Arbeitsgemeinschaft mit aktuellem Thema:

TOPOLOGICAL CYCLIC HOMOLOGY

Mathematisches Forschungsinstitut Oberwolfach

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Introduction:

Cyclic homology was introduced by Connes and Tsygan in the early eighties to serve as an extension of de Rham cohomology to a non-commutative setting. The negative version of cyclic homology receives a trace map from algebraic K-theory, which extends the classical Chern character and roughly records traces of powers of matrices. This trace map is a powerful rational invariant of algebraic K-theory. Indeed, a theorem of Goodwillie from 1986 shows that, rationally, the discrepancy for K-theory to be invariant under nilpotent extentions of rings agrees with that for negative cyclic homology; and a theorem of Cortiñas from 2006 shows similarly that, rationally, the discrepancy for K-theory to preserve cartesian squares of rings agrees with that for negative cyclic homology.

In the early seventies, Boardman and Vogt had planted the seeds for the higher algebra that was only fully developed much later by Joyal and Lurie, and later in the decade, Waldhausen had extended Quillen's definition of algebraic K-theory from the rings of algebra to the (connective $\mathbb{E}_1$-)rings of higher algebra. Waldhausen advocated that the initial ring $\mathbb{S}$ of higher algebra be viewed as an object of arithmetic and that the cyclic homology of Connes and Tsygan be developed with the ring $\mathbb{S}$ as its base. In his philosophy, such a theory should be meaningful integrally as opposed to rationally.

In 1985, Bökstedt carried out Waldhausen's vision as far as Hochschild homology is concerned, and he named this new theory topology Hochschild homology. (A similar construction had been made by Breen ten years earlier.) He also made the fundamental calculation that, as a graded ring,

$$\mathrm{THH}_*(\mathbb{F}_p) = \mathrm{HH}_*(\mathbb{F}_p/\mathbb{S}) = \mathbb{F}_p[x]$$

is a polynomial algebra on a generator x in degree two. By comparison,

$$\mathrm{HH}_*(\mathbb{F}_p/\mathbb{Z}) = \mathbb{F}_p\langle x\rangle$$

is the corresponding divided power algebra, so Bökstedt's theorem supports Waldhausen's vision that passing from the base $\mathbb{Z}$ to the base $\mathbb{S}$ eliminates denominators. In fact, the base-change map $\mathrm{HH}_*(\mathbb{F}_p/\mathbb{S}) \to \mathrm{HH}_*(\mathbb{F}_p/\mathbb{Z})$ can be identified with the edge homomorphism of a spectral sequence

$$E^2_{i,j} = \mathrm{HH}_i(\mathbb{F}_p/\pi_*(\mathbb{S}))_j \Rightarrow \mathrm{HH}_{i+j}(\mathbb{F}_p/\mathbb{S}),$$

so apparently the higher stable homotopy groups of spheres, which Serre had proved to be finite, are exactly the right size to eliminate the denominators in the divided power algebra.

The appropriate definition of cyclic homology relative to $\mathbb{S}$ was given by Bökstedt-Hsiang-Madsen in 1993. It involves a new ingredient, not present in the Connes-Tsygan cyclic theory: a Frobenius. The nature of this Frobenius is now much better understood thanks to the work of Nikolaus-Scholze [18], and we will use this work as our basic reference. As in the Connes-Tsygan theory, the circle group $\mathbb{T}$ acts on topological Hochschild homology, and by analogy, we may define negative topological cyclic homology and periodic topological cyclic homology to be the homotopy fixed points and the Tate construction of this action, respectively:

$$\mathrm{TC}^-(A) = \mathrm{THH}(A)^{h\mathbb{T}} \qquad \text{and} \qquad \mathrm{TP}(A) = \mathrm{THH}(A)^{t\mathbb{T}}.$$

There is always a canonical map from homotopy fixed points to the Tate construction, but, after p-completion, the Frobenius gives rise to another such map and the Bökstedt-Hsiang-Madsen topological cyclic homology is the homotopy equalizer of these two maps:

$$\mathrm{TC}(A) \longrightarrow \mathrm{TC}^-(A) \underset{\mathrm{can}}{\overset{\varphi_p}{\rightrightarrows}} \mathrm{TP}(A).$$

Topological cyclic homology receives a trace map from algebraic K-theory, which is called the cyclotomic trace map, and Dundas-McCarthy-Goodwillie showed that the discrepancy for K-theory to be invariant under nilpotent extensions agrees integrally with that for topological cyclic homology. Similarly, by work of Geisser-Hesselholt and Dundas-Kittang, the discrepancy for K-theory to preserve cartesian squares of rings agrees integrally with that for topological cyclic homology.

Calculations of algebraic K-groups, or rather the homotopy groups of the p-adic completion of the K-theory spectrum, by means of the cyclotomic trace begin with the calculation that said trace map

$$K(\mathbb{F}_p) \to \mathrm{TC}(\mathbb{F}_p)$$

induces an isomorphism of p-adic homotopy groups in non-negative degrees. The Dundas-McCarthy-Goodwillie theorem together with continuity results of Suslin and Hesselholt-Madsen then show that the same is true for

$$K(\mathbb{Z}_p) \to \mathrm{TC}(\mathbb{Z}_p)$$

and, more generally, for finite algebras over the ring of Witt vectors in a perfect field of characteristic p. This was used by Hesselholt-Madsen in 2003 to verify the Lichtenbaum-Quillen conjecture for p-adic fields, by evaluating the relevant topological cyclic homology, and one of the goals of the Arbeitsgemeinschaft is to understand this calculation.

The theories TC^- and TP are of significant independent interest, since they are closely related to interesting p-adic cohomology theories, both new and old. The precise relationship was established only recently by work of Bhatt-Morrow-Scholze that defines "motivic filtrations" on THH and related theories, the graded pieces of which are p-adic cohomology theories such as crystalline cohomology and the $\mathbb{A}\Omega$-theory of [2]. For example, if X is a scheme smooth over a perfect field of characteristic p, then the j'th graded pieces of TC, TC^-, and TP form a homotopy equalizer

$$\mathbb{Z}_p(j) \longrightarrow \mathrm{Fil}^j R\Gamma_{\mathrm{crys}}(X/W(k)) \underset{\mathrm{can}}{\overset{\text{“}\frac{\varphi_p}{p^j}\text{”}}{\rightrightarrows}} R\Gamma_{\mathrm{crys}}(X/W(k)).$$

A second goal of the Arbeitsgemeinschaft is to understand these filtrations.

Since the questions we consider are in the p-complete setting and for $\mathbb{E}_\infty$-algebras (in fact, usual commutative rings!), we will largely restrict our attention to this case, and in particular work with p-typical cyclotomic spectra.

Talks:

Day 1: Definition of THH

1. **Cyclotomic spectra and the Tate construction**
   Define the category of cyclotomic spectra following [18, Section II.1]. As preparation, introduce the Tate construction for finite groups, and recall some of its properties, including that it is lax symmetric monoidal, cf. [18, Section I.3].

2. **Genuine cyclotomic spectra**
   Using the Tate Orbit lemma, [18, Section I.2], explain how to construct a genuine cyclotomic spectrum from a cyclotomic spectrum, by defining TR^r with its Frobenius and Restriction maps, following [18, Section II.4]. Explain the necessary background on genuine equivariant homotopy theory (only) as needed.

3. **The Tate diagonal**
   Construct the Tate diagonal for spectra, [18, Section III.1] and discuss its properties, including that it is lax symmetric monoidal. Prove that on bounded below spectra, it is equivalent to the p-completion map.

4. **Topological Hochschild homology**
   Define topological Hochschild homology for $\mathbb{E}_\infty$-algebras following [18, Section IV.2], and relate it with the Tate-valued Frobenius of [18, Section IV.1].

Day 2: Examples

5. **Dyer–Lashof operations**
   If A is an $\mathbb{E}_\infty$-algebra in Sp, then the beginning of the cocyclic spectrum

$$A \rightrightarrows A \otimes A \rightrightarrows A \otimes A \otimes A \;\vdots\; \cdots$$

   defines a co-groupoid structure on $(\pi_*(A), \pi_*(A \otimes A))$, provided that one of $d^0, d^1 \colon \pi_*(A) \to \pi_*(A \otimes A)$ is flat. Explain Milnor's calculation of this for $A = \mathbb{F}_p$ following [17]. And if R and A are two $\mathbb{E}_\infty$-rings, then every $e \in \pi_k(((\Sigma^m \mathbb{S})^{\otimes n})_{h\Sigma_n} \otimes A)$ gives a power operation

$$\pi_m(R \otimes A) \xrightarrow{Q_e} \pi_k(R \otimes A),$$

first considered by Araki-Kudo and Dyer-Lashof. Review the general setup following [1] and Steinberger's calculation of these operations for $R = A = \mathbb{F}_p$ following [8, Chapter 3]; see also [12, Lecture 5].

6. **Bökstedt's computation of** $\mathrm{THH}(\mathbb{F}_p)$
   If A is an $\mathbb{E}_\infty$-ring and R an $\mathbb{E}_\infty$-A-algebra, then the split equalizer

$$R \rightleftarrows R \otimes A \overset{\longrightarrow}{\underset{\longleftarrow}{\longrightarrow}} R \otimes A \otimes A$$

   defines a split equalizer

$$\pi_*(R) \rightleftarrows \pi_*(R \otimes A) \overset{\longrightarrow}{\underset{\longleftarrow}{\longrightarrow}} \pi_*(R \otimes A) \otimes_{\pi_*(A)} \pi_*(A \otimes A),$$

   provided that $d^0 \colon \pi_*(A) \to \pi_*(A \otimes A)$ is flat, so $\pi_*(R)$ can be calculated from $\pi_*(R \otimes A)$ by descent. For $R = \mathrm{THH}(\mathbb{F}_p)$ and $A = \mathbb{F}_p$, Bökstedt uses the spectral sequence

$$E^2 = \mathrm{HH}_*(\pi_*(A \otimes A)/\pi_*(A)) \Rightarrow \pi_*(R \otimes A)$$

   to show that $\pi_*(R \otimes A) = \pi_*(A \otimes A)[x]$ with x of degree 2 and concludes that $\pi_*(R) = \mathbb{F}_p[x]$. Review the essential steps in this calculation, beginning with the fact that the map "σ" defined as the composite

$$\pi_j(A \otimes A) \xrightarrow{\eta \otimes A} \pi_j(R \otimes A) \xrightarrow{d} \pi_{j+1}(R \otimes A)$$

   commutes with power operations following [6]; see also [15, Section 5.2] and [12, Lecture 6]. Finally, show that, as an $\mathbb{E}_\infty$-ring in cyclotomic spectra, $\mathrm{THH}(\mathbb{F}_p) \simeq \tau_{\geq 0}(\mathbb{Z}^{tC_p})$ following [18, Section IV.4].

7. THH **for associative rings and categories**
   Define THH for $\mathbb{E}_1$-algebras in Sp by means of the cyclic bar-construction following [18, Sections III.1–III.3]. Define THH for stable ∞-categories following [4, Section 10] and [5].

8. THH **of loop spaces**
   An $\mathbb{E}_1$-monoid M in pointed spaces gives rise to an $\mathbb{E}_1$-algebra $\mathbb{S}[M]$ in spectra. Following [18, Section IV.3], show that as cyclotomic spectra $\mathrm{THH}(\mathbb{S}[M]) = \mathbb{S}[B^{\mathrm{cy}}(M)]$ and discuss the cases $M = \mathbb{N}$ and $M = \mathbb{Z}$. For M an $\mathbb{E}_1$-group, evaluate the p-completion of $\mathrm{TC}(\mathbb{S}[M])$.

Day 3: Relation to algebraic K-theory

9. **Construction of the cyclotomic trace**
Construct the cyclotomic trace map $\mathrm{tr}\colon K(\mathcal{C}) \to \mathrm{TC}(\mathcal{C})$ for stable ∞-categories following [4, Section 10]. Show that for an $\mathbb{E}_1$-algebra A in spectra, $\mathrm{THH}(A)$ and $\mathrm{THH}(\mathrm{Perf}(A))$ are equivalent as cyclotomic spectra.

10. **The theorem of Dundas–Goodwillie–McCarthy**
The theorem states that if $f\colon A \to B$ is a map of connective $\mathbb{E}_1$-rings such that the induced map $\pi_0(f)\colon \pi_0(A) \to \pi_0(B)$ is a surjection with nilpotent kernel, then the diagram

$$\begin{array}{ccc} K(A) & \xrightarrow{\mathrm{tr}} & \mathrm{TC}(A) \\ \big\downarrow{\scriptstyle f^*} & & \big\downarrow{\scriptstyle f^*} \\ K(B) & \xrightarrow{\mathrm{tr}} & \mathrm{TC}(B) \end{array}$$

is (homotopy) cartesian. Following [9], explain the steps in the proof and discuss the example of the Hurewicz map $h\colon \mathbb{S} \to \mathbb{Z}$.

Day 4: K-theory of p-adic fields

11. **Logarithmic** THH
The work [16] of Hesselholt–Madsen works with logarithmic THH for the ring of integers A in a p-adic field K. In general, if (A, M) is a ring with (pre-)log structure, then its logarithmic THH can be defined in the language of the Arbeitsgemeinschaft as

$$\mathrm{THH}(A, M) = \mathrm{THH}(A) \otimes_{\mathbb{S}[B^{\mathrm{cy}}(M)]} \mathbb{S}[M \times_{M^{\mathrm{gp}}} B^{\mathrm{cy}}(M^{\mathrm{gp}})]\ ,$$

using the identification $\mathrm{THH}(\mathbb{S}[M]) = \mathbb{S}[B^{\mathrm{cy}}(M)]$ from Talk 8. Here, everything in sight is an $\mathbb{E}_\infty$-algebra in cyclotomic spectra. Prove [16, Theorem B], including the formula $d\kappa = \kappa \cdot d\log(-p)$.

12. **The de Rham–Witt complex**
Define the absolute de Rham–Witt complex $W\Omega_{(A,M)}$ of a ring with log structure (A, M) following [16, Section 3] or [14]. Explain the structure of $W\Omega_{(A,M)}/pW\Omega_{(A,M)}$ for (A, M) a ring of integers in a p-adic field with log structure $M = A \cap K^* \hookrightarrow A$ following [16, Section 3]; see also [13, Section 1].

13. **The Tate spectral sequence**
Set up the Tate spectral sequence

$$E^2_{i,j} = \hat{H}^{-i}(C_{p^n}, \pi_j(\mathrm{THH}(A,M), \mathbb{Z}/p\mathbb{Z})) \\ \Rightarrow \pi_{i+j}(\mathrm{THH}(A,M)^{tC_{p^n}}, \mathbb{Z}/p\mathbb{Z})$$

and evaluate the differentials therein following [16, Section 4 and 5].

14. **Conclusion**
Deduce [16, Theorem C] from the structure of the Tate spectral sequences in Talk 13, and deduce [16, Theorem A]. The latter uses a continuity result in algebraic K-theory due to Suslin [15, Appendix B].

Day 5: A motivic filtration on THH

15. **Flat descent for** THH
By relating THH to HH and then to (wedge powers of) the cotangent complex, prove that THH satisfies flat descent, cf. [3].

16. **Construction of the filtration**
Following [3], discuss the notion of regular semiperfect algebras. Show that THH (and its variants) are concentrated in even degrees for regular semiperfect algebras. Use the Postnikov filtration and flat descent to define a filtration on THH (and its variants) for smooth algebras.

17. **Identification of graded pieces**
Following [3], identify for regular semiperfect rings R the homotopy groups of $\mathrm{TP}(R) = \mathrm{THH}(R)^{t\mathbb{T}}$ of [3] as the A_{crys}-construction of Fontaine.

18. **Conclusion**
Recall the syntomic definition of crystalline cohomology due to Fontaine–Messing, [10], and deduce that if R is smooth, then $\mathrm{TP}(R)$ is, after inverting p, given by (a sum of) crystalline cohomology groups.

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Participation:

The idea of the Arbeitsgemeinschaft is to learn by giving one of the lectures in the program.

If you intend to participate, please send your full name and full postal address to

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by December 1, 2017 at the latest.

You should also indicate which talk you are willing to give:
First choice: talk no. . . .
Second choice: talk no. . . .
Third choice: talk no. . . .

You will be informed shortly after the deadline if your participation is possible and whether you have been chosen to give one of the lectures.

The Arbeitsgemeinschaft will take place at Mathematisches Forschungsinstitut Oberwolfach, Lorenzenhof, 77709 Oberwolfach-Walke, Germany. The institute offers accomodation free of charge to the participants. Travel expenses cannot be covered. Further information will be given to the participants after the deadline.