$$(4d-2)\lim_{u\to 0} \frac{\log \mathbb{P}\big(\omega_b \in [u, 2u]\big)}{\log u} < \alpha, \tag{C}$$

Then, there exists $c > 0$, such that $\mathbb{P}_o$-a.s. for any n large enough, we have

$$\mathsf{P}^{2n}_\omega(o,o) \geq c\,\pi(o)\, n^{-(2+\alpha(d-1))} \tag{4}$$

Remarks. *(i) The lower bound (4) is to be compared with general upper bounds from* [1, Theorem 2.1]*, and the general louwer bound $\mathsf{P}^{2n}_\omega(o,o) \geq c\, n^{-d/2}$ (see for example* [2, Remark 1.3]*). Then we observe that we always have a normal decay in $d = 2, 3$ and our lower bound (4) is interesting for $d \geq 5$ and $2\alpha < (d-4)/(d-1)$.*

(ii) In the special case with i.i.d. polynomial lower tail conductances, i.e.

$$\mathbb{P}(\omega_e < u) = u^\gamma(1 + o(1)), \quad u \to 0, \tag{LP}$$

one can easily see that the condition (C) becomes $\gamma < \alpha/(4d-2)$. This last condition on γ and the estimate (4) yield

$$\liminf_n \frac{\log \mathsf{P}^{2n}_\omega(o,o)}{\log n} \geq -2 - (d-1)(4d-2)\gamma \tag{5}$$

(iii) I believe that we can delete the term $(d-1)$ in (4)–(5) by improving a little more the techniques used for the proof of this result, which gives for the polynomial model the critical value

$$\gamma_{AHK} = \frac{1}{8}\frac{d-2}{d-1/2}$$

This is not far from the critical value for the local CLT found in [3]*, that is*

$$\gamma_{LCLT} = \frac{1}{8}\frac{d}{d-1/2}$$

I dare not conjuncture but I like to believe that for $\gamma < \gamma_{AHK}$, we have

$$\frac{\log \mathsf{P}^{2n}_\omega(o,o)}{\log n} = -2 - (4d-2)\gamma \tag{6}$$

For the proof of the upper bound in (6), I think that Flegel's spectral analysis [4] *of the RCM Laplacian could be very useful.*

Sketch of the Proof of Theorem 1. Let $\alpha \in (0, 1/2)$. Set $r = n^\alpha$ and partition $B_r = [-r, r]^d \cap \mathbb{Z}^d$ into annuli of diameter 3, $A_k = B_{3(k+1)} \setminus B_{3k}, k = 0, \dots, [r/3]$.

Let $\mathscr{T}_n$ be the event in the environment defined as follows: *an edge $e = \{y, z\}$ with conductance $\omega_e \geq c > 0$, which we call the strong edge, is such that for any $e' \neq e$ incident with either y or z, $\omega_{e'} \in [1/n, 2/n]$.* The configuration $\mathscr{T}_n$, called *trap*, is then made up of a strong edge and of $4d-2$ weak edges with conductances of order $1/n$. Let $\mathscr{T}_n(x)$ be the event on the space of environments that x is a vertex neighboring a trap edge, which trap is situated outside the hypercubic box $B_{\max x_i}$, where the $x_i, i = 1, \dots, d$ are the associated coordinates of x.

First we prove with an argument which improves that in [2] that the random walk will meet a.s. a trap $\mathscr{T}_n$ before getting outside B_r when it hits one of the annuli $A_k, k = 0, \dots, r-3$. At this time, one can oblige the random walk to get into the trap, which costs a probability of order $1/n$, and then spend a time of order n in it. In fact, as a kee step, we prove the following. Set $H_r = \inf\{k \geq 0 : X_k \in \partial B_r\}$ and define K as the first rank such that $\mathscr{T}_n(X_{H_{3k}})$ happens, i.e.

$$K = \inf\{k \in \{0, \dots, r/3 - 1\} : \mathscr{T}_n(X_{H_{3k}})\}; \quad \text{with } \inf \emptyset = \infty.$$

Then, we have

$$P^o_\omega(X_n \in A_K) \geq c\, n^{-1} \tag{7}$$

Now observe that by the Markov property and reversibility, we have

$$\mathsf{P}^{2n}_\omega(o, o) \geq \sum_{k=0}^{r/3-1} \sum_{x \in A_k} \mathsf{P}^n_\omega(o, x)\mathsf{P}^n_\omega(x, o) \geq \sum_{k=0}^{r/3-1} \sum_{x \in A_k} \mathsf{P}^n_\omega(o, x)^2 \frac{\pi(o)}{\pi(x)} \tag{8}$$

Bounding $\pi(x) \leq 2d$ and using CauchySchwarz, we get

$$\mathsf{P}^{2n}_\omega(o, o) \geq \frac{\pi(o)}{2d} \sum_{k=0}^{r/3-1} |A_k|^{-1} P^o_\omega\big(X_n \in A_k\big)^2. \tag{9}$$

Since for all $k \in \{0, \dots, r/3 - 1\}$, $|A_k| \leq cn^{\alpha(d-1)}$, and using K, we obtain that

$$\text{the r.h.s. of (9)} \geq c\, \pi(o)\, n^{-\alpha(d-1)}\, P^o_\omega\big(X_n \in A_K\big)^2. \tag{10}$$

The estimate (7) yields the result. □

References

[1] N. Berger, M. Biskup, C. E. Hoffman, G. Kozma, Anomalous heat-kernel decay for random walk among bounded random conductances, Ann. Inst. Henri Poincaré Probab. Stat. **44** (2008), no. 2, 374–392.

[2] O. Boukhadra, Heat-kernel estimates for random walk among random conductances with heavy tail, Stochastic Process. Appl. **120** (2010), no. 2, 182–194.

[3] O. Boukhadra T. Kumagai and P. Mathieu, Local CLT for the polynomial lower tail RCM, J. Math. Soc. Japan Vol. 67, No. 4 (2015) pp. 14131448.

[4] Franziska Flegel, Localization of the principal Dirichlet eigenvector in the heavy-tailed random conductance model, arXiv:1801.05684.

Stochastic perturbation of the ergodic constant in homogenization of Hamilton-Jacobi equations

Pierre Cardaliaguet
(joint work with Claude Le Bris and Panagiotis E. Souganidis)

We study the behavior of the ergodic constant associated with convex and superlinear Hamilton-Jacobi (HJ for short) equations in a periodic environment which is perturbed either by medium with increasing period which is a multiple of the original one or by a random Bernoulli perturbation with small parameter. The result is a first-order Taylor's expansion for the ergodic constant which depends on the dimension d. Our results are the first of this kind for nonlinear problems. The arguments, which rely on viscosity solutions and the weak KAM theory, also raise several new and challenging questions.

The motivation for this work came from the recent studies by Anantharaman and Le Bris [1, 2] and Duerinckx and Gloria [3], who considered similar questions for linear uniformly elliptic operators (and systems in [3]). The former paper considered Bernoulli perturbations of a periodic environment, while the latter reference, which complemented and generalized the work of the former, considered Bernoulli perturbations of a stationary ergodic medium and provided, taking strong advantage of the linearity of the equation, a full expansion.

We now describe in a somewhat informal way the results of the paper. Let $H := H(p,x)$ be a Hamiltonian which is coercive in p and $\mathbb{Z}^d-$periodic in x. It was shown by Lions, Papanicolaou and Varadhan [4] that there exists a unique $\overline{H}$, often referred to as the effective Hamiltonian or the ergodic constant, such that the cell problem

$$H(D\chi, x) = \overline{H} \text{ in } \mathbb{R}^d, \tag{1}$$

has a continuous, $\mathbb{Z}^d-$periodic (viscosity) solution χ known as a corrector.

The randomly perturbed Hamiltonian H_η is given by

$$H_\eta(p,x) := H(p,x) - \zeta_\eta(x)$$

where

$$\zeta_\eta(x) := \sum_{k\in\mathbb{Z}^d} \zeta(x-k)X_k,$$

with $\zeta : \mathbb{R}^d \to \mathbb{R}$ nonnegative, Lipschitz continuous and compactly supported and $(X_k)_{k\in\mathbb{Z}^d}$ is a family of i.i.d. Bernoulli random variables of parameter η.

Contrary to the periodic setting, in random media the effective Hamiltonian is not characterized by the cell-problem. The reason is that to guarantee its uniqueness, it is necessary to have correctors which are strictly sub-linear at infinity. As shown in Lions and Souganidis [5], in general, this is not possible.

The effective constant $\overline{H}_\eta$ is defined, for instance, through the discounted problem

$$\delta v^{\eta,\delta} + H_\eta(Dv^{\eta,\delta}, x) = 0 \text{ in } \mathbb{R}^d$$

which has unique bounded solution $v^{\eta,\delta}$, as the almost sure limit (see [6])

$$\overline{H}_\eta := \lim_{\delta\to 0} -\delta v^{\eta,\delta}(0)$$

Note that, as $\eta \to 0$, the probability that there is a bump in a fixed ball becomes smaller and smaller. So it is natural to expect that $\overline{H}_\eta$ converges to $\overline{H}$ as $\eta \to 0$ and we want to understand at which rate this convergence holds.

We establish two types of results. The first is an estimate of the difference between $\overline{H}_\eta$ and $\overline{H}$. We prove that, if $H = H(p, x)$ is convex and coercive in p and $\mathbb{Z}^d-$periodic in x, then there exists $C > 0$ depending only on ζ such that

$$0 \le \overline{H} - \overline{H}_\eta \le C\eta \text{ for all } \eta \in (0, 1), \tag{2}$$

and, in particular, $\lim_{\eta\to 0} \overline{H}_\eta = \overline{H}$.

The result is unusual in the homogenization of Hamilton-Jacobi equations because the perturbations do not vanish in the L^∞ norm and relies strongly on the fact that the bumps are nonnegative. In general the convergence does not hold otherwise.

In view of the above discussion, it is natural, and this is the second type of results in this paper, to identify the limit

$$\lim_{\eta\to 0} \eta^{-1}(\overline{H}_\eta - \overline{H}). \tag{3}$$

It turns out that is much more complicated than proving (2) and we only have a complete answer under some additional assumptions. These assumptions, which will not be discussed here, require H to be in a neighborhood of an integrable hamiltonian.

Our main result says that, in dimension $d \ge 2$, the limit in (3) vanishes. This conclusion is in stark contrast with what is happening for uniformly elliptic divergence form operators where the first term in the expansion is nonzero. The heuristic explanation for this difference is that in the Hamilton-Jacobi setting information is propagated along curves which are lower dimensional objects when $d \ge 2$, while for the elliptic problem the information is obtained by averaging. At a very intuitive level, the proof of the result consist in showing that the characteristics of the perturbed problem stay close to those of (1) on large time intervals but eventually manage to avoid the bumps.

The general behavior of the perturbed ergodic constant in more general contexts remains an open question. If the estimate (2) easily generalizes (to second order Hamilton-Jacobi equations for instance, and probably to other perturbations), we do not really know what to expect for the more subtle limit (3).

References

[1] A. Anantharaman and C. Le Bris. *A numerical approach related to defect-type theories for some weakly random problems in homogenization.* Multiscale Modeling & Simulation. A SIAM Interdisciplinary Journal, 9(2): 513–544, 2011.

[2] A. Anantharaman and C. Le Bris. *Elements of mathematical foundations for numerical approaches for weakly random homogenization problems.* Communications in Computational Physics, 11(4): 1103–1143, 2012.

[3] M. Duerinckx and A. Gloria. *Analyticity of homogenized coefficients under Bernoulli perturbations and the Clausius-Mossotti formulas.* Arch. Ration. Mech. Anal., 220(1), 297–361, 2016.

[4] P.-L. Lions, G. Papanicolaou, and S. R. S. Varadhan. *Homogenization of Hamilton-Jacobi equations.* Unpublished preprint, 1987.

[5] P.-L. Lions and P. E. Souganidis. *Correctors for the homogenization of Hamilton-Jacobi equations in the stationary ergodic setting.* Communications on pure and applied mathematics, 56(10):1501–1524, 2003.

[6] P. E. Souganidis. *Stochastic homogenization of Hamilton-Jacobi equations and some applications.* Asymptot. Anal., 20(1):1–11, 1999.

The shape of the emerging condensate in effective models of condensation

Steffen Dereich

(joint work with Volker Betz, Peter Mörters)

We consider effective models of condensation where the condensation occurs as time t goes to infinity. We provide natural conditions under which the build-up of the condensate occurs on a spatial scale of $1/t$ and has the universal form of a Gamma density. The exponential parameter of this density is determined only by the equation and the total mass of the condensate, while the power law parameter may in addition depend on the decay properties of the initial condition near the condensation point.

More explicitly, we consider solutions $(p_t)_{t\geq 0}$ of equations of the form

$$\partial_t p_t(dx) = \mathbf{B}[p_t]\, p_t(dx) + x^\alpha\, \mathbf{C}[p_t]\, dx.$$

Here, $\alpha > 0$ is fixed, each p_t takes values in the space

$$\mathcal{M}_0 := \{\rho\, \delta_0 + p\, dx : \rho \geq 0,\, p \in L^1([0,\infty))\}$$

and $\mathbf{B} : \mathcal{M}_0 \to C([0,\infty)$ and $\mathbf{C} : \mathcal{M}_0 \to C([0,\infty))$ are operators.

In what follows the solution $(p_t)_{t\geq 0}$ will have no atom at 0 and with slight misuse of notation we refer to the respective Lebesgue density by the same identifier p_t and represent the equation in the form

$$\partial_t p_t(x) = \mathbf{B}[p_t]\, p_t(x) + x^\alpha\, \mathbf{C}[p_t]. \tag{1}$$

For the statement of our main result we rely on the following definition.

Definition. *(1) A solution $(p_t)_{t\geq 0}$ of (1)* converges regularly *to an element $p_\infty \in \mathcal{M}_0$ if*

(i): $p_t \to p_\infty$ weakly as $t \to \infty$ as measures;

(ii): the following two equations hold for a $\delta > 0$:

$$\lim_{t\to\infty} \|\mathbf{B}[p_t] - \mathbf{B}[p_\infty]\|_{C^1([0,\delta])} = 0,$$
$$\lim_{t\to\infty} \|\mathbf{C}[p_t] - \mathbf{C}[p_\infty]\|_{C([0,\delta])} = 0,$$

where for $f \in L^1([0,\infty))$

$$\|f\|_{C^1([0,\delta])} = \sup\{|f(x)| + |f'(x)| : 0 \leq x \leq \delta\}$$

and

$$\|f\|_{C([0,\delta])} = \sup\{|f(x)| : 0 \leq x \leq \delta\}$$

with the convention that the norm is infinite in the case that the functoin is not in C^1 or C respectively.

(2) An element $q \in \mathcal{M}_0$ is called stationary *if*

$$\mathbf{B}[q](x)\, q(dx) + x^\alpha\, \mathbf{C}[q](x)\, dx = 0.$$

The main theorem is as follows.

Theorem. *Assume that $(p_t)_{t\geq 0}$ is a solution to equation (1) that converges regularly to a stationary limit $p_\infty = \rho\delta_0 + q(x)\, dx \in \mathcal{M}_0$ with*

- $\rho > 0$,
- $c_1 := \mathbf{C}[p_\infty](0) > 0$,
- $c_2 := \lim_{x\to 0} \frac{x^{\alpha-1}}{q(x)} > 0$ *exists for a $\alpha > 0$.*

Suppose further that $p_0(x) = x^\alpha \eta(x) \in \mathcal{M}_0$ (no atom) with η being continuous with $\eta(0) = 0$ (p_0 is of lower order).

Then we have, uniformly on compact intervals of $\mathbb{R}_0^+$, that

$$\lim_{t\to\infty} \tfrac{1}{t} p_t(\tfrac{x}{t}) = Ce^{-\gamma x} x^\alpha. \tag{2}$$

where $\gamma := c_1 c_2$, and $C = \rho\gamma^\alpha/\Gamma(\alpha)$, where Γ is the Gamma function.

Note that in our main theorem the constant C is such that the right hand side of (2) integrates to ρ so that the full mass of the condensate is covered. The condensation always acts on the scale $1/t$ and has a Gamma-distributed shape.

Selection mutation equations. A natural example for our theory is Kingman's model of selection and mutation which was originally introduced in [7] in discrete time. We introduce a continuous version.

Given

- a *mutation rate* $\beta \in (0,1)$,
- an *initial fitness density* p_0, which corresponds to a probability measure on $(0,1)$, and
- a *mutant fitness density* u , which corresponds to a probability measure on $(0,1)$ with essential supremum one

we consider the equation

$$\partial_t p_t(dx) = \Big((1-\beta)\frac{x}{w[p_t]} - 1\Big)p_t(dx) + \beta u(dx), \tag{3}$$

where $w[p_t] = \int_0^1 x p_t(x)\,dx$. This models a population where at rate β spontaneous mutations destroy the individuals' biochemical 'house of cards' so that the mutant fitness distribution u does not depend on their previous fitness. Further the remaining offspring is generated with a selective advantage proportional to the fitness.

An application of a substitution $y = 1 - x$ leads to an equation covered by the theorem. For sake of simplicity we give the results in terms of the original variable x. In the case where

$$\beta \int_0^1 \frac{u(dx)}{1-x} < 1,$$

we have $p_t \to p_\infty$ weakly for the stationary solution p_∞ given by

$$p_\infty(dx) = \Big(1 - \beta \int_0^1 \frac{u(x)}{1-x}\,dx\Big)\delta_1 + \beta\frac{u(x)}{1-x}\,dx..$$

Under the appropriate decay assumptions on p_0 and u in the upper tale one one can easily verify the other assumptions and gets validity of an analogues version of (2) around one.

Further examples. Two further examples are treated in [1]. The main theorem can also be applied for an effective model for Bosons in contact with a bath of Fermions in thermal equilibrium introduced and analysed by Escobedo and Mischler [4, 5, 6]. A third example is a simple model for the emergence of a condensate in a Bose gas in contact with a heat bath, which was developed by Buffet, de Smedt and Pulé in [2, 3].

References

[1] V. Betz, S. Dereich and P. Mörters, *The shape of the emerging condensate in effective models of condensation*, to appear in Ann. Inst. Henri Poincaré.

[2] E. Buffet, P. de Smedt, and J.V. Pulé, *On the dynamics of Bose-Einstein condensation*, Annales de l'Institut Henri Poincaré (C) Analyse non linéaire **1** (1984), 413–451.

[3] E. Buffet, P. de Smedt, P. and J.V. Pulé, *The dynamics of the open Bose gas*, Ann. Physics **155** (1984), 269–304.

[4] M. Escobedo and S. Mischler, *Equation de Boltzmann quantique homogene: existence et comportement asymptotique*, C. R. Acad. Sci. Paris **329** Serie I (1999), 593–598.

[5] M. Escobedo and S. Mischler, *On a quantum Boltzmann equation for a gas of photons*, J. Math. Pures Appl. **80** (2001), 471–515.

[6] M. Escobedo, S. Mischler and J.J.L. Velazquez, *Asymptotic description of Dirac mass formation in kinetic equations for quantum particles*, Journal of Differential Equations **202** (2004), 208–230.

[7] Kingman, J.F.C., *A simple model for the balance between selection and mutation*, J. Appl. Prob. **15** (1978), 1–12.

Harnack inequality in degenerated i.i.d. balanced environments

Jean-Dominique Deuschel
(joint work with Noam Berger, Moran Cohen and Xiaoqin Guo)

We consider random walks in an i.i.d. balanced environment that is not necessarily elliptic but d-dimensional. We will prove an elliptic Harnack inequality at large scale.

To be specific, let $\mathcal{M}$ be the set of probability measures on $\{e \in \mathbb{Z}^d : |e| = 1\}$, $d \geq 2$. A **balanced environment** is an element $\omega \in \Omega := \mathcal{M}^{\mathbb{Z}^d}$ with

$$\omega = \{\omega(x)\}_{x\in\mathbb{Z}^d} \;=\; \{\omega(x,e) : |e| = 1\}_{x\in\mathbb{Z}^d}$$

and

$$\omega(x,e) \;=\; \omega(x,-e).$$

Let P be a probability measure on Ω which is i.i.d. and *genuinely d-dimensional*. That is, $\mathbb{P}[\omega(0,e_i) > 0] > 0$ for all $i = 1,\dots,d$. For a given environment ω, the random walk $(X_n)_{n\geq 0}$ is the Markov chain with law

$$P_\omega[X_{n+1} = x + e \mid X_n = e] = \omega(x,e).$$

Recently Berger and Deuschel [1] proved the *quenched invariance principle*. Namely, for $\mathbb{P}$-almost every ω, the law of the rescaled process

$$(X_{\lfloor tn\rfloor}/\sqrt{n})_{t\geq 0}$$

converges weakly to a Brownian motion with a *deterministic* nondegenerate diffusion matrix Σ. This invariance principle generalizes previous results in the non degenerate cases, cf. [3], [2] and, for diffusions in non-divergence form, in [4].

Our main result shows an elliptic Harnack inequality for this model. More precisely, a function $u\colon B_R \to \mathbb{R}$ is called ω-harmonic on the ball $B_R = \{z \in \mathbb{Z}^d : |z| \leq R\}$ when

$$\sum_{|e|=1} \omega(x,e)(u(x+e) - u(x)) \;=\; 0, \qquad x \in B_R.$$

Then, there exists a constant C depending on Σ only, such that every non-negative ω-harmonic u in a large ball B_R with $R \geq R_0(\omega)$ satisfies

$$\max_{B_{R/2}} u \;\leq\; C \min_{B_{R/2}} u.$$

Moreover, the (random) $R_0(\omega)$ has stretched exponential tails:

$$P[R_0 > L] \;\leq\; \mathrm{e}^{-L^\alpha}$$

for some $\alpha \in (0,1)$.

Our proof relies on a detailed analysis of the corresponding infinite directed percolation cluster, a quantitative estimate for the invariance principle and an oscillation inequality. The later follows from a coupling argument within a multiscale structure.

References

[1] N. Berger and J.-D. Deuschel, *A quenched invariance principle for non-elliptic random walk in i.i.d. balanced random environment*, Probab. Theory Relat. Fields **158** (2014), 91-126 .
[2] X. Guo and O. Zeitouni, *Quenched invariance principle for random walk in balanced random environment*, Probab. Theory Relat. Fields, **152** (2010), 207–230.
[3] G. Lawler, *Weak convergence of a random walk in a random environment*, Comm. Math. Phys. **87** (1982), 81–87, .
[4] G. Papanicolaou and S.R.S. Varadhan, *Diffusions with random coefficients*, Statistics and probability: essays in honor of C.R. Rao, pp. 547-552, North-Holland, Amsterdam, (1982).

Linear response, current fluctuations and uncertainty relations in periodically driven Markov processes

Alessandra Faggionato

(joint work with A.C. Barato, L. Bertini, R. Chetrite, D. Gabrielli, P. Mathieu)

Periodically driven Markov processes have many applications. We focus here on the statistical physics of small systems, including molecular motors. Molecular motors are proteins, working as machines inside the cell. Their size is of order 1 nm=10^{-9} m. They are essential for cell division, cellular transport, muscle contraction, genetic transcription... Simply, they are at the basis of our life. They use chemical energies from ATP hydrolysis to produce mechanical work and are very efficient machines despite the very noisy environment in which they operate. Unlike their biological counterparts, artificial molecular machines are generally non-autonomous: they are manipulated by varying the external parameters or stimuli such as temperature, the chemical environment, or laser light. Often, the external parameters/stimuli vary in a time-periodic way, hence periodically driven Markov processes have received much attentions in the last years also inside stochastic thermodynamics, which is a statistical physics theory developed for the analysis of small systems [6].

In the first part of the talk we concentrate on continuous–time Markov chains with time-periodic jump rates. For these models we derive large deviations principles for the empirical measure, flow and current [1]. These results extend in part the analysis already performed for time–homogenous continuous–time Markov chains [2, 3, 4]. As an application we derive Gallavotti-Cohen duality relations for the fluctuating entropy flux and we also derive lower bounds on the variance of antisymmetric functionals in terms of entropy production (the so called "uncertainty relations"). We then investigate the probabilistic structure behind linear response w.r.t. the oscillatory steady state, enlarging the discussion also to diffusions on a torus (or a generic compact manifold) with time-periodic coefficients. We show that the linear response of the system can be formulated in terms of suitable covariances. Moreover, we analyze the complex mobility matrix and give a probabilistic representation [5].

(Joint works with A.C. Barato, L. Bertini, R. Chetrite, D. Gabrielli for LDP's and uncertainty relations, and with P. Mathieu for linear response)

References

[1] L. Bertini, R. Chetrite, A. Faggionato, D. Gabrielli, *Level 2.5 large deviations for continuous time Markov chains with time periodic rates*, arXiv:1710.08001 (2017).

[2] L. Bertini, A. Faggionato, D. Gabrielli, *From level 2.5 to level 2 large deviations for continuous time Markov chains*, Markov Processes and Related Fields **20** (2014), 545–562.

[3] L. Bertini, A. Faggionato, D. Gabrielli, *Large deviations of the empirical flow for continuous time Markov chains*. Ann. Inst. H. Poincaré Probab. Statist. **51** (2015), 867–900.

[4] L. Bertini, A. Faggionato, D. Gabrielli, *Flows, currents, and cycles for Markov Chains: large deviation asymptotics*, Stochastic Processes and their Applications **125**, (2015) 2786–2819.

[5] A. Faggionato, P. Mathieu, *Linear response, Nyquist relation and complex mobility in periodically driven Markov processes*, forthcoming (hopefully).

[6] U. Seifert, *Stochastic thermodynamics, fluctuation theorems and molecular machines*, Rep. Prog. Phys. *75* (2012) 126001.

Homogenization vs. localization in the random conductance model

Franziska Flegel

(joint work with M. Heida and M. Slowik)

Our aim is to understand the asymptotic behavior of the top eigenvectors and eigenvalues of the random conductance Laplacian in a large domain of $\mathbb{Z}^d$ ($d \geq 2$) with zero Dirichlet conditions. That is, we consider the spectral problem

$$\begin{aligned} -\mathcal{L}_w\psi &= \lambda\psi && \text{on } (-n,n)^d \cap \mathbb{Z}^d\,, \\ \psi &= 0 && \text{else,} \end{aligned}$$

where

$$(\mathcal{L}_w u)(x) = \sum_y w_{xy}\,(u(y) - u(x))\,, \qquad x \in \mathbb{Z}^d, u \in \ell^2\left(\mathbb{Z}^d\right),$$

is the random conductance Laplacian and the w's are the random conductances, which are symmetric in the sense that $w_{xy} = w_{yx}$. We fix a realization of conductances on the whole lattice (i.e., we fix a realization of the environment) and then we let the box size n tend to infinity.

In the special case where only nearest-neighbor conductances are positive and the conductances are independent and identically distributed, there is a dichotomy between a parameter regime where the first k eigenvectors strongly localize and a regime where the first k eigenvectors homogenize. Then we show that the spectrum of the Laplacian displays a sharp transition between a completely localized and a completely homogenized phase. A simple moment condition distinguishes between the two phases. To be more precise: If $\gamma = \sup\{q \geq 0\colon \mathbb{E}[w^{-q}] < \infty\} < 1/4$ and certain regularity assumptions apply, then we show that for almost every environment the kth Dirichlet eigenvector $\psi_k^{(n)}$ asymptotically concentrates in a sequence of single site $(z_{(k,n)})_{n\in\mathbb{N}}$ and the corresponding eigenvalue $\lambda_k^{(n)}$ is asymptotically equivalent to the local speed measure $\pi_z = \sum_{x\colon x\sim z} w_{xz}$ in the site $z_{(k,n)}$ [4, 5]. In fact, the site $z_{(k,n)}$ is the location of the kth minimum of π_z over the box B_n. The proof for this result is based on a spatial extreme value analysis of the local

speed measure, Borel-Cantelli arguments, the Rayleigh-Ritz formula, results from percolation theory, and path arguments à la [2] and the Bauer-Fike theorem.

In the homogenized phase we can even generalize our results to stationary and ergodic conductances with additional jumps of arbitrary length. This part is joint work with M. Slowik and M. Heida [3]. In order to prove spectral homogenization, we first prove homogenization of the discrete Poisson equation and then infer the spectral result by [6]. For the homogenized phase we assume the following.

Assumption.

(i) The law $\mathbb{P}$ is stationary and ergodic with respect to spatial translations in $\mathbb{Z}^d$.
(ii) $\mathbb{E}\left[\sum_{z\in\mathbb{Z}^d} w_{0,z}|z|^2\right] < \infty$.
(iii) For $\mathbb{P}$–a.e. w, the set of open edges contains the set of nearest-neighbor edges of $\mathbb{Z}^d$.
(iv) There exists $q > d/2$ such that for any nearest-neighbor edge e we have $\mathbb{E}\left[w(e)^{-q}\right] < \infty$.

In case of i.i.d. nearest-neighbor conductances the last condition can be improved to the condition that there exists $\gamma > 1/4$ such that $\mathbb{E}\left[w(e)^{-\gamma}\right] < \infty$. Together with the localization result, this gives the dichotomy.

To obtain the homogenization result, we introduce the rescaled operator $\mathcal{L}_w^\epsilon$ by

$$(\mathcal{L}_w^\epsilon u)(x) \;=\; \epsilon^{-2} \sum_{z\in\epsilon\mathbb{Z}^d} w_{\frac{x}{\epsilon},\frac{z}{\epsilon}} \left[f\left(z\right) - f\left(x\right)\right], \qquad \left(x \in \epsilon\mathbb{Z}^d\right), u \in \ell^2\left(\epsilon\mathbb{Z}^d\right)$$

and the operator $\mathcal{R}_\epsilon^*$ that translates between the discrete functions living on $\epsilon\mathbb{Z}^d$ and functions living on the entire $\mathbb{R}^d$ by a simple extension into the ϵ-unit cells. Then we prove that if

$$\begin{aligned} -\mathcal{L}_w^\epsilon u^\epsilon \;&=\; f^\epsilon && \text{on } (-1,1)^d \cap \epsilon\mathbb{Z}^d \\ u^\epsilon \;&=\; 0 && \text{else,} \end{aligned}$$

and the sequence $\mathcal{R}_\epsilon^* f^\epsilon$ converges weakly to some f in L^2, then $\mathbb{P}$–a.s. the sequence $\mathcal{R}_\epsilon^* u^\epsilon$ of solutions converges to strongly to the solution of the homogenized equation $-\nabla\cdot(A_{\text{hom}}\nabla u) = 2f$ almost everywhere with A_{hom} the usual homogenized matrix, see e.g. [3, (5.11)]. Spectral homogenization then follows by [6, Chapter 11.1].

Without the long-range connections, the integrability condition on the lower tail is optimal for spectral homogenization apart from logarithmic corrections, see e.g. the counter example constructed for the proof of [1, Theorem 5.4], which can easily be adapted to our specific choice of Laplace operator. It coincides with a necessary condition for the validity of a local central limit theorem for the random walk among random conductances [1]. Our proofs are based on a compactness result for the Laplacian's Dirichlet energy, Poincaré inequalities, Moser iteration and two-scale convergence à la [7].

References

[1] S. Andres, J.-D. Deuschel, M. Slowik. Harnack inequalities on weighted graphs and some applications to the random conductance model. *Probability Theory and Related Fields*, 164(3):931–977, 2016.

[2] O. Boukhadra, T. Kumagai, and P. Mathieu. Harnack inequalities and local central limit theorem for the polynomial lower tail random conductance model. *Journal of the Mathematical Society of Japan*, 67(4):1413–1448, 2015.

[3] F. Flegel, M. Heida, and M. Slowik. Homogenization theory for the random conductance model with degenerate ergodic weights and unbounded-range jumps. *Preprint, available at arXiv:1702.02860*, 2017.

[4] F. Flegel. Localization of the principal Dirichlet eigenvector in the heavy-tailed random conductance model. *Preprint, available at arXiv:1608.02415*, 2016.

[5] F. Flegel. Eigenvector localization in the heavy-tailed random conductance model. *Preprint, available at arXiv:1801.05684*, 2018.

[6] V. V. Jikov, S. M. Kozlov, and O. A. Oleinik. *Homogenization of differential operators and integral functionals*. Springer Berlin Heidelberg, 1994.

[7] V. V. Zhikov, A. L. Pyatnitskii. Homogenization of random singular structures and random measures. *Izvestiya: Mathematics*, 70(1):19–67, 2006.

Periodic striped ground states in Ising models with competing interactions

Alessandro Giuliani

(joint work with J. Lebowitz, E. Lieb, R. Seiringer)

In this talk, I will review some selected results obtained in the last few years on the existence of periodic minimizers in two- and three-dimensional spin systems with competing interactions. The model that we consider is an Ising model in dimension d (the most interesting cases being $d = 2$ and $d = 3$), with short range ferromagnetic and long range, power-law decaying, anti-ferromagnetic interactions. The Hamiltonian describing the energy of the system is

$$H = -J \sum_{\langle x,y\rangle} (\sigma_x\sigma_y - 1) + \sum_{\{x,y\}:\, x\neq y} \frac{(\sigma_x\sigma_y - 1)}{|x-y|^p}, \tag{1}$$

where $J > 0$ is the ratio between the strengths of the ferromagnetic and of the anti-ferromagnetic interaction, and $p > d$ is the decay exponent of the long-range interaction. The first sum ranges over pairs of nearest-neighbor sites in the discrete torus $\mathbb{T}_L^d := \mathbb{Z}^d/L\mathbb{Z}^d$, while the second over pairs of distinct sites in $\mathbb{T}_L^d$. The spins σ_x, $x \in \mathbb{T}_L^d$, take values in $\{\pm 1\}$, and the constant -1 appearing in the two terms is chosen in such a way that the energy of the homogeneous configuration $\sigma_x \equiv +1$, is equal to zero. A physically relevant case is $d = 2$ and $p = 3$, in which case (1) models the low-temperature equilibrium properties of thin magnetic films, embedded in the three-dimensional space, with the easy-axis of magnetization coinciding with the axis orthogonal to the film; in this case, the long range term models the dipolar interaction among the localized magnetic moments, while the short-range term models a ferromagnetic exchange interaction.

The goal is to characterize the structure of the ground states of the system, for any (even, sufficiently large) $L \in \mathbb{N}$. Ideally, one would also like to characterize the low-temperature infinite volume Gibbs states, but this is beyond our current abilities. Note that the short-range interaction favors a homogeneous state, that is $\sigma_x \equiv +1$ or $\sigma_x \equiv -1$, while the long-range term favors an anti-ferromagnetic 'Nèel' state, that is $\sigma_x = (-1)^{x_1+\cdots+x_d}$ or $\sigma_x = (-1)^{x_1+\cdots+x_d+1}$. The fact that the long-range contribution to the energy is minimized by the Nèel state is not obvious, and was proved in [2] by Reflection Positivity (RP) methods.

In the presence of both terms, the competition between the short-range ferromagnetic and the long-range anti-ferromagnetic interaction induces the system to form domains of minus spins in a background of plus spins, or vice versa. This happens in an intermediate range of values of J: in fact, if J is sufficiently small, the ground state is the same as for $J = 0$, that is, it is the Nèel state [2]; if J is sufficiently large and $p > d + 1$[1], the ground state is the same as for $J = +\infty$, that is, it is the homogeneous state. For intermediate values of J the ground state is characterized by non-trivial structures, whose typical length scale diverges as $J \to J_c(p)$ from the left; here $J_c(p)$ is the critical value of J, beyond which the ground state is homogeneous. It coincides with the value of J at which the surface tension of an infinite straight domain wall, separating a half space of minuses from a half space of pluses, vanishes [5]. It is expected that, for values of J close to $J_c(p)$ and slightly smaller than it, all the ground states are quasi-one-dimensional (i.e., they are translationally invariant in $d-1$ directions), and periodic, provided the box size L is an integer multiple of an 'optimal period' $2h^*$, which can be explicitly computed. We shall refer to these expected ground states as the 'optimal periodic striped states': they consist of 'stripes' (in $d = 2$, or 'slabs', in $d = 3$) of spins all of the same sign, arranged in an alternating way (that is, neighbouring stripes have opposite magnetization), and all of the same width h^*.

The conjecture that optimal periodic striped states are ground states of (1) has been first proved in [3, 4], via a generalization of the standard RP technique, which we named 'block reflection positivity', because the reflections are performed across the bonds that separate a block of plus spins from a block of minus spins. The same proof shows that in any dimension, optimal periodic striped states are the states of minimal energy, among all the possible quasi-one-dimensional states.

More recently, in a work in collaboration with R. Seiringer, we succeeded in proving this conjecture [6], for all dimensions $d \geq 1$ and sufficiently large decay exponents, namely $p > 2d$. The result has been recently extended to the continuum setting and $p > d + 2$ [1]. The proof is based on the following main steps:

[1] for $p \leq d + 1$, the homogeneous state is *not* the ground state, for any finite value of J. In these cases, which include the case $d = 2$, $p = 3$ mentioned above, it would be interesting to characterize the ground states for J sufficiently large; unfortunately, we do not have rigorous results to report on this case yet, with the only exception of the one-dimensional case $d = 1$.

(1) We re-express the energy of the spin configuration as the energy of an equivalent droplet configuration. Here the droplets are the connected regions of minus spins, in a background of plus spins. The energy, if expressed in terms of droplets, consists of (i) a sum of droplet self-energies, which include the ferromagnetic contribution to the surface tension, plus the long range interaction of the minus spins in each droplet δ with a 'sea' of plus spins in the complement of the droplet $\delta^c = \mathbb{T}_L^d \setminus \delta$, and (ii) a droplet-droplet pair interaction, which is repulsive. Remarkably, the long range contribution to the self-energy of a droplet δ behaves (for the purpose of a lower bound) as $-2J_c(p)|\partial\delta|$, where $|\partial\delta|$ is the length (if $d = 2$, or area, if $d = 3$) of the boundary of the droplet, plus a positive constant times the number of corners, that is, the points where the domain walls bend by 90^o. In this respect, the corners look like the elementary excitations of the system.
(2) We localize the droplet energy in bad boxes, characterized by a local 'atypical' configuration (which either has corners, or too large uniformly magnetized regions – called 'holes'), and good regions, which are the connected components of the complement of the union of the bad boxes. By 'localizing', we mean here that the original energy is bounded from below in terms of a sum of local energy functionals, each depending only on the local droplet configuration (supported either in a bad box or in a good region). By construction, the configuration in a good region is quasi-one-dimensional, and consists of stripes all in the same direction, but not necessarily all of the same width.
(3) We use our lower bound on the self-energy of the droplets, to infer that the localized energy in a bad box is much larger than the energy of an optimal striped configuration in the same box. The energy difference scales like the number of corners contained in the bad box, plus the volume of the holes. We shall refer to this energy difference as the energy gain associated with each bad box.
(4) We use a slicing procedure, combined with block RP and an optimal control of the boundary errors, to derive an optimal lower bound on the localized energy in a good region. Such a lower bound scales like the energy of the optimal striped configuration in the same region, minus a boundary error, which is so small that it can be over- compensated by the energy gains of the bad boxes at the boundary of the good region (note that every boundary portion of a good region borders on a bad box).

Our result provides the first rigorous proof of the formation of mesoscopic periodic structures in $d \geq 2$ systems with competing interactions. It leaves a number of important problems open:

(1) Extend the result of [6] to smaller decay exponents. In particular, prove that the ground states of (1) with $d = 2$ and $p = 3$ are periodic and striped, for all sufficiently large J.

(2) Prove that there are at least d infinite volume Gibbs states at low temperatures, which are translationally invariant in $d-1$ coordinate directions. Depending on the dimension, prove the existence of Long-Range Striped Order (LRSO), or of quasi-LRSO a'la Kosterlitz-Thouless, in the last coordinate direction.
(3) Extend these results to the continuum setting, for an effective free energy functional that is rotationally invariant. In particular, prove the onset of continuous symmetry breaking, both in the ground state and in the low-temperature Gibbs states.

References

[1] S. Daneri, E. Runa: *Exact periodic stripes for a minimizers of a local/non-local interaction functional in general dimension*, arXiv:1702.07334.
[2] J. Fröhlich, R. B. Israel, E. H. Lieb, and B. Simon: *Phase Transitions and Reflection Positivity. II. Lattice Systems with Short-Range and Coulomb Interactions*, J. Stat. Phys. **22**, 297 (1980).
[3] A. Giuliani, J. Lebowitz, E. Lieb: *Ising models with long-range dipolar and short range ferromagnetic interactions*, Phys. Rev. B **74**, 064420 (2006).
[4] A. Giuliani, J. Lebowitz, E. Lieb: *Striped phases in two-dimensional dipole systems*, Phys. Rev. B **76**, 184426 (2007).
[5] A. Giuliani, J. Lebowitz, E. Lieb: *Checkerboards, stripes and corner energies in spin models with competing interactions*, Phys. Rev. B **84**, 064205 (2011).
[6] A. Giuliani, R. Seiringer: *Periodic Striped Ground States in Ising Models with Competing Interactions*, Comm. Math. Phys. **347**, 983-1007 (2016).

Mean-field equations for stochastic particle systems

Stefan Grosskinsky
(joint work with Watthanan Jatuviriyapornchai)

The derivation of effective single-particle dynamics from interacting many-particle systems has a long history in the context of kinetic theory, and can pose challenging mathematical problems with the Boltzmann equation as a classical example. In the physics literature, stochastic particle systems in a limit of large system size are often described by a mean-field master equation for the time evolution of a single lattice site [1, 2, 3]. For conservative systems, these equations are very similar to mean-field rate or kinetic equations in the study of cluster growth models. We focus on systems where only one particle jumps at a time, which corresponds to monomer exchange in cluster growth models as studied in [4], and also in the well-known Becker-Döring model [5, 6]. While these mean-field equations often provide the starting point for the analysis and have an intuitive form, to our knowledge their connection to underlying microscopic particle systems has not been rigorously established so far. Details of the following can be found in [7].

We consider a stochastic particle system $(\eta(t) : t > 0)$ on a complete graph Λ of size $|\Lambda| = L$. Configurations are denoted by $\eta = (\eta_x : x \in \Lambda)$ where $\eta_x \in \mathbb{N}_0$ is the

number of particles on site x, and a particle jumps from site x to any $y \neq x$ with rate $c(\eta_x, \eta_y)/(L-1)$. The dynamics of the process is defined by the generator

$$(\mathcal{L}g)(\eta) = \sum_{x,y\neq x\in\Lambda} \frac{1}{L-1} c(\eta_x, \eta_y)(g(\eta^{x\to y}) - g(\eta)) \ , \tag{1}$$

with the usual notation $\eta^{x\to y}$ for a configuration where one particle has moved from site x to y, i.e. $\eta_z^{x\to y} = \eta_z - \delta_{z,x} + \delta_{z,y}$. To ensure that the process is non-degenerate, the jump rates are strictly positive, except for $c(0,l) = 0$, $l \geq 0$. Our main assumption on the dynamics is that the rates grow sublinearly, such that

$$c(k,l) \leq C_1 k(l + C_2) \quad \text{for constants } C_1, C_2 > 0 \ . \tag{2}$$

We study the empirical processes $t \mapsto F_k^L(\eta(t))$ defined by the test functions

$$F_k^L(\eta) := \frac{1}{L} \sum_{x\in\Lambda} \delta_{\eta_x,k} \in [0,1], \tag{3}$$

counting the fraction of lattice sites for each occupation number $k \geq 0$.

In the following, we consider initial conditions $\eta(0)$ whose distribution converges as $L \to \infty$ to a probability distribution $f(0)$ on $\mathbb{N}_0$ with finite first and second moments, such that we have a weak law of large numbers

$$F_k^L(\eta(0)) \to f_k(0) \quad \text{in distribution for all } k \geq 0 \ . \tag{4}$$

A further technical assumption concerns a bound on first and second moments uniformly in $\eta(0)$ and L, which could be replaced by a less restrictive tail condition on $f(0)$. Simple choices that fulfill all conditions are for example product measures with a finite maximal occupation number per site.

Our main result is a weak law of large numbers for the empirical processes $t \mapsto F_k^L(\eta(t))$ which holds pointwise in k or, equivalently, in a weak sense, where we use the notation

$$\langle F^L(\eta), h\rangle = \sum_{k\geq 0} h_k F_k^L(\eta) \ , \tag{5}$$

for all bounded functions $h : \mathbb{N}_0 \to \mathbb{R}$.

Theorem. *Consider a process with generator* (1) *on the complete graph with sublinear rates (2) and initial conditions satisfying the above assumpations. Then we have a weak law of large numbers, i.e. for all bounded $h : \mathbb{N}_0 \to \mathbb{R}$,*

$$\big(\langle F^L(\eta(t)), h\rangle\big)_{t\geq 0} \to \big(\langle f(t), h\rangle\big)_{t\geq 0} \quad \textit{weakly on path space as } L \to \infty \ , \tag{6}$$

where $t \mapsto (f_k(t) : k \in \mathbb{N}_0)$ is the unique solution of the ***mean-field equation***

$$\begin{aligned} \frac{df_k(t)}{dt} &= \sum_{l\geq 0} c(k+1,l) f_l(t) f_{k+1}(t) + \sum_{l\geq 0} c(l,k-1) f_l(t) f_{k-1}(t) \\ &\quad - \bigg(\sum_{l\geq 0} c(k,l) f_l(t) + \sum_{l\geq 0} c(l,k) f_l(t) \bigg) f_k(t) \quad \textit{for all } k \geq 0, \end{aligned} \tag{7}$$

with initial condition $f(0)$ given by (4). Here we use the convention $f_{-1}(t) \equiv 0$ for all $t \geq 0$ and recall that $c(0,l) = 0$ for all $l \geq 0$.

Note that this result implies in particular existence and uniqueness of the solution to (7) for all $t \geq 0$, which has been shown independently in a recent preprint [8]. Existence of limits follows from standard tightness arguments, and the deterministic limit arises from a vanishing martingale part for the empirical processes.

Suppose a further symmetry assumption on the initial conditions,

$$\{\eta_x(0) : x \in \Lambda\} \quad \text{is permutation invariant for all } L \geq 1 \ . \tag{8}$$

Then by symmetry of the dynamics, this holds also for the full process $(\eta(t) : t \geq 0)$ and in particular at all fixed times $t \geq 0$. Then the weak law of large numbers for the empirical measures implies that for all $m \geq 1$, and $t \geq 0$ as $L \to \infty$

$$(\eta_1(t), \eta_2(t), \ldots, \eta_m(t)) \quad \text{converge weakly to iidrv's with distribution } f(t) \ .$$

This is a standard result in propagation of chaos and a recent exposition of a proof can be found in [9]. Since the law of large numbers holds not only for time marginals but for the full process, we can lift it on path space and also establish propagation of chaos on the level of processes.

Corollary. *Consider the process with generator* (1) *and conditions as in the Theorem together with* (8). ***Propagation of chaos*** *holds, i.e. for all $m \geq 1$, and $T \geq 0$ as $L \to \infty$ the finite dimensional processes $\big((\eta_1(t), \eta_2(t), \ldots, \eta_m(t)) : t \in [0,T]\big)$ converge weakly on path space to independent, identical birth death chains on $\mathbb{N}_0$ with distribution $f(t)$ and master equation given by (7).*

The limit equation (7) therefore describes the dynamics of the fraction $f_k(t) \in [0,1]$ of lattice sites with a given occupation number k, and also provides the master equation of a birth death chain for the limiting single site dynamics

$$(\eta_x(t) : t \geq 0) \quad \text{for any fixed } x \in \Lambda \text{ (with } \Lambda \text{ big enough)} \tag{9}$$

under additional assumptions on the initial condition. Note that the chain and its master equation are non-linear since the birth rates $\sum_{l \geq 0} c(l,k) f_l(t)$ and death rates $\sum_{l \geq 0} c(k,l) f_l(t)$ depend on the distribution $f(t)$. Even though the limiting birth death dynamics is irreducible under any non-degenerate initial conditions, the non-linearity leads to conservation of the first moment, resulting in a continuous family of stationary distributions and interesting ergodic behaviour. This includes e.g. the coarsening dynamics of condensing particle systems [10, 7], such as zero-range processes of the type introduced in [11], inclusion processes with a rescaled system parameter [12, 13] and explosive condensation models [14, 15].

References

[1] C. Godrèche, *Dynamics of condensation in zero-range processes*, J. Phys. A: Math. Gen. **36** (23) (2003), 6313.

[2] C. Godrèche, J. Luck, *Dynamics of the condensate in zero-range processes*, J. Phys. A: Math. Gen. **38** (33) (2005), 7215.

[3] M. Evans, B. Waclaw, *Condensation in stochastic mass transport models: beyond the zero-range process*, J. Phys. A: Math. Gen. **47** (9) (2014), 095001.
[4] E. Ben-Naim, P. Krapivsky, *Exchange-driven growth*, Phys. Rev. E **68** (3) (2003), 031104.
[5] R. Becker, W. Döring, *Kinetische behandlung der Keimbildung in übersättigten Dämpfen*, Ann. Physik **416** (8) (1935), 719–752.
[6] J. Ball, J. Carr, O. Penrose, *The becker-döring cluster equations: basic properties and asymptotic behaviour of solutions*, Comm. math. phys. **104** (4) (1986), 657–692.
[7] W. Jatuviriyapornchai, S. Grosskinsky, *Derivation of mean-field equations for stochastic particle systems*, arXiv:1703.08811.
[8] E. Esenturk, *Mathematical theory of exchange-driven growth*, arXiv:1710.08375.
[9] P. Dai Pra. *Stochastic mean-field dynamics and applications to life sciences*. CIRM lecture notes (2017).
[10] W. Jatuviriyapornchai, S. Grosskinsky, *Coarsening dynamics in condensing zero-range processes and size-biased birth death chains*, J. Phys. A: Math. Gen. **49** (18) (2016), 185005.
[11] M. R. Evans, *Phase transitions in one-dimensional nonequilibrium systems*, Braz. J. Phys. **30** (1) (2000), 42–57.
[12] S. Grosskinsky, F. Redig, K. Vafayi, *Dynamics of condensation in the symmetric inclusion process*, EJP **18** (66) (2013), 1–23.
[13] J. Cao, P. Chleboun, S. Grosskinsky, *Dynamics of condensation in the totally asymmetric inclusion process*, J. Stat. Phys. **155** (3) (2014), 523–543.
[14] B. Waclaw, M. R. Evans, *Explosive condensation in a mass transport model*, Phys. rev. lett. **108** (7) (2012), 070601.
[15] Y.-X. Chau, C. Connaughton, S. Grosskinsky, *Explosive condensation in symmetric mass transport models*, JSTAT **2015** (11) (2015), P11031.

Convergence of the squareroot approximation sceme to the Fokker–Planck equation

Martin Heida

Let Q be a bounded domain with a family of points $(P_{m,i})_{i=1,\dots m}$. From these points we construct a Voronoi tessellation of cells $G_{m,i}$ that correspond to $P_{m,i}$ for every i. We write $i \sim j$ if the cells $G_{m,i}$ and $G_{m,j}$ are neighbored. Thus, the finite volume space for the discretization $(G_{m,i})_{i=1,\dots m}$ is isomorphic to $\mathbb{R}^m$. Given a potential $V \in C^2(\overline{Q})$ and writing $v_i^m := \exp\left(-\frac{1}{2}\beta V(P_{m,i})\right)$, the squareroot approximation operator on $P_{m,i}$ is then defined as

$$(\mathcal{F}_m u)_i := C_m \sum_{i\sim j} \left(u_j \frac{v_i^m}{v_j^m} - u_i \frac{v_j^m}{v_i^m} \right), \tag{1}$$

where C_m is a normalizing constant.

The discretization scheme (1) proposed by Lie, Fackeldey and Weber [2] is implemented and applied to alanine dipeptide (Ac–A–NHMe) in a recent work [1]. The operator $\mathcal{F}_m$ has precisely one eigenvector u^0 to the eigenvalue 0, namely $u_i = v_i^2$. Hence, writing $\pi_i^m := \exp\left(-\beta V(P_{m,i})\right) = v_i^2$, we obtain

$$(\mathcal{F}_m u)_i := C_m \sum_{i\sim j} \left(u_j \frac{\sqrt{\pi_i^m}}{\sqrt{\pi_j^m}} - u_i \frac{\sqrt{\pi_j^m}}{\sqrt{\pi_i^m}} \right), \qquad \mathcal{F}_m \pi^m = 0.$$

Hence, the coefficients can be written in terms of the square roots of the stationary solution, which is the reason the method is called *squareroot approximation*. As boundary conditions one usually uses Dirichlet conditions in space variables on periodic boundary conditions for angles.

It turns out that this normalizing constant can be estimated from the case $V \equiv 0$, i.e. from the discrete Laplace operator $\mathcal{L}_m$ which is given as

$$(\mathcal{L}_m u)_i := C_m \sum_{j \sim i} (u_j - u_i) \ . \tag{2}$$

More precisely, the main Theorem states that the convergence behavior of $\mathcal{F}_m$ is mostly characterized by the convergence behavior of $\mathcal{L}_m$: If $\mathcal{L}_m$ is G-convergent (in the discrete sense) to $\mathcal{L}u = \nabla \cdot (A_{\hom} \nabla u)$, the solutions u_m of the equation $\mathcal{F}_m u_m = f_m$ converge to solutions $\mathcal{F}u := \nabla \cdot (A_{\hom} \nabla u) + \nabla \cdot (u A_{\hom} \nabla V) = f$, provided $f_m \to f$ in a weak sense. Note that the opposite direction is trivial: If the SQRA converges for all $V \in C^2(\overline{Q})$ then $\mathcal{L}_m \rightsquigarrow \nabla \cdot A_{\hom} \nabla$.

As a further contribution, we show that the class of admissible discretization, i.e. discretizations such that $\mathcal{L}_m$ is G-convergent, is not empty. For this purpose, we study periodizations of stationary ergodic Voronoi-Tessellations.

References

[1] L. Donati, M. Heida, M. Weber, and B. Keller. Estimation of the initesimal generator by square-root approximation. *In preparation.*

[2] Han Cheng Lie, Konstantin Fackeldey, and Marcus Weber. A square root approximation of transition rates for a markov state model. *SIAM Journal on Matrix Analysis and Applications*, 34:738–756, 2013.

Rigorous derivation of density functionals for classical systems

Sabine Jansen

(joint work with Tobias Kuna, Dimitrios Tsagkarogiannis)

1. Motivation: Onsager functional for liquid crystals

Consider a system of N thin rods with centers $x_1, \dots, x_N \in \Lambda$, $\Lambda := [0, L]^d \subset \mathbb{R}^d$ and orientations $n_1, \dots, n_N \in \mathbb{S}$, interacting via some pair potential $v(q_i, q_j)$ where $q_i = (x_i, n_i)$. In the theory of liquid crystals it is of interest to systematically derive a free energy functional $\mathcal{F}$ which assigns to a density profile $\rho : \Lambda \times \mathbb{S} \to \mathbb{R}_+$ with $\int_{\Lambda \times \mathbb{S}} \rho(q) \mathrm{d}q = N$ a free energy $\mathcal{F}[\rho]$. A commonly employed functional is

$$\mathcal{F}[\rho] = \int_{\Lambda \times \mathbb{S}} \rho(q) \log(\rho(q) - 1) \mathrm{d}q + \frac{1}{2} \int_{(\Lambda \times \mathbb{S}^2} \rho(q) \rho(q') f(q, q') \mathrm{d}q \mathrm{d}q', \tag{1}$$

where $f(q, q') = \exp(-v(q, q')) - 1$ may capture, for example, excluded volume between rods. Onsager [4] gave a microscopic derivation starting from statistical mechanics, building on the theory of cluster expansions and virial inversions and with the "artifice... of viewing rods of different orientations as different types of particles". A corollary of our main result is a fully rigorous derivation, in

the following sense: Consider density profiles of the form $\rho(x,n) = \rho_L(x,n) = \rho_0(x/L,n)$ with $\rho_0 : [0,1]^d \times \mathbb{S} \to \mathbb{R}_+$ and $\int_{[0,1]^d\times\mathbb{S}} \rho_0(x',n)\mathrm{d}x'\mathrm{d}n = N/L^d$. Then, under suitable conditions on the profile ρ_0, we have

$$\frac{1}{N!}\int_{(\Lambda\times\mathbb{S})^N} \mathrm{e}^{-\sum_{1\le i<j\le N} v(q_i,q_j)} \mathbf{1}_{\{\frac{1}{L^d}\sum_{i=1}^N \delta_{(x_i/L,n_i)} \approx \rho_0(x',n)\mathrm{d}x'\}} \mathrm{d}^N\boldsymbol{q}$$
$$\approx \exp\Big(-L^d\mathcal{F}_0[\rho_0]\Big) \approx \exp\Big(-\mathcal{F}_L[\rho_L]\Big)$$

in the sense of large deviations as $N, L \to \infty$ at fixed N/L^d, where $\mathcal{F}_0[\rho_0] = \lim_{L\to\infty} L^{-d}\mathcal{F}_L[\rho_L]$ and

$$\mathcal{F}_L[\rho_L] = \int_{\Lambda\times\mathbb{S}} \rho_L(q)\log(\rho_L(q)-1)\mathrm{d}q + \sum_{k=2}^\infty \frac{1}{k!}\int_{(\Lambda\times\mathbb{S})^k} D_k(q_1,\dots,q_k)\prod_{i=1}^n \rho_L(q_i)\mathrm{d}^k\boldsymbol{q},$$

with explicitly known kernels D_n and absolute convergence of the series. Keeping only the quadratic term in the series we recover the functional (1).

2. Cluster expansion

More generally, let $(\mathbb{X},\mathcal{X})$ be some measurable space, λ a reference measure, $v : \mathbb{X}\times\mathbb{X} \to \mathbb{R}\cup\{\infty\}$ a pair interaction, and $z : \mathbb{X} \to \mathbb{R}_+$ an activity profile. For simplicity we assume that the pair potential is non-negative. For $\mathbb{X}_L \subset \mathbb{X}$ with $\int_{\mathbb{X}_L} z\mathrm{d}\lambda < \infty$, let

$$\Xi_L(z) := 1 + \sum_{N=1}^\infty \frac{1}{N!}\int_{\mathbb{X}_L^N} \mathrm{e}^{-\sum_{1\le i<j\le N} v(x_i,x_j)} \prod_{i=1}^N z(x_i)\mathrm{d}\lambda^N(\boldsymbol{x}),$$
$$\rho_L(q;z) := z(q)\frac{\delta}{\delta z(q)}\log\Xi_L(z)$$

with $\frac{\delta}{\delta z(q)}$ a variational derivative. The profile $\mathbb{X}_L \ni q \mapsto \rho_L(q;z)$ is the density of the grand-canonical Gibbs measure at activity z; in probabilistic terms, the measure $\rho_L(q;z)\mathrm{d}\lambda(q)$ is the intensity measure of the Gibbs point process and $z\mathrm{d}\lambda$ is the intensity measure of an a priori Poisson point process. Cluster expansions [5] guarantee that if for some function $a : \mathbb{X} \to \mathbb{R}_+$ and all $x \in \mathbb{X}$ we have

$$\int_{\mathbb{X}^2} |\mathrm{e}^{-v(x,y)} - 1|\,|z(y)|\mathrm{e}^{a(y)}\mathrm{d}\lambda(y) \le a(x),$$

then

$$\rho_L(q;z) = z(q)\exp\Big(-\sum_{n=1}^\infty \frac{1}{n!}\int_{\mathbb{X}_L^n} A_n(q;x_1,\dots,x_n)\prod_{i=1}^n z(x_i)\mathrm{d}\lambda^n(\boldsymbol{x})\Big) \tag{2}$$

with

$$\sum_{n=1}^\infty \frac{1}{n!}\int_{\mathbb{X}^n} \big|A_n(q;x_1,\dots,x_n)\big|\prod_{i=1}^n |z(x_i)|\mathrm{d}\lambda^n(\boldsymbol{x}) \le a(q) < \infty$$

and explicitly known kernels A_n that depend on the pair interaction only. The bounds allow us to exchange summation and the limit $\mathbb{X}_L \nearrow \mathbb{X}$; henceforth we

drop the index L from (2), thus obtaining a functional mapping activity profiles $z : \mathbb{X} \to \mathbb{R}_+$ to density profiles $\rho : \mathbb{X} \to \mathbb{R}_+$.

Our task is to invert this functional, expressing the activity z as a function of the density ρ. For homogeneous, single-species systems, z and ρ become numbers and the inversion is a classical result [3]. For countably many variables, the required inversion can be performed with contour integrals and Lagrange-Good inversion [2]. The principal challenge is that for the natural Banach spaces at hand, the functional mapping z to ρ is not necessarily Fréchet-differentiable.

3. An inversion theorem

To avoid confusion between maps and variables, let us write $\tilde{\rho}$ and $\tilde{z}$ for functionals and ρ and z for the variables of the functionals. Thus $\rho(q; z)$ becomes $\tilde{\rho}[z](q)$.

Theorem. *There exists a uniquely defined family of kernels $t_n(q; x_1, \dots, x_n)$, depending only on the kernels A_n, such that the following holds:*

(a) If $\rho(\cdot)$ is such that for some $b : \mathbb{X} \to \mathbb{R}_+$ and all $q \in \mathbb{X}$, we have

$$\sum_{n=1}^{\infty} \int_{\mathbb{X}^n} \prod_{i=1}^{n} |A_n(q; x_1, \dots, x_n)| \prod_{i=1}^{n} |\rho(x_i)| \mathrm{e}^{b(x_i)} \mathrm{d}\lambda^n(\boldsymbol{x}) \le b(q),$$

then we also have

$$1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbb{X}^n} \big|t_n(q; x_1, \dots, x_n)\big| \prod_{i=1}^{n} |\rho(x_i)| \mathrm{d}^n(\boldsymbol{x}) \le \mathrm{e}^{b(q)} < \infty$$

and may define

$$\tilde{z}[\rho](q) := \rho(q)\Big(1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbb{X}^n} t_n(q; x_1, \dots, x_n) \prod_{i=1}^{n} \rho(x_i) \mathrm{d}\lambda^n(\boldsymbol{x})\Big).$$

(b) If ρ is as in (a), then $\tilde{\rho}[\tilde{z}[\rho]] = \rho$.

The theorem works for general kernels A_n that need not arise from statistical mechanics. In the concrete context of Section 2, a sufficient condition for the convergence condition from part (a) of the theorem to hold true is that

$$\int_{\mathbb{X}} |\mathrm{e}^{-v(x,y)} - 1||\rho(y)| \mathrm{e}^{a(y)+b(y)} \mathrm{d}\lambda(y) \le a(y)$$

for all $q \in \mathbb{X}$ and some $a, b : \mathbb{X} \to \mathbb{R}_+$ with $a \le b$. As a corollary, we obtain a condition for the homogeneous gas of hard spheres of radius R in $\mathbb{R}^d$: if $\rho > 0$ satisfies

$$\rho |B(0, 2R)| \le \sup\{a\mathrm{e}^{-a-b} \mid 0 \le a \le b\} = \frac{1}{2\mathrm{e}} \simeq 0.1839$$

then the virial series converges. This bound improves available bounds.

For the proof we generalize results on Lagrange-Good inversion and combinatorics of trees [1] to a setting with uncountably many variables, i.e., uncountable color or type space for vertices of trees.

The results generalize to attractive interactions v, classical formulas for the coefficients as sums of doubly connected graphs are rigorously proven.

References

[1] I. J. Good, *Generalizations to several variables of Lagrange's expansion, with applications to stochastic processes*, Proc. Cambridge Philos. Soc. **56** (1960), 367–380.

[2] S. Jansen, S. Tate, D. Tsagkarogiannis, D. Ueltschi, *Multispecies virial expansions*, Comm. Math. Phys. **330** (2014), 801–817.

[3] J. Lebowitz, O. Penrose, *Convergence of virial expansions*, J. Mathematical Phys. **5** (1964), 841–847.

[4] L. Onsager, *The effects of shape on the interaction of colloidal particles*, Ann. New York Acad. Sci. **51** (1949), 627–659.

[5] D. Ueltschi, *Cluster expansions and correlation functions*, Mosc. Math. J. **4** (2004), 511–522.

Quenched invariance principle for random walks among random conductances with stable-like jumps

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(joint work with Xin Chen, Jian Wang)

Consider random conductances that allow long range jumps. In particular we consider conductances $C_{xy} = w_{xy}|x-y|^{-d-\alpha}$ for distinct $x, y \in \mathbb{Z}^d$ and $0 < \alpha < 2$, where $\{w_{xy} = w_{yx} : x, y \in \mathbb{Z}^d\}$ are positive random variables. We prove that under some 'mixing conditions' for w, suitably rescaled Markov chains among the random conductances converge to a rotationally symmetric α-stable process almost surely w.r.t. the randomness of the environments. Our results hold for a class of 'nice' graphs with polynomial volume growth.

To clarify our results, we present a statement about the quenched invariance principle on a half/quarter space $F := \mathbb{R}^{d_1}_+ \times \mathbb{R}^{d_2}$ where $d_1, d_2 \in \mathbb{N} \cup \{0\}$. Let $\mathbb{L} := \mathbb{Z}^{d_1}_+ \times \mathbb{Z}^{d_2}$ and let μ be a measure on $\mathbb{L}$ such that $\mu_x := \mu(\{x\})$ satisfies for some constant $c_1 \geq 1$ and all $x \in \mathbb{L}$ that $c_1^{-1} \leq \mu_x \leq c_1$. Consider a Markov generator

$$L^\omega_{\mathbb{L}} f(x) = \sum_{y \in \mathbb{L}} (f(y) - f(x)) \frac{w_{x,y}(\omega)}{|x-y|^{d+\alpha}} \mu_y, \quad x \in \mathbb{L}, \tag{1}$$

where $\alpha \in (0,2)$ and $\{w_{x,y}(\omega) : x, y \in \mathbb{L}\}$ is a sequence of random variables such that $w_{x,y}(\omega) = w_{y,x}(\omega) > 0$ for all $x \neq y$. We write $w_{x,x}(\omega) = w^{-1}_{x,x}(\omega) = 0$ for all $x \in \mathbb{L}$. Let $\{X^\omega_t\}_{t \geq 0}$ be the corresponding Markov process. For every $n \geq 1$ and $\omega \in \Omega$, we define a process $X^{(n),\omega}_\cdot$ on $V_n = n^{-1}\mathbb{L}$ by $X^{(n),\omega}_t := n^{-1} X^\omega_{n^\alpha t}$ for any $t > 0$. Let $\mathbb{P}^{(n),\omega}_x$ be the law of $X^{(n),\omega}_\cdot$ with initial point $x \in V_n$.

Theorem. *Let $d := d_1 + d_2 > 4 - 2\alpha$. Suppose that $\{w_{x,y} : x, y \in \mathbb{L}\}$ is a positive sequence of independent random variables such that $\mathbb{E} w_{x,y} = 1$ for all $x, y \in \mathbb{L}$,*

$$\sup_{x,y \in \mathbb{L}} \mathbb{E}[w^{2p}_{x,y}] < \infty, \quad \sup_{x,y \in \mathbb{L}} \mathbb{E}[w^{-2q}_{x,y}] < \infty \tag{2}$$

for $p, q \in \mathbb{Z}_+$ with

$$p > \max\big\{(d+2)/d, (d+1)/(2(2-\alpha))\big\}, \quad q > (d+2)/d. \tag{3}$$

Then the quenched invariance principle holds for $X^\omega_\cdot$ with the limit process being a symmetric α-stable Lévy process Y on F with jumping measure $|z|^{-d-\alpha}\,dz$. Namely, for any $\{x_n \in V_n : n \geq 1\}$ such that $\lim_{n\to\infty} x_n = x$ for some $x \in F$, it holds that for $\mathbb{P}$-a.s. $\omega \in \Omega$ and every $T > 0$, $\mathbb{P}^{(n),\omega}_{x_n}$ converges weakly to $\mathbb{P}^Y_x$ on the space of all probability measures on $\mathcal{D}([0,T];F)$, the collection of càdlàg F-valued functions on $[0,T]$ equipped with the Skorohod topology.

Remark. *When $\alpha \in (0,1)$, the conclusion still holds true for $d > 2-2\alpha$, if $p > \max\big\{(d+1)/(2(1-\alpha)), (d+2)/d\big\}$ and $q > (d+2)/d$.*

Open Problem. *The integrability condition* (3) *is far from optimal. What is the optimal integrability condition?*

Example. *The following example satisfies* (2)*: for each distinct $x, y \in \mathbb{Z}^d$,*

$$\mathbb{P}(w_{x,y} = |x-y|^{\varepsilon}) = (3|x-y|^{2p\varepsilon})^{-1},\ \mathbb{P}(w_{x,y} = |x-y|^{-\delta}) = (3|x-y|^{2q\delta})^{-1},$$
$$\mathbb{P}(w_{x,y} = g(x,y)) = 1 - (3|x-y|^{2p\varepsilon})^{-1} - (3|x-y|^{2q\delta})^{-1},$$

where $\varepsilon, \delta > 0$ and $g(x,y)$ is chosen so that $\mathbb{E}w_{x,y} = 1$. (It is easy to see that $c^{-1} \leq g(x,y) \leq c$ for some $c > 1$.)

While detailed heat kernel estimates and Harnack inequalities (de Giorgi-Nash-Moser theory) are recently established for uniformly elliptic α-stable-like processes (see [1, 2, 3] etc.), the arguments rely on pointwise estimates of the jumping density (conductance in this setting), which cannot hold in our setting unless we assume uniform ellipticity of $w_{x,y}(\omega)$ in (1). Furthermore, Harnack inequalities do not hold (even for large enough balls) in general on long range random conductance models. By these reasons, it requires hard work to obtain the results in our random conductance setting. There are two essential ingredients in the proof, namely the tightness estimates and the Hölder regularity of parabolic functions for non-elliptic α-stable-like processes on graphs.

References

[1] Z.Q. Chen and T. Kumagai, *Heat kernel estimates for stable-like processes on d-sets*, Stoch. Proc. Appl. **108** (2003) 27–62.

[2] Z.Q. Chen and T. Kumagai, *Heat kernel estimates for jump processes of mixed types on metric measure spaces*, Probab. Theory Relat. Fields **140** (2008) 277–317.

[3] Z.Q. Chen, T. Kumagai and J. Wang, *Stability of heat kernel estimates for symmetric non-local Dirichlet forms*, Preprint 2016, available at `arXiv:1604.04035`.

Loop models related to quantum spin systems

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(joint work with Volker Betz and Johannes Ehlert)

We consider a class of probability models on graphs where Poisson point processes (PPP) on edges produce random geometric objects (loops) according to a set of rules. Similar models find their origin in the work of Tóth [4] and Aizenman and

Nachtergaele [1] who used these representations to study the quantum Heisenberg ferromagnet and antiferromagnet, respectively. These representations were combined and extended by Ueltschi [5] in order to study various quantum spin systems for spins $S \in \frac{1}{2}\mathbb{N}$. This model involved objects, called *links*, on edges that were either crosses or bars. Loops are constructed by attaching an interval $[0, \beta)$ to each edge and placing the links according to a PPP, we then follow the crosses and bars as in the example figure 1. Realisations are given a weighting of $\theta^{\#\text{loops}}$ for $\theta \geq 1$. It is shown that, in the thermodynamic limit, macroscopic loops will occur at sufficiently low temperature if we take our graph to be a d-dimensional box ($d \geq 3$) [5]. This result corresponds to the famous result of Dyson, Lieb and Simon [3] in the case $S = \frac{1}{2}$. Both these results rely on the methods of *reflection positivity* and *infrared bounds*.

Our current work in progress with Volker Betz and Johannes Ehlert considers these loop models on d-regular trees for $\theta > 1$. The case of $\theta = 1$ was previously studied by Björnberg and Ueltschi [2] where it was shown that there is a critical inverse temperature β_c such that, above this inverse temperature the loop from the root of the tree will reach infinitely far down the tree with positive probability and below this inverse temperature all loops will be finite. To first order in d^{-1} β_c is precisely the percolation threshold however the behaviour in order d^{-2} is different and depends on the relative intensity of the bars and crosses. We extend this result to the case $\theta > 1$ to first order in d^{-1} using some relatively simple estimates. The proof is remarkably simple given the *a priori* difficult problem of dealing with quantum spin systems and phase transitions. Perhaps more interestingly, we are also able to obtain results in the case of a supercritical Galton-Watson tree, provided the offspring distribution satisfies certain conditions. These conditions are satisfied by several common distributions.

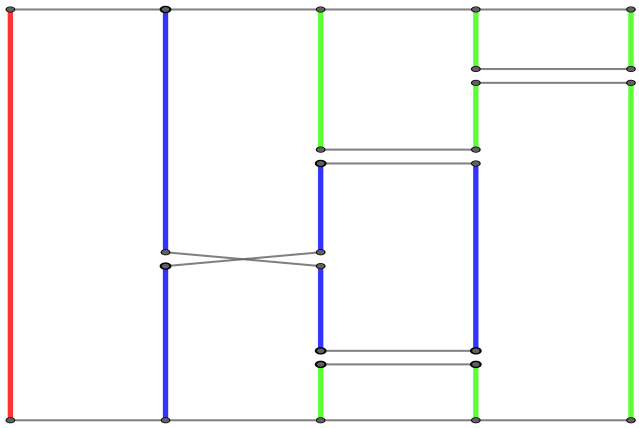

FIGURE 1. A simple example of a realisation, ω, with three loops.

References

[1] *Aizenman, M. and Nachtergaele, B.* Geometric aspects of quantum spin states. *Comm. Math. Phys.*, 164(1):17–63, 1994.

[2] *Björnberg, J. E. and Ueltschi, D.* Critical parameter of random loop model on trees. arXiv:1608.08473 2017.

[3] *Dyson, F. J., Lieb, E. H. and Simon, B.* Phase transitions in quantum spin systems with isotropic and nonisotropic interactions. *J. Stat. Phys.*, 18(4):335–383, 1978.

[4] *Tóth, B.* Improved lower bound on the thermodynamic pressure of the spin 1/2 Heisenberg ferromagnet. *Lett. Math. Phys.*, 28(1):75–84, 1993.

[5] *Ueltschi, D.* Random loop representations for quantum spin systems. *J. Math. Phys.*, 54, 083301, 2013; arXiv:1301.0811.

Metastability: a journey from probability to semi-classical analysis

Tony Lelièvre

(joint work with G. Di Gesù, D. Le Peutrec and B. Nectoux)

We present recent results [1] concerning the precise description of the exit event from a metastable state $D \subset \mathbb{R}^d$ for the overdamped Langevin dynamics:

$$dX_t = -\nabla f(X_t)\, dt + \sqrt{h}\, dW_t, \tag{1}$$

where $f : \mathbb{R}^d \to \mathbb{R}$ is a smooth function, and W_t is a d-dimensional Brownian motion. The objective is to show that, in the small temperature regime $h \to 0$, one can use a simple jump Markov model parameterized by the Eyring-Kramers formulas to describe the exit event (τ, X_τ) from D, where $\tau = \inf\{t > 0,\, X_t \notin D\}$.

In a jump Markov model, the exit event from a state is modeled as follows: (i) the residence time T is exponentially distributed: $T \sim \mathcal{E}(\sum_{i=1}^n k_i)$; (ii) the next visited state I is independent of T and (iii) the law of I is given by: $\forall i \in \{1, \ldots, n\}$, $\mathbb{P}(I = i) = \frac{k_i}{\sum_{j=1}^n k_j}$. Here $(k_i)_{i=1,\ldots,n}$ denote the rates associated with exits through one of the n possible exit events. In the framework of the harmonic transition state theory, these rates are parameterized as follows. One considers the local minima $(z_1, \ldots, z_n)$ of f on ∂D and the rates are defined by:

$$k_i = A_i e^{-\frac{2}{h}(f(z_i) - f(x_0))} \tag{2}$$

where x_0 is the unique global minimum of f in $\overline{D}$ and A_i is a prefactor which depends on the underlying dynamics. For example, for the overdamped Langevin dynamics (1), if z_i is a saddle point of f (which is indeed the case if D is the basin of attraction of x_0 for the gradient dynamics $\dot{x} = -\nabla f(x)$), then

$$A_i = \frac{|\lambda(z_i)|}{2\pi} \frac{\sqrt{\mathrm{detHess} f(x_0)}}{\sqrt{\mathrm{detHess} f(z_i)}}. \tag{3}$$

Such jump Markov models to describe exit events from metastable states are used to simulate metastable dynamics over very long times, and to accelerate the simulations of molecular dynamics trajectories, see for example [2, 3].

The question we would like to address is the following: is it possible to make a link between the law of the exit event (τ, X_τ) for the overdamped Langevin

dynamics (1) and the exit event (T, I) for the jump Markov model described above? The cornerstone of our analysis is the quasi-stationary distribution (QSD, denoted by ν in the following) which can be defined as the law of X_t conditioned to $\{t < \tau\}$, in the limit $t \to \infty$. The QSD thus describes the law of the process when it remains for a very long time in D before exiting: this is in essence the meaning of a metastable state. From a partial differential equation (PDE) viewpoint, the QSD writes

$$\nu = Z^{-1} u(x) e^{-\frac{2}{h} f(x)} \, dx,$$

where $Z = \int_D u(x) e^{-\frac{2}{h} f(x)} \, dx$ is the normalizing constant and u is the principal eigenfunction of the infinitesimal generator $L = -\nabla f \cdot \nabla + \frac{h}{2} \Delta$ with Dirichlet boundary conditions on ∂D:

$$\begin{cases} Lu = -\lambda u \text{ in } D, \\ u = 0 \text{ on } \partial D. \end{cases} \tag{4}$$

The domain D is assumed to be smooth and bounded. The operator $-L$ with Dirichlet boundary conditions on ∂D is positive and has compact resolvent: (λ, u) is its first eigenvalue-eigenfunction pair. An important property of the QSD which shows its interest with respect to our aim is the following: if X_0 is distributed according to the QSD ν in D, then (i) τ is exponentially distributed: $\tau \sim \mathcal{E}(\lambda)$; (ii) X_τ is independent of τ and (iii) the law of X_τ is given by: for any bounded measurable test function $\varphi : \partial D \to \mathbb{R}$,

$$\mathbb{E}(\varphi(X_\tau)) = \frac{\displaystyle\int_{\partial D} \varphi \, u \, e^{-\frac{2}{h} f} \, d\sigma}{\displaystyle\int_{\partial D} u \, e^{-\frac{2}{h} f} \, d\sigma}$$

where σ is the Lebesgue measure on ∂D. To make the connection with the jump Markov model above complete, it remains to show that the parameter λ and the law of X_τ can be related to the rates $(k_i)_{i=1,\dots,n}$. In [1], we prove the following result:

Theorem 1. *Let us assume that:*

- *The functions $f : \overline{D} \to \mathbb{R}$ and $f|_{\partial D} : \partial D \to \mathbb{R}$ are Morse functions. Moreover, $|\nabla f|(x) \neq 0$ for all $x \in \partial D$.*
- *The function f has a unique global minimum x_0 in $\overline{D}$ and $\min_{\partial D} f > f(x_0)$. Moreover, x_0 is the unique critical point of f in $\overline{D}$. The function $f|_{\partial D}$ has exactly n local minima $(z_i)_{i=1,\dots,n}$ which are assumed to be numbered such that*

$$f(z_1) \leq f(z_2) \leq \dots \leq f(z_n).$$

- *For all $x \in \partial D$, $\partial_n f(x) > 0$ (where ∂_n denotes the outward normal derivative to D).*
- *$f(z_1) - f(x_0) > f(z_n) - f(z_1)$ and for all $i \in \{1, \dots, n\}$,*

$$d_a(z_i, B_{z_i}^c) > \max(f(z_n) - f(z_i), f(z_i) - f(z_1)) \tag{5}$$

where d_a denotes the Agmon distance defined by

$$d_a(x,y) = \inf_{\substack{\gamma:[0,1]\to\overline{D} \text{ Lipschitz} \\ \text{and s.t. } \gamma(0)=x,\gamma(1)=y}} \int_0^1 g(\gamma(t))|\gamma'(t)|\,dt$$

with $g(x) = |\nabla f|(x)1_D(x) + |\nabla_T f|(x)1_{\partial D}(x)$, and $B_{z_i} \subset \partial D$ is the basin of attraction of z_i for the dynamics $\dot{x} = -\nabla_T f(x)$, where $\nabla_T f$ denotes the tangential gradient of f on ∂D.

Then, if $X_0 \sim \nu$, in the limit $h \to 0$,

$$\tau \sim \mathcal{E}\left(\sum_{j=1}^n \tilde{k}_j\right)$$

and for all $i \in \{1,\dots,n\}$, for all $\Sigma_i \subset \partial D$ such that $z_i \in \Sigma_i$ and $\overline{\Sigma_i} \subset B_{z_i}$,

$$\mathbb{P}(X_\tau \in \Sigma_i) = \frac{\tilde{k}_i}{\sum_{j=1}^n \tilde{k}_j}$$

where for $i \in \{1,\dots,n\}$,

$$\tilde{k}_i = \frac{\partial_n f(z_i)}{\sqrt{\pi h}} \frac{\sqrt{\det \mathrm{Hess} f(x_0)}}{\sqrt{\det \mathrm{Hess} f|_{\partial D}(z_i)}} e^{-\frac{2}{h}(f(z_i)-f(x_0))}(1+O(h)). \tag{6}$$

This result thus gives a first answer to the question raised above. The proof crucially relies on previous works by B. Helffer, F. Nier and J. Sjöstrand, see in particular [4]. Let us make a few remarks to conclude. First, the prefactor in the rates $\tilde{k}_i$ differ from the prefactors (3) since the local minima z_i are not saddle points of f in the geometric setting of Theorem 1. We are currently working on generalizations to deal with saddle points of f on ∂D. Second, we have checked numerically that the assumption (5) indeed seems necessary to get the correct prefactors, see [1]. Third, similar results have been obtained starting from a point $x \in D$ rather than from the QSD ν if $f(x)$ is sufficiently small, and also for more general subsets Σ of ∂D, see [1]. Fourth, our approach based on the QSD provides more precise results than those obtained using strandard techniques from large deviations [5] (in particular prefactors and error estimates).

Currently, we are working in three directions to extend these results: (i) considering saddle points of f on ∂D; (ii) working with the Langevin dynamics rather than the overdamped Langevin dynamics; (iii) studying the exit event for non reversible dynamics.

References

[1] G. Di Gesù, D. Le Peutrec, T. Lelièvre, B. Nectoux, Precise asymptotics of the first exit point density, `https://arxiv.org/abs/1706.08728` (2017).

[2] M. Sørensen, A. Voter, Temperature-accelerated dynamics for simulation of infrequent events, J. Chem. Phys. 112 (21) (2000) 9599–9606.

[3] T. Lelièvre, Mathematical foundations of accelerated molecular dynamics methods, `https://arxiv.org/abs/1801.05347` (2018).

[4] B. Helffer, F. Nier, Quantitative analysis of metastability in reversible diffusion processes via a Witten complex approach: the case with boundary, Mémoire de la Société mathématique de France 105 (2006) 1–89.
[5] M. Freidlin, A. Wentzell, Random Perturbations of Dynamical Systems, Springer-Verlag, 1984.

Measure-valued Pólya processes

Cécile Mailler
(joint work with Jean-François Marckert)

A Pólya urn is a Markov process describing the contents of an urn containing balls of different colours. In the d-colour case, the process $(U(n))_{n\geq 1}$ takes values in $\mathbb{N}_0^d$, and the i-th coordinate of $U(n)$ is the number of balls of colour i in the urn at time n. The process is defined by two parameters: the initial composition vector $U(0) \in \mathbb{N}_0^d$, and the $d \times d$ replacement matrix R with coefficients in $\mathbb{N}_0$. We denote by $R_1, \dots, R_d$ the lines of R. Given these two parameters, the process evolves as follows: given $U(n)$, we set $U(n+1) = U(n) + R_{\xi_{n+1}}$, where ξ_{n+1} is a random variable of distribution

$$\mathbb{P}_n(\xi_{n+1} = i) = \frac{U_i(n)}{\sum_{j=1}^d U_j(n)}.$$

In other words, at every discrete time step, we draw a ball uniformly at random in the urn (denote its colour by i), and replace it in the urn together with $R_{i,j}$ balls of colour j, for all $1 \leq j \leq d$.

A vast literature is dedicated to understanding the asymptotic behaviour of these urn processes; this talk focuses on the following law of large numbers proved by Athreya and Karlin in 1968:

Theorem 1 (Athreya and Karlin [1])**.** *Assume that $\sum_{j=1}^d U_i(0) > 0$, and that the replacement matrix R is irreducible, then, almost surely when $n \to \infty$, we have*

$$\frac{U(n)}{n} \to v,$$

where v is a left eigenvector of the replacement matrix R associated to its Perron-Frobenius eigenvalue.

This law of large numbers actually holds under weaker assumptions on R, and if the replacement matrix is re-sampled at every time step in an i.i.d. fashion (see Janson [7]). One can also prove convergence results about the fluctuations around this almost sure limit: the fluctuations can be Gaussian or not, depending on the spectral gap of R (also see Janson [7]).

In this talk, I present a very recent extension of the Pólya urn model to infinitely-many colours that we call "measure-valued Pólya processes" (MVPPs); this work was inspired by a series of papers by Bandyopatyay and Thacker [2, 3, 4] where a similar model is defined and studied. I show how MVPPs can be coupled with branching Markov chains (BMCs) on the random recursive tree (RRT), and how

we can use the ergodicity of the underlying Markov chain and the typical shape of the RRT to prove a (weak) law of large numbers for a wide class of MVPPs.

1. Model and main result

A measure-valued Pólya process is a Markov chain $(\mathcal{M}_n)_{n\geq 0}$ taking values in the set of measures on a Polish space $\mathcal{P}$ (the set of *colours*). It is defined by two parameters: the initial composition measure $\mathcal{M}_0$, and the replacement measures $(\mathcal{R}_x)_{x\in\mathcal{P}}$, being a family of measures on $\mathcal{P}$. Given $\mathcal{M}_n$, we set $\mathcal{M}_{n+1} = \mathcal{M}_n + \mathcal{R}_{\xi_{n+1}}$, where ξ_{n+1} is a random variable of distribution $\mathcal{M}_n/\mathcal{M}_n(\mathcal{P})$. One can think of ξ_{n+1} as the colour of the ball drawn at random in the urn at time $n+1$, although we allow the *balls* to be infinitesimal in this model.

The following definition of ergodicity is needed for our main result:

Definition. *A Markov chain $(W_n)_{n\geq 0}$ on $\mathcal{P}$ is (a_n, b_n)-ergodic if*

$$\frac{W_n - b_n}{a_n} \Rightarrow \gamma, \text{ in distribution when } n \to \infty,$$

and if the limit distribution γ does not depend on the distribution of W_0.

Theorem 2 ([8])**.** *Assume that*

(a) $0 < \mathcal{M}_0(\mathcal{P}) < \infty$,

(b) the family of measures $(\mathcal{R}_x)_{x\in\mathcal{P}}$ is a probability Kernel on $\mathcal{P}$,

(c) the Markov chain W of Kernel $(\mathcal{R}_x)_{x\in\mathcal{P}}$ is (a_n, b_n)-ergodic for some sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$, and

(d) for any sequence $(\varepsilon_n)_{n\geq 0}$ such that $\varepsilon_n = o(\sqrt{n})$ when $n \to \infty$, for all $w \in \mathbb{R}$, we have

$$\lim_{n\to\infty} \frac{b_{n+w\sqrt{n}+\varepsilon_n} - b_n}{a_n} =: g(w) \text{ and } \lim_{n\to\infty} \frac{a_{n+w\sqrt{n}+\varepsilon_n}}{a_n} =: f(w)$$

both exist. Then, in probability when $n \to \infty$, the MVPP $\mathcal{M}$ of replacement Kernel $\mathcal{R}$ satisfies

$$n^{-1}\mathcal{M}_n(a_{\log n} \ \cdot \ + b_{\log n}) \to \nu,$$

for the weak topology on the set of measures on $\mathcal{P}$, where ν is the distribution of $f(\Lambda)\Gamma + g(\Lambda)$, where $\Lambda \sim \mathcal{N}(0,1)$ and $\Gamma \sim \gamma$ are two independent random variables.

Remark. *Our model is indeed a generalisation of the d-colour case of Athreya and Karlin, and one can check that our result applies and gives that $U(n)/n \to v$ in probability (and not almost surely as proved by Athreya and Karlin).*

Remark. *The take-home message is that any ergodic Markov chain gives an example of a convergent MVPP. For example, the MVPP on $\mathbb{N}_0$ of replacement Kernel given by*

$$\mathcal{R}_x = \frac{\lambda}{\lambda + x\mu}\delta_{x+1} + \frac{x\mu}{\lambda + x\mu}\delta_{x-1},$$

for all $x \geq 1$ and $\mathcal{R}_0 = \delta_1$ corresponds to the discrete-time $M/M/\infty$ queue. Therefore, our main result applies as soon as $\lambda < \mu$ and the MVPP converges in probability as follows: $n^{-1}\mathcal{M}_n \to \gamma$, where

$$\gamma(x) = \frac{(\lambda/\mu)^x e^{-\lambda/\mu}}{x!}.$$

2. An almost surely convergent case

Theorem 3 ([8])**.** *Let $(\mathcal{M}_n)_{n\geq 0}$ be the MVPP on $\mathcal{P} = \mathbb{R}$ of replacement Kernel given by $\mathcal{R}_x = x + \Delta$ for all $x \in \mathbb{R}$, where Δ is a random variable of finite mean m and variance σ^2. Assume that there exists $\delta > 0$ such that $\mathbb{E}e^{\delta\Delta} < \infty$, then, almost surely when $n \to \infty$,*

$$n^{-1}\mathcal{M}_n(\sqrt{\log n}\ \cdot\ + m \log n) \to \mathcal{N}(0, m^2 + \sigma^2),$$

for the weak topology on the set of measures on $\mathbb{R}^d$.

Remark. *Note that Theorem 2 implies a weaker version of Theorem 3 where the stated convergence is in probability and not almost surely. The proof of Theorem 2 relies on a coupling with a branching Markov chain on the random recursive tree whereas the almost sure convergence of Theorem 3 is proved using martingale techniques. Such martingale techniques are standard on the literature; they are used to prove almost sure convergence of the profile of different random trees (see, e.g.* [5]*), and of the occupation measure of branching random walks on different random trees (see, e.g.* [6]*).*

Remark. *On a private communication, S. Janson informed us that the exponential moment condition is superfluous in Theorem 3 and that the result would hold just with the second moment assumption.*

3. Some open problems

I believe that the most urgent open problem about this model is to remove the "balance assumption", namely (b) in Theorem 2. Another interesting problem is to understand better which MVPPs converge almost surely, and which don't. In an ongoing work with D. Villemonais, we prove almost sure convergence for a large class of MVPPs, although our approach seems to be restricted to the case when the underlying Markov chain is $(1, 0)$-ergodic.

References

[1] K. Athreya and S. Karlin, *Embedding of urn schemes into continuous time Markov branching processes and related limit theorems*, The Annals of Mathematical Statistics, **39(6)** (1968), 1801–1817.

[2] A. Bandyopadhyay and D. Thacker, *On Pólya Urn Schemes with Infinitely Many Colors*, Bernoulli, **23(4B)** (2017), 3243–3267.

[3] A. Bandyopadhyay and D. Thacker, *Rate of convergence and large deviation for the infinite color Pólya urn schemes*, Statistics & Probability Letters, **92** (2014), 232–240.

[4] A. Bandyopadhyay and D. Thacker, *A New Approach to Pólya Urn Schemes and Its Infinite Color Generalization*, ArXiV:1606.05317.

[5] B. Chauvin, T. Klein, J.-F. Marckert and A. Rouault, *Martingales and profile of binary search trees*, Electronic Journal of Probability, **10** (2005), 420–435.
[6] E. Fekete, *Branching random walks on binary search trees: convergence of the occupation measure*, ESAIM: Probability and Statistics, **14** (2010), 286–298.
[7] S. Janson, *Functional limit theorems for multitype branching processes and generalized Pólya urns*, Stochastic Processes and Applications, **110(2)** (2004), 177–245.
[8] C. Mailler and J.-F. Marckert, *Measure-valued Pólya urn processes*, Electronic Journal of Probability, **22** (2017).

Metastability of the contact process on evolving scale-free networks

Peter Mörters
(joint work with Emmanuel Jacob and Amitai Linker)

The aim of this research project is to investigate the possible influence of time-variability of a network on transport or spreading processes taking place on the network. We present a particular example, namely the contact process acting on scale-free networks evolving by stationary vertex updating.

In our context an *evolving network* is a (random) family $(\mathcal{G}_t^N : t \geq 0, N \in \mathbb{N})$ of graphs with fixed vertex set $\{1, \dots, N\}$. Given the network, the *contact process* on $(\mathcal{G}_t^N : t \geq 0)$ is a time-inhomogeneous Markov process that can be defined as follows: Every vertex may be healthy or infected. Infected vertices infect healthy neighbours at rate λ and recover at rate one. When vertices recover they are again susceptible to infection.

We start the process with every vertex infected and ask for the size of the *extinction time* T, the first time of entry in the absorbing state when every vertex is healthy. We say that the system experiences *fast extinction* if, for some sufficiently small infection rate $\lambda > 0$, the expected extinction time is bounded by a power of the network size. We say that we have *slow extinction* if, for every infection rate $\lambda > 0$, the expected extinction time is at least exponential in the network size with high probability.

Slow extinction is a phenomenon of *metastability*, a physical system reaching its equilibrium very slowly because it spends a lot of time in states which are local energy minima, the so-called metastable states. Metastability in our model suggests, informally, that starting from all vertices infected the density of infected vertices is likely to decrease rapidly to a *metastable density*, and stay close to this density up to the exponential survival time of the infection. When the metastable density decays like $\lambda^{\xi+o(1)}$ we call ξ the *metastability exponent*. Our interest in metastability exponents stems from the fact that they reflect which is the optimal survival strategy for the infection.

Assume now that $\mathcal{G}_0^N$ is an inhomogeneous random graph, i.e. edges exist independently with the probability of an edge $\{i, j\}$ given as $\frac{1}{N}\, p(i/N, j/N) \wedge 1$ for a suitable kernel $p\colon (0,1] \times (0,1] \to (0, \infty)$. We focus on two universal types of kernel which produce scale-free networks, the *factor kernel* given by

$$p(x, y) = \beta x^{-\gamma} y^{-\gamma},$$

and the *preferential attachment kernel* given by

$$p(x,y) = \beta(x \wedge y)^{-\gamma}(x \vee y)^{\gamma-1},$$

for some $\beta > 0$ and $0 < \gamma < 1$. It is easy to see that the inhomogeneous networks with kernel p are scale free with *power-law exponent* $\tau = 1 + \frac{1}{\gamma}$.

For both kernels we have slow extinction of the contact process on the static network $\mathcal{G}_0^N$. This changes when the networks undergo a stationary dynamics. Each vertex i updates independently with rate

$$\kappa_i \;=\; \kappa_0 \left(\frac{N}{i}\right)^{\gamma\eta} \qquad \text{for } i \in \{1,\dots,N\},$$

where $\eta \in \mathbb{R}$ and $\kappa_0 > 0$ are fixed constants. When vertex i updates, every unordered pair $\{i,j\}$, for $j \neq i$ forms an edge with probability $p_{i,j}$, independently of its previous state and of all other edges. The remaining edges $\{k,l\}$ with $k,l \neq i$ remain unchanged.

The parameter $\eta \in \mathbb{R}$ regulates the speed of the network dynamics. When $\eta \to -\infty$ we slow it down and approach the static case, for $\eta = 0$ we have network and process dynamics on the same scale and for $\eta \to \infty$ we approach a mean-field scenario where edges are independently resampled whenever the infection wants to use them. Our main theorem describes the phases of the system in the case of fast network evolution, i.e. for $\eta \geq 0$.

Theorem. (a) *Suppose p is the factor kernel.*

(i) *If $0 \leq \eta < \frac{1}{2}$ and $\gamma < \frac{1}{3-2\eta}$, or if $\eta \geq \frac{1}{2}$ and $\gamma < \frac{1}{2}$, there is fast extinction.*

(ii) *If $0 \leq \eta < \frac{1}{2}$ and $\gamma > \frac{1}{3-2\eta}$, or if $\eta \geq \frac{1}{2}$ and $\gamma > \frac{1}{2}$, there is slow extinction and the metastability exponent is*

$$\xi \;=\; \begin{cases} \frac{2-2\gamma\eta}{3\gamma-2\gamma\eta-1} & \text{if } \gamma < \frac{2}{3+2\eta}, \\ \frac{\gamma}{2\gamma-1} & \text{if } \gamma > \frac{2}{3+2\eta}. \end{cases}$$

(b) *Suppose p is the preferential attachment kernel.*

(i) *If $\eta \geq \frac{1}{2}$ and $\gamma < \frac{1}{2}$, there is fast extinction.*

(ii) *If $0 \leq \eta < \frac{1}{2}$, or if $\eta \geq \frac{1}{2}$ and $\gamma > \frac{1}{2}$, there is slow extinction and the metastability exponent is*

$$\xi \;=\; \begin{cases} \frac{3-2\gamma-2\gamma\eta}{\gamma-2\gamma\eta} & \text{if } \eta < \frac{1}{2} \text{ and } 0 < \gamma < \frac{3}{5+2\eta}, \\ \frac{3-\gamma-2\gamma\eta}{3\gamma-2\gamma\eta-1} & \text{if } \eta < \frac{1}{2} \text{ and } \frac{3}{5+2\eta} < \gamma < \frac{1}{1+2\eta}, \\ \frac{1}{2\gamma-1} & \text{if } \frac{1}{1+2\eta} < \gamma. \end{cases}$$

The figure below shows the different phases in a diagram. Each phase of slow extinction corresponds to a different survival strategy for the contact process.

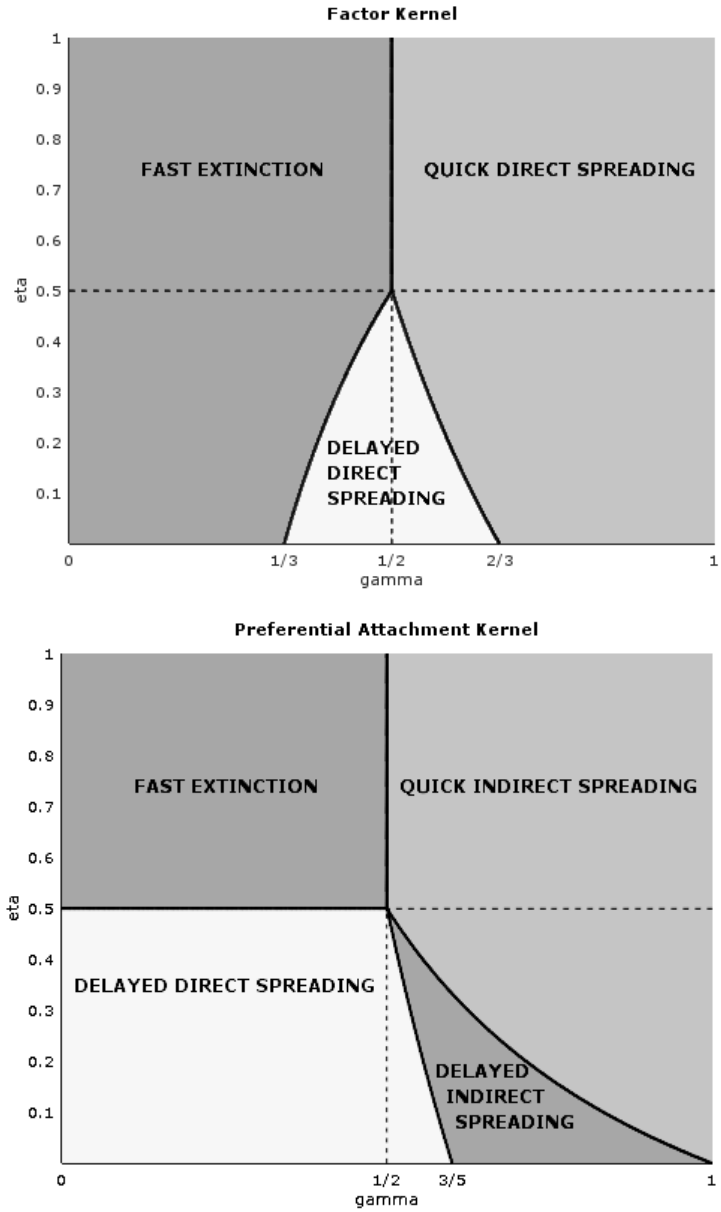

FIGURE 1. The figures summarise the theorem in the form of phase diagrams for the factor kernel (top) and the preferential attachment kernel (bottom).

The analogous problems for *slow dynamics*, i.e. the case $\eta < 0$, are not yet fully understood and subject of our ongoing research.

Stochastic homogenization of discrete energies with degenerate growth

Stefan Neukamm
(joint work with Mathias Schäffner, Anja Schlömerkemper)

Let $(\mathcal{L},\mathcal{E})$ denote a $\mathbb{Z}^d$-periodic, locally finite, connected graph with vertices $\mathcal{L}\subset\mathbb{R}^d$ and oriented edges $\mathcal{E}\subset\big(\mathcal{L}\times\mathcal{L}\big)\setminus\{e=[x,x]\,:\,x\in\mathcal{L}\}$ – for simplicity, set $\mathcal{L}:=\mathbb{Z}^d$ and $\mathcal{E}:=\{e=[x,x+e_i]\,:\,x\in\mathbb{Z}^d\}$ where $e_1,\dots,e_d$ denotes the canonical basis of $\mathbb{R}^d$. For a scaling parameter $0<\varepsilon\ll 1$ and a (macroscopic) domain $A\subset\mathbb{R}^d$ we consider the energy functional

$$H_\varepsilon(u):=\varepsilon^d\sum_{e\in\varepsilon\mathcal{E}\cap A}V(\tfrac{e}{\varepsilon},\nabla u(e)),$$

where $u:\varepsilon\mathcal{L}\to\mathbb{R}^n$ denotes a possibly vector-valued state variable and $\nabla u(e):=\frac{u(y_e)-u(x_e)}{|y_e-x_e|}$ denotes the discrete gradient of u at the edge $e=[x_e,y_e]\in\varepsilon\mathcal{E}$. Above, $V:\mathcal{E}\times\mathbb{R}^n\to(0,\infty)$ denotes a *random interaction potential* which we assume to be stationary and ergodic w.r.t. the action of $\mathbb{Z}^d$ on $\mathcal{E}$ by shifting.

Different discrete models of mechanics and physics can be phrased in this form, in particular:

- In the scalar case, i.e. co-dimension $n=1$, and for the quadratic potential $V(e,\xi):=\omega(e)|\xi|^2$ we recover the *random conductance model* with stationary and ergodic conductances $\{\omega(e)\in(0,\infty)\}_{e\in\mathcal{E}}$.
- In the vectorial case with $d=n\geq 2$, and non-convex potential $V(e,\xi):=k(e)(|\xi|-|e|)^2$ we recover a nonlinear elasticity model describing *a network of harmonic springs* with random spring constants $\{k(e)\in(0,\infty)\}_{e\in\mathcal{E}}$.

We are interested in the homogenization limit $\varepsilon\downarrow 0$ (in the sense of a discrete-to-continuum Γ-limit) in the case when the interaction potentials satisfy the *degenerate growth condition*

$$\forall e\in\mathcal{E},\xi\in\mathbb{R}^n\,:\qquad \lambda(e)(\tfrac{1}{c}|\xi|^p-c)\leq V(e,\xi)\leq c(\lambda(e)|\xi|^p+1),$$

where $1<p<\infty$ and $c\in\mathbb{R}$ are deterministic constants and $\lambda:\mathcal{E}\to(0,\infty)$ is a random, stationary & ergodic weight satisfying the moment condition

$$\forall e\in\mathcal{E}\,:\qquad \mathbb{E}[\lambda^\alpha(e)]+\mathbb{E}[\lambda^{-\beta}(e)]<\infty,$$

with exponents α,β satisfying as a minimal assumption the condition

$$1\leq\alpha\leq\infty,\qquad \frac{1}{p-1}\leq\beta\leq\infty. \tag{1}$$

The continuum limit invokes the deterministic energy density $W_{\rm hom}:\mathbb{R}^{n\times d}\to[0,\infty)$ defined by the multi-cell homogenization formula

$$W_{\rm hom}(F):=\lim_{k\to\infty}\mathbb{E}\Big[\inf_{\substack{\phi:\mathcal{L}\to\mathbb{R}^n\\ \phi\text{ is }k\mathbb{Z}^d\text{-periodic}}}\frac{1}{k^d}\sum_{e\in\mathcal{E}\cap[0,k)^d}V(e,\nabla(F+\phi)(e))\Big],$$

where $(F+\phi)$ stands short for the function $\mathcal{L}\ni x\mapsto Fx+\phi(x)\in\mathbb{R}^n$.

The problem of deriving continuum models from discrete models has a long tradition in rational mechanics and the calculus of variations, e.g. see [2, 1] for models describing elastic solids. In a similar spirit, in [4] we study the impact of degenerate growth. We make the following observations:

(a). The moment condition (1) is the minimal assumption required to ensure that W_{hom} satisfies a non-degenerate p-growth condition of the form

$$\forall F \in \mathbb{R}^{n\times d} \,:\, \frac{1}{c'}|F|^p - c' \leq W_{\mathrm{hom}}(F) \leq c'(|F|^p + 1),$$

see [4, Lemma 11,Remark 5].

(b). In the scalar case, i.e. for co-dimension $n = 1$, and under the assumption of a "convexity at ∞" assumption for V, we prove that the functional H_ε (almost surely) Γ-converges in the L^1-topology to the continuum, deterministic energy functional

$$H_{\mathrm{hom}}(u) := \int_A W_{\mathrm{hom}}(\nabla u), \tag{2}$$

see [4, Theorem 4]. Moreover, if the potential V is convex, the formula for W_{hom} simplifies to a single-cell homogenization formula

$$W_{\mathrm{hom}}(F) := \mathbb{E}\Big[\inf_{\substack{\phi:\mathcal{L}\to\mathbb{R}^n \\ \phi \text{ is } \mathbb{Z}^d\text{-periodic}}} \sum_{e\in\mathcal{E}\cap[0,1)^d} V(e, \nabla(F+\phi)(e))\Big].$$

For given F, the minimization problem for ϕ can be rephrased with help of the associated Euler-Lagrange equations. In particular, in the case of the random conductance model (i.e. if V is quadratic and convex) we recover the corrector problem of stochastic homogenization, see [4, Remark 4].

(c). In the vectorial, non-convex case, we need to replace (1) by the stronger moment condition

$$\alpha > 1, \qquad \frac{1}{\alpha} + \frac{1}{\beta} \leq \frac{p}{d}, \tag{3}$$

and prove Γ-convergence (in L^1) to the functional H_{hom} defined in (2), see [4, Theorem 4].

(d). The convergence statements of (b) and (c) can be combined with the following compactness statement (for sequences of functions $u_\varepsilon : \varepsilon\mathcal{L} \to \mathbb{R}^n$):

$$\begin{gathered} \text{If } u_\varepsilon \rightharpoonup u_0 \text{ weakly in } L^1 \text{ and } \limsup_{\varepsilon\downarrow 0} H_\varepsilon(u_\varepsilon) < \infty, \\ \text{then } u_\varepsilon \to u_0 \text{ strongly in } L^q, \end{gathered} \tag{4}$$

for all $1 \leq q < \infty$ satisfying

$$\begin{cases} \dfrac{1}{q} \geq \dfrac{\beta+1}{\beta}\dfrac{1}{p} - \dfrac{1}{d} & \text{if } \beta < \infty, \\ \dfrac{1}{q} > \dfrac{1}{p} - \dfrac{1}{d} & \text{if } \beta = \infty, \end{cases} \tag{5}$$

see [4, Lemma 6]. This observation allows to lift the convergence statements of (b) and (c) to Γ-convergence in L^q. Thus, a standard argument of Γ-convergence implies that Γ-convergence of H_ε is stable under (additive) perturbation by functionals that are continuous w.r.t. strong convergence in L^q, e.g. we may conclude that *almost) minimizers to the energy $H_\varepsilon(u) + \varepsilon^d \sum_{x\in\varepsilon\mathcal{L}\cap A} f_\varepsilon(x)\cdot u(x)$ with $f_\varepsilon \rightharpoonup f$ weakly in $L^{q'}(A)$, $q' = \frac{q}{q-1}$ converge in L^q to minimizers of $H_{\hom}(u) + \int_A f\cdot u$, see [4, Corollary 7].

(e). In the case of the random conductance model — i.e. V is quadratic & convex, $n = 1$, and $p = 2$ — it is especially interesting to recover compactness in the sense of (d) for $q = 2$. In view of (5) this leads to the moment condition

$$1 \le \alpha \le \beta, \qquad \frac{d}{2} \le \beta \le \infty.$$

In that case we recover that H_ε Γ-converges to $H_{\hom}$ in L^2, and in the stronger sense of Mosco convergence. As a consequence, by classical results, we obtain convergence (of finite dimensional distributions) of the associated evolution equation (L^2-gradient flow), and spectral convergence (i.e. convergence of the associated Eigenspaces).

(f). If we further restrict to the special case of the random conductance model on the nearest-neighbour lattice with i.i.d. conductances $\omega(e)$, $e \in \{[x, x+e_i] : x \in \mathbb{Z}^d, i = 1,\dots,d\}$, then we can relax the condition on β, see [4, Section 3.2]. In particular, in view of (e) we recover Mosco convergence in L^2 (for all dimensions $d \ge 2$) under the moment condition

$$\alpha = 1, \qquad \beta > \frac{1}{4}.$$

This is optimal in the sense that for $\beta < \frac{1}{4}$ Mosco convergence in L^2 breaks down due to the localization of eigenvalues in the limit $\varepsilon \downarrow 0$, see [3].

References

[1] R. Alicandro, M. Cicalese and A. Gloria, *Integral representation results for energies defined on stochastic lattices and application to nonlinear elasticity*, Arch. Ration. Mech. Anal., 200, 881-943, 2011.

[2] A. Braides and M. S. Gelli, *Limits of discrete systems with long range interactions*, J. Convex Anal., 9, 363-399, 2002.

[3] F. Flegel, *Localization of the principal Dirichlet eigenvector in the heavy-tailed random conductance model*, arXiv:1608:02415v1

[4] S. Neukamm, M. Schäffner, A. Schlömerkemper, *Stochastic homogenization of nonconvex discrete energies with degenerate growth*, SIAM Journal on Mathematical Analysis **49** (3), 1761-1809, 2017.

Fluid limit analysis for Hastings–Levitov planar growth

James Norris
(joint work with Vittoria Silvestri and Amanda Turner)

Conformal maps provide an effective way to encode subsets of the complex plane and thus to describe planar growth processes. We have considered in particular the map

$$F(z) = e^c z \exp\left\{\frac{2}{\gamma z - 1}\right\}$$

defined on $\{|z| > 1\}$. Here $c > 0$ and $\gamma = 1 + c + \sqrt{c^2 + 2c}$. Fix $\eta \in \mathbb{R}$ and $\sigma > 0$. Set $\Phi_0(z) = z$ and define recursively a sequence of random conformal maps on $\{|z| > 1\}$ as follows: given Φ_n, choose a random angle Θ_{n+1} such that

$$\mathbb{P}(\Theta_{n+1} \in d\theta | \mathcal{F}_n) \propto |\Phi_n'(e^{\sigma+i\theta})|^{-\eta} d\theta$$

and set

$$\Phi_{n+1} = \Phi_n \circ F_{\Theta_{n+1}}.$$

Here

$$F_\theta(z) = e^{i\theta} F(e^{-i\theta} z) = e^c z \exp\left\{\frac{2}{\gamma z e^{-i\theta} - 1}\right\}$$

and $\mathcal{F}_n = \sigma(\Theta_1, \ldots, \Theta_n)$. Write K_n for the complement of the range of Φ_n. Then $(K_n)_{n\geq 0}$ is a non-decreasing random sequence of compact sets in the plane. The set K_{n+1} may be considered as obtained from K_n as follows: first map the complementary domain D_n conformally to the reference domain D_0, then attach the particle

$$P_{n+1} = \{z \in D_0 : z \notin F_{\Theta_{n+1}}(D_0)\}$$

to K_0, then map back to D_n, thereby adding the new particle $\Phi_n(P_{n+1})$.

In the case $\eta = 0$, the new particle can be thought of as attached at a point chosen according to harmonic measure. This corresponds to the mechanism used for diffusion limited aggregation (DLA). The resulting dynamics do not behave like DLA, however, because the attached particles are distorted by the conformal map Φ_n, which has the effect of magnifying particles attached at points where the density of arc length with respect to harmonic measure is high – that is in the 'fjords' of the current cluster. On the other hand, particles attached at the tips of 'fingers', where harmonic measure is large, are scaled down. This introduces a negative feedback, which keeps the clusters disc-like, in contrast to the fractal behaviour seen in DLA.

The scaling limit $c \to 0$ with $cn \to t$ for $\eta = 0$ was analysed in [1]. Fluctuations around this limit were analysed in [2]. In the case $\eta = 0$, analysis is maded easier by the fact that, for all $z \in D_0$, the process $X_n = \Phi_n^{-1}(z)$ is Markov.

The parametrized family of models described above, for η in the range $[0, 1]$ allow the negative feedback to be progressively removed, with $\eta = 1$ thought to be the critical value where the scaling limit will cease to be a disc.

We have shown [3] that, for all $\eta \in [0, 1]$, for c small and $cn = t$, the cluster is close to a disc of radius e^t, just as in the case $\eta = 0$. The fluctuations around this

fluid limit remain Gaussian for $\eta \in [0,1)$ with a covariance structure depending on η which diverges as $\eta \to 1$.

References

[1] James Norris and Amanda Turner, *Hastings-Levitov aggregation in the small-particle limit*, Communications in Mathematical Physics **316** (2012), 809–841.

[2] Vittoria Silvestri *Fluctuation results for Hastings-Levitov planar growth*, Probability Theory and Related Fields, **167** (2015), 417–460.

[3] James Norris, Vittoria Silvestri and Amanda Turner, to appear.

Characterizing fluctuations in stochastic homogenization

Felix Otto

(joint work with Mitia Duerinckx)

Let $\boldsymbol{a}$ be a uniformly elliptic random coefficient field, which is stationary and ergodic. Given a macroscopic r.h.s. $f = \hat{f}(\frac{\cdot}{L})$, $\hat{f} \in C_0^\infty(\mathbb{R}^d)^d$ deterministic, we consider the equation

$$\nabla \cdot (\boldsymbol{a}\nabla u + f) = 0 \qquad \text{in } \mathbb{R}^d, \tag{1}$$

and we study macroscopic observables of the form $\int g \cdot \nabla u$ with $g = \hat{g}(\frac{\cdot}{L})$, $\hat{g} \in C_0^\infty(\mathbb{R}^d)^d$ deterministic. Qualitative homogenization theory states that almost surely $L^{-d}\int g\cdot\nabla u - L^{-d}\int g\cdot\nabla\bar{u} \to 0$ as $L \uparrow \infty$, where $\bar{u}$ solves the (deterministic) homogenized equation

$$\nabla \cdot (\bar{\boldsymbol{a}}\nabla \bar{u} + f) = 0 \qquad \text{in } \mathbb{R}^d,$$

where the homogenized coefficient $\bar{\boldsymbol{a}} \in \mathbb{R}^{d\times d}$ is given by $\bar{\boldsymbol{a}}e_i = \mathbb{E}\left[\boldsymbol{a}(e_i + \nabla\varphi_i)\right]$ in terms of the corrector φ, that is, the solution of $\nabla \cdot \boldsymbol{a}(e_i + \nabla\varphi_i) = 0$ in $\mathbb{R}^d$. A natural concept in homogenization is to compare u to its "two-scale expansion" $(1+\varphi_i\partial_i)\bar{u}$ (using Einstein's summation convention), which captures the oscillations of u to order $O(L^{-1})$, in the sense that the difference between the gradients is of (relative) order $O(L^{-1})$. Such expansions can be pursued to higher order: while φ is characterized by $(1+\varphi_i\partial_i)\bar{\ell}$ being $\boldsymbol{a}$-harmonic for all affine functions $\bar{\ell}$, the second-order corrector φ' (throughout the talk, a prime denotes a second-order object) is characterized by the property that $(1+\varphi_i\partial_i+\varphi'_{ij}\partial^2_{ij})\bar{q}$ is $\boldsymbol{a}$-harmonic for all $\bar{\boldsymbol{a}}$-harmonic quadratic polynomials $\bar{q}$. The second-order two-scale expansion $(1+\varphi_i\partial_i+\varphi_{ij}\partial^2_{ij})\bar{u}'$ then captures the oscillations of u at order $O(L^{-2})$, where $\bar{u}' := \bar{u} + \tilde{u}'$ with $\tilde{u}'$ given by $\nabla\cdot(\bar{\boldsymbol{a}}\nabla\tilde{u}' + \bar{\boldsymbol{a}}'_i\nabla\partial_i\bar{u}) = 0$ and where $\bar{\boldsymbol{a}}'_i \in \mathbb{R}^{d\times d}$ is the second-order homogenized coefficient, see below for a definition. While these error estimates are classical in the periodic setting, they also hold in the random setting for large enough dimension: $O(L^{-1})$ for $d > 2$, when φ is stationary; and $O(L^{-2})$ for $d > 4$, when φ' is stationary [3]. Here and in the following we assume that $\boldsymbol{a}$ has integrable correlations.

Periodic homogenization is about understanding the oscillations of u by means of two-scale expansions, random homogenization means in addition studying the

random fluctuations of the macroscopic observable $\int g \cdot \nabla u$. It was recently shown that the rescaled observable $L^{-d/2} \int g \cdot (\nabla u - \mathbb{E}[\nabla u])$ converges in law to a Gaussian. We may naturally look for a finer description of this convergence by means of a two-scale expansion. As first observed in [2], however, the limiting variance of $L^{-d/2} \int g \cdot \nabla u$ generically differs from that of $L^{-d/2} \int g \cdot \nabla (1 + \varphi_i \partial_i) \bar{u}$: when it comes to fluctuations, the two-scale expansion cannot be applied naively. In [1], we unravelled the mechanism behind this observation by means of the "homogenization commutator", which led to a new pathwise theory of fluctuations (see also the pathwise heuristics in [2]). In the present talk we explain how this approach naturally extends to higher orders, in parallel with the known theory of oscillations. For simplicity of exposition, we focus on second order, which is the relevant order for dimension $d = 3$.

Key is the homogenization commutator, which on first-order level takes the form

$$\Xi_k[u] := e_k \cdot (\boldsymbol{a} - \bar{\boldsymbol{a}}) \nabla u.$$

This expression is natural: H-convergence is equivalent to convergence of $L^{-d} \int g \cdot \Xi[u]$ to 0. This is made quantitative with help of the flux corrector, a skew-symmetric matrix field σ_i with $\boldsymbol{a}(e_i + \nabla \varphi_i) = \bar{\boldsymbol{a}} e_i + \nabla \cdot \sigma_i$. Indeed, Leibniz' rule yields $\Xi_k[u] = -\nabla \cdot ((\varphi_k^* \boldsymbol{a}^* + \sigma_k^*) \nabla u)$ for any $\boldsymbol{a}$-harmonic u, where φ_k^*, σ_k^* are the correctors for the pointwise transpose field $\boldsymbol{a}^*$. As the r. h. s. is in divergence form and φ_k^*, σ_k^* are stationary for $d > 2$, it is of order $O(L^{-1})$ with g. For a higher-order theory, we need a second-order extension of Ξ:

$$\Xi_k'[u] := e_k \cdot (\boldsymbol{a} - \bar{\boldsymbol{a}}) \nabla u + \bar{\boldsymbol{a}}_k^{*\prime} e_l \cdot \nabla \partial_l u,$$

which, for $\boldsymbol{a}$-harmonic u, indeed satisfies the corresponding identity $\Xi_k'[u] = \partial_l \nabla \big((\varphi_{kl}^{*\prime} \boldsymbol{a}^* + \sigma_{kl}^{*\prime}) \nabla u \big)$, where the r.h.s. is now of order $O(L^{-2})$ for dimension $d > 4$, when also $\varphi^{*\prime}, \sigma^{*\prime}$ are stationary. The identity follows from the characterizing property of φ', σ', and $\bar{a}'$, namely $(\phi_i a - \sigma_j) e_j = \bar{\boldsymbol{a}}_i' e_j - \boldsymbol{a} \nabla \varphi_{ij}' + \nabla \cdot \sigma_{ij}'$, which also yields $\bar{\boldsymbol{a}}_i' e_j = \mathbb{E}\left[(\phi_i a - \sigma_j) e_j + \boldsymbol{a} \nabla \varphi_{ij}' \right]$. Next, we define suitable two-scale expansions of these objects. For the first order, we simply inject the first-order two-scale expansion of ∇u into $\Xi[\cdot]$ and set

$$\Xi_k^\circ[\bar{u}] := e_k \cdot (\boldsymbol{a} - \bar{\boldsymbol{a}})(e_i + \nabla \varphi_i) \partial_i \bar{u},$$

which alternatively is characterized by $\Xi^\circ[\bar{u}](x) = \Xi[(1 + \varphi_i \partial_i) T_x \bar{u}](x)$, where $T_x \bar{u}$ denotes the first-order Taylor polynomial of $\bar{u}$ at x. For the second order, we similarly define

$$\Xi^{\circ\prime}[\bar{u}'](x) := \Xi[(1 + \varphi_i \partial_i + \varphi_{ij} \partial_{ij}) T_x' \bar{u}'](x),$$

where $T_x' \bar{u}'$ is the second-order Taylor polynomial of $\bar{u}'$ at x. The above defined $\Xi[\cdot]$ and $\Xi^\circ[\cdot]$ (resp. $\Xi'[\cdot]$ and $\Xi^{\circ\prime}[\cdot]$) are viewed as a first-order (resp. second-order) differential operators with (distributional) stationary random coefficients.

Theorem. *It holds*

$$\mathrm{Var}\left[L^{-\frac{d}{2}}\int g\cdot\nabla u - L^{-\frac{d}{2}}\int \nabla\bar v'\cdot\Xi'[u]\right]^{\frac12}$$
$$+\,\mathrm{Var}\left[L^{-\frac{d}{2}}\int g\cdot\Xi'[u] - L^{-\frac{d}{2}}\int g\cdot\Xi^{\circ\prime}[\bar u']\right]^{\frac12} \lesssim_{\hat f,\hat g} \begin{cases} L^{-\frac32}\log L & : \ d=3;\\ L^{-2}\log L & : \ d=4;\\ L^{-2} & : \ d>4;\end{cases}$$

where $\bar v' := \bar v + \tilde v'$ *with* $\nabla\cdot(\bar{\boldsymbol a}^*\nabla\bar v + g) = 0$ *and* $\nabla\cdot(\bar{\boldsymbol a}^*\nabla\tilde v' + \bar{\boldsymbol a}_i^{*\prime}\nabla\partial_i\bar v) = 0$.

The above result splits into two parts: 1) The fluctuations of macroscopic observables can be recovered from those of $\Xi'[\cdot]$ by a suitable Helmholtz-type projection with an error of order $O(L^{-\frac{d}{2}})$ up to logarithmic corrections (the stated estimate saturates at $d=4$, starting from $d>4$, third-order correctors should be taken into account and so forth). 2) The second-order two-scale expansion $\Xi^{\circ\prime}[\cdot]$ of the homogenization commutator $\Xi'[\cdot]$ is accurate in the fluctuation scaling at order $O(L^{-\frac{d}{2}})$. We focus here on the second part, the first part follows from a direct computation. Combining the two parts leads to a second-order pathwise theory of fluctuations: the fluctuations of all macroscopic observables are almost surely determined up to order $O(L^{-\frac{d}{2}}\log L)$ (here only for $d\le 4$) by the fluctuations of the new intrinsic quantity $\Xi^{\circ\prime}[\cdot]$. In dimension $d=3$, the above yields a full pathwise description of the fluctuations of $L^{-d}\int g\cdot\nabla u$ with accuracy $O(L^{-d}\log L)$, that is, the square of the CLT scaling! In upcoming work we establish this result in any dimension, and that fluctuations of $\Xi^{\circ\prime}[\cdot]$ are asymptotically Gaussian.

For the proof, we focus on the model setting $\boldsymbol a(x) := h(G(x))$ for some smooth map h and Gaussian random field G with integrable covariance function, in which case a Malliavin calculus is available on the probability space and substantially simplifies the analysis. In particular, for any random variable X, a Poincaré inequality holds in the form $\mathrm{Var}[X]\le C\mathbb{E}\big[\int|\delta X/\delta\boldsymbol a|^2\big]$, where $\delta X/\delta\boldsymbol a$ denotes the functional (Malliavin) derivative of X with respect to $\boldsymbol a$. Key is a representation formula for the infinitesimal variation of the two-scale expansion error $\Xi'[u]-\Xi^{\circ\prime}[\bar u']$. We start with the infinitesimal variation of $\Xi'[u]$:

$$\delta\Xi_k'[u] = (e_k+\nabla\varphi_k^*)\cdot\delta\boldsymbol a\nabla u - \partial_l\big((\varphi_k^* e_l+\nabla\varphi_{kl}^{*\prime})\cdot\delta\boldsymbol a\nabla u\big)$$
$$+\,\partial_l\nabla\cdot\big((\varphi_{kl}^{*\prime}\boldsymbol a+\sigma_{kl}^{*\prime})\nabla\delta u\big) + \partial_l\nabla\cdot(\varphi_{kl}^{*\prime}\delta\boldsymbol a\nabla u\big). \tag{2}$$

We argue that the last two terms lead to a contribution of order $O(L^{-2})$. First note that (1) yields $\nabla\cdot(\boldsymbol a\nabla\delta u+\delta\boldsymbol a\nabla u)=0$, so that $\nabla\delta u$ essentially behaves like $\delta\boldsymbol a\nabla u$, and hence we may focus on the last term in (2). Applying Poincaré's inequality to $X := L^{-d/2}\int g\cdot\Xi'[u]$, its contribution is estimated by $L^{-d}\mathbb{E}\big[\int|\nabla^2 g|^2|\varphi^{*\prime}|^2|\nabla u|^2\big]$. Using the stationarity of the corrector $\varphi^{*\prime}$ for $d>4$ and the equation for u, this is essentially estimated by $L^{-d}\int|\nabla^2 g|^2|f|^2 \lesssim_{\hat f,\hat g} (L^{-2})^2$ as claimed. The only important terms in (2) are thus the first two. Next, applying identity (2) to the two-scale expansions $(1+\varphi_i\partial_i)\bar\ell$ and $(1+\varphi_i\partial_i+\varphi_{ij}'\partial_{ij})\bar q$ with first- and second-order

polynomials $\bar{\ell}$ and $\bar{q}$, and suitably arranging the terms, we find

$$\delta\Xi_k^{\circ\prime}[\bar{u}'] = (e_k + \nabla\varphi_k^*)\cdot\delta\boldsymbol{a}\nabla(1+\varphi_i\partial_i+\varphi_{ij}'\partial_{ij})\bar{u}' \\ - \partial_l\big((\varphi_k^* e_l + \nabla\varphi_{kl}^{*\prime})\cdot\delta\boldsymbol{a}\nabla(1+\varphi_i\partial_i)\bar{u}\big) + O(L^{-2}).$$

Subtracting this identity from (2), and recognizing the two-scale errors $\nabla u - \nabla(1+\varphi_i\partial_i)\bar{u} = O(L^{-1})$ and $\nabla u - \nabla(1+\varphi_i\partial_i+\varphi_{ij}'\partial_{ij})\bar{u}' = O(L^{-2})$, the conclusion follows in the form $\mathrm{Var}\left[L^{-d/2}\int g\cdot(\Xi'[u]-\Xi^{\circ\prime}[\bar{u}'])\right] \lesssim_{\hat{f},\hat{g}} (L^{-2})^2$.

References

[1] M. Duerinckx, A. Gloria, F. Otto, *The pathwise structure of fluctuations in stochastic homogenization*, arXiv:1602.01717v3.

[2] Y. Gu, J.-C. Mourrat, *Scaling limit of fluctuations in stochastic homogenization*, Multiscale Model. Simul., **14** (2016), 452–481.

[3] P. Bella, B. Fehrman, J. Fischer, F. Otto, *Stochastic homogenization of linear elliptic equations: higher-order error estimates in weak norms via second-order correctors*, SIAM J. Math. Analysis, **49**, 2016.

Large deviations for reaction fluxes

Robert I. A. Patterson

(joint work with D. R. Michiel Renger)

Mean field particle systems are a common model for chemical reactions in well mixed containers. Molecules (and ions, ...) are modelled as particles and jumps model reactions in which atoms are reorganised into new molecules. The model is then a family of Markov processes indexed by $V > 0$, which may heuristically be understood as size of the well mixed container. One studies the empirical measure of the particle system, which should be identified with the concentration vector. The innovation in the work reported here and presented in more detail in [1] is to study reaction fluxes and not just concentrations. The reaction fluxes are the (rescaled) reaction counts and so the initial condition and the fluxes imply the state of the particle system, but they contain more information since multiple sequences of reactions may produce the same change in concentrations.

1. The stochastic model

To make ideas precise, let $\mathcal{Y}$ be the set of possible molecules, for example $\mathcal{Y} = \{\mathrm{H_2O}, \mathrm{H_2}, \mathrm{O_2}\}$. For fixed V the particle system state can be represented by a concentration vector

$$c = (c_y)_{y\in\mathcal{Y}} \in l^1_{\geq 0}(\mathcal{Y}), \quad c_y = \frac{1}{V}\#\left\{\text{particles of type } y\right\},$$

where $l^1_{\geq 0}$ is the space of non-negative, summable sequences, which can be identified with the space of finite measures on an underlying space, here $\mathcal{Y}$. Let $\mathcal{R}$ be the reaction set, for example $\mathcal{R} = \{(2\mathrm{H_2O} \to 2\mathrm{H_2} + \mathrm{O_2}), (2\mathrm{H_2} + \mathrm{O_2} \to 2\mathrm{H_2O})\}$ and let the rate of reaction r be $V\bar{k}^{(r)}(c)$ whenever the system state is c. In fact,

once should take rates $\bar{k}^{(r,V)}$ such that $\frac{1}{V}\bar{k}^{(r,V)} \to \bar{k}^{(r)}$ locally uniformly, for details see [1], but this detail is ignored in the interests of brevity.

Like the concentrations, the fluxes can be represented by a vector:

$$w(t) = \left(w^r(t)\right)_{r\in\mathcal{R}} \in l^1_{\geq 0}(\mathcal{R}), \quad w^r(t) = \frac{1}{V}\#\left\{\text{occurrences of reaction } r \text{ in } (0,t]\right\}.$$

Finally write $\gamma^{(r)}$ for the vector of molecules consumed and created by a single instance of reaction r so that we have Markov jump processes with generators $Q^{(V)}$ acting on bounded measurable test functions $\phi\colon l^1(\mathcal{Y}) \times l^1(\mathcal{R}) \to \mathbb{R}$ as

$$\left(Q^{(V)}\phi\right)(c,w) = V\sum_{r\in\mathcal{R}} \bar{k}^{(r)}(c)\left[\phi\left(c + \frac{1}{V}\gamma^{(r)}, w + \frac{1}{V}1_r\right) - \phi(c,w)\right], \tag{1}$$

where 1_r is the vector with 1 at position r and 0 elsewhere. Under suitable assumptions on the $\bar{k}^{(r)}$, which ensure that the processes cannot blow up these generators define laws $\mathbb{P}^{(V)}$ on càdlàg, bounded variation paths from compact time intervals $[0,T]$ into $l^1(\mathcal{Y}) \times l^1(\mathcal{R})$.

2. Results

A functional law of large numbers for the concentration process (not the fluxes) goes back to Kurtz [2]. The present work incidentally extends this to the combined concentration and flux process, showing that, provided the initial conditions converge weakly the $\mathbb{P}^{(V)}$ converge weakly to the measure concentrated on the (unique) solution to

$$\dot{c}(t) = \Gamma\dot{w}(t), \quad \dot{w}(t) = \left(k^{(r)}\left(c(t)\right)\right)_{r\in\mathcal{R}} \tag{2}$$

with the limiting initial condition and where $\Gamma w := \sum_{r\in R}\gamma^{(r)}w^{(r)}$.

The main result of this work is that, provided the initial concentrations satisfy a sufficiently regular LDP on the scale V with rate function $\mathcal{I}_0$, the $\mathbb{P}^{(V)}$ also satisfy a large deviation principle on the scale V with rate functional $\mathcal{J}(c,w)$ given by

$$\begin{aligned}\mathcal{I}_0\left(c(0)\right) + \sup_{\zeta\in C^1_c((0,T);l^\infty(\mathcal{R}))}\int_0^T\left(\zeta(t)\cdot\dot{w}(t) - \sum_{r\in\mathcal{R}}k^{(r)}\left(c(t)\right)\left[e^{\zeta^{(r)}} - 1\right]\right)\mathrm{d}t \\ = \mathcal{I}_0\left(c(0)\right) + \int_0^T\sum_{r\in\mathcal{R}}\left(\dot{w}^{(r)}(t)\log\left(\frac{\dot{w}^{(r)}(t)}{k^{(r)}\left(c(t)\right)}\right) - \dot{w}^{(r)}(t) + k^{(r)}\left(c(t)\right)\right)\mathrm{d}t\end{aligned} \tag{3}$$

when (c,w) is absolutely continuous with $\Gamma\dot{w} \equiv c$ and otherwise $+\infty$.

An application of the contraction principle yields as a corollary an LDP for the concentration process extending the range of validity of LDPs from [3] and [4].

3. Remarks

The initial large deviation is a technical necessity for the current proof and cannot be trivial, but small fluctuations of the initial condition must be possible for finite V.

The proofs are currently presented for $\mathcal{Y}$ and $\mathcal{R}$ finite, but under modest assumptions an extension to the countable setting, for example for pure coagulation, is possible.

This appears to be the first use of fluxes in studying LDPs for general models of chemical reaction systems. For the special case of reactions that only involve one particle changing its type fluxes were studied by Renger [6] and Kraaij [5]. For further references on the use of fluxes in an LDP setting the reader is referred to [1] as well as to the abstracts by Bertini and Faggionato in this volume.

References

[1] R. Patterson and M. Renger, *Large deviations of reaction fluxes*, arxiv:1802.02512 (2018).

[2] T. G. Kurtz, *The relationship between stochastic and deterministic models for chemical reactions.*, J. Chem. Phys. **57** (1972), 2976–2978.

[3] P. Dupuis *et. al.*, *Large deviation principle for finite-state mean field interacting particle systems*, arxiv:1601.06219 (2016).

[4] A. Agazzi *et. al.*, *Large deviations theory for Markov jump models of chemical reaction networks*, arxiv:1701.02126 (2017).

[5] R. C. Kraaij, *Flux large deviations of weakly interacting jump processes via well-posedness of an associated Hamilton-Jacobi equation*, arxiv:1711.00274 (2017).

[6] D. R. M. Renger *Flux large deviations of independent and reacting particle systems, with implications for macroscopic fluctuation theory.* WIAS Preprint 2375, (2017).

Homogenization of convolution type periodic operators

Andrey Piatnitski

(joint work with Elena Zhizhina, Moscow)

The talk focuses on homogenization problem for the operator

$$\mathcal{L}^\varepsilon u(x) = \frac{1}{\varepsilon^{d+2}} \int\limits_{\mathbb{R}^d} \lambda\big(\frac{x}{\varepsilon}\big)\mu\big(\frac{y}{\varepsilon}\big)a\big(\frac{x-y}{\varepsilon}\big)\big(u(y)-u(x)\big)\,dx$$

in $L^2(\mathbb{R}^d)$, $d \ge 1$. Here ε is a small positive parameter, $\mu(z)$ and $\lambda(z)$ are periodic functions in $\mathbb{R}^d$ with period one in each coordinate direction such that

$$0 < \Lambda^- \le \lambda(z),\ \mu(z) \le \Lambda^+.$$

The function $a(z)$ possesses the following properties:

$$a(z) = a(-z), \qquad a(z) \ge 0, \qquad a \in L^1(\mathbb{R}^d) \cap L^2_{\mathrm{loc}}(\mathbb{R}^d),$$

$$\int_{\mathbb{R}^d} a(z)\,dz = 1, \qquad \int_{\mathbb{R}^d} |z|^2 a(z)\,dz \le +\infty.$$

Under these conditions $\mathcal{L}^\varepsilon$ is the generator of a continuous time jump Markov process in a periodic environment.

Lemma. *For any $m > 0$ and any $f \in L^2(\mathbb{R}^d)$ the equation $(m - \mathcal{L}^\varepsilon)u^\varepsilon = f$ has a unique solution. Moreover, $\|u^\varepsilon\|_{L^2(\mathbb{R}^d)} \le \frac{c}{m}\|f\|_{L^2(\mathbb{R}^d)}$ with a constant c that does not depend on ε.*

We turn to our homogenization result.

Theorem. *There exists a symmetric positive definite matrix* Θ *such that for any* $m > 0$ *and any* $f \in L^2(\mathbb{R}^d)$ *the solution* u^ε *of equation* $(m - \mathcal{L}^\varepsilon)u^\varepsilon = f$ *converges in* $L^2(\mathbb{R}^d)$*, as* $\varepsilon \to 0$*, to the solution* u^0 *of equation* $\big(m - \Theta^{ij}\frac{\partial^2}{\partial x_i \partial x_j}\big)u^0 = f$.

The matrix Θ can be constructed in terms of solutions of auxiliary periodic problems.

The results presented here can be found in the paper [1].

References

[1] A. Piatnitski; E. Zhizhina, Periodic homogenization of nonlocal operators with a convolution-type kernel. *SIAM J. Math. Analysis*, **49**(1), (2017), 64–81.

A non-local Fokker-Planck equation related to nucleation and coarsening

André Schlichting

(joint work with Joseph G. Conlon)

We shall be concerned with a non-linear non-local problem associated to the Fokker-Planck equation on the half line $\mathbf{R}^+ = [0, \infty)$,

$$\partial_t c(x,t) + \partial_x\big(a(x)\big(\theta(t)W'(x) - V'(x)\big)c(x,t)\big) \;=\; \partial_x^2\big(a(x)c(x,t)\big)\;. \tag{1}$$

We shall assume that a is differentiable and strictly positive, $V, W \in \mathcal{C}^1$ and $\theta : [0,T] \to \mathbf{R}$ is continuous. Together with suitable Dirichlet boundary condition and a conservation law, the evolution (1) may be considered a continuous version of the discrete Becker-Döring model [2].

At this point, the equation (1) is a Fokker-Planck equation with time and space dependent coefficients. In particular, if the function $\theta(\cdot)$ is constant $\theta(\cdot) \equiv \theta$, then $c(x,t) = c_\theta^{\mathrm{eq}}(x)$ with $c_\theta^{\mathrm{eq}}(x) = a(x)^{-1}\exp\big(-V(x) + \theta W(x)\big)$, is a steady state solution of (1). In the problem we study here $\theta(\cdot)$ is non-constant in time and is determined by the conservation law

$$\theta(t) + \int_0^\infty W(x)c(x,t)\,\mathrm{d}x = \;\rho\;, \qquad \text{where } \rho > 0 \text{ is constant.} \tag{2}$$

In the application of this model to coarsening, θ models the gaseous phase and c is the volume cluster density, the constraint (2) corresponds to the conservation of total mass and makes the Fokker-Planck equation non-local and non-linear. Additionally, we impose a Dirichlet boundary condition which is consistent with the requirement that $c_\theta^{\mathrm{eq}}(x)$ is a stationary solution to (1). The Dirichlet condition is therefore given by

$$c(0,t) \;=\; c_{\theta(t)}^{\mathrm{eq}}(0) \;=\; a(0)^{-1}\exp\big(-V(0) + \theta(t)W(0)\big)\;, \quad t > 0\;. \tag{3}$$

It turns out that the above Dirichlet condition (3) is also thermodynamic consistent, since the system (1), (2), (3) has a free energy functional acting as Lyapunov function for the evolution.

To specify the long-time limit, we observe that if W is assumed to be a positive function such that $\rho_s = \int_0^\infty W(x)a(x)^{-1}\exp[-V(x)]\,\mathrm{d}x < \infty$, then $W(\cdot)c_\theta^{\mathrm{eq}}(\cdot)$ is integrable for $\theta \le 0$. Furthermore, the function $\theta \mapsto \theta + \|W(\cdot)c_\theta^{\mathrm{eq}}(\cdot)\|_1$ is strictly increasing and maps $(-\infty, 0]$ to $(-\infty, \rho_s]$. We denote by $\theta_{\mathrm{eq}}(\cdot)$ the inverse function with domain $(-\infty, \rho_s]$. Evidently $\theta_{\mathrm{eq}}(\rho_s) = 0$, and so we may extend $\theta_{\mathrm{eq}}(\cdot)$ in a continuous way to have domain $\mathbf{R}$ by setting $\theta_{\mathrm{eq}}(\rho) = 0$ for $\rho > \rho_s$.

For the specific set of assumptions, we refer to [5, Assumption 1.1] and illustrate here an admissible set of assumptions on a, V, W in terms of power laws

$$W(x) = (1+x)^\kappa \qquad a(x) = (1+x)^\alpha \qquad \text{and} \qquad V(x) = (1+x)^\gamma. \tag{4}$$

The admissible range of exponents is given by

$$0 < \kappa \le 2, \qquad \max\{2-2\kappa, 0\} \le \alpha \le 2-\kappa \qquad \text{and} \qquad 0 < \gamma < \min\{2-\alpha, \kappa\}\ .$$

Under the above set of assumptions, we can state the first main result of the presented work [5] on the well-posendess and convergence to equilibrium, which can be seen as the analog of the one of [1] for the Becker-Döring model.

Theorem 1. *Let $c(x,0)$, $x > 0$, be a non-negative measurable function such that*

$$\int_0^\infty W(x)c(x,0)\,\mathrm{d}x \ < \ \infty\ . \tag{5}$$

Then there exists a unique solution $c(\cdot,t)$, $t > 0$, to the Cauchy problem (1), (2), (3) with initial condition $c(\cdot,0)$.

For all $t > 0$ the function $c(\cdot,t) \in \mathcal{C}^1([0,\infty))$ and $\theta \in \mathcal{C}^1([0,\infty))$.

For any $L > 0$ the solution $c(\cdot,t)$ converges uniformly on the interval $[0,L]$ as $t \to \infty$ to the equilibrium $c_\theta^{\mathrm{eq}}(\cdot)$ with $\theta = \theta_{\mathrm{eq}}(\rho)$. If $\rho \le \rho_s$ then also

$$\lim_{t\to\infty} \int_0^\infty W(x)|c(x,t) - c_\theta^{\mathrm{eq}}(x)|\,\mathrm{d}x \ = \ 0\ . \tag{6}$$

In addition to the well-posedness, we derive in the subcritical case $\rho < \rho_s$ a quantified rate of convergence to equilibrium. The proof relies on the entropy method and the convergence statement is shown with respect to a free energy, which is decreasing along the solution and adapts ideas for the proof of convergence established for the Becker-Döring model [9, 3], but also for gradient-flows with constraints from [6]. The following energy dissipation estimate is deduced

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\mathcal{G}\big(c(\cdot,t),\theta(t)\big) \le -\mathcal{D}\big(c(t),\theta(t)\big)\ , \tag{7}$$

where $\mathcal{G}$ is a suitable free energy and $\mathcal{D}$ is a dissipation funtional. The free energy $\mathcal{G}$ is proven to be convex with a unique minimizer satisfying the constraint (2) given by $\mathcal{G}(c^{\mathrm{eq}}_{\theta_{\mathrm{eq}}}, \theta_{\mathrm{eq}})$, with $\theta_{\mathrm{eq}} = \theta_{\mathrm{eq}}(\rho)$ as before. This allows to define the normalized free energy functional

$$\mathcal{F}_\rho(c) = \mathcal{G}(c,\theta) - \mathcal{G}(c^{\mathrm{eq}}_{\theta_{\mathrm{eq}}}, \theta_{\mathrm{eq}}) \qquad \text{with} \qquad \theta = \rho - \int W(x)c(x)\,\mathrm{d}x. \tag{8}$$

Therewith, we can state the second main result of [5] on the rate of convergence.

Theorem 2. *Let $\rho < \rho_s$. In addition, assume for some $\beta \in (0,1]$ and constants $0 < c_0 < C_0 < \infty$ holds*

$$c_0 W^{1-\beta}(x) \le a(x) W'(x)^2 \qquad \text{for } x \in \mathbf{R}^+. \tag{9}$$

Let c be a solution to (1), (2), (3) with initial condition $c(\cdot,0)$ satisfying (5) and for some C_0 and $k > 0$ the moment condition

$$\int W(x)^{1+k\beta}\, c(x,0)\,\mathrm{d}x \le C_0, \tag{10}$$

Then there exists λ and C depending on $a, V, W, \theta_{\mathrm{eq}}, C_0, k$ such that for all $t \ge 0$

$$\mathcal{F}_\rho(c(t)) \le \frac{1}{(C+\lambda t)^k}\ .$$

Moreover, if (9) holds with $\beta = 0$, that is $c_0 W(x) \le a(x) W'(x)^2 \le C_0 W(x)$ for $x \in \mathbf{R}^+$, then there exists $C > 0$ and $\lambda > 0$ such that

$$\mathcal{F}_\rho(c(t)) \le C e^{-\lambda t}.$$

By a suitable Pinsker inequality, the convergence of Theorem 2 also implies the quantified version of the statement (6) of Theorem 1 as well as a quantified convergence statement for $\theta(t)$. Let us emphasize, that the rates given in (4) satisfy the refined assumption (9) with $\beta = \frac{2-\alpha-\kappa}{\kappa} \in [0,1]$.

In future work the connection of the evolution in the super critical case and Lifshitz-Slyozov-Wagner model of coarsening [8, 14] will be investigated. This will continue the studies along the lines of [11, 7, 10, 4]. In the present situation, we plan to take advantage of the the variational structure based on the gradient flow formulation similar as it is done in [12] for the Becker-Döring model.

References

[1] Ball, J.M.; Carr, J.; Penrose, O. The Becker Döring cluster equations: basic properties and asymptotic behavior of solutions. *Comm. Math. Phys.* **104** (1986), 657-692, MR 0841675.

[2] Becker, R.; Döring, W. Kinetische Behandlung der Keimbildung in übersättigten Dämpfen. *Ann. Der Phys.* **24** (1935) 719-752.

[3] Cañizo, J.A.; Einav, A.; Lods, B. Trend to Equilibrium for the Becker-Döring Equations: An Analogue of Cercignanis Conjecture. *Anal. PDE. 10* **10** (2017), 1663-1708.

[4] Conlon, J.G. On a Diffusive Version of the Lifschitz-Slyozov-Wagner Equation, *J. Nonlinear Sci.* **20** (2010) 463-521. doi:10.1007/s00332-010-9065-y.

[5] Conlon, J.G.; Schlichting, A. A non-local problem for the Fokker-Planck equation related to the Becker-Döring model. *arXiv: 1711.00782* (2017).

[6] Eberle, S.; Niethammer, B.; Schlichting, A. Gradient flow formulation and longtime behaviour of a constrained Fokker-Planck equation, *Nonlinear Anal.* **158** (2017) 142-167.

[7] Laurençot, P.; Mischler, S. From the Becker-Döring to the Lifshitz-Slyozov-Wagner Equations, *J. Stat. Phys.* **106** (2002) 957-991. doi:10.1023/A:1014081619064.

[8] Lifshitz, I.M.; Slyozov, V. V. The kinetics of precipitation from supersaturated solid solutions, *J. Phys. Chem. Solids.* **19** (1961) 35-50. doi:10.1016/0022-3697(61)90054-3.

[9] Jabin, P.-E.; Niethammer, B. On the rate of convergence to equilibrium in the Becker-Döring equations. *J. Differ. Equ.* **191** (2003), 518-543, doi:10.1016/S0022-0396(03)00021-4.

[10] Niethammer, B. On the Evolution of Large Clusters in the Becker-Döring Model. *J. Nonlinear Sci.* **13** (2003), 115-155, doi:10.1007/s00332-002-0535-8.

[11] Penrose, O. The Becker-Döring equations at large times and their connection with the LSW theory of coarsening, *J. Stat. Phys.* **89** (1997) 305-320.
[12] Schlichting, A. Macroscopic limit of the Becker-Döring equation via gradient flows. *to appear ESAIM:COCV*, arXiv: 1607.08735, 2018.
[13] Velázquez, J.J.L. The Becker-Döring equations and the Lifschitz-Slyozov theory of coarsening. *J. Stat. Phys.* **92**, 195-236 (1998).
[14] Wagner, C. Theorie der Alterung von Niederschlägen durch Umlösen (Ostwald-Reifung), *Z. Elektrochem.* **65** (1961) 581-591.

Local limit theorem for random walks among time-dependent ergodic degenerate weights

Martin Slowik
(joint work with Alberto Chiarini)

Random walks and their scaling limits provide a simple yet powerful model to describe transport processes through a medium in a large variety of systems. In many situations of practical interest the medium is highly irregular, and it is natural to model such a disordered medium as a realization of a random environment. A specific model of a symmetric random walk in a random environment is the *random conductance model* (RCM) that has been intensively studied in the past 10-15 years, see e.g. the survey [6] and references therein. Of particular interest is the question under what kind of conditions a quenched invariance principle and a quenched local limit theorem hold.

The invariance principle is a *functional* version of the *central limit theorem* that has first been proven by Donsker [9] for simple symmetric random walks on the Euclidean lattice $\mathbb{Z}^d$. For any fixed realization of the environment, it describes how to rescale a random walk in space in time in order to obtain a Brownian motion in the limit. The *local limit theorem* however provides a much finer result, namely that the transition probabilities of the random walk properly rescaled converge to the Gaussian transition density of the limiting Brownian motion.

We are interested in establishing a quenched local limit theorem for the time-dependent random conductance model on the d-dimensional Euclidean lattice, $d \geq 2$. This model is a time-inhomogeneous Markov process $X \equiv (X_t : t \geq 0)$ on $(\mathbb{Z}^d, E^d)$ with instantaneous generator, $\mathcal{L}_t^\omega$, (in the L^2 sense) which acts on bounded functions $f\colon \mathbb{Z}^d \to \mathbb{R}$ as

$$\big(\mathcal{L}_t^\omega f\big)(x) \;=\; \sum_{y\sim x} \omega_t(\{x,y\})\,\big(f(y)-f(x)\big),$$

where $\omega \equiv \{\omega_t(e) : t \in \mathbb{R}, e \in E^d\} \in [0,\infty]^{\mathbb{R}\times E^d} =: \Omega$ is a family of non-negative weights (also called *conductances*). Further, we denote by $\mathrm{P}_{s,x}^\omega$ the law of X on the space of $\mathbb{Z}^d$-valued càdlàg functions on $\mathbb{R}$ when starting at time s in x. For $x, y \in \mathbb{Z}^d$ and $s, t \in \mathbb{R}$ with $s \leq t$ the transition density (or heat kernel) of the Markov process X is given by

$$p_{s,t}^\omega(x,y) \;:=\; \mathrm{P}_{s,x}^\omega\big[X_t = y\big]$$

Note that the counting measure, independent of t, is an invariant measure for X.

Of particular interest is the case when the conductances are itself random variables with law $\mathbb{P}$.

Assumption. *Assume that the law $\mathbb{P}$ of the conductances on $(\Omega, \mathcal{F})$ satisfies:*

(i) $\mathbb{P}$ is stationary and ergodic with respect to space-time shifts.

(ii) For every $A \in \mathcal{F}$ the mapping $(\omega, t, x) \mapsto \mathbb{1}_A(\tau_{t,x}\omega)$ is jointly measurable with respect to the σ-algebra $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{P}(\mathbb{Z}^d)$.

For the static random conductances model with i.i.d. environments, i.e. the conductances are constant in time and $\mathbb{P}$ is a product measure, a local limit theorem has first been proven by Barlow and Hambly. They assumed that either $\omega(e) \in \{0, 1\}$ with $\mathbb{P}[\omega(e) > 0] > p_c$ for all $e \in E^d$ [5, Theorem 5.2] (supercritical percolation model) or that the conductances are uniformly elliptic [5, Theorem 5.7], i.e. there exists $c \in (0, \infty)$ such that $c^{-1} \leq \omega(e) \leq c$ for all $e \in E^d$. In case of i.i.d. conductances that are uniformly bounded from below has later been treated in [4, Theorem 5.14].

For general ergodic but static conductances, a quenched local limit theorem has been proven in [12, Theorem 1.19] for supercritical percolation clusters and in [2, Theorem 1.11] for elliptic conductances, i.e. $\mathbb{P}[0 < \omega(e) < \infty] = 1$ for all $e \in E^d$, under the additional (optimal) integrability condition that $\mathbb{E}[\omega(e)^p] < \infty$ and $\mathbb{E}[1/\omega(e)^q] < \infty$ for $p, q \in [1, \infty]$ such that $1/p + 1/q < 2/d$.

For general time-dependent ergodic conductances, a quenched local limit theorem for uniformly elliptic conductances has been proven by Andres, see [1, Theorem 1.6], under the additional assumption that the law $\mathbb{P}$ satisfies a certain mixing condition.

Hence, it is clear that some moment conditions are needed.

Theorem. *Suppose that $d \geq 2$ and that the above assumptions hold. For any $p, q \in [1, \infty]$ satisfying*

$$\frac{1}{p-1} \cdot \frac{q+1}{q} + \frac{1}{q} < \frac{2}{d}$$

assume that $\mathbb{E}[\omega_t(e)^p] < \infty$ and $\mathbb{E}[1/\omega_t(e)^q] < \infty$ for all $e \in E^d$ and $t \in \mathbb{R}$. Then, for any $T_1 > 0$ and $K \in (0, \infty)$

$$\lim_{n \to \infty} \sup_{|x| \leq K} \sup_{t \geq T_1} \left| n^d \, p^{\omega}_{0, tn^2}\big(0, \lfloor nx \rfloor\big) - k^{\Sigma}_t(0, x) \right| = 0, \qquad \mathbb{P}\text{-a.s.}$$

where k^{Σ} is the heat kernel of the limiting Brownian motion with deterministic non-degenerate covariance matrix $\Sigma^{\mathrm{T}} \cdot \Sigma$.

The Method. The proof of the local limit theorem is based on the approach in [5] and [7]. The two main ingredients are

1. a quenched functional central limit theorem (QFCLT) and
2. a Hölder-continuity estimate on the heat kernel,

which enables us to replace the weak convergence given by the QFCLT by the pointwise convergence in the theorem. The QFCLT has been established in [3].

In order to derive the Hölder-continuity estimate, we prove a *parabolic maximal inequality* and an *oscillation lemma* using the de Giorgi iteration scheme. Since the pioneering works of de Giorgi, Moser and Nash [8, 11, 10] iteration techniques are by far the best-established tools in order to prove both elliptic and parabolic maximal inequalities and regularity estimates. The de Giorgi's iteration is based on three ideas: (1) a Sobolev-type inequality which allows to control the ℓ^r-norm with $r = r(d) = d/(d-2) > 1$ in terms of the Dirichlet form, (2) a control of the Dirichlet energy of the truncation $(u-k)_+$, $k \geq 0$, of a given caloric function u, and (3) an iteration lemma. In our case where the conductances are unbounded from above and below and time-dependent, we need to work with a dimension dependent weighted Sobolev inequality, which we obtain from an isoperimetric inequality of the underlying graph and Hölder's inequality. Moreover, assuming a strong local ℓ^1-Poincaré inequality, we also show that the parabolic regularity can be obtained without going through any kind of John-Nirenberg or Bombieri-Giusti type argument.

References

[1] S. Andres, *Invariance principle for the random conductance model with dynamic bounded conductances*, Ann. Inst. Henri Poincaré Probab. Stat. **50**(2) (2014), 352–374.

[2] S. Andres, J.-D. Deuschel, M. Slowik, *Harnack inequalities on weighted graphs and some applications to the random conductance model*, Probab. Theory Related Fields **164**(3-4) (2016), 931–977.

[3] S. Andres, A. Chiarini, J.-D. Deuschel, M. Slowik, *Quenched invariance principle for random walks with time-depenent ergodic degenerate weights*, Ann. Probab. **46**(1) (2018), 302–336.

[4] M. T. Barlow and J.-D. Deuschel, *Invariance principle for the random conductance model with unbounded conductances*, Ann. Probab. **38**(1) (2010), 234–276.

[5] M. T. Barlow and B. M. Hambly, *Parabolic Harnack inequalities and local limit theorem for percolation clusters*, Electron. J. Probab. **14** (2009), 1–27.

[6] M. Biskup, *Recent progress on the random conductance model*, Probab. Surv. **8** (2011), 294-373.

[7] D. A. Croydon and B. M. Hambly, *Local limit theorems for sequences of simple random walks on graphs*, Potential Anal. **29** (2008), 351–389.

[8] E. De Giorgi, *Sulla analiticità e la differenziabilità delle estremali degli integrali multipli regolari*, Mem. Acc. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **3**(3) (1957), 25–43.

[9] M. D. Donsker, *Justification and extension of Doob's heuristic approach to the Komogorov-Smirnov theorems*, Ann. Math. Statistics **23** (1952), 277–281.

[10] J. Moser, *On Harnack's theorem for elliptic differential equations*, Comm. Pure Appl. Math. **14** (1961), 577–591.

[11] J. Nash, *Continuity of solutions of parabolic and elliptic equations*, Am. J. Math. **80** (1958), 931-954.

[12] A. Sapozhnikov, *Random walks on innite percolation clusters in models with long-range correlations*, Ann. Probab. **45**(3) (2017), 1842–1898.

Intermittent regularization and long-time behavior of Hamilton-Jacobi equations with rough multiplicative time dependence and convex Hamiltonians

Panagiotis E. Souganidis
(joint work with Pierre-Louis Lions)

We consider Hamilton-Jacobi equations with convex Hamiltonians and rough multiplicative time dependence. We prove a new and surprising result that shows that at times when the path does not equal its running maximum and minimum the solution is actually in $C^{1,1}$ in space. In the case of Brownian paths, the result implies that the stochastic viscosity solution is $C^{1,1}$ off a set of times of Hausdorff dimensions $1/2$. The estimate is new even for the deterministic case. We then use this intermittent regularization to prove that as time goes to infinity, the solutions converge to a constant.

Random permutations without macroscopic cycles

Dirk Zeindler
(joint work with Volker Betz and Helge Schäfer)

We consider in this talk uniform random permutations $\sigma \in S_n$ conditioned to have no cycles of length larger or equal to $\alpha(n)$ with $n^{a_1} \leq \alpha(n) \leq n^{a_2}$ and $a_1, a_2 \in (0,1)$. For cycles of length $o(\alpha(n)/\log n)$, we find that they behave just like those of unconstrained permutations. At the scale $\alpha(n)/\log n$, the influence of the restriction starts to manifest itself in the sense that, as $n \to \infty$, the expected cycle numbers converge to zero more slowly than they would in unrestricted permutations. At the scale $c\alpha(n)$, $0 \leq c < 1$, the restriction starts to become manifest, and if $\alpha(n)$ diverges more slowly than $\sqrt{n}$, diverging numbers of cycles occur for lengths corresponding to sufficiently large c. In these cases, a central limit theorem holds. Finally, we investigate the scale where most of the cycles live. Due to the length constraint, there must be at least $n/\alpha(n)$ cycles, and we show that almost all of them live on the scale $\alpha(n) + \alpha(n)\frac{\log t}{\log n}$, $0 < t < 1$. On that scale, the cumulative cycle numbers satisfy a limit shape theorem, and their fluctuations around that limit shape satisfy a functional central limit theorem towards a Brownian bridge. We get immediately from this result that the length the longest cycles are asymptotically $\alpha(n)$.

References

[1] V. Betz, H. Schäfer and D. Zeindler, *Random permutations without macroscopic cycles*, Preprint 2017, arxiv.org/abs/1712.04738

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