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Mini-Workshop: Arithmetic Geometry and Symmetries around Galois and Fundamental Groups

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Abstract. The geometric study of the absolute Galois group of the rational numbers has been a highly active research topic since the first milestones: Hilbert's Irreducibility Theorem, Noether's program, Riemann's Existence Theorem. It gained special interest in the last decades with Grothendieck's "Esquisse d'un programme", his "Letter to Faltings" and Fried's introduction of Hurwitz spaces. It grew on and thrived on a wide range of areas, *e.g.* formal algebraic geometry, Diophantine geometry, group theory. The recent years have seen the development and integration in algebraic geometry and Galois theory of new advanced techniques from algebraic stacks, ℓ-adic representations and homotopy theories. It was the goal of this mini-workshop, to bring together an international panel of young and senior experts to draw bridges towards these fields of research and to incorporate new methods, techniques and structures in the development of geometric Galois theory.

Mathematics Subject Classification (2010): 12F12, 14G32, 14H30 (14F05, 14F22, 55Pxx).

Introduction by the Organisers

The workshop *Arithmetic Geometry and Symmetries around Galois and Fundamental Groups* dealt with recent progress in the study of the absolute Galois group of the rational numbers based on geometric representations through étale fundamental groups. This includes various approaches which aim at moduli properties of algebraic spaces *via* their arithmetic-geometric interpretations, and which translate into the study of the finite quotients of the absolute Galois group – the *Inverse Galois Problem.* IGP has been one key thread of the workshop, a second one being the *symmetries of the spaces*, which the automorphisms of the structures reflect.

Overview

The traditional *Geometric Galois & Inverse Galois* approaches – abelian, geometric, *via* Galois representation – have grown into new branches leading to striking results: conditions on Galois realizations expressed in terms of rational properties of Hurwitz space towers (Fried, Dèbes, Cadoret, Tamagawa), arithmetic properties of the stack structures of moduli spaces of curves (Schneps, Nakamura, Collas, Maugeais), extension of anabelian results to higher-dimension (Hoshi, Schmidt, Stix), realization of new groups as Galois groups (Dettweiler, Reiter), specialization properties of geometric Galois realizations (Dèbes, Legrand, König, Neftin), contributions to Colliot-Thélène's program on G-torsors (Harari, Wittenberg).

The workshop focused on this progress as organized under 3 *hot topics*:

(1) *Abelian approach to Inverse Galois.* After the completion of the Shafarevich solution to IGP for solvable groups, the Colliot-Thélène approach to the Noether program and the Grunwald problem, *via* the study of rational points on rationally connected varieties, has become a leading project.
(2) *Geometric Galois Theory*, which investigates the arithmetic of finite Galois covers of the projective line and their specializations and has led to the study of their moduli spaces and their towers — Hurwitz spaces and Modular Towers, has been the only key to the non-solvable territory.
(3) *Galois Anabelian and Homotopical Geometry*, which deals with Galois properties of the étale fundamental group as supported by the seminal example of the moduli stack of curves, has been, since Grothendieck, one of the most influential set of ideas.

One goal of the workshop has been to take full advantage of the bridges between the three topics, notably by considering the *geometric and arithmetic higher symmetries* of the objects *via* homotopical methods. The automorphisms of families at the 2-categorical level of Hurwitz and moduli stacks, and the higher cohomological obstructions to rationality are two successful examples of this approach. The introductory talk referred to this quote by S. Lefschetz *"It was my lot to plant the harpoon of algebraic topology into the body of the whale of algebraic geometry"*. Taking over this mission for Arithmetic was inspirational for the workshop.

The program of the mini-workshop consisted of 18 eighty-minutes lectures. Participants introduced their knowledge and shared their progress in a lively atmosphere of stimulating exchanges. Informal sessions crystallizing the connections between classical problems and some of the new expertises revealed quite promising. For example the embedding problem paired with the spectral étale Brauer-Manin obstruction, Regular Inverse Galois Theory with approximation properties on the Noether variety, anabelian geometric properties with étale homotopy type.

This mini-workshop renewed the long and strong tradition of fruitful exchanges between Arithmetic and Galois theories in such famous places as Oberwolfach, Luminy, Seattle, Red Lodge, and Kyoto. Because of the strong support and feedback of the participants, agreement has been made to meet again in a similar event within the next two years for sharing the progress of the research directions initiated during this mini-workshop.

1. Abelian Inverse Galois Theory

The three talks on this topic related to the Brauer-Manin obstruction to existence or density of rational points on a variety over a number field. A classical conjecture due to Colliot-Thélène – the BM obstruction is the only one for geometrically rationally connected varieties (*e.g.* unirational) – link this issue to Inverse Galois Theory. Indeed it is well-known that this conjecture, applied to the Noether variety $\mathbb{A}^d/G$ (with $d = |G|$), or its variant SL_m/G (for some embedding $G \hookrightarrow \mathrm{SL}_m$), leads to a solution of IGP for the group G in question. Furthermore this approach generally also provides some answers to the local-global Grunwald problem – lift a finite set of local Galois extensions of group embedded in G to some global Galois extension of group G. Recently new ingredients coming from homotopy theory, the notion of spectras, have appeared in this topic.

In the first part of his talk, Harari reviewed the classical material around the Brauer-Manin obstruction (over number fields). Then he explained that a big part extends to global fields of positive characteristic, *e.g.* the function field K of a curve over a finite field, or even over more complicated fields, provided some good arithmetic duality properties remain. In this context, he finally presented some recent results obtained (some jointly with Szamuely and some by Izquierdo) for some K-algebraic groups.

Schlank reported on how modern homotopy theory gives – in terms of étale topological type, spaces, and spectras – a potential finer context where to express fundamental arithmetic questions. He first discussed the problems of Grothendieck section conjecture, of Skorobogatov's étale-BM obstruction to rational points, and of Galois embeddings (jw. Carlson) in terms of cohomological methods. He then presented recent developments of this approach within the stable motivic homotopy theory, which reveal a higher cohomological obstruction (jw. Stojanoska).

Wittenberg, after recalling the Colliot-Thélène conjecture and its connection to Inverse Galois Theory, presented his recent joint result with Harpaz: a proof of the conjecture when the variety is a smooth proper model of an homogeneous space V of SL_m with finite and supersolvable stabilizers. In the case $V = \mathrm{SL}_m/G$ with G a finite group, embedded in some SL_m, they obtain as a corollary that every supersolvable group G is a Galois group over any given number field, and even that every Grunwald problem for G, for places not dividing $|G|$, has a solution. He also explained the strategy of the proof and its main ingredients.

2. Geometric Galois Theory

The most basic *Inverse Galois Problem* version is to show that $G_{\mathbb{Q}}$ has every finite group as a quotient. Some significant success has, however, come through the *Regular Inverse Galois Problem* (RIGP) for which the basic tools are sophisticated versions of *Riemann's existence theorem* followed by specialization (Hilbert's irreducibility theorem). The regular approach is driven by the moduli of covers of the projective line $\mathbb{P}^1$ – Hurwitz spaces. The profinite nature of Galois groups leads to their organization in towers – Modular Towers, which also takes us back to fundamental ℓ-adic representation issues. Recently there has also been a focus on the specialization process itself aimed at assessing the difference between the two inverse Galois problems.

Fried's talk applied his **M**(odular)**T**(ower) generalization of modular curve towers to the **R**(egular)**I**(nverse)**G**(alois)**P**(roblem) and expanding Serre's **O**(pen) **I**(mage)**T**(heorem). From any finite ℓ-perfect (ℓ prime) group G, a characteristic extension, $V_\ell \stackrel{\text{def}}{=} (\mathbb{Z}_\ell)^{\nu(G,\ell)} \to \tilde{G}_\ell \to G$, leads to towers of Hurwitz spaces based on the finite group quotients $\tilde{G}_\ell/\ell^{k+1}V_\ell \stackrel{\text{def}}{=} {}^k_\ell G$, $k \geq 0$. An example used his formula for computing all expected properties – genus, cusps, degree, fine moduli properties – of j-line covers by reduced Hurwitz spaces of 4-branch point covers. His concluding examples showed **MT**s to be a *seam* between the **OIT** and the **RIGP** enhancing Fried's two main **OIT** conjectures.

Dèbes followed Raynaud's "one-slide" tradition to present a diagram showing the state of the art in Inverse Galois Theory and structuring it in three categories of problems: realizing, lifting, parametrizing. He then used the same diagram to recast a series of recent results from a joint program with Koenig, Legrand and Neftin on the specialization process. In various situations, they show that the sets of Galois extensions obtained by specialization from natural sets of Galois covers of the line of fixed group G (singletons, moduli spaces) are big (in some density sense), but also cannot be too big (*e.g.* they generally do not contain all Galois extensions of group G). More detailed talks by his co-authors were to follow.

Legrand presented in more details some of the specialization results mentioned in Dèbes' talk. He emphasized his results on the parametricity property. No group G had failed having a parametric extension over a given number field k: a regular Galois extension $F/k(T)$ that parametrizes, *via* specialization of T in $\mathbb{P}^1(k)$, all Galois extensions E/k of group $G = \text{Gal}(F/k(T))$. A joint work of König and his now offers many such groups: abelian $\neq C_4, C_p$, S_n ($n \geq 4$), *etc.* He also discussed analogous results for the *regular* type of specialization, for which T is specialized in $k(U)$, and so the outcome is really a rational pull-back cover.

König came back to the specialization approach to the Grunwald problem alluded to in Dèbes's talk, which consists in using the specialized extensions of some regular realization of a group G to solve Grunwald problems for this group. After recalling the unramified case (after Dèbes-Ghazi), he showed, based on a joint result of Legrand, Neftin and his on decomposition groups of specialized extensions, that the Grunwad problem cannot be handled by the specialization approach in general, but that promising perspectives exist if a 1-parameter family of regular realizations of G is available. He also explained how to use their local work to produce new groups with no parametric extensions, *e.g.* A_n ($n \geq 4$).

Neftin focused on an old famous problem related to Hilbert's Irreducibility Theorem, which is to investigate situations for which the exceptional "reducible set" in Hilbert's theorem is finite. A breakthrough in this problem, due to Fried, was to understand how it is governed by group theory, *via* the monodromy groups of the associated covers. Neftin recalled the special problem where the initial polynomial is of the form $p(Y)-T$ for which the expected conclusions (the reducible set is finite except for the values of p) have been obtained if p is indecomposable and $\deg(p) \neq 5$ (Fried). Neftin explained new results in this context, obtained by Zieve and him, together with some further results, by König and him, on the decomposable case using Ritt's theorems on decompositions of rational functions.

Dettweiler presented a recent joint work with Collas and Reiter on the category of perverse sheaves over elliptic curves which is Tannakian with respect to the convolution product. He showed how this allows some classical Galois realization methods to go *beyond the rational rigidity barrier.* After presenting an alternative to Hilbert's Irreducibility Theorem in terms of a Mordell-Weil rank criterion for local systems, he explained how the convolution approach relies on computations of the monodromy within elliptic braid groups, and provided some examples.

Cadoret reported on a variant with ultraproduct coefficients of the fundamental theorem of *Weil II for curves.* She first recalled Deligne's theory of Frobenius weights for lisse $\overline{\mathbb{Q}}_\ell$-coefficients and its application to the semisimplicity of geometric monodromy. Considering the issue of extending the semisimplicity of geometric monodromy to integral and modulo-ℓ coefficients arising from arbitrary compatible systems of lisse $\overline{\mathbb{Q}}_\ell$-sheaves, she motivated the introduction of a category of "almost-curve tame" ultraproduct coefficients. She explained to what extend this category is well-behaved and why it can be used to develop a theory of Frobenius weights paralleling the one of $\overline{\mathbb{Q}}_\ell$-coefficients. Finally, she elaborated on further applications of her theory such as torsion freeness and unicity of integral models in arbitrary compatible systems of lisse $\overline{\mathbb{Q}}_\ell$-sheaves or the construction of the automorphic to Galois direction of a Langlands correspondence with ultraproduct coefficients.

3. Galois Anabelian and Homotopical Geometry

Broadly, anabelian geometry deals with the arithmetic properties of finite étale covers of a space, which relies on the study of Geometric Galois representations of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ in the étale fundamental group of spaces. With the goal of identifying arithmetic invariants within topological constructions, it relies on the essential example of moduli stack of curves which provides a computational context – for example via the Grothendieck-Teichmüller theory–, and also a connection to the theory of motives. New fruitful research directions to be pursued appeared during the workshop, which includes simplical and homotopical methods via étale topological type, (unstable) motivic homotopy theory and operads, as well as higher genus and stack considerations.

Nakamura reported on problems and recent progress of the study of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ via the tower of the universal monodromy representations in the moduli spaces of marked curves. After some reminder on Anderson-Ihara Beta function, Soulé and Kummer characters in meta-abelian quotients and their relation in genus 0 to the profinite Grothendieck-Teichmüller group $\widehat{GT}$, he presented his recent work in genus 1 on Eisenstein cocycles and Enriquez' group $\widehat{GT}_{ell}$, and how they are the analogs of the genus 0 constructions.

Schmidt and Stix presented their joint work: they showed how to use étale homotopic methods and Mochizuki's work to deduce the existence of anabelian Zariski-neighbourhoods in smooth variety of any dimension. Schmidt first explained the necessary requirements and difficulties in Artin-Mazur-Friedlander pointed-unpointed étale homotopy theory, then Stix presented the proof based on Tamagawa's idea of Jacobian approximation of rational points via the existence of a certain retract.

Collas presented the divisorial and stack inertia arithmetic contexts of the moduli stacks of unordered marked curves, and their key role in Geometric Galois representations, anabelian geometry, and mixed Tate motivic theory. After discussing his joint work with Maugeais on the Tate-like Galois action on cyclic stack inertia, its connection to Inverse Galois theory via Hurwitz spaces and the fundamental role of Harris/Deligne-Mumford compactification, he presented a work in progress on how the homotopical approach leads to stacky constructions in Morel-Voevodsky's motivic homotopy category, as well as to computable stack periods.

Litt presented his work on étale Geometric Galois representations via the Malčev completion of the pro-ℓ-completion of fundamental group of algebraic varieties. He reported on a joint work with Betts on the semisimplicity of Frobenius actions on ℓ-adic and p-adic (log-crystalline) pro-unipotent fundamental groups, with application to the irreducibility of Kim-Selmer varieties in Chabauty-Kim theory, and results on the representation theory of arithmetic fundamental groups. As an archimedean analogue, he produced and explained the role of canonical paths in the computation of iterated integrals, and used them to recover various special functions of Bloch-Ramakrishnan-Zagier.

Borne discussed some joint works with Biswas and Vistoli on the construction of cyclic ramified covers of curves via the stack of roots and the notion of weighted parabolic sheaves. After a recollection on Mehta-Seshadri's work on weighted parabolic bundle, Nori's fundamental group scheme, and Noohi's automorphisms uniformisation criterion for stacks, he presented his result in terms of Nori uniformization Tannakian criterion.

Quick presented his results on obstructions for the algebraicity of topological cycles via cobordism and simplicial homotopy theory of presheaves. He first showed how the stable motivic homotopy theory for smooth varieties over finite fields allows the construction of cobordism invariants that can be used to detect and construct non-algebraic classes. He also mentioned some arithmetic prospects in Arakelov theory.

Wickelgren reported on a joint work in progress with Westerland on the π_1-sections for configuration spaces. Their approach relies on the use of tangential base points in connection to the parenthesized braid group operad PaB. After presenting how they are linked together, she recalled the Drinfel'd-Fresse definition of $\widehat{GT}$ as homotopy automorphism group of $\widehat{PaB}$. This enriches the monoidal structure on Hurwitz spaces given by juxtaposing conjugacy classes with the intent to algorithmically produce families of special points on Hurwitz spaces.

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Mini-Workshop: Arithmetic Geometry and Symmetries around Galois and Fundamental Groups

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Abstracts

Specialization and Localization in Inverse Galois Theory

Pierre Dèbes

Specialization and localization have always been at the core of Inverse Galois Theory (IGT): Hilbert's Irreducibility Theorem, the Noether Program, the Grunwald Problem, the Hurwitz moduli space approach are prominent milestones.

We focus on the situation that the base field is a number field. The goal of the talk, based on the diagram in §3, was twofold. First, explain how we see the two operations, specialization and localization, and the somehow inverse ones that are parametrization and lifting still structure the area. Second, present a series of new results which are part of a joint program with J. Koenig, F. Legrand and D. Neftin. In various situations, these results roughly show that the sets of Galois extensions obtained by specialization or/and localization from natural sets of geometric Galois covers of fixed group G (singletons, families, moduli spaces) are big (in some density sense), but also cannot be too big (e.g. they generally do not contain all Galois extensions of group G).

The diagram in §3 displays a number of IGT properties for a finite group G over a given number field k. The abbreviations used for these properties refer to our two part glossary where they are fully defined: §1 for the classical ones and §2 for the more recent ones. For example:

IGP (Inverse Galois Problem): *There is a Galois extension E/k of group G.*

Left side of our diagram is more geometric than the right side; indeterminates are the recognition sign. Specialization connects the two. We specialize a k-regular Galois extension $F/k(T)$ or the corresponding k-cover $f : X \to \mathbb{P}^1_T$ in two ways:
- for $t_0 \in k$, F_{t_0}/k, also denoted by f_{t_0}, is the classical specialized extension of F at t_0: the residue field extension at some prime ideal above t_0 in the extension $F/k(T)$. As number fields are Hilbertian (HIT), the extension F_{t_0}/k is Galois of group G for "many" $t_0 \in k$.
- if $T_0 \in k(U) \setminus k$, $f_{T_0} : X_{T_0} \to \mathbb{P}^1_U$ is the pull-back of f along $T_0 : \mathbb{P}^1 \to \mathbb{P}^1$. As $k(U)$ is Hilbertian, for "many" $T_0 \in k(U)$, X_{T_0} is connected and the function field extension $k(X_{T_0})/k(U)$, which is equivalently obtained by specializing T to $T_0(U)$ in $k(X)/k(T)$, is Galois of group G.

1. Classical properties

RIGP (Regular IGP): *There is a k-regular Galois extension $F/k(T)$ (k-regular: $F \cap \overline{k} = k$), or, equivalently, a k-regular Galois cover $f : X \to \mathbb{P}^1_k$, of group G.*

HIT (Hilbert Irreducibility Thm): *For every polynomial $P(T, Y)$, irreducible in $k(T)[Y]$, there are infinitely many $t_0 \in k$ such that $P(t_0, Y)$ is irreducible in $k[Y]$.*

G **has a parametric extension** $F/k(T)$: *There is a Galois extension $F/k(T)$ of group G that is k-parametric, i.e., every Galois extension E/k of group contained in G is the specialized extension F_{t_0}/k of $F/k(T)$ at some point $\mathrm{t}_0 \in k$.*

G **has a generic extension** $F/k(T)$: *There is a Galois extension* $F/k(T)$ *of group* G *such that* $FK/K(T)$ *is* K*-parametric for every field extension* K/k.

G **has a generic extension** $F/k(T_1, \dots, T_s)$: as above with T replaced by $T_1, \dots, T_s$ and t_0 by $\mathbf{t}_0 = (t_{01}, \dots, t_{0s})$.

Noether: *If* $\mathbf{Y} = Y_1, \dots, Y_d$ *are* $d = |G|$ *indeterminates, the fixed field* $k(\mathbf{Y})^G$ *of* G *in* $k(\mathbf{Y})$*, with* $G \hookrightarrow S_d$ *acting via its regular representation, is a pure transcendental extension of* k*. Equivalently the* Noether variety $\mathbb{A}^d/G$ *is* k*-rational.*

Noether has WWA: *The variety* $V = \mathbb{A}^d/G$ *has the* Weak Weak Approximation property*: there is a finite set* $\mathcal{S}_{\text{exc}}$ *of finite places of* k *such that for every finite set* $\mathcal{S}$ *of finite places of* k *s.t.* $\mathcal{S} \cap \mathcal{S}_{\text{exc}} = \emptyset$*, the set* $V(k)$ *is dense in* $\prod_{v \in S} V(k_v)$.

2. More recent properties

W-Grunwald: *There is a finite set* $\mathcal{S}_{\text{exc}}$ *of finite places of* k *such that for every finite set of Galois extensions* E_i/k_{v_i} *of group* $H_i \subset G$ *with* $v_i \notin \mathcal{S}_{\text{exc}}$ $(i = 1, \dots, m)$*, there is a Galois extension* E/k *of group* G *such that* $Ek_{v_i}/k_{v_i} = E_i/k_{v_i}$*,* $i = 1, \dots, m$. (The original property, with $\mathcal{S}_{\text{exc}} = \emptyset$, is denoted by **Grunwald**).

W-Grunwald$^{\text{ur}}$: *The property* **W-Grunwald** *above but with the additional condition that the extensions* E_i/k_{v_i}*,* $i = 1, \dots, m$*, are unramified.*

BB-N (Beckmann-Black lifting property): *For the given* $N \geq 1$ *and every* N *Galois extensions* $E_1/k, \dots, E_N/k$ *of group contained in* G*, there is a* k*-regular Galois extension* $F/k(T)$ *of group* G *that specializes to the extensions* $E_1/k, \dots, E_N/k$.

G **has a regularly parametric extension** $F/k(T)$: *The corresponding* k*-regular Galois cover* $f : X \to \mathbb{P}^1_k$ *(of function field extension* $F/k(T)$*) has this property: every* k*-regular Galois cover* $g : Y \to \mathbb{P}^1_k$ *of group* G *is some rational pullback* $f_{T_0} : X_{T_0} \to \mathbb{P}^1_k$ *of* f *(for some* T_0 *in* $k(U) \setminus k$*). Equivalently, every* k*-regular Galois extension* $L/k(U)$ *of group* G *can be obtained from the* k*-regular Galois extension* $F/k(T)$ *by specializing* $F(U)/k(U,T)$ *at some* T_0 *in* $k(U) \setminus k$.

reg-BB-N (Regular Beckmann-Black lifting property): *For the given* $N \geq 1$ *and every* N k*-regular Galois covers* $g_1, \dots, g_N$ *of* $\mathbb{P}^1$ *of group* G*, there is a* k*-regular Galois cover* f *of* $\mathbb{P}^1$ *of group* G *such that* $g_1, \dots, g_N$ *are rational pullbacks of* f.

Malle: *The number* $N(G, y)$ *of sub-Galois extensions* E/k *of* $\overline{k}$ *of group* G *and discriminant of norm* $|N_{k/\mathbb{Q}}(d_E)| \leq y$ *satisfies*

$$c_1\, y^{a(G)} \leq N(G, y) \leq c_2\, y^{a(G)+\varepsilon} \qquad \textit{for every } y \geq y_0$$

Here $a(G) = (|G|(1 - 1/\ell))^{-1}$ *with* ℓ *the smallest prime divisor of* $|G|$ *and* c_1*,* c_2*,* $y_0 > 0$ *depend on* G *for* c_1 *and on* G*,* ε *for* c_2 *and* y_0.

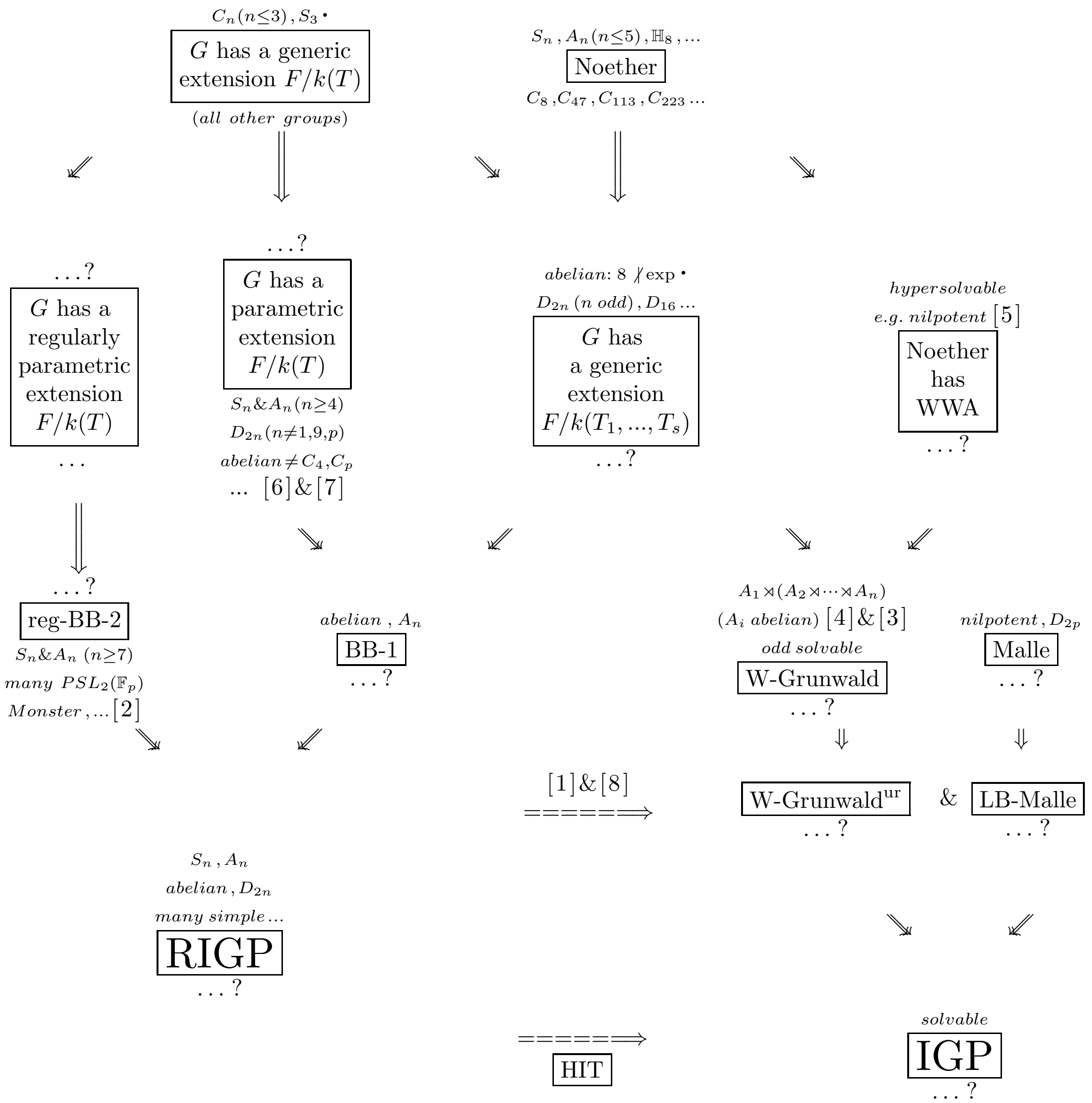
LB-Malle (Lower Bound part of **Malle** conjecture):

$$N(G, y) \geq y^{\alpha(G)} \qquad \text{for every } y \geq y_0$$

Here $\alpha(G)$ *and* y_0 *are positive constants depending on* G.

3. The diagram

Groups appearing above a given box satisfy the corresponding property, those appearing below do not, both over $k = \mathbb{Q}$. The symbol ... (resp. $\bullet$) means that the list is open (resp. closed), possibly as a question if used with a question mark.



The main recent results are those assertions about groups satisfying or not a property which come with a reference. References are given below.

Complement: We refer to `http://math.univ-lille1.fr/~pde/pub.html` – item 57 for the sequence of slides (converging to the diagram) used during the talk and for a more detailed description of our research project.

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Arithmetic and Homotopy of Moduli Stacks of Curves

Benjamin Collas

Let $\mathcal{M}_{g,[m]}$ be the moduli stacks of genus g curves with m-unordered marked points, that we consider endowed with their complementary divisorial and stack stratifications. The former is a *stratification at infinity* and is given by the topological type (g', m') of curves in the Deligne-Mumford compactification of stable curves $\mathfrak{M}_{g,[m]}$, while the later is local and is given by the flat *stratification by the automorphisms of curves.*

As $\mathbb{Q}$-stacks, the moduli spaces accept some Geometric Galois Representations

$$\text{(GGR)} \qquad \rho_{\vec{s}}\colon \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Aut}[\pi_1^{et}(\mathcal{M}_{g,[m]} \otimes \bar{\mathbb{Q}}, \vec{s})]$$

where $\vec{s}\colon \bar{\mathbb{Q}}\{\{\mathbf{q}\}\} \to \mathcal{M}_{g,[m]}$ is a tangential base point associated to a chosen $\mathbb{Q}$-rational structure on $\mathcal{M}_{g,[m]}$ – tangential base points define at once some geometric base points for the fundamental group and some $\mathbb{Q}$-rational π_1-sections. They allow to bypass Falting's limitation on rational markings on curves, and also to benefit for the study of (GGR) from the rational Knudsen-Mumford lower dimensional (g', m')-embeddings in terms of limit Galois representations. The stack structure, via Hurwitz spaces, draw some connections between Geometric Galois Representations and the Regular Inverse Problem.

By providing accessible geometries that capture key arithmetic properties, the moduli stacks of curves are fundamental spaces in arithmetic geometry, in anabelian geometry – e.g. the unordered marked $\mathcal{M}_{0,m}$ are anabelian–, and in motivic theory – see the category of Mixed Tate motives.

We report on recent results on the stack arithmetic of these spaces, and on works in progress on the use of homotopical methods: how this leads to a motivic interpretation of these higher symmetries, and to a finer understanding of the operadic and arithmetic properties of the divisorial stratification.

1. Stack Arithmetic of Curves (joint with S. Maugeais.)

The Deligne-Mumford stack structure of $\mathcal{M}_{g,[m]}$ is recovered through the *inertia group sheaf* $I_{\mathcal{M},x}$, which for a given point $x\colon \mathrm{Spec}\,\mathbb{Q} \to \mathcal{M}_{g,[m]}$ geometrically identifies with the finite automorphism group $I_{\mathcal{M},\bar{x}} \simeq Aut_{\mathbb{C}}(C_{\bar{x}})$ of a Riemann surfaces represented by $\bar{x}$. By Noohi's uniformization Theorem, it follows that

$$I_{\mathcal{M},\bar{x}} \hookrightarrow \pi_1^{et}(\mathcal{M}_{g,[m]} \otimes \bar{\mathbb{Q}}, \vec{s}),$$

i.e. the automorphisms of curves form some local *ghost loops subgroups* of the étale fundamental group. This raises the question of *describing the* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$*-action of* (GGR) *on the stack inertia groups of* $\mathcal{M}_{g,[m]}$, which is indeed of local-vs-global nature.

Because they form the first non-trivial stack stratas, we consider in what follows the case of the cyclic stack inertia group $I_{\mathcal{M},\bar{x}} \simeq \langle\gamma\rangle$ that we study via their associated special loci.

1.1. Special Loci, Irreducible Components of Hurwitz Spaces.

Let thus $\mathcal{M}_{g,[m]}(G)$ denote the special loci attached to a finite order group G: $\mathcal{M}_{g,[m]}(G)$ is the $\mathbb{Q}$-stacks of curves C/S that admit a faithful G-action $G \hookrightarrow Aut_S(C)$. In terms of Galois action, we notice that an irreducible components of $\mathcal{M}_{g,[m]}(G)$ is *a priori* defined over a number field K: this implies the stability of the $\pi_1^{et}(\mathcal{M}_{g,[m]} \otimes \bar{\mathbb{Q}})$-conjugacy classes of G under the action of the absolute Galois group $\mathrm{Gal}(\bar{K}/K)$. In genus 0 and for G cyclic, one proves that every such component is of the form $\mathcal{M}_{0,[m]+k}$ – with m permuted points and k fixed points, $k \in \{0,1,2\}$ – thus defined over $\mathbb{Q}$. This implies that the $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$*-action stabilizes the conjugacy classes of the cyclic stack inertia in genus* 0.

In higher genus, this raises the question of finding *arithmetic coarse invariants* of the irreducible components of $\mathcal{M}_{g,[m]}(\gamma)$. The identification of the normalization of the special loci $\widetilde{\mathcal{M}}_{g,[m]}(G) \simeq \mathcal{M}_{g,[m]}[G]/Aut(G)$ as quotient of the Hurwitz space of G-covers $\mathcal{M}_{g,[m]}[G]$ reduces this question to the characterization of S-families of G-cover, which draws a first connection with the geometry of Hurwitz spaces.

For $G = \langle\gamma\rangle$, an answer is provided in terms of étale cohomology with the definition on the geometric fibers of some branching datas **kr**, which allows to establish:

Theorem ([5] - Th. 4.3)**.** *The stack of* γ*-special loci admits a finite decomposition in irreducible components given by:*

$$\mathcal{M}_{g,[m]}(\gamma) = \coprod \mathcal{M}_{g,[m],\mathbf{kr}}(\gamma),$$

where $\mathcal{M}_{g,[m],\mathbf{kr}}(\gamma)$ *denotes the* $\mathbb{Q}$*-stack of curves inducing* γ*-covers with given branching datas* **kr***.*

In a similar way to the Deligne-Mumford proof of the irreducibility of $\mathcal{M}_{g,[m]}$, this result relies on the existence of a Teichmüller space that parametrizes unmarked curves with given **kr**-datas. An arithmetic property of G-covers appears for the general case, to ensure that *the field of moduli K of certain γ-covers is indeed a field of definition* – see (Seq/Split)-condition of [8].

As in genus 0, this implies this time in every genus the conjugacy-stability of $I_{\mathcal{M}} = \langle\gamma\rangle$ under a certain local $\mathrm{Gal}(\bar{K}/K)$-action. The comparison with the global $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-action of (GGR) and its complete description requires the use of the divisorial stratification.

1.2. **Inertial Limit Galois Actions, a Tate-like Action.** Let η denote the generic point of an irreducible component $\mathcal{M}_{g,[m],\mathbf{kr}}(G)$ of a special loci $\mathcal{M}_{g,[m]}(G)$, and let $\kappa(\eta)$ denote its residue field. After rigidification, one obtains an *inertial Galois action* ρ^I_η: $\mathrm{Gal}[\overline{\kappa}(\eta)/\kappa(\eta)] \to Aut(I_\eta)$ on the generic stack inertia group $I_\eta > G$ of the component. Since a tangential structure on $\mathcal{M}_{g,[m]}$ is a formal neighbourhood of a singular stable curve, the comparison of this local Galois action to the global (GGR) is provided by a specialization result for Deligne-Mumford stacks – see §3.2 and §4.2 of [6]:

> *For any component of cyclic special loci, there exist a K-point of $\mathcal{M}_{g,[m],\mathbf{kr}}(G)$ and a tangential base point $\vec{s}$ of $\mathcal{M}_{g,[m]}$, such that the* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$*-action $\rho_{\vec{s}}$ of* (GGR) *induces the inertial action ρ^I_η.*

For G cyclic, the local inertial Galois action can indeed be proven to be given by a certain extension of the field of definition K discussed in the previous section. From the $\mathbb{Q}$-definition of the cyclic irreducible components given by the Theorem above, one establishes more precisely:

Theorem ([5] – Th. 5.4 & [6] – Cor. 4.6, Th. 4.8)**.** *The* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$*-representation of* (GGR) *induces a χ-conjugacy* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$*-action on the cyclic stack inertia of $\mathcal{M}_{g,[m]}$. For $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$:*

$$\sigma.\gamma = h^{-1}_{\gamma,\sigma}.\gamma^{\chi_\sigma}.h_{\gamma,\sigma} \text{ where } h_{\gamma,\sigma} \in \pi_1^{et}(\mathcal{M}_{g,[m]} \otimes \bar{\mathbb{Q}}, \vec{s}). \tag{1}$$

This result surprisingly depends on the geometry of the Hurwitz components and of their stable compactification: if the **kr**-data corresponds to a class of G-covers whose G-isotropy groups span G, the result then follows from Fried's branch cycle argument; the general case requires a fine deformation argument of G-covers which allows to *compare G-stratas of $\mathfrak{M}_{g,[m],\mathbf{kr}}$ and $\mathfrak{M}_{g-1,[m]+2,\mathbf{kr}'}$ of different topological and ramification types.* This comparison requires the choice of tangential $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-actions which are compatible at the level of the stack inertia groups. We refer to this process under the term *inertial limit Galois action.*

1.3. **Towards the Stack Arithmetic of Higher Stratas.** By analogy with the divisorial arithmetic of $\mathcal{M}_{g,[m]}$, this Tate-like action motivates further studies of the higher arithmetic of the stack stratification. The inertial limit Galois action provides for example a mean of comparing the conjugacy factors of the $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-action between *stratas of different topological types.*

We mention two immediate directions of research:

(i) determines some discrete arithmetic invariants *of the irreducible components of $\mathcal{M}_{g,[m]}(G)$ for non-abelian groups G;*

(ii) complete the description of the $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$*-action Eq.* (1) *by determining* relations for the conjugacy factors.

As shown in the case of the cyclic strata, progress will certainly rely on a fine understanding of the arithmetic of G-covers and of the Hurwitz spaces. As more concrete examples, let us mention for (i) the use in [7] of H^2-data in terms of mixed cohomology that complete the monodromy invariants for cyclic extensions. For (ii), relations could come either from the comparison of different stack inertia groups and follow either from (i), or from the use of the topological type: equations ($\star$)-($\star\star$) of [21], and their generalization (R) in [3], are some examples of such comparisons – in these cases of $\mathcal{M}_{0,[5]}(\mathbb{Z}/2\mathbb{Z})$ with respect to $\mathcal{M}_{0,[4]}$, of $\mathcal{M}_{0,[6]}(\mathbb{Z}/3\mathbb{Z})$ with respect to $\mathcal{M}_{0,[4]}$, and of $\mathcal{M}_{1,[2]}(\mathbb{Z}/2\mathbb{Z})$ with respect to $\mathcal{M}_{1,1}$ –, see also [19].

In another direction, and always by analogy with the Galois divisorial arithmetic, the Tate-like action of Eq. (1) raises the question of a motivic interpretation of this result.

2. Motivic Stack Considerations

Let k be a number field, and let $MT(k)$ denote the category of Mixed Tate motives over k. This is a Tannakian category of group G_{MT}, neutralized by the canonical Adams weight fibre functor, whose properties are tightly related to the divisorial arithmetic of the moduli schemes $\mathcal{M}_{0,m}$: it is motivically generated by the $\mathcal{M}_{0,m}$s [2], a $p+2q$-motivic weight comes from a p-codimensional component of $\mathfrak{M}_{0,m}$ and of a q-Tate twist, relation between periods are induced by the Knudsen morphisms [22].

Motivated by the Tate-like $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-action on the cyclic stack inertia, we present how Morel-Voedsky's motivic homotopy provides a convenient context where to *develop motivic properties for the stack inertia that are Tate-compatible.* We present the main difficulties, illustrate the approach in the case of representable, and generally refer to the forthcoming [4] for the case of stacks.

2.1. **Motivic Homotopy Theory and Stacks.** The motivic context is given by the unstable-stable motivic categories $\mathcal{H}(\mathbb{Q}) \rightleftarrows \mathcal{SH}(\mathbb{Q})$, respectively defined as the homotopy categories of spaces $Sp(\mathbb{Q}) = sPr(Sm_{\mathbb{Q}})$ and $\mathbb{P}^1$-spectras over $Sp(\mathbb{Q})$ endowed with their $\mathbb{A}^1$-local injective model category with respect to the étale topology. The category $\mathcal{SH}(\mathbb{Q})$ is triangulated, equivalent to Voevodsky's $DM(\mathbb{Q})$, and has the Lefschetz motive inverted at the level of morphisms as a result of the $\Sigma_{\mathbb{P}}$-stabilization.

A first connection between the $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-stack inertia framework and the $\mathcal{H}(\mathbb{Q})$ context is given by replacing the Galois category formalism of SGA1 by Artin-Mazur and Friedlander [12] simplicial étale topological type one: we attache to the stack $\mathcal{M}_{g,[m]}$ – and similarly to its stack inertia $I_{\mathcal{M}}$ – first a pro-space $\{\mathcal{M}_{g,[m]}\}_{et}$, then by Isaksen's étale realization functor [16] a $\mathbb{A}^1$-type $\{\mathcal{M}_{g,[m]}\}_{\mathbb{A}^1}$ in $sPr(Sm_{\mathbb{Q}})$.

A more intrinsic context is indeed given by considering $\mathcal{M}_{g,[m]}$ as a specific object of $sPr(Sm_{\mathbb{Q}})$, whose Giraud's descent property is characterized in terms of hypercovers [10]: within this context, the (group) stack inertia identifies to a derived loop space $I_{\mathcal{M}} \simeq RHom(\mathbb{S}^1, \mathcal{M})$ (resp. to a homotopy group sheaf).

This approach places in particular the stack motivic study of $\mathcal{M}_{g,[m]}$ within Toën-Vezzosi's Homotopical Algebraic Geometry theory (HAG) [24]. Since $\mathbb{P}^1 \simeq \mathbb{S}^1 \wedge \mathbb{G}_m$ within $\mathcal{H}(\mathbb{Q})$, and since $\mathbb{G}_m$ is the Lefschetz motivic divisorial monodromy of $\mathcal{M}_{g,[m]}$, one concludes that:

> *the motivic homotopy theory of $\mathcal{M}_{g,[m]}$ gives a favourable context where to illustrate how the $\mathbb{S}^1$-loops encode the ghost 2-structure of motivic spectras.*

2.2. **Mixed Tate Motives and Beyond.** By contrast, we now illustrate the relevance and the non-triviality of this approach on the specific examples of representable stacks $\mathcal{M}_{0,m} \in Sm_{\mathbb{Q}}$. On one side, the homotopical Mixed Tate framework is given by [17] and follows Spitzweck's representation theorem for cells modules over an Adams graded cycles algebra, as provided by Bloch-Kriz's $\mathcal{N}_{BK}$ – see op.cit.

On the other side, the HAG context is provided by Toën's Spec-functor of [23] Spec : $Alg_{\mathbb{Q}}^{\triangle^o} \to sPr(\mathbb{Q})$, and by Hitchin's Quillen equivalence $Alg_{\mathbb{Q}}^{\triangle^o} \rightleftarrows cdga_{\mathbb{Q}}$. As a result, since the Bar complex is an homotopy colimit of diagrams, one obtains that (the prounipotent part of) G_{MT} *is weakly equivalent to the derived loop space of* $Spec\mathcal{N}_{BK}$. Since the prounipotency of the homotopy group sheaf characterizes the schematic image in $sPr(Sch_Q)$, notice that a similar construction for $\mathcal{M}_{g,[m]}$ that realizes $I_{\mathcal{M}}$ as motivic object requires to enlarge the aforementioned Quillen equivalence. This lead to the DAG-context that allows to capture the $\mathbb{S}^1$-motivic inertia.

Despite the relevance of this approach, a final and fundamental difficulty is still given by the *question of a neutralizing fibre functor*, which must induces *non-2-trivial* geometrical de Rham-Betti comparison isomorphisms: as an étale-locally quotient stack, the rational cohomology of $\mathcal{M}_{g,[m]}$ is whose of its coarse scheme $M_{g,[m]}$. An answer is here again provided by the HAG context, that gives computable *stack inertia periods* in terms of iterated integrals that are compatible with the Tannakian weight – via $\mathcal{M}_{0,m} \to \mathcal{M}_{0,[m]}$ and the choice of tangential structures as involved for Galois and $\mathcal{M}_{g,[m],\mathbf{kr}}(\gamma)$ see [4].

3. Arithmetic of Operads

We present how the divisorial stratification of $\mathcal{M}_{g,[m]}$ supports some fundamental arithmetic and geometric properties. The question is two fold: from the geometric point of view, it is related to the fundamental question of defining *on smooth objects an operadic structure that is given by singular degeneracies*; from the arithmetic point of view it is related to Grothendieck-Teichmüller theory that is to determine *how* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ *is encoded within the geometric symmetries of* $\mathcal{M}_{g,[m]}$. More precisely, GT theory provides a finitely presented group $\widehat{GT}$ that contains $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and factorizes the (GGR) [9, 14].

We report on how the homotopy theory of spaces and the notion of tangential structures on $\mathcal{M}_{0,m}$ provides some insight on these questions. We were recently informed that K. Wickelgren and C. Westerland developed an independent and similar approach in the case of configuration spaces – see *K. Wickelgren, Operad Structure on* $Conf_n$ in this volume.

3.1. **Genus Zero Moduli Spaces.** Motivation for this work comes from the recent operadic result of B. Fresse and G. Horel [11, 13], that interprets Drinfel'd definition of $\widehat{GT}$ in terms of operad in prospaces.

Theorem (Fresse, Horel)**.** *The group* $\widehat{GT}$ *is isomorphic to the homotopy group of (pro) little* 2*-discs operads* $E_2^\wedge$*.*

Here $E_2^\wedge$ denotes either the Sullivan rational model or the completion in prospaces. Their fundamental groups are respectively given by the Malčev and the profinite completion of the parenthesized braid operad in groupoids $PaB^\wedge$. Since the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is contained in $\widehat{GT}$, this induces a $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-action on $PaB^\wedge$ that is group-theoretically defined and from topological origin.

We deal with the question of recovering this result – more precisely the refinement of [1] – in terms of an arithmetic $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-action at the level of a $\mathbb{Q}$-operadic structure on $\mathcal{M}_{0,m}$ then $\mathcal{M}_{0,[m]}$. To fix some (GGR) or $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-actions requires to specify some $\mathbb{Q}$-tangential structures on $\mathcal{M}_{0,m}$, i.e. to choose a formal neighbourhood $\mathrm{Spec}\,\mathbb{Q}[[\mathbf{q}]] \to \mathfrak{M}_{0,m}$ of some singular curves in the Deligne-Mumford compactification of $\mathcal{M}_{0,m}$ [15, 6].

In terms of operads, the choice of a tangential structure on the spaces defines the geometric operadic composition morphisms and ensures that they are $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-equivariant. We obtain a refinement of the Theorem above, that is from arithmetic origin.

Theorem (conjecture)**.** *The set of* $\mathbb{Q}$*-tangential structures* $\{\vec{s}\}$ *over* $\mathcal{M}_{0,m}$ *defines an operad* $\mathcal{M}_{\mathbb{Q}} = \{(\mathcal{M}_{0,m}, \vec{s})\}_{m,\vec{s}}$ *in* $\mathbb{Q}$*-Proschemes whose geometric étale homotopy type is endowed with a* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$*-action – see Eq.* (GGR)*.*

The approach is based on Grothendieck's formal-algebraization deformation theory for curves; the operad $\mathcal{M}_{\mathbb{Q}}$ is defined in terms of Friedlander Artin-Mazur étale topological type. As a result, we obtain after completion an operad in prospaces $\mathcal{M}_{\mathbb{Q}}(\bar{\mathbb{Q}})^\wedge$ which encodes some arithmetic a priori not distinguished by $\widehat{GT}$. More precisely, $\mathcal{M}_{\mathbb{Q}}(\bar{\mathbb{Q}})^\wedge$ is weakly equivalent to the framed little 2-discs operads $FE_2^\wedge$, while $\widehat{GT} \simeq Aut^h(E_2^\wedge)$ and $Aut^h(E_2^\wedge) \simeq Aut^h(FE_2^\wedge)$.

This property can already be seen at the level of configuration spaces and braids groups – PaB is a model for E_2 –, by providing a $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-action on an extension of $PaB^\wedge$ that descends to the classical $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-action of Ihara-Matsumoto on Braids groups [14].

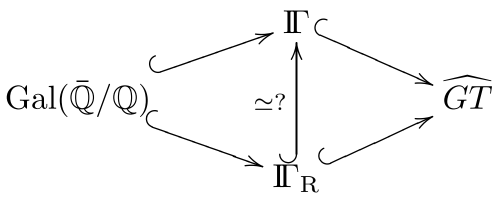
Following Mac Lane's coherence Theorem, this approach provides in particular a computable $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-action which is entirely defined in 1, 2 and 3-arity. Let us mention that Serre's anabelian bonté of $\mathcal{M}_{0,m}$ plays a key role in defining the operadic composition on $\mathcal{M}_{\mathbb{Q}}$.

3.2. **Towards Higher Symmetries.** Because this approach is close to the geometry of curves and already provides some refinement in genus zero, it motivates and gives access to further developments in the direction of *stack and higher genus symmetries.*

In terms of stack, this motivates our work in progress on defining a similar rational operad for the genus zero moduli stack of curves with unordered marked points $\mathcal{M}_{0,[m]}$. The ∞-model category of [20] provides the necessary context to connect the tangential arithmetic and the 2-structure of $\mathcal{M}_{0,[m]}$. In this case, the homotopy groups are *not torsion-free* – unlike the braid groups in the previous case – but contains some stack torsion like $\pi_1^{orb}(\mathcal{M}_{0,[m]}(\mathbb{C}))$ does.

Regarding the moduli spaces in higher genus, the operad $\mathcal{M}_{\mathbb{Q}}$ already comes with additional commutativity- and associativity-like constraints at the level of braided monoidal category. This provides additional GT-like equations, which while already included in the pentagon-hexagon equations I, II and III defining $\widehat{GT}$, motivates in higher genus the study of a potential refinement of the original group $\mathbb{\Gamma}$ of [18].

(3.2.1) Define in higher genus a Grothendieck Teichmüller group $\mathbb{\Gamma}_{\mathrm{R}}$ *given by relations based on the tangential refined associativity and commutativity constraints:*



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ℓ-adic representationa, the O(pen)I(mage)T(heorem) and the R(egular) I(nverse)G(alois)P(roblem)

Michael D. Fried

The talk started with the fiber product construction of both the *arithmetic* and *geometric* Galois closures of a cover of the projective line $\mathbb{P}^1$. At the beginning that line was $\mathbb{P}^1_z$, the z-line from Graduate complex variables.

At the end of the talk, the line was $\mathbb{P}^1_j$, the classical j-line used as a surrogate for the compactification of $U_r/\mathrm{PGL}_2(\mathbb{C})$, with U_r the space of distinct unordered points on $\mathbb{P}^1_z$. These we regarded as the configuration space for reduced Hurwitz spaces of r-branch point covers.

In between the beginning and end we took two of the premier unsolved problems in the last few decades – the **RIGP** and generalizing Serre's **OIT** and showed how these two problems are halves of a paradigm that joines by a generalization – **M**(odular)**T**(ower)s – of the classical tower of modular curves that comes from the universal abelianized cover of any finite group G with at least one prime ℓ for which it is ℓ-perfect.

The talk featured the value of **MT**s as a gadget, appropriate to the **RIGP**, for visualizing, connecting and analyzing arithmetic geometry problems.

1. Nielsen Classes and Galois Closures of Covers

Applications often require sophisticated understanding Galois closures and moduli properties of collections of sphere covers. The key interpretation tool is the action of the *braid group* on *Nielsen classes* of covers attached to any finite group G and collection C of ($r \geq 3$) conjugacy classes of G. The notation for Nielsen classes is $\mathrm{Ni}(G,\mathrm{C})$ Depending on the equivalence relation on the covers, the result of this braid action by through the Hurwitz monodromy group, H_r, is a Hurwitz space, denoted $\mathrm{H}(G,\mathrm{C})$. Extra decoration indicates the equivalence choice. The two main types of equivalence are *absolute* (the one most are accustomed to) and *inner* (which figures in the **RIGP**); then *reduced* versions (modding out by $\mathrm{PGL}_2(\mathbb{C})$ as above) for both.

Lots of definitions call for an example that illustrates the theory – especially around that braid group action – that has been developed since 2002. We used $(A_4, \mathrm{C}_{\pm 3^2})$ as an example. By §3, this appears as a good choice; the conjugacy class notation indicates the repetition twice of each of the 3-cycle classes in A_4.

The H_4 (in this case) orbits, denoted by O, and cusp orbits, denoted by ${}_{\mathbf{c}}O$, orbits of q_2 – where q_i is the image in H_4 of the ith string twist braid element from B_4, $i = 1, 2, 3$ – acting on *reduced Nielsen classes* combine in a graphical device. We call this the **sh***(ift)-incidence matrix*, named for the shift

$$(g_1, \dots, g_r) \mapsto (g_2, \dots, g_r, g_1) \text{ on product-one } r\text{-tuples comprising } \mathrm{Ni}(G,\mathrm{C}).$$

The device pairs *cusps* of reduced Hurwitz spaces illustrating the following theorem [3, §4.2]. Here $\mathcal{Q}'' = \langle q_1 q_3^{-1}, \mathbf{sh}\rangle$ and the † superscript indicates the equivalence choice of absolute or inner.

Ni_0^+ Orbit	${}_{\mathbf{c}}O^4_{1,1}$	${}_{\mathbf{c}}O^2_{1,2}$	${}_{\mathbf{c}}O^3_{1,3}$
${}_{\mathbf{c}}O^4_{1,1}$	1	1	2
${}_{\mathbf{c}}O^2_{1,2}$	1	0	1
${}_{\mathbf{c}}O^3_{1,3}$	2	1	0

Ni_0^- Orbit	${}_{\mathbf{c}}O^4_{2,1}$	${}_{\mathbf{c}}O^1_{2,2}$	${}_{\mathbf{c}}O^1_{2,3}$
${}_{\mathbf{c}}O^4_{2,1}$	2	1	1
${}_{\mathbf{c}}O^1_{2,2}$	1	0	0
${}_{\mathbf{c}}O^1_{2,3}$	1	0	0

Theorem 1 (Riemann-Hurwitz for j-line covers)**.** *Suppose a component* $\overline{H}'$*, of* $\overline{H}(G,C)^{\dagger,\mathrm{rd}}$ *corresponds to a braid orbit,* O*, on* $Ni(G,C)^{\dagger,\mathrm{rd}} = Ni(G,C)^{\dagger}/\mathcal{Q}''$.

Ramified points, respectively over $0,1,\infty$*, of* $\overline{H}' \to \mathbb{P}^1_j \Leftrightarrow$ *disjoint cycles of*

$$\gamma_0 = q_1q_2, \gamma_1 = q_1q_2q_1, \gamma_\infty = q_2 \text{ (cusps).}$$

The genus of $\mathbf{g}_{\bar{H}'}$*, appears from the formula*

$$2(|O| + \mathbf{g}_{\bar{H}'} - 1) = \mathrm{ind}(\gamma_0)+\mathrm{ind}(\gamma_1)+\mathrm{ind}(\gamma_\infty).$$

Applying this, I computed the genus of the two reduced Hurwitz space components of 4-branch point covers; a graphic look at very different components of one Hurwitz space. Each cusp ${}_{\mathbf{c}}O^k_{i,j}$ has $\boldsymbol{g}$ with entries $\{g, g^{-1}\}$ or $\{g, g\}$ resp. **HM** in Ni_0^+, or **D**(ouble) **I**(dentity) in Ni_0^-.

- ${}_{\mathbf{c}}O^4_{1,1} = (g_{1,1})^{q_2,\bullet}, \boldsymbol{g}_{1,1} = ((1\,2\,3),(1\,3\,2),(1\,3\,4),(1\,4\,3))$
  ${}_{\mathbf{c}}O^3_{1,3} = (\boldsymbol{g}_{1,3})^{q_2,\bullet}, \boldsymbol{g}_{1,3} = ((1\,2\,3),(1\,3\,2),(1\,4\,3),(1\,3\,4))$.
- ${}_{\mathbf{c}}O^4_{2,1} = (g_{2,1})^{q_2,\bullet}, \boldsymbol{g}_{2,1} = ((1\,2\,3),(1\,3\,4),(1\,2\,4),(1\,2\,4))$.
  ${}_{\mathbf{c}}O^1_{2,2}$ and ${}_{\mathbf{c}}O^1_{2,3}$ seeded by **DI**s repeated in positions 2 and 3.

After the well-known tests for $\dagger$ fine moduli, reduced fine moduli for a braid orbit requires that $\mathcal{Q}''$ acts as the *Klein* 4-group on that orbit. Then that neither γ_0 or γ_1 have fixed points. You can read the degree (resp. 9 and 6) of the components over $\mathbb{P}^1_j$, their genuses (both 0) and precisely whether the components have *reduced fine moduli* (they don't) directly from the blocks.

Denote $(\gamma_0,\gamma_1,\gamma_\infty)$ on Ni_0^+ (resp. Ni_0^-) orbit by $(\gamma_0^+,\gamma_1^+,\gamma_\infty^+)$ (*resp.* $(\gamma_0^-,\gamma_1^-,\gamma_\infty^-)$). The distinction between the two components is in their cusp types, here called **HM** that appeared in the Ni^+ orbit and **DI** that appeared in the Ni^- orbit.

2. The Geometry Behind **RIGP** Unknowns

Problem: Most groups are neither simple nor solvable. Example: Take G, ℓ-perfect and centerless. Then, their exists $\nu(G,\ell) > 0$ (> 1, outside supersolvable G) and an extension

$$1 \to (\mathbb{Z}_\ell)^{\nu(G,\ell)} \to {}_\ell\tilde{G}_{\mathrm{ab}} \xrightarrow{{}_\ell\tilde{\psi}_{\mathrm{ab}}} G \to 1 :$$

This ${}_\ell\tilde{G}_{\mathrm{ab}}$ is universal for covers of G with abelian ℓ group kernel.

Subex.: Even for $G = A_5$, where $v(G,2) = 5$, for no $k > 0$ has

$${}_2\tilde{A}_{5,\mathrm{ab}}/2^k \ker({}_\ell\tilde{\psi}_{\mathrm{ab}}) = {}^k_2A_5$$

been realized (regularly or not) over $\mathbb{Q}$. Assume r conjugacy classes, C of G; all containing elements of order ℓ' (prime to ℓ).

Schur-Zassenhaus lifts these classes uniquely to ${}_{\ell}\tilde{G}_{\rm ab}$.

From this we form the Nielsen classes that gives **MT**s:

This makes sense of ${}_{\ell}\mathbb{H}(G,\mathrm{C})^{\rm in,rd} \stackrel{\rm def}{=} \{\mathrm{H}({}^{k}_{\ell}G,\mathrm{C})^{\rm in,rd}\}_{k=0}^{\infty}$.

A **MT** is a projective sequence of components on ${}_{\ell}\mathbb{H}(G,\mathrm{C})^{\rm in,rd}$.

Assume: For a finite group G, any prime ℓ for which G is ℓ-perfect, and any bound B on the number of branch points, that each ${}^{k}_{\ell}G_{\rm ab}$ (analog of the group for A_5 above) has a $\mathbb{Q}$ regular realization with no more than B branch points.

A "Yes" answer implies there exists r ($\le B$) conjugacy classes C of G with ℓ' elements , and a natural **MT** of spaces constructed from (G,ℓ,C);

with each tower level having a $\mathbb{Q}$ *point.*

The Main Conjecture: High **MT** tower levels have *general type* and no $\mathbb{Q}$ points. A special case (joint with Pierre Debes) is $G = D_\ell$, ℓ an odd prime. That interprets as existence of ℓ^{k+1} cyclotomic points for each k, on hyperelliptic jacobians of a fixed dimension d (independent of k, but the Jacobian may change with k).

Theorem 2 (Outlined in [5]). *The Main Conjecture is true for* $r = 4$*, based on the genus formula and methods for distinguishing different types of cusps. It suffices to show the genus rises with the* **MT** *levels.*

[2] proved the disappearance of rational points at high levels, without engaging the reduced Hurwitz spaces or their cusps. [1] showed the Torsion Conjecture on abelian varieties $\Longrightarrow$ $\mathbb{Q}$ statement of the Main Conjecture in general.

3. Using MTs for Generalizing Serre's OIT

Serre's case is on decomposition groups of projective sequences of points on the modular curve tower $\{X_1(\ell^{k+1})\}_{k=0}^{\infty}$. We interpret that as the **MT** attached to D_ℓ with C four repetitions of the involution conjugacy class.

I used the Fried-Serre lift invariant for A_n and odd order cycles on the $(A_4,\mathrm{C}_{\pm 3^2})$ case to explain the two components as having different lift – these are braid – invariants, computable just from any cusp orbit in a braid orbit.

The phrasing of a general **OIT** based on **MT**s requires this definition. An *eventually (ℓ)-Frattini* sequence of group covers, $\{H_k\}_{k=0}^{\infty}$:

$$\exists k_0 \text{ with } H_{k_0+k} \to H_{k_0} \text{ Frattini (resp. } \ell\text{-Frattini) for } k \ge 0.$$

Weak OIT Conj.: For a **MT** $\{\mathrm{H}_k\}_{k=0}^{\infty} \le {}_{\ell}\mathbb{H}(G,\mathrm{C})$, $\Phi_k : \mathrm{H}_k \to J_r$, with $H_k = G_{\Phi_k}$ (geom. monodromy of Φ_k), then

$${}_{\ell}G_j(\Phi) \stackrel{\rm def}{=} \lim_{\infty \leftarrow k} H_k \text{ is eventually } \ell\text{-Frattini.}$$

Weak OIT Conclusion: Then, the *decomposition group* ${}_{\ell}\hat{G}_{j'}(\Phi)$ of a *general* $j' \in \bar{\mathbb{Q}}$ equals ${}_{\ell}\hat{G}_j(\Phi)$ (arith. mon.).

In Serre's system, $\mathrm{Ni}_{\ell^{k+1},2} \stackrel{\rm def}{=} \mathrm{Ni}(\mathbb{Z}/\ell^{k+1} \times^s \mathbb{Z}/2, \mathrm{C}_{2^4}), k \ge 0$:

2 types of decomposition groups: CM and GL_2.

I concluded with the $(\mathbb{Z}/\ell)^2 \times^s \mathbb{Z}/3$ example. The A_4 example above is level 0 for $\ell = 2$, with C two repetitions each of the class of the two 3-cycles.

We know the tower levels in the $(\mathbb{Z}/\ell)^2 \times^s \mathbb{Z}/3$ and the definition fields of all their components. I explained the level 0 case for all ℓ (joint with Mark Hoeij).

Theorem 3 (Level 0 Main Result)**.** *For $\ell > 3$ prime and level $k = 0$:*

- *K_ℓ braid orbits with trivial (0) lift invariant. All* **HM** *orbits.*
- *Braid orbits with nontrivial lift invariant consist of* **DI** *cusps. each such braid orbit distinguished by its lift invariant.*

All **DI** *components are conjugate over $\mathbb{Q}(e^{2\pi i/\ell})$.*

The lift invariant – which in these cases comes from two different Heisenberg groups – and often its relation with the *Weil Pairing*, explains all the definition fields of components in these cases.

The Weil Pairing (which manifests in several ways), appears as a deus ex machina in Serre's theory. Here it has a direct interpretation from **MT**s, which I hadn't quite grasped until the end of our Oberwolfach conference. Also, as here, it common that cusps have structure coming from their braid orbits that is not visible, say, just from the Hurwitz space components (when $r = 4$) being upper half-plane quotients by finite index subgroups of $\mathrm{PSL}_2(\mathbb{Z})$ [3, §2.2].

[8], either reference, has a much more comprehensive bibliography.

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Lifting Problems and Specializations

François Legrand

In inverse Galois theory, specializing a finite Galois extension of the function field $\mathbb{Q}(T)$ is a central tool, going back to Hilbert. Namely, suppose a given finite group G is a *regular Galois group over* $\mathbb{Q}$, i.e., there is a Galois extension $E/\mathbb{Q}(T)$ of group G that is $\mathbb{Q}$-*regular* (i.e., such that $E \cap \overline{\mathbb{Q}} = \mathbb{Q}$). Then, by *Hilbert's irreducibility theorem*, there are infinitely many $t_0 \in \mathbb{Q}$ such that the *specialization of* $E/\mathbb{Q}(T)$ *at* t_0 (i.e., the residue extension at any prime ideal above $\langle T - t_0 \rangle$ in the extension $E/\mathbb{Q}(T)$) is of group G; thus solving the *inverse Galois problem* for the group G. Many finite groups have been realized by this method; see, e.g., [8].

The *Beckmann-Black problem* is a classical issue on the converse: can every finite Galois extension $F/\mathbb{Q}$ be lifted to a $\mathbb{Q}$-regular Galois extension $E/\mathbb{Q}(T)$ (possibly depending on $F/\mathbb{Q}$) with the same Galois group? That is, is $F/\mathbb{Q}$ a specialization of $E/\mathbb{Q}(T)$? This question remains widely open: it has a positive answer for a few groups (see, e.g., the survey paper [1] for more details and references) and has no known counter-example.

Another classical lifting issue asks whether a given $\mathbb{Q}$-regular Galois extension $E/\mathbb{Q}(T)$ of group G is *generic*, that is, whether every Galois extension of any field of characteristic zero of group G occurs as a specialization of $E/\mathbb{Q}(T)$ (after proper scalar extension). In fact, this very strong property occurs very rarely: only the subgroups of the symmetric group S_3 have a generic extension $E/\mathbb{Q}(T)$; see [4].

The main aim of the talk was to explore the gap between the genericity property and the Beckmann-Black problem, by discussing some intermediate new lifting issues studied in a series of works by P. Dèbes, J. König, D. Neftin, and the speaker. These issues were briefly presented during the introductory talk of P. Dèbes to the subtopic "Specialization and Lifting Problems" of the mini-workshop.

We first proposed the following terminology (introduced in [5]):

Definition 1. *Say that a (non-empty) set S of $\mathbb{Q}$-regular Galois extensions of $\mathbb{Q}(T)$ of group G is N-parametric if any given N Galois extensions $F_1/\mathbb{Q}, \dots, F_N/\mathbb{Q}$ of group G occur as specializations of some extension $E/\mathbb{Q}(T) \in S$.*

Remark 1.

(1) If S consists of a single extension $E/\mathbb{Q}(T)$, then, the above definition does not depend on N; we then say that $E/\mathbb{Q}(T)$ is a parametric extension.
(2) The Beckmann-Black problem over $\mathbb{Q}$ for the group G has a positive answer iff G has a 1-parametric set over $\mathbb{Q}$.

(3) If G has a generic polynomial over $\mathbb{Q}$ (see [4]), then, the set of all $\mathbb{Q}$-regular Galois extensions of $\mathbb{Q}(T)$ of group G is N-parametric for every $N \geq 1$.

The first part was devoted to the study of the case of a single $\mathbb{Q}$-regular quadratic extension $E/\mathbb{Q}(T)$. In this situation, a diophantine viewpoint can be used. Namely, if $E = \mathbb{Q}(T)(\sqrt{P(T)})$ for a separable polynomial $P(T) \in \mathbb{Z}[T]$, then, $E/\mathbb{Q}(T)$ is parametric iff, for every squarefree integer d, the twisted hyperelliptic curve $C_d : y^2 = dP(t)$ has a "non-trivial" $\mathbb{Q}$-rational point.

While the situation is well-understood in the genus at most 1 case, it is unknown in general whether at least one quadratic twist of a given hyperelliptic curve over $\mathbb{Q}$ of genus at least 2 has only trivial $\mathbb{Q}$-rational points. By results of A. Granville [3], under some conjectures (e.g., the *abc-conjecture*), this property always holds (and the number of quadratic twists with a non-trivial $\mathbb{Q}$-rational point is "small").

We then presented this theorem [7], which gives an evidence for this conclusion:

Theorem 2. *Given an even integer $N \geq 4$, the proportion $f(H)$ of all $\mathbb{Q}$-regular quadratic extensions of $\mathbb{Q}(T)$ with N branch points, "height" at most H, and which are not parametric tends to 1 as H tends to ∞. In fact, one has $f(H) = 1 - O(\log(H)/\sqrt{H})$ as $H \to \infty$.*

Whether the error term can be unconditionally removed remains to be seen.

The second part was devoted to the presentation of the following theorem, which provides the first examples of finite groups with no finite 1-parametric set over $\mathbb{Q}$:

Theorem 3. *These finite groups G have no finite 1-parametric set over $\mathbb{Q}$:*

(1) G is of order prime to 6 but not of prime order,
(2) G is abelian, not of prime order and not $\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}, \mathbb{Z}/12\mathbb{Z}$,
(3) $G = D_n$ if $n \neq 9$ and n is not a prime,
(4) $G = S_n$ or A_n with $n \geq 4$,
(5) $G = \mathrm{GL}_n(\mathbb{F}_q)$ with $n \geq 2$, $q \geq 3$, and $(n, q) \neq (2, 3)$.

Determining the list of all finite groups G with a finite 1-parametric set over $\mathbb{Q}$ (or with a parametric extension over $\mathbb{Q}$) seems to be a more challenging problem. Also, let us mention that no finite $\mathbb{Q}$-regular Galois extension of $\mathbb{Q}(T)$ that is parametric, but not generic, seems to be known.

Theorem 3 is the outcome of two different methods, developed independently in [5] and [6]. In [5], the "global" method consists in showing that, under the assumption that there is a finite 1-parametric set over $\mathbb{Q}$ for some groups G, then, via the *twisting lemma*, there are smooth projective curves over $\mathbb{Q}$ of genus at least 2 and with infinitely many $\mathbb{Q}$-rational points, which cannot happen by Faltings' theorem. This approach was presented in details, as well as further consequences. In particular, we gave a variant of Theorem 3, with "1-parametric" replaced by "N-parametric for large N", for sets of regular realizations of a given group G with bounded number of branch points, conditionally under a "uniform Faltings' theorem" (asserting that the number of $\mathbb{Q}$-rational points on a smooth projective curve over $\mathbb{Q}$ of genus at least 2 depends only on the genus). As to the method developed in [6], that is called "local" as it relies on a study of decomposition groups in specializations, it was presented in details in J. König's talk.

The last part was devoted to a geometric variant of the specialization notion. For a field k, a k-regular Galois cover $f : X \to \mathbb{P}^1_k$ of group G, and $T_0 \in k(U)\setminus k$, the *pullback of f along T_0* is the cover $f_{T_0} : X_{T_0} \to \mathbb{P}^1_k$ obtained by pulling back f along the rational map $T_0 : \mathbb{P}^1_k \to \mathbb{P}^1_k$. If X_{T_0} is connected (which is true for "almost all" T_0), then, f_{T_0} is Galois of group G and can be viewed as a specialization: the function field extension $k(X_{T_0})/k(U)$ of f_{T_0} is the specialization of $k(U)(X)/k(U)(T)$ at $T = T_0$, where $k(X)/k(T)$ denotes the function field extension of f.

Definition 2. *Given a finite group G, say that a subset H of the set $\mathsf{H}_G(k)$ of all k-regular Galois covers $f : X \to \mathbb{P}^1_k$ of group G is k-regularly parametric if $\mathsf{H}_G(k) \subseteq \mathrm{PB}(\mathsf{H}) = \{f_{T_0} \mid f \in \mathsf{H},\ T_0 \in k(U) \setminus k\}$.*

We then presented the following result, due to P. Dèbes [2]:

Theorem 4.

(1) Suppose k is algebraically closed of characteristic zero. Then, finite subgroups of $\mathrm{PGL}_2(\mathbb{C})$ have a k-regularly parametric cover $f : X \to \mathbb{P}^1_k$.

(2) Suppose k is of characteristic zero. Then, the following finite groups G have no k-regularly parametric cover $f : X \to \mathbb{P}^1_k$: S_n with $n \geq 6$, A_n with $n \geq 7$, $\mathrm{PSL}_2(\mathbb{F}_p)$ with either $(2/p) = -1$ or $(3/p) = -1$, the Monster group M, etc.

A main tool to prove (2) (see [2]) is to show that the number of branch points cannot drop by taking pullbacks of a given k-regular Galois cover $f : X \to \mathbb{P}^1_k$ of group G. Consequently, to be k-regularly parametric, f should have "a few" branch points. However, by using a classical result of Beckmann about inertia groups in specializations, the *ramification type* of f should contain "many" conjugacy classes of G, thus implying that f should have "many" branch points.

Finally, we briefly mentioned a work in progress, joint with P. Dèbes, J. König and D. Neftin, where it is shown that, if k is algebraically closed of characteristic zero and $G \not\subset \mathrm{PGL}(\mathbb{C})$, then, G has no k-regularly parametric cover $f : X \to \mathbb{P}^1_k$ (thus providing the converse of (1) in Theorem 4). In fact, for every $r_0 \geq 1$, the set of all k-regular Galois covers $f : X \to \mathbb{P}^1_k$ of group G with at most r_0 branch points is not k-regularly parametric. In particular, for $G \not\subset \mathrm{PGL}_2(\mathbb{C})$, letting the number of branch points grow provides an endless source of "new" Galois covers of group G (i.e., not mere rational pullbacks of some with a bounded number of branch points), and so truely new candidates to be defined over $\mathbb{Q}$.

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Some Aspect of Arithmetic of Profinite Fundamental Groups

Hiroaki Nakamura

The Grothendieck-Teichmüller tower of moduli stacks $M_{g,n}$ (of smooth curves of genus g with n-ordered marked points) is a complex of two types of exact sequences of profinite fundamental groups for $g, n \in \mathbb{Z}_{\geq 0}$, $2-2g-n<0$:

$$(A) \quad 1 \to \hat{\Gamma}_{g,n} \to \pi_1(M_{g,n}) \to G_{\mathbb{Q}} \to 1;$$
$$(B) \quad 1 \to \hat{\pi}_{g,n} \to \pi_1(M_{g,n+1}) \to \pi_1(M_{g,n}) \to 1,$$

where $\hat{\Gamma}_{g,n}$, $\hat{\pi}_{g,n}$ are respectively the profinite completion of the mapping class group resp. of the surface group of type (g, n), and $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) = \pi_1(M_{0,3})$. Noting that (A) splits according to a choice of $\mathbb{Q}$-rational (tangential) base point on $M_{g,n}$, we obtain from (B) the universal monodromy representation

$$\varphi_{g,n} : \pi_1(M_{g,n}) = \hat{\Gamma}_{g,n} \rtimes G_{\mathbb{Q}} \longrightarrow Out(\hat{\pi}_{g,n})$$

which is conjectured to be injective (true for $g = 0, 1$ by Belyi [B79], Asada [A01], Matsumoto-Tamagawa [MT00]). In fact, the above (A) and (B) can be regarded as special cases of the exact sequence

$$(C) \quad 1 \to \hat{\pi}^{(r)}_{g,n} \to \pi_1(M_{g,n+r}) \to \pi_1(M_{g,n}) \to 1$$

with $\hat{\pi}^{(r)}_{g,n}$ the profinite braid group with r strings on an n-point punctured Riemann surface of genus g. It is observed from (C) that the associated universal representations into certain 'special' outer automorphism subgroups $Out^*(\hat{\pi}^{(r)}_{g,n})$ $(r>1)$ that respect fiber subgroups $ker(\hat{\pi}^{(r)}_{g,n} \to \hat{\pi}^{(r-1)}_{g,n})$ form the following (conjectually injective) sequence of homomorphisms factoring through the above $\varphi_{g,n}$:

$$\varphi^{(*)}_{g,n} : \pi_1(M_{g,n}) \to \cdots \to Out^*(\hat{\pi}^{(r)}_{g,n}) \to Out^*(\hat{\pi}^{(r-1)}_{g,n}) \to \cdots \to Out(\hat{\pi}_{g,n}).$$

One may expect the stable image $\widehat{GT}_{g,n} := \bigcap_r Out^*(\hat{\pi}^{(r)}_{g,n})$ in $Out(\hat{\pi}_{g,n})$ to be a combinatorial model approximating the arithmetic fundamental group $\pi_1(M_{g,n})$.

In this talk, I focused on the meta-abelian reductions of $\varphi_{g,n}$ (viz. modulo the double commutator subgroup $\hat{\pi}''_{g,n}$) in the special cases $(g, n) = (0, 3)$, $(1, 1)$. For $\varphi''_{0,3} : G_{\mathbb{Q}} \to Out(\hat{\pi}_{0,3}/\hat{\pi}''_{0,3})$, Anderson-Ihara theory created the adelic beta function $\mathbb{B} : \widehat{GT} \to \hat{\mathbb{Z}}[[\hat{\mathbb{Z}}^2]]^\times$ that recovers $\varphi''_{0,3}$ and holds a number of arithmetic properties related to Jacobi sums, ℓ-adic Soule characters on $G_\mathbb{Q} \subset \widehat{GT}$.

For the case of $\varphi''_{1,1} : \pi_1(M_{1,1}) = \widehat{SL_2(\mathbb{Z})} \rtimes G_\mathbb{Q} \to Out(\hat{\pi}_{1,1}/\hat{\pi}''_{1,1})$, I introduced an adelic Eisenstein periods in the form $\mathbb{E} : \widehat{GT}_{ell} \times \mathbb{Q}_f^2 \to \hat{\mathbb{Z}}$, where $\widehat{GT}_{ell}(= \widehat{GT}_{1,\vec{1}})$ is Enriquez's *elliptic Grothendieck-Teichmüller group*, and $\mathbb{Q}_f = \mathbb{Q} \otimes \hat{\mathbb{Z}}$ is the ring of finite adeles. It recovers $\varphi''_{1,1}$, and has explicit formulas on components of $\hat{B}_3 \rtimes G_\mathbb{Q} \subset \widehat{GT}_{ell}$ in relation to generalized Dedekind sums and ℓ-adic Soule characters. At a technical point, the property "$\widehat{GTK} = \widehat{GT}$" (which was posed by Ihara [I00] and settled by Enriquez [E07]) was applied for the extension of our invariant $\mathbb{E}$ from $\pi_1(M_{1,\vec{1}}) = \hat{B}_3 \rtimes G_\mathbb{Q}$ to $\widehat{GT}_{ell}$,

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Cohomological Obstructions over Function Fields

David Harari

Let X be an algebraic variety defined over a field K. In the classical case when K is a number field, deciding whether X has a K-point is in general a difficult task. An obvious necessary condition is that X has a point over every completion K_v of K, but except in very specific cases (e.g. quadrics) this "local" condition is not sufficient. Indeed obstructions related to étale cohomology of the variety have been put forward in the last 40 years : some of them (like the Brauer-Manin obstruction) are of abelian nature, others (like the descent obstruction [3], which is related to the geometric fundamental group of X) are not. Some of these obstructions can also be linked (at least conjecturally) to *homotopic* obstructions, see Schlank's talk.

An important question is to determine classes of varieties such that these obstructions to the existence of a K-point or to approximation properties (like weak approximation) are the only ones. For example, a conjecture due to Colliot-Thélène predicts that for a geometrically *rationally connected* (e.g. unirational) variety X, Brauer-Manin obstruction to weak approximation is the only one; in particular such an X should satisfy weak approximation outside a "bad" set of places of K and applying this to quotients of SL_n by a finite group G, this would imply a positive answer to the inverse Galois problem for G (see Wittenberg's talk for recent results about this special case, and also for an extension of Colliot-Thélène's conjecture to zero-cycles). Actually it is even sufficient to have "hyper-weak approximation" on SL_n/G as defined in [1] (roughly speaking, this means approximation for *integral* local points) to conclude that G is a Galois group over K.

It is of course possible to extend this framework to global fields of characteristic p, that is: function fields K of a curve over a finite field k. In the last few years, wide generalisations have been developped when the base field k is more complicated, but has good arithmetic duality properties. In this talk, I explain recent results (joint work with Szamuely [4] and work by Izquierdo [5], [6]) about local-global principles and approximation properties on K-algebraic groups for various k. The methods strongly rely on new arithmetic duality theorems and higher degree cohomological obstructions. For example, we describe (with a detailed proof of the main result) the obstruction to the local-global principle for a principal homogeneous space under a linear algebraic group in terms of a degree 3 étale cohomology group when k is a p-adic field. This can be viewed as an extension of a classical theorem by Sansuc [7] over global fields, where the obstruction (Brauer-Manin) is of degree 2. We also describe the adelic space of a K-torus when k is algebraically closed of characteristic zero [2], which has a flavor of global class field theory in this context. Many open problems about local-global questions over function fields remain, like relating these cohomological methods to *patching techniques* (developped by Harbater, Hartmann, Krashen) and describing the defect of weak approximation for arbitrary (non commutative) algebraic groups.

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The Fundamental Theorem of Weil II for Curves with Ultraproduct Coefficients

Anna Cadoret

Let k_0 be a finite field of characteristic $p > 0$ with geometric Frobenius F_0. Fix an algebraic closure k of k_0. In this note a variety over k_0 always means a reduced scheme separated and of finite type over k_0. For a variety X_0 over k_0, write $X := X_0 \times_{k_0} k$.

Given a field K of characteristic 0, an embedding $\iota : K \hookrightarrow \mathbb{C}$ and $q \in \mathbb{R}_{>1}$, one defines the ι-weights (with respect to q) of an automorphism F of a finite-dimensional K-vector space V to be the $w \in \mathbb{R}$ such that $|\iota\alpha| = q^{\frac{w}{2}}$ for α describing the set of eigenvalues of F acting on $V \otimes \overline{K}$.

Given a prime $\ell(\neq p)$, we always denote by Q_ℓ a finite extension of $\mathbb{Q}_\ell$ and by Z_ℓ, λ_ℓ and F_ℓ the corresponding ring of integers, uniformizer and residue field .

1.1. Fix an infinite set of primes $\mathcal{L}$ not containing p. For a map $\underline{n} : \mathcal{L} \to \mathbb{Z}_{\geq 1}$, $\ell \to n_\ell$, set $\underline{F_{\underline{n}}} := \prod_{\ell \in \mathcal{L}} \mathbb{F}_{\ell^{n_\ell}}$, $\underline{F} := \prod_{\ell \in \mathcal{L}} \overline{\mathbb{F}}_\ell = \varinjlim \underline{F_{\underline{n}}}$. Given a (non principal)[1] ultrafilter $\mathcal{U}$ on $\mathcal{L}$, let $\underline{F_{\underline{n}}} \twoheadrightarrow F_{\underline{n},\mathcal{U}}$ and $\underline{F} \twoheadrightarrow F_{\mathcal{U}} = \varinjlim F_{\underline{n},\mathcal{U}}$ denote the corresponding ultraproducts. One has the following parallelism

	$\overline{\mathbb{Q}}_\ell$	$F_{\mathcal{U}}$
torsion coefficients	Z_ℓ/λ_ℓ^n, $n \geq 1$	$\mathbb{F}_{\ell^{n_\ell}}$, $\underline{n} : \mathcal{L} \to \mathbb{Z}_{\geq 1}$
$\varprojlim$ (to char 0 ring)	Z_ℓ	$\underline{F_{\underline{n}}}$
localization (exact) (to char 0 field)	$Z_\ell \hookrightarrow Q_\ell$	$\underline{F_{\underline{n}}} \twoheadrightarrow F_{\underline{n},\mathcal{U}}$
$\varinjlim$ (to alg. closed char 0 field $\simeq \mathbb{C}$)	$Q_\ell \hookrightarrow \overline{\mathbb{Q}}_\ell$	$F_{\underline{n},\mathcal{U}} \hookrightarrow F_{\mathcal{U}}$

Recall also that the kernel of $\underline{F_{\underline{n}}} \to \prod_{\mathcal{U}} F_{\underline{n},\mathcal{U}}$ is the ideal of elements with finite support. This translates to the general principal that a property which, for every ultrafilter $\mathcal{U}$, holds over a set $S \in \mathcal{U}$ actually holds for all but finitely many $\ell \in \mathcal{L}$.

[1] In this note, an ultrafilter always means a non principal ultrafilter.

1.2. Let X_0 be a smooth and geometrically connected variety over k_0. For $\underline{n} : \mathcal{L} \to \mathbb{Z}_{\geq 1}$, let $S_{lcc,\underline{n}}(X_0)$ denote the abelian (not full!) subcategory of étale torsion sheaves whose objects $\underline{\mathcal{F}}$ are direct products of locally constant constructible (lcc for short) sheaves $\mathcal{F}_\ell$ of $\mathbb{F}_{\ell^{n_\ell}}$-modules and let $S_{lcc}(X_0) := 2\text{-}\varinjlim S_{lcc,\underline{n}}(X_0)$. For every ultrafilter $\mathcal{U}$ on $\mathcal{L}$ let $S^t_{\mathcal{U}}(X_0) \subset S_{lcc}(X_0)$ denote the full subcategory of *almost* $\mathcal{U}$*-tame* sheaves that is of those $\underline{\mathcal{F}}$ such that (1) $\underline{\mathcal{F}}_{\overline{x},\mathcal{U}} := \underline{\mathcal{F}}_{\overline{x}} \otimes F_{\mathcal{U}}$ has finite $F_{\mathcal{U}}$-rank and (2) there exists a connected étale cover $X'_0 \to X_0$ for which the set of primes $\ell \in \mathcal{L}$ such that $\mathcal{F}_\ell|_{X'_0}$ is curve-tame is in $\mathcal{U}$. Here, 'curve-tame' means that for every smooth curve C over k and morphism $C \to X$, $\mathcal{F}_\ell|_C$ is tamely ramified in the usual sense. $S^t_{\mathcal{U}}(X_0)$ is an abelian category admitting internal Hom, $\otimes$, stable under arbitrary pull-back and finite direct image and $S^t_{\mathcal{U}}(X_0) \otimes F_{\mathcal{U}}$ is Tannakian with fiber functor $\underline{\mathcal{F}} \to \underline{\mathcal{F}}_{\overline{x},\mathcal{U}}$. In contrast $S^t_{\mathcal{U}}(X_0)$ is not stable under higher direct image by smooth-proper morphisms.

1.3. The finiteness condition (1) allows to define Frobenius weights. Given an isomorphism $\iota : F_{\mathcal{U}} \tilde{\to} \mathbb{C}$, the $(\mathcal{U}, \iota)$-weights of $\underline{\mathcal{F}}$ at $x_0 \in |X_0|$ are the ι-weights with respect to $|k(x_0)|$ of F_{x_0} acting on $\underline{\mathcal{F}}_{x,\mathcal{U}}$. If for every $x_0 \in |X_0|$ the $(\mathcal{U}, \iota)$-weights of $\underline{\mathcal{F}}$ at x_0 are all equal to $w \in \mathbb{R}$ one says that $\underline{\mathcal{F}}$ is $(\mathcal{U}, \iota)$-pure of weight w.

1.4. The tameness condition (2) ensure that the $F_{\mathcal{U}}$-vector spaces $\mathrm{H}^i_{?,\mathcal{U}}(X, \underline{\mathcal{F}}) := (\prod_{\ell \in \mathcal{L}} \mathrm{H}^i_c(X, \mathcal{F}_\ell)) \otimes F_{\mathcal{U}}$, $? = c, \emptyset$, $i \geq 0$ are finite-dimensional, which is enough to get the cohomological interpretation of attached L-functions:

$$\prod_{x_0 \in |X_0|} \det(1 - TF_{x_0} | \underline{\mathcal{F}}_{x,\mathcal{U}})^{-1} = \prod_{i \geq 0} \det(1 - TF_0 | \mathrm{H}^i_{c,\mathcal{U}}(X, \underline{\mathcal{F}}))^{(-1)^{i+1}}.$$

Condition (2) also ensures that the global and local unipotent monodromy theorems hold and that the canonical functor $S^t_{\mathcal{U}}(X_0) \to S^t_{\mathcal{U}}(X_0) \otimes F_{\mathcal{U}}$ is essentially surjective.[2]

1.5. With these tools in hands, one can adjust Deligne's proof of [D80, Thm. (3.2.1)] to $F_{\mathcal{U}}$-coefficients. Fix an ultrafilter $\mathcal{U}$ on $\mathcal{L}$ and an isomorphism $\iota : F_{\mathcal{U}} \tilde{\to} \mathbb{C}$. Let X_0 be a smooth curve over k_0 and $\underline{\mathcal{F}} \in S^t_{\mathcal{U}}(X_0)$.

Theorem 1. *If $\underline{\mathcal{F}}$ is $(\mathcal{U}, \iota)$-pure of weight w then, for every $i \geq 0$, $\mathrm{H}^i_{c,\mathcal{U}}(X, \underline{\mathcal{F}})$ has ι-weights $\leq w + i$. Equivalently, $\mathrm{H}^i_{\mathcal{U}}(X, \underline{\mathcal{F}})$ has ι-weights $\geq w + i$.*

1.6. Combined with geometric method (Bertini, Lefschetz pencils), Theorem 1 is enough for most applications. Let X_0 be a smooth, geometrically connected variety and let $\underline{\mathcal{F}} \in S^t_{\mathcal{U}}(X_0)$ be $(\mathcal{U}, \iota)$-pure of weight w.

- **(Purity)** Assume furthermore X_0 is proper over k_0. Then for every $i \geq 0$, $H^i(X, \underline{\mathcal{F}})$ is $(\mathcal{U}, \iota)$-pure of weights $w + i$.
- **(Geometric semisimplicity)** $\pi_1(X)$ acts semisimply on $\underline{\mathcal{F}}_{x,\mathcal{U}}$ (equivalently, the set of primes $\ell \in \mathcal{L}$ such that $\mathcal{F}_\ell|_X$ is semisimple is in $\mathcal{U}$).

[2] The delicate point is to show that subobjects lift; this requires the fact that the tame étale fundamental group in the sense of Kerz-Schmidt is topologically finitely generated.

- **(Cebotarev)** Let $\underline{\mathcal{F}}, \underline{\mathcal{F}}' \in S^t_{\mathcal{U}}(X_0)$ be $(\mathcal{U}, \iota)$-pure. Assume that for every closed point $x_0 \in |X_0|$, $tr(F_{x_0}, \underline{\mathcal{F}}_{x,\mathcal{U}}) = tr(F_{x_0}, \underline{\mathcal{F}}_{x,\mathcal{U}})$. Then the set of primes $\ell \in \mathcal{L}$ such that $\mathcal{F}^{ss}_\ell \simeq \mathcal{F}'^{ss}_\ell$ is in $\mathcal{U}$.

1.7. **Integral models in E-RCS.** Let X_0 be a smooth variety. Sheaves $\underline{\mathcal{F}}$ in $S^t_{\mathcal{U}}(X_0)$ naturally arise when taking Z_ℓ-models and reducing modulo λ_ℓ in pure E-RCS of lcc $\overline{\mathbb{Q}}_\ell$-sheaves.

1.7.1. *E-RCS.* Given a number field E, an E-RCS of lcc $\overline{\mathbb{Q}}_\ell$-sheaves on X_0 is a system $\mathcal{F}_{\ell^\infty}$, $\ell \in \mathcal{L}$ of lcc $\overline{\mathbb{Q}}_\ell$-sheaves on X_0 such that for every $x_0 \in |X_0|$ the characteristic polynomial of F_{x_0} acting on $\mathcal{F}_{\ell^\infty,x}$ is in $E[T]$ and independent of ℓ. If for every $x_0 \in |X_0|$ and isomorphism $\iota : \overline{\mathbb{Q}}_\ell \tilde{\to} \mathbb{C}$, the ι-weights with respect to $|k(x_0)|$ of F_{x_0} acting on $\mathcal{F}_{\ell^\infty,x}$ are all equal to w one says that $\mathcal{F}_{\ell^\infty}$ is pure of weight w.

By definition, a lcc $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}_{\ell^\infty}$ on X_0 is obtained as $\mathcal{F}_{\ell^\infty} = \mathcal{H}_{\ell^\infty} \otimes \overline{\mathbb{Q}}_\ell$ for some lcc sheaf of Z_ℓ-modules $\mathcal{H}_{\ell^\infty}$. Call such an $\mathcal{H}_{\ell^\infty}$ a Z_ℓ-model for $\mathcal{F}_{\ell^\infty}$ and write $\mathcal{H}_\ell := \mathcal{H}_{\ell^\infty} \otimes F_\ell$ for its reduction modulo λ_ℓ. Given an E-RCS $\mathcal{F}_{\ell^\infty}$, $\ell \in \mathcal{L}$ and a choice of Z_ℓ-models $\mathcal{H}_{\ell^\infty}$, $\ell \in \mathcal{L}$, write $\underline{\mathcal{H}} = (\mathcal{H}_\ell) \in S_{lcc}(X_0)$. Then for every ultrafilter $\mathcal{U}$ on $\mathcal{L}$, $\underline{\mathcal{H}}$ is in $S^t_{\mathcal{U}}(X_0)$ and for every closed point $x_0 \in |X_0|$, the characteristic polynomials of F_{x_0} acting on $\underline{\mathcal{F}}_{x,\mathcal{U}}$ and $\mathcal{F}_{\ell^\infty,x}$ coincide.

1.7.2. *Arbitrary coefficients.* E-RCS provide the right setting to define 'arbitrary coefficients' in the category of lcc $\overline{\mathbb{Q}}_\ell$-sheaves. More precisely, consider the following isomorphism classes

(G) irreducible lcc $\overline{\mathbb{Q}}_\ell$-sheaf of rank r with finite determinant on X_0;
(RCS) irreducible E-RCS of lcc $\overline{\mathbb{Q}}_\ell$-sheaves pure of weight 0.

If X_0 is a curve

(A) cuspidal automorphic representations of $\mathrm{GL}_r(\mathbb{A})$, unramified on X_0 and whose central character is of finite order (where $\mathbb{A}$ denotes the adèle ring of $k(X_0)$)

Then

(1) Up to semisimplification, twist and isomorphism every lcc $\overline{\mathbb{Q}}_\ell$-sheaf on X_0 is a direct sum of objects in (G).
(2) There is a (necessarily unique) 1-1 correspondence $(G) \longleftrightarrow (RCS)$ and, if X_0 is a smooth curve, there is also a (necessarily unique) 1-1 correspondence $(A) \longleftrightarrow (G)$. Both correspondences are characterized by the fact that the local factors of the involved objects coincide.

While (1) is formal, (2) when X_0 is a curve follows from the Langlands correspondance for GL_r [L02]; the higher dimensional case of (2) reduces to the case of curves by geometric arguments [Dr12]. To sum it up, one has the following parallelism

	(G)	E-RCS	Z_ℓ-models	reduction
constant coefficients	$\mathbb{Q}_\ell$	$\mathbb{Q}_\ell$, $\ell \in \mathcal{L}$	$\mathbb{Z}_\ell$, $\ell \in \mathcal{L}$	$\mathbb{F}_\ell$, $\ell \in \mathcal{L}$
arbitrary coefficients	$\mathcal{F}_{\ell^\infty}$	$\mathcal{F}_{\ell^\infty}$, $\ell \in \mathcal{L}$	$\mathcal{H}_{\ell^\infty}, \ell \in \mathcal{L}$	$\mathcal{H}_\ell$, $\ell \in \mathcal{L}$

Note that in the case of arbitrary coefficients, there is *a priori* no canonical choice for the Z_ℓ-models hence for their reduction. But see 1.7.3 (5) below.

1.7.3. The following summarizes the main applications of Theorem 1 to E-RCS. Let $\mathcal{F}_{\ell^\infty}$, $\ell \in \mathcal{L}$ be a pure E-RCS of lcc $\overline{\mathbb{Q}}_\ell$-sheaves on X_0. There exists a prime ℓ_0 such that for $\ell \geq \ell_0$ and every system of integral models $\mathcal{H}_{\ell^\infty}$, $\ell \in \mathcal{L}$ the following holds.

(1) $\mathcal{H}_\ell|_X$ is semisimple on X;
(2) $\mathcal{H}_{\ell^\infty,\eta}^{\pi_1(X)} \otimes_{Z_\ell} F_\ell = \mathcal{H}_{\ell,\eta}^{\pi_1(X)}$ (equivalently, $\mathrm{H}^1(X, \mathcal{H}_{\ell^\infty})$ is torsion-free);
(3) The Zariski-closure of $\pi_1(X)$ acting on $\mathcal{H}_{\ell^\infty,\eta}$ is a semisimple group-scheme over Z_ℓ;
(4) $\mathcal{F}_{\ell^\infty}|_X$ is irreducible (resp. $\mathcal{F}_{\ell^\infty}$ is semisimple, resp. irreducible) (if and) only if $\mathcal{H}_\ell \otimes \overline{\mathbb{F}}_\ell|_X$ is irreducible (resp. $\mathcal{H}_\ell$ is semisimple, resp. $\mathcal{H}_\ell \otimes \overline{\mathbb{F}}_\ell$ is irreducible).
(5) (Resp. if $\mathcal{F}_{\ell^\infty}$ is semisimple for $\ell \gg 0$) for any two Z_ℓ-models $\mathcal{H}_{\ell^\infty}$, $\mathcal{H}'_{\ell^\infty}$ of $\mathcal{F}_{\ell^\infty}$, $\mathcal{H}_{\ell^\infty}|_X \simeq \mathcal{H}'_{\ell^\infty}|_X$ (resp. $\mathcal{H}_{\ell^\infty} \simeq \mathcal{H}'_{\ell^\infty}$).
(6) For every Z_ℓ-model $\mathcal{H}_{\ell^\infty}$ of $\mathcal{F}_{\ell^\infty}$, $\mathrm{H}^i(X, \mathcal{H}_{\ell^\infty})$ is torsion-free, $i \geq 0$.

(1), (2), (3) (resp. (6)) (reprove and) extend the main results of [CHT17a] (resp. of Gabber's torsion-freeness theorem [G83]) to arbitrary coefficients. The fact that ℓ_0 can be taken independently of the choice of system of Z_ℓ-models and the asymptotic unicity of such in (5) are formal output of the definition of ultraproducts. (5) shows in particular that the correspondence (G) $\longleftrightarrow$ (RCS) automatically extends at the level of systems of integral models modulo 'asymptotic' isomorphisms and that for every ultrafilter $\mathcal{U}$ on $\mathcal{L}$ the sheaf $\underline{\mathcal{F}} := \underline{\mathcal{H}} \in S^t_{\mathcal{U}}(X_0)$ is well-defined independently of the choice of the system of Z_ℓ-models. When X_0 is a smooth curve, this combined with (5) and weak Cebotarev provides a unique and well-defined injective map (A) $\longrightarrow$ (G) satisfying the expected local compatibility conditions of a Langlands correspondence for GL_r with $F_{\mathcal{U}}$-coefficients.

1.8. **Questions.**

(1) Simplify the proof of Theorem 1 following Laumon's strategy.
(2) Prove the Langlands correspondence for GL_r with $F_{\mathcal{U}}$-coefficients (namely that the injective map (A) $\longrightarrow$ (G) is surjective);
(3) Define a good notion of constructibility (see [O17]) so that one can develop a systematic formalism of ultraproduct coefficients paralleling the one of $\overline{\mathbb{Q}}_\ell$-coefficients (and in particular, get a relative theory of Frobenius weights).

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Brauer-Wall Group Poitou-Tate Sequence in Spectra and Zero Cycles

Tomer Schlank
(joint work with Vesna Stojanoska)

We started by reviewing classical problems in arithmetic geometry and Galois theory specifically Diophantine equations and embedding problems. We also discussed the "linearised" versions which are the existence of degree one zero cycles for Diophantine equations and the index problem for an embedding problem. We showed how translating these problems to question about sections in Topoi theory allows one to use homotopical theoretic methods. On the other hand we discussed related classical local -global methods. Specifically the Brauer-Manin obstruction is probably the best-known obstruction to the existence of points on an algebraic variety. The Brauer-Manin obstruction can also be used to obstruct the existence of zero cycles (. For rational points, stronger obstruction exists. In 99' Skorobogatov defined the more refined étale-Brauer-Manin obstruction, which is a finer obstruction to the existence of such points. However, this obstruction cannot be applied to zero cycles. The theory of étale homotopy gives us a way to understand this fact. The difference between Brauer-Manin and étale-Brauer-Manin lies in the difference between homotopy and homology, and it is homology's abelian nature and relation to complexes that allows extending the obstruction to zero-cycles. In the eyes of a homotopy theorist this suggests a natural question: can one give a stronger obstruction to zero cycles that exist in stable étale homotopy?

Indeed, Modern Homotopy theory teaches us that spectra rather than complexes are the correct "higher" abelian objects in homotopy theory and this observation can indeed be used to defined a "Spectral" analog of the Brauer-Manin obstruction. To drive the analogy closer and really compare that homological and spectral versions one needs to produce analogs to different characters in the Brauer-Manin story. These include Spectral analogs of the Poitou-Tate 9-terms exact sequence in spectra and the Brauer group itself. This allows to explicitly describe the additional information detected by the spectral obstruction.

The additional spectral "Arithmetic" information can be seen to relate to the Brauer wall groups and to maps in the stable motivic homotopy category. The main additional challenge in the way to exploiting this information then lies with analysing bounded ramification results, which are "free" in the homological version but raises natural and non-trivial questions about the behaviour of bounded ramification in Chow groups and their twisted counterparts in the spectral realms. Additionally the definitions at hand can be used not only of number fields but also for fields of higher cohomological dimension. This higher cohomological dimension allows to differentiate even more between the unstable, stable and homological obstructions. Much of this is a work in progress with V. Stojanoska.

Anabelian Geometry with Étale Homotopy Types I

Alexander Schmidt
(joint work with Jakob Stix)

Let k be a finitely generated field extension of $\mathbb{Q}$. Grothendieck's anabelian philosophy (developed in a letter to G. Faltings [Gr83]) predicts the existence of a class of *anabelian* varieties X/k that are reconstructible from their étale fundamental group $\pi_1^{\text{et}}(X, \bar{x})$. He made the following conjectures:

Conjecture 1. *C smooth, hyperbolic curve $\Rightarrow$ C anabelian.*

Conjecture 2. *X smooth $\Rightarrow$ every $x \in X$ has basis of Zariski neighbourhoods which are anabelian.*

Conjecture 3. *X anabelian: $X(k) = (X \overset{s}{\leftrightarrows} \operatorname{Spec} k)$ is reconstructible from $S(\pi_1^{\text{et}}(X)) = (\pi_1^{\text{et}}(X) \overset{s}{\leftrightarrows} G_k)$.*

What does "reconstructible from $\pi_1^{\text{et}}(X)$" mean? Here are three conditions of increasing intricacy:

a) $\pi_1^{\text{et}}(Y) \underset{G_k}{\cong} \pi_1^{\text{et}}(X) \Rightarrow Y \underset{k}{\cong} X$.
b) $\operatorname{Iso}_k(Y, X) \cong \operatorname{Iso}_{G_k}(\pi_1(Y), \pi_1(X))_{\pi_1(X_{\bar{k}})}$.
c) $\forall$ Y smooth: $\operatorname{Hom}_k^{\text{dom}}(Y, X) \cong \operatorname{Hom}_{G_k}^{\text{open}}(\pi_1(Y), \pi_1(X))_{\pi_1(X_{\bar{k}})}$.

Amongst others, the following results were achieved so far:

- Nakamura [Na90]: Conjecture 1 with a) for $C \subseteq \mathbb{P}_k^1 \smallsetminus \{0, 1, \infty\}$
- Tamagawa [Ta97] /Mochizuki [Mo96]: Conjecture 1 with b)
- Mochizuki [Mo99] Conjecture 1 with c) and Conjecture 2 with b) in dimension ≤ 2
- Hochi [Ho14]: Conjecture 2 with b) in dimension ≤ 4.

Today (joint work with J. Stix):

- There are "many" anabelian varieties of higher dimension (in the sense of a)).
- Conjecture 2 with b) in arbitrary dimension.

The topological point of view suggests that reconstruction from $\pi_1^{et}(X)$ is only plausible if X is of type $K(\pi, 1)$. Therefore one should ask the modified question: Can X be reconstructed from its étale homotopy type $X_{\mathrm{et}} \in \mathrm{Ho}(\text{pro-spc})$ (defined by Artin-Mazur [AM69], Friedlander [Fr82])? Here "spc" means "simplicial sets" and the model structure is that defined by Isaksen [Is01]. Notation:

$$[X, Y] = \mathrm{Hom}_{\mathrm{Ho}(\text{pro-spc})}(X, Y).$$

The present talk will provide necessary background on pro-spaces to translate Mochizuki's result [Mo99] to this setting.

For $(X, x) \in \text{pro-spc}_*$ we put $\pi_n^{\mathrm{top}}(X, x) \overset{\mathrm{df}}{=} [(S^n, *), (X, x)]$.

Theorem 1 (Schmidt-Stix [SS16]). *Let $(X, x), (Y, y) \in \text{pro-spc}_*$ and assume that $\pi_0^{\mathrm{top}}(Y, y) = 0$. Then there is a natural isomorphism*

$$[(X, x), (Y, y)]_{\pi_1^{\mathrm{top}}(Y,y)} \xrightarrow{\sim} [X, Y].$$

Theorem 2 (Schmidt-Stix [SS16]). *Assume that $\pi_0(X, x) = 0$ and $\pi_n(X, x)$ is profinite $\forall n \geq 1$. Then*

$$\pi_n^{\mathrm{top}}(X, x) \xrightarrow{\sim} \lim \pi_n(X, x).$$

Theorem 3 (Schmidt-Stix [SS16]). *(Existence of classifying spaces) Let $G \in$ pro-gps. There exists a (canonical) object $BG \in \mathrm{Ho}(\text{pro-spc}_*)$ with*

$$[X, BG] = \mathrm{Hom}_{\text{pro-gps}}(\pi_1(X), G)$$

for all connected $X \in \text{pro-spc}_*$.

Recall that a profinite group G is called *strongly center-free* if every open subgroup of G has a trivial center.

Theorem 4 (Schmidt-Stix [SS16]). *Let k be a field with G_k strongly center-free, X, Y connected and of finite type $/k$, Y normal and geometrically connected. Let K/k be a separably closed field extension and let $\bar{x} : \mathrm{Spec}(K) \to X$, $\bar{y} : \mathrm{Spec}(K) \to Y$ be geometric points. Let $\bar{k}$ be the separable closure of k in K. Then*

$$\mathrm{Hom}_{\mathrm{Ho}(\text{pro-spc}_*)\downarrow(k_{et}, \bar{k}_{\mathrm{et}})}((X_{\mathrm{et}}, \bar{x}_{\mathrm{et}}), (Y_{\mathrm{et}}, \bar{y}_{\mathrm{et}}))_{\pi_1^{et}(Y_{\bar{k}}, \bar{y})} \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Ho}(\text{pro-spc})\downarrow k_{\mathrm{et}}}(X_{\mathrm{et}}, Y_{\mathrm{et}}).$$

The above (and some more) allow the following reformulation of Mochizuki's theorem:

Theorem 5 (Mochizuki [Mo99] + Schmidt & Stix [SS16]). *Let p be a prime number, k a sub-p-adic field, X a smooth, connected k-variety and Y a smooth, hyperbolic curve over k. Then*

$$\mathrm{Hom}_k^{\mathrm{dom}}(X, Y) \xrightarrow{\sim} \mathrm{Hom}^{\pi_1\text{-open}}_{\mathrm{Ho}(\text{pro-spc})\downarrow k_{\mathrm{et}}}(X_{\mathrm{et}}, Y_{\mathrm{et}}).$$

Here "dom" means dominant and a map is "π_1-open" if it induces a homomorphism with open image on π_1 for any choice of base points. In particular, we have

$$\mathrm{Isom}_k(X, Y) \xrightarrow{\sim} \mathrm{Isom}_{\mathrm{Ho}(\text{pro-spc})\downarrow k_{\mathrm{et}}}(X_{\mathrm{et}}, Y_{\mathrm{et}}).$$

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Anabelian Geometry with Étale Homotopy Types II

Jakob Stix

(joint work with Alexander Schmidt)

We report on the second of two consecutive talks. Therefore we keep the notation of [Sch18]. In particular X_{et} denotes the étale homotopy type of X and $\mathrm{Ho}(\text{pro-spc}) \downarrow k_{et}$ denotes the over-category of the homotopy category of pro-spaces over $\mathrm{Spec}(k)_{\mathrm{et}}$. The first result is a weakly anabelian statement:

Theorem 1 (Schmidt-Stix, [SS16])**.** *Let $k/\mathbb{Q}$ be a finitely generated field. Let X and Y be smooth geometrically connected varieties over k such that for both X and Y we have a locally closed embedding into a product of hyperbolic curves over k. Then there is a map*

$$r : \mathrm{Isom}(X_{\mathrm{et}}, Y_{\mathrm{et}}) \longrightarrow \mathrm{Isom}_k(X, Y)$$

such that

(i) r is a retraction: $r(f_{\mathrm{et}}) = f$ for all isomorphisms $f : X \cong Y$ over k,

(ii) r is functorial: if Z is a further such variety and $\gamma : X_{\mathrm{et}} \cong Y_{\mathrm{et}}$ and $\delta : Y_{\mathrm{et}} \cong Z_{\mathrm{et}}$ are isomorphisms in the homotopy category over k_{et}, then $r(\delta\gamma) = r(\delta)r(\gamma)$,

(iii) for all dominant maps $f : Y \to C$ with C a hyperbolic curve over k, we have for all $\gamma : X_{\mathrm{et}} \cong Y_{\mathrm{et}}$

$$f_{\mathrm{et}}\gamma = f_{\mathrm{et}} r(\gamma)_{\mathrm{et}}.$$

Note that property (iii) determines the retraction r uniquely, and that (i) and (ii) follow at once from (iii).

Corollary 2. *Let X and Y be as in Theorem 1. Then X and Y are isomorphic as varieties over k if and only if $X_{\mathrm{et}} \cong Y_{\mathrm{et}}$ as pro-spaces over k_{et} up to homotopy.*

The proof of Theorem 1 proceeds in several steps. Let $\gamma : X_{\mathrm{et}} \cong Y_{\mathrm{et}}$ be an isomorphism. We first embed $Y \hookrightarrow W := \prod C_i$ in a product of hyperbolic curves such that each projection $\mathrm{pr}_i : Y \to C_i$ is dominant. Applying Mochizuki's anabelian theorem to $\mathrm{pr}_i\gamma$ in the form of Theorem 5 of [Sch18] we obtain a map $f : X \to W$. Spreading out over a scheme S of finite type over $\mathrm{Spec}(\mathbb{Z})$ as $\mathcal{Y} \hookrightarrow \mathcal{W}$ and $f : \mathcal{X} \to \mathcal{W}$ we must show that $\mathcal{Y} \times_{\mathcal{W}} \mathcal{X} \to \mathcal{X}$ is surjective on $\mathbb{F}_q$-points for sufficiently many q. This follows by applying a technique pioneered by Tamagawa in [Ta97]: auxilliary étale covers of $\mathcal{W}$ separate rational points, and the existence of rational points can be computed by the Lefschetz trace formula (a special *harpoon in the body of the whale of algebraic geometry* as mentioned several times during the workshop). Here we use crucially that the étale homotopy type of the special fibre of $\mathcal{X} \to S$ determines the étale cohomology also of finite étale covers as a Galois representation, hence the isomorphism γ induces an isomorphism with the corresponding Galois representation for $\mathcal{Y} \to S$. The factorization of f yields the desired map $r(\gamma) : X \to Y$.

A refined version based on Chebotarev's theorem yields some modest control about the extent to which the retraction r might actually be an inverse.

Theorem 3 (Schmidt-Stix, [SS16]). *Let X be as in Theorem 1, and let γ be an automorphism of X_{et} as pro-spaces over k_{et} up to homotopy with $r(\gamma) = \mathrm{id}_X$. Then (upon choosing base points and lifting to a pointed homotopy, see* [Sch18]*)*

$$\varphi = \pi_1(\gamma)$$

is a class-preserving automorphism of $\pi_1(X)$, i.e., $\varphi(\sigma)$ is conjugate to σ for all $\sigma \in \pi_1(X)$.

Recall that a *good Artin neighbourhood* is a smooth variety X that admits the structure of an iterated fibration

$$X = X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = \mathrm{Spec}(k),$$

such that for all i the fibration $X_i \to X_{i-1}$ is an elementary fibration in hyperbolic curves. In characteristic 0, good Artin neighbourhoods are $K(\pi, 1)$-spaces. We define a *strongly hyperbolic Artin neighbourhood* to be a good Artin neighbourhood such that furthermore all X_i admit locally closed embeddings into a product of hyperbolic curves. Refining the classical argument, we obtain that for a variety over an infinite field any smooth point admits a Zariski open neighbourhood that is also a strongly hyperbolic Artin neighbourhood. Therefore the following strong anabelian statement establishes a positive answer to Conjecture 2 mentioned in [Sch18].

Theorem 4 (Schmidt-Stix, [SS16]). *For strongly hyperbolic Artin neighbourhoods X and Y the retraction r of Theorem 1 is a bijection.*

Note that Theorem 4 was obtained by Hoshi [Ho14] by different means and restricted to $\dim(X) \leq 4$.

The proof proceeds by induction on the dimension. The compatibility with a choice of a fibration structure follows from Theorem 3 above. The induction hypothesis simplifies the situation of the last fibration $X_n \to X_{n-1}$ to the extent that we have an induced map on homotopy types for the generic fibres. Here we are back in the case of hyperbolic curves and Mochizuki's theorem applies again in the form of Theorem 5 of [Sch18]. This shows that all homotopy equivalences are geometric and a posteriori that the retraction r is an inverse.

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Arithmetic of Elliptic Function Fields and Elliptic Convolution

Michael Dettweiler

(joint work with Benjamin Collas, Stefan Reiter)

The use of Hilbert's irreducibility theorem has been proven to be very fruitful in hindsight of the inverse Galois problem: if $L/\mathbb{Q}(x)$ is a finite Galois extension of the rational function field then one may specialise this at infinitely many points in a way that the Galois group is preserved. In many cases it has been proven easier to obtain a regular realisation of a given group G as $G = \mathrm{Gal}(L/\mathbb{Q}(x))$ and then to specialise it in order to realise G over $\mathbb{Q}$.

This has resulted in Galois extensions of $\mathbb{Q}$ with finite groups of Lie type [5]. In particular, a result by H. Völklein [6], S. Reiter and M. Dettweiler [2] is the statement that the general linear groups $\mathrm{GL}_n(\mathbb{F}_q)$ is the Galois group of a regular Galois extension $L/\mathbb{Q}(x)$ if $n > c \cdot q$ (where c is a constant independent of n, q). The approach is given by Katz' algorithm for the middle convolution product of perverse sheaves on $\mathbb{P}^1$ [4, 3]. Unfortunately, in this context one has a certain *rational rigidity barrier:* due to the branch cycle argument, one can not extend these results to $n \leq cq$ without violating rationality.

We can consider a similar context by replacing the function field of the projective line as above by those of an elliptic curve: for elliptic function fields $\mathbb{Q}(E)$ one may also specialise a given finite Galois extension $L/\mathbb{Q}(E)$ at infinitely many rational points as soon as the Mordell-Weil rank is positive. If the Galois group $G = \mathrm{Gal}(L/\mathbb{Q}(E))$ is perfect then it can be shown that the Galois group is preserved for almost all specialisations. The convolution $K * L$ of perverse sheaves K, L on an elliptic curve E is the pushforward along the addition map. In this way, one can construct from easy to handle perverse sheaves (e.g. those which come from étale local systems of rank one) more complicated ones, resulting in nontrivial Galois extensions of $\mathbb{Q}(E)$. The geometric monodromy of such convolutions can be determined following [1].

In this talk we presented that one can use these methods in order to break the above mentioned rational rigidity barrier in some cases, leading to Galois realizations of certain groups of the type $PSU(4, p^6)$ and $PGL(4, p^6)$. This is done by using convolutions of perverse sheaves related to the existence of rational p-torsion (Mazur's theorem) in conjunction with a positive Mordell-Weil rank.

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Hilbert Irreducibility and Ritt Decompositions

Danny Neftin

Let $f(t, x) \in \mathbb{Q}[t, x]$ be an irreducible polynomial and $n := \deg_x f$ its x-degree. For simplicity assume $\mathbb{Q}$ is algebraically closed in the Galois closure Ω of f over $\mathbb{Q}(t)$. A fundamental question arising from Hilbert's irreducibility theorem is to describe the sets $Red_f = \{a \in \mathbb{Q} \mid f(a, x) \in \mathbb{Q}[x] \text{ is reducible}\}$ and $Red_f \cap \mathbb{Z}$. In particular when are those sets finite?

Hilbert's proof shows that the set Red_f is thin. That is, there are finitely many maps $\phi_i : X_i \to \mathbb{P}^1$ (resp., polynomials $f_i \in \mathbb{Q}[t, x]$), $i \in I$, such that

$$Red_f \subseteq \bigcup_{i \in I} \phi_i(X_i(\mathbb{Q})) \cup \text{finite set}.$$

However, it is unknown how many ϕ_i's are needed and what are their degrees. The latter is need to determine the growth rate of $Red_f \cap \mathbb{Z}$. Note that the case where Red_f is finite corresponds to $I = \emptyset$.

Previous results mainly concern the indcomposable case and separate between the low and high genus cases. Let $\phi_f : X_f \to \mathbb{P}^1$ be the natural projection $(x, t) \to t$ from the curve X_f corresponding to $f(x, t) = 0$. In low genus, the results typically show that Red_f or $Red_f \cap \mathbb{Z}$ is the union of one value set and a finite set. For example, Fried's theorem assumes that ϕ_f is an indecomposable polynomial map $\mathbb{P}^1 \to \mathbb{P}^1$ of degree > 5, and asserts that $Red_f \cap \mathbb{Z}$ is $\phi_f(\mathbb{Z})$ union a finite set. In higher genus, these result typically show that Red_f or $Red_f \cap \mathbb{Z}$ is finite. For example, Mueller's theorem assumes $g(X_f) \geq 1$ and either that n is prime or the "general case" where the monodromy group of ϕ_f is S_n, and asserts that $Red_f \cap \mathbb{Z}$ is finite. In fact, an extensive work of Mueller studies $Red_f \cap \mathbb{Z}$ in the indecomposable case, in particular describing it in the case of simple monodromy groups.

In the indecomposable case, the passage from $Red_f \cap \mathbb{Z}$ to the entire set Red_f and from polynomials to the general low genus case became possible in view of the classification of monodromy groups. The latter program, which was initiated by Guralnick–Thompson, has been recently completed by the author and Zieve for coverings $\phi : X \to \mathbb{P}^1$ of large degree in comparison to the genus of X. As a consequence of this result, we get that if n is sufficiently large in comparison to the genus of X_f, then Red_f is the union of $\phi_f(X_f(\mathbb{Q}))$ and a finite set. Note that $\phi_f(X_f(\mathbb{Q}))$ is finite if the genus of X_f is at least 2.

In contrast to previous work which mainly focuses on the indecomposable case, we describe here recent advancements in the decomposable case obtained jointly with König. Namely, we show that for a polynomial map ϕ_f which decomposes as $\phi_f = \phi_1 \circ \cdots \circ \phi_r$ for indecomposable polynomials $\phi_1, \ldots, \phi_r$ with noncyclic and nondihedral monodromy, degree > 9, and $\deg \phi_1 > 20$. Then either Red_f is the union of $\phi_1(\mathbb{Q})$ and a finite set, or the ramification of ϕ_1 is in an explicit list and $Red_f = \phi_1(\mathbb{Q}) \cup \tilde{\phi}_1(\mathbb{Q}) \cup$ finite set, for some rational function $\tilde{\phi}_1$ whose ramification is determined by that of ϕ_1. The result partially extends to low genus coverings, with the "bottle neck" of the proof being the availability of analogues of Ritt's decomposition theorem for such coverings.

Zero-cycles on Homogeneous Spaces and the Inverse Galois Problem

Olivier Wittenberg
(joint work with Yonatan Harpaz)

Let Γ be a finite group and k be a number field. It has been understood since Hilbert and Noether that the problem of realising Γ as a Galois group over k is really a problem about the arithmetic of the unirational variety $\mathbf{A}^m_k/\Gamma$, where Γ acts by permuting the coordinates once an embedding $\Gamma \hookrightarrow \mathfrak{S}_m$ has been chosen. Equivalently, it is really a problem about the arithmetic of the homogeneous space $\mathrm{SL}_{n,k}/\Gamma$, where Γ acts by multiplication on the right, an embedding $\Gamma \hookrightarrow \mathrm{SL}_n(k)$ having been chosen. (Indeed, by Speiser's lemma, these two varieties are stably birational.) Letting X denote a smooth compactification of $\mathrm{SL}_{n,k}/\Gamma$, a positive answer to the inverse Galois problem for Γ would follow, in particular, from the density of $X(k)$ in the Brauer–Manin set $X(\mathbf{A}_k)^{\mathrm{Br}(X)}$.

Theorem 1. *Let X be a smooth compactification of a homogeneous space V of a connected linear algebraic group G over k. Let $\bar{x} \in V(\bar{k})$ be a geometric point and let $H_{\bar{x}} \subseteq G(\bar{k})$ denote its stabiliser.*

(1) The image of the natural map $\mathrm{CH}_0(X) \to \prod_{v\in\Omega} \mathrm{CH}_0(X \otimes_k k_v)$*, where* Ω *denotes the set of places of* k*, consists of those families* $(z_v)_{v\in\Omega}$ *in the kernel of the Brauer–Manin pairing such that* $\deg(z_v)$ *is independent of* v*. In particular, if* $X(\mathbf{A}_k)^{\mathrm{Br}(X)} \neq \varnothing$*, then* X *admits a zero-cycle of degree* 1*.*

Assume now that G *is semi-simple and simply connected (*e.g.*, that* $G = \mathrm{SL}_{n,k}$*) and that* $H_{\bar{x}}$ *is finite.*

(2) Assume that [4, Conjecture 9.1] *is true. If* $H_{\bar{x}}$ *is solvable, then* $X(k)$ *is dense in* $X(\mathbf{A}_k)^{\mathrm{Br}(X)}$*.*
(3) If $H_{\bar{x}}$ *admits a filtration, with cyclic graded quotients, by normal subgroups of* $H_{\bar{x}}$ *which are stable under the natural outer action of* $\mathrm{Gal}(\bar{k}/k)$*, then* $X(k)$ *is dense in* $X(\mathbf{A}_k)^{\mathrm{Br}(X)}$*.*

The density of $X(k)$ in $X(\mathbf{A}_k)^{\mathrm{Br}(X)}$ may well hold for all rationally connected varieties (a conjecture put forward by Colliot-Thélène) and was previously known for smooth compactifications of homogeneous spaces of linear algebraic groups with connected geometric stabiliser and for smooth compactifications of homogeneous spaces of semi-simple simply connected linear algebraic groups with finite abelian geometric stabiliser (Sansuc, Borovoi). Assertion (1) was previously known under these same assumptions (Liang). Furthermore, Neukirch had proved the density of $X(k)$ in $X(\mathbf{A}_k)$ when $V = \mathrm{SL}_{n,k}/\Gamma$ and the order of Γ and the number of roots of unity contained in k are coprime (a condition which excludes, *e.g.*, 2-groups).

Assertion (3) applies to $V = \mathrm{SL}_{n,k}/\Gamma$ when Γ is supersolvable (*e.g.*, when it is nilpotent). Both (2) and (3) rely on an induction on the order of $H_{\bar{x}}$. Even if one focuses on the inverse Galois problem, and therefore on homogeneous spaces of the form $\mathrm{SL}_{n,k}/\Gamma$, it is an essential point for this induction that the theorem can be applied to homogeneous spaces of $\mathrm{SL}_{n,k}$ which need not have a rational point and whose geometric stabiliser need not be constant.

Assertions (2) and (3) thus recover Shafarevich's theorem on the inverse Galois problem for finite solvable groups, in the supersolvable case (and, conditionally, in general). They also go further, as they yield answers to Grunwald's problem, which asks for Galois extensions of k with group Γ whose completions at a finite set S of places of k are prescribed. In particular, combining (2) and (3) with recent work of Lucchini Arteche on the bad places for the Brauer–Manin obstruction on such X shows that Grunwald's problem admits a positive answer when Γ is supersolvable (conditionally, when Γ is solvable) and S avoids the places dividing the order of Γ. We recall that some condition on S must appear even in the simplest situations: Wang gave a counterexample for $k = \mathbf{Q}$, $\Gamma = \mathbf{Z}/8\mathbf{Z}$, $S = \{2\}$.

The strategy of the proof of Theorem 1 relies on a geometric dévissage of these homogeneous spaces, which we now explain. For the sake of simplicity, we focus on (2) and assume that $G = \mathrm{SL}_{n,k}$ and that [4, Conjecture 9.1] holds. In this situation, we establish the following inductive step, which does not require any assumption on the finite group $H_{\bar{x}}$ and which clearly implies (2): *if* $X'(k)$ *is dense in* $X'(\mathbf{A}_k)^{\mathrm{Br}(X')}$ *for any smooth compactification* X' *of any homogeneous space* *of G with geometric stabiliser isomorphic to the derived subgroup of $H_{\bar{x}}$, then $X(k)$ is dense in $X(\mathbf{A}_k)^{\mathrm{Br}(X)}$.*

The starting point is a version, for rationally connected varieties, of the theory of descent that Colliot-Thélène and Sansuc developed in the 80's for geometrically rational varieties. Just as in the theory of descent for elliptic curves, the idea is to reduce questions about the rational points of a given variety X to the same questions for auxiliary varieties which lie above it. The auxiliary varieties we use are the *universal torsors* of Colliot-Thélène and Sansuc; for a smooth and proper variety X over k such that $\mathrm{Pic}(X \otimes_k \bar{k})$ is torsion-free, they are torsors $Y \to X$ under the torus over k whose character group is $\mathrm{Pic}(X \otimes_k \bar{k})$.

Proposition 2. *If X is a smooth and proper variety over k such that $\mathrm{Pic}(X \otimes_k \bar{k})$ is torsion-free, then $X(\mathbf{A}_k)^{\mathrm{Br}(X)}$ is contained in the union, over all universal torsors $f : Y \to X$, of $f'\big(Y'(\mathbf{A}_k)^{\mathrm{Br}(Y')}\big)$, where Y' denotes a smooth compactification of Y such that f extends to a morphism $f' : Y' \to X$.*

This statement, which can also be deduced from recent work of Cao, was first proved by Colliot-Thélène and Sansuc when X is geometrically rational, a generality which is insufficient for our purposes since there exist finite nilpotent groups Γ such that $\mathrm{SL}_{n,k}/\Gamma$ fails to be geometrically rational (Saltman, Bogomolov). Here, the point is that the full Brauer group of X is taken into account. Dealing with the *algebraic* subgroup $\mathrm{Br}_1(X) = \mathrm{Ker}\big(\mathrm{Br}(X) \to \mathrm{Br}(X \otimes_k \bar{k})\big)$ is not enough, as recent examples of transcendental Brauer–Manin obstructions to weak approximation on $\mathrm{SL}_{n,k}/\Gamma$ for certain p-groups Γ show (Demarche, Lucchini Arteche, Neftin [2]).

The next result elucidates the geometry of the universal torsors in our situation.

Proposition 3. *Let V and X be as in Theorem 1, with $G = \mathrm{SL}_{n,k}$ and $H_{\bar{x}}$ finite. Let $f : Y \to X$ be a universal torsor of X. There exist a dense open subset $W \subseteq Y$, a quasi-trivial torus Q and a morphism $\pi : W \to Q$ whose fibres are homogeneous spaces of G with geometric stabiliser isomorphic to the derived subgroup of $H_{\bar{x}}$.*

Putting together the above two propositions, we see that in order to prove assertion (2) of Theorem 1, the only missing ingredient is a positive answer to the following question, where Z, B are smooth compactifications of W, Q such that π extends to a morphism $p : Z \to B$.

Question 4. *Let $p : Z \to B$ be a dominant morphism between smooth and proper varieties over k. Assume that B and the generic fibre of p are rationally connected. If $B(k)$ is dense in $B(\mathbf{A}_k)^{\mathrm{Br}(B)}$ and $Z_b(k)$ is dense in $Z_b(\mathbf{A}_k)^{\mathrm{Br}(Z_b)}$ for all rational points b of a dense open subset of B, is $Z(k)$ dense in $Z(\mathbf{A}_k)^{\mathrm{Br}(Z)}$?*

An answer in a very special case (the case in which $B = \mathbf{P}^1_k$ and in which for any $b \in \mathbf{P}^1(k) \setminus \{0, \infty\}$, the variety Z_b is smooth and the map $\mathrm{Br}(k) \to \mathrm{Br}(Z_b)$ is onto) was a key step in Hasse's proof of the Hasse–Minkowski theorem on the rational points of quadrics—namely, the step which relies on Dirichlet's theorem on primes in arithmetic progressions (see [5, Proposition 3.17]). More general situations have been considered ever since. The general case, however, seems entirely out of reach. In [4], we provide a positive answer when B is rational over k, assuming the validity of [4, Conjecture 9.1], and we provide a positive answer to the analogous question for zero-cycles, under the same assumption on B, unconditionally.

As the torus Q is quasi-trivial, it is rational over k, and so is B. This completes the strategy for establishing the first two assertions of the theorem in the case of a homogeneous space of $\mathrm{SL}_{n,k}$ with finite solvable geometric stabiliser. To deduce (1) in full generality, a first reduction, using Sylow subgroups, allows us to deal with homogeneous spaces of $\mathrm{SL}_{n,k}$ with arbitrary finite geometric stabiliser; the general case then follows thanks to the work of Demarche and Lucchini Arteche [1]. The proof of (3) rests on a variant of this strategy. Universal torsors are replaced with torsors of a certain intermediate type built out of the cyclic quotients appearing in (3); they lead to fibrations $\pi : W \to Q$ admitting a rational section over $\bar{k}$. On the other hand, for fibrations into rationally connected varieties which are smooth over a quasi-trivial torus and which admit a rational section over $\bar{k}$, we provide an unconditional positive answer to Question 4, by performing a descent (in the form of a slight generalisation of Proposition 2) and applying the work of Harari [3].

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Canonical Paths on Algebraic Varieties

Daniel Litt
(joint work with Alexander Betts)

Let K be a local field (archimedean or non-archimedean) and X a normal variety over K. Then, in several settings, there exist canonical linear combinations of paths between any two points in X. I explained several applications of this observation.

1. Galois actions on fundamental groups

In joint work with Alexander Betts, I prove new structural results about Galois actions on the fundamental group of X, in both the ℓ-adic and p-adic setting; for example

Theorem 1 (Betts, L-). *Let X be a smooth variety over a p-adic field K. Then*

(1) Let ℓ be a prime different from p. Then any Frobenius element of the absolute Galois group of K acts semi-simply on the Lie algebra of the $\mathbb{Q}_\ell$-pro-unipotent fundamental group of X.
(2) Suppose X admits a simple normal crossings compactification with semi-stable reduction. Then the crystalline Frobenius acts semi-simply on the Lie algebra of the log-crystalline pro-unipotent fundamental group of the special fiber of X.

This theorem is a generalization of a theorem of Weil/Tate, essentially; i.e. that Frobenius acts semi-simply on $H^1(X_{\bar{K}}, \mathbb{Q}_\ell)$. Applications include results on the representation theory of fundamental groups, related to work of Cadoret and to the geometric torsion conjecture. For example (see [L]):

Theorem 2 (L–)**.** *Let X be a normal, geometrically connected variety over a finitely generated field k of characteristic zero, and let ℓ be a prime. Then there exists an integer $N = N(X, \ell)$ such that for any non-trivial, semi-simple representation*

$$\rho : \pi_1(X_{\bar{k}}) \to GL_m(\mathbb{Z}_\ell)$$

which extends to a finite index subgroup of $\pi_1^{\acute{e}t}(X)$, one has that ρ is non-trivial mod ℓ^N.

2. Results on the Chabauty-Kim method (in particular, the Chabauty-Kim method at primes of bad reduction)

After some work, Theorem 1 above implies that Kim's Selmer varieties (see e.g. [K]) are irreducible at all primes (the interesting case being primes of bad reduction). This is the zero-th step in trying to run the non-Abelian Chabauty method at a prime of bad reduction. Some motivation for wanting to do this: (a) in real life, one likes to work with small primes (which may be primes of bad reduction) and (b) more seriously, if one hopes to prove the uniform Mordell conjecture (a reasonable end goal for the non-Abelian Chabauty method), one needs to be able to work at a small prime, regardless of reduction type. For a recent success of the (Abelian) Chabauty method in proving uniform bounds (by using primes of bad reduction), see recent work of Katz-Rabinoff-Zureick-Brown [KZB].

3. Applications of p-adic Hodge theory to "explicit" computation of Galois actions on fundamental groups

The theorem I mentioned above indicates that certain operators (either Galois Frobenii or crystalline Frobenii) act diagonalizably on some infinite-dimensional vector spaces — the Lie algebras of some pro-unipotent fundamental group. Knowing this, the natural questions are (a) what are the eigenvalues? (this question is easy), and (b) what are the eigenvectors? This latter question is very hard and interesting. In the crystalline case, I can give an explicit description of the eigenvectors of the crystalline Frobenius (say, if $X = \mathbb{P}^1 \setminus D$ for some divisor D) in terms of p-adic iterated integrals on X (some caution is needed here, since the crystalline

Frobenius is only semi-linear). Via p-adic Hodge theory, this gives a complete description of the action of the absolute Galois group of K on the $\mathbb{Q}_p$-pro-unipotent (etale) fundamental group of X, and with some integral p-adic Hodge theory, quite a bit of information about the Galois action on the usual pro-p fundamental group of X (no pro-unipotent completion). This is related to work of Nakamura.

4. Monodromy-free iterated integrals

Finally, I explained an archimedean analogue of some of the results above, which gives a monodromy-free theory of iterated integrals on complex varieties. Indeed, I described the following so-called "canonical path":

Theorem 3 (L–). *Let X be a normal, connected complex algebraic variety, and let $x, y \in X$. Then there exists a unique element $p(x,y) \in \mathbb{C}[[\pi_1(X;x,y)]]$, such that:*

(1) $\epsilon(p(x,y)) = 1$,
(2) $p(x,y) \in F^0\mathbb{C}[[\pi_1(X;x,y)]]$, *and*
(3)
$$\overline{p(x,y)} \in \bigcap_{i>0}(W^{-i-1} + F^{-i+1}).$$

Here $\overline{p(x,y)}$ is the complex conjugate of $p(x,y)$.

Here $\mathbb{C}[[\pi_1(X;x,y)]]$ is the Mal'cev completion of the space of (based homotopy classes of) paths from x to y, and F^i, W^j are the Hodge and weight filtrations, respectively. I explained how to construct, for each r-tuple of holomorphic forms $\omega_1, \cdots, \omega_r$ on a Riemann surface, a single-valued function

$$f(z) = \int_{\overline{p(x_0,z)}} \omega_1 \cdots \omega_r$$

and how this recovers many classical real-analytic special functions constructed by Bloch, Wigner, Ramakrishnan, Zagier, etc. (the single-valued polylogarithms) [Z].

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Tamely Ramified Torsors and Parabolic Bundles

Niels Borne
(joint work with Indranil Biswas)

Mehta and Seshadri have shown that a unitary representation of the topological fundamental group of a punctured Riemann surface gives rise to a (polystable) parabolic vector bundle (of degree 0): roughly, an ordinary vector bundle, and for each cusp, a weighted filtration of the corresponding stalk (see [4]). This association is explicit, algebraic in nature, and, in fact, one to one. This gives an algebraic approach of étale fundamental groups, namely if X/k is a proper scheme over an algebraically closed field of characteristic 0, endowed with a (simple) normal crossings divisor, representations of the étale fundamental group of the complement are identified with the category of essentially finite parabolic vector bundles. This holds in positive characteristic as well if one replaces the étale fundamental group by a tamely ramified version of Nori's fundamental group scheme ([6, 2]).

More or less by construction, if one fixes a tamely ramified Galois cover, the denominators of the weights of the associated parabolic vector bundles divide the ramification indices. The other way round, if one starts from a finite set of essentially finite parabolic vector bundles, one can ask how the weights relate to the ramification indices of the minimal tamely ramified Galois cover trivializing them all. In a recent joint work with Indranil Biswas, we give an answer to this question in the abelian case.

More precisely, starting from X/k a base scheme over a field, endowed with a simple normal crossings divisor, we define tamely ramified torsors $Y \to X$ under an abelian finite group scheme G as fppf locally induced by Kummer covers. We relate the existence of such torsors with prescribed ramification data (but allowing G to vary) with the existence of essentially finite parabolic vector bundles with prescribed weights along the ramification locus.

Let us now give precise definitions and our statement. We fix a scheme X of finite type over a field k, and D a simple normal crossings divisor on X, meaning $D = \cup_{i\in I} D_i$ is the union of a finite family of irreducible, smooth divisors, crossing normally. We denote the corresponding family by $\mathbf{D} = (D_i)_{i\in I}$ and add to our data a family $\mathbf{r} = (r_i)_{i\in I}$ of positive integers. Given a closed point x of X, we set $(\mathbf{r}_x)_i$ as r_i if x belongs to D_i and 1 otherwise; this defines a local multi-index $\mathbf{r}_x$.

Definition 1. *Let G/k be a finite abelian group scheme. A* tamely ramified G-torsor *with ramification data $(\mathbf{D}, \mathbf{r})$ is the data of a scheme Y endowed with an action of G and a finite and flat G-invariant morphism $Y \to X$ such that for each closed point x of X, there exists a monomorphism $\mu_{\mathbf{r}_x} \to G$ defined over an extension k'/k such that in a* fppf *neighbourhood of x in X, the morphism $Y \to X$ is isomorphic to $Z \times^{\mu_{\mathbf{r}_x}} G$, where $Z \to Spec\, R$ is a Kummer cover locally defined by the choice of equations of $\mathbf{D}$ at x.*

Definition 2 (C.Simpson around 1990)**.** *A parabolic bundle $\mathcal{E}_\cdot$ on $(X, \mathbf{D})$ with weights in $\prod_{i\in I} \frac{1}{r_i}\mathbb{Z}$ is the data*

(1) *For all* $\mathbf{m} \in \prod_{i\in I} \frac{1}{r_i}\mathbb{Z}$ *of a locally free sheaf* $\mathcal{E}_{\mathbf{m}}$*, verifying* $\mathcal{E}_{\mathbf{m}'} \subset \mathcal{E}_{\mathbf{m}}$ *for* $\mathbf{m} \leq \mathbf{m}'$ *(for the component-wise partial order)*
(2) *For* $\mathbf{m} \in \prod_{i\in I} \frac{1}{r_i}\mathbb{Z}$*, and* $\mathbf{n} \in \mathbb{Z}^I$*, of pseudo-period isomorphisms*

$$\mathcal{E}_{\mathbf{m}+\mathbf{n}} \simeq \mathcal{E}_{\mathbf{m}} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-\sum_{i\in I} n_i D_i)$$

compatible between themselves, and with inclusions above.

The category $\mathrm{Par}(X, \mathbf{D})$ of parabolic vector bundles with arbitrary rational weights form an abelian tensor category, the tensor product being given by a convolution formula. Nori defines *finite* parabolic vector bundles as objects of this category satisfying a non trivial tensor relation ([6]). In order to get an abelian category in positive characteristic as well, one introduces *essentially finite* parabolic vector bundles as the kernels of morphisms between two finite parabolic vector bundles ([3]). Under the assumptions of our theorem, the corresponding full subcategory $\mathrm{EF}\,\mathrm{Par}(X, \mathbf{D})$ is even tannakian, so that it makes sense to consider the monodromy group of an essentially finite parabolic vector bundle.

Given a closed point x of D, and $\mathbf{l} \in \mathbb{Z}^I$, we will write that a parabolic vector bundle $\mathcal{E}.$ on $(X, \mathbf{D})$ *admits* $\mathbf{l}/\mathbf{r}$ *as a weight at* x if $\left(\mathcal{E}_{(\mathbf{l}+\mathbf{1})/\mathbf{r}}\right)_x \subset \left(\mathcal{E}_{\mathbf{l}/\mathbf{r}}\right)_x$ is not an equality.

Theorem 1 (I.Biswas-B., 2017)**.** *Assume that* X/k *is proper, of finite type, geometrically* $\{$*connected and reduced*$\}$*. The two following statements are equivalent :*

(1) *There exists a finite abelian group scheme* G/k *and a tamely ramified* G*-torsor* $Y \to X$ *with ramification data* $(\mathbf{D}, \mathbf{r})$*,*
(2) *for each closed point* x *of* D*, and for all* $\mathbf{l} \in \mathbb{Z}^I$*, such that* $0 \leq \mathbf{l} < \mathbf{r}_x$*, there exists an object* $\mathcal{E}.$ *in* $\mathrm{EF}\,\mathrm{Par}(X, \mathbf{D})$ *with abelian monodromy and weights in* $\frac{1}{\mathbf{r}}\mathbb{Z}^I$*, such that* $\mathcal{E}.$ *admits* $\mathbf{l}/\mathbf{r}_x$ *as a weight at* x*.*

It is certainly natural to ask if the theorem holds for G/k finite but possibly non abelian, but this is still an open question.

The main difficulty is to identify tamely ramified torsors with ordinary torsors on natural orbifolds associated to X and the ramification data, the so-called stack of roots, that are fppf locally quotient stacks of Kummer covers. Some tools we use are closely related to the main topics of the conference. For instance, to avoid assuming that X/k has a rational point, or having to pick up one, we use Nori fundamental gerbe ([3]), whose rational points are the sections of the section conjecture (see reports by A. Schmidt and J. Stix). Another key ingredient is a Nori version ([1]) of Noohi's uniformization criterion for algebraic stacks ([5]), that was used by Sylvain Maugeais and Benjamin Collas in their study of the Galois action of the inertia stack of the moduli spaces of curves (see report by B. Collas and references therein).

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Grunwald Problems and Specialization of Galois Covers

Joachim König
(joint work with François Legrand, Danny Neftin)

This talk summarized the results of the two papers [8] and [6] on the local behaviour of specializations of Galois covers. The underlying question, informally stated, is: Given a number field k and a finite group G, to which extent is it possible to impose local conditions on a Galois extension F/k with group G, with the extra requirement that F/k be a specialization of some prescribed k-regular G-extension $E/k(t)$? The topic is part of a larger ongoing project investigating in various ways the structure of the sets of specializations of a Galois cover.

Grunwald problems. The Grunwald problem (over a number field k) is a strengthening of the inverse Galois problem, asking about the existence of Galois extensions of k with prescribed Galois group which approximates finitely many prescribed local extensions. There are several variants which can be considered as Grunwald problems. The most classical is the following one:

Definition (Grunwald problem)**.** *Let k be a number field, G be a finite group and S be a finite set of primes of k. For each $p \in S$, let F_p/k_p be a Galois extension of k_p whose Galois group embeds into G. Does there exist a Galois extension of k with group G whose completion at p equals F_p/k_p for all $p \in S$?*

If we consider only sets S which are disjoint from a certain finite set S_0 of primes of k (depending on G), we speak of a weak Grunwald problem.
Weak Grunwald problems are known to have positive answers for several important classes of groups. In particular, the Grunwald-Wang theorem gives a positive answer for all abelian groups, exempting at most the primes of k extending the rational prime 2 (see [13]). Results by Harari ([4]) give positive answers for weak Grunwald problems for groups which are iterated semidirect products of abelian groups. Recent work of Harpaz and Wittenberg ([5]) gives a positive answer for all supersolvable groups (in particular, for all nilpotent groups). A different direction was exhibited by Saltman, who showed that all Grunwald problems have a positive answer if the group G has a generic Galois extension over k ([10]).

The specialization approach. We consider the solvability of Grunwald problems via specialization of Galois covers. This approach has two motivations: Firstly, for many non-solvable groups, specializations of regular Galois realizations (i.e., of Galois covers) using Hilbert's irreducibility theorem are the only available Galois realizations. Secondly, the question which extensions can appear as specializations of one or several given Galois covers is of interest by itself. Concrete forms of this question include e.g. the problem of existence of parametric Galois extensions.

A central result in favor of the specialization approach is Dèbes' and Ghazi's result about solvability of unramified Grunwald problems for a group G via specialization of any given G-cover ([3]). More precisely, for k a number field and G a finite group, one has the following property, denoted in [3], Thm.1.2, as the *Hilbert-Grunwald property*:

(HG) Let $E/k(t)$ be a k-regular G-extension. Then there exists a finite set S_0 of primes of k such that all unramified Grunwald problems with a set S of primes which is disjoint from S_0 are solvable infinitely many times via specialization from $E/k(t)$. I.e., there exist infinitely many (linearly disjoint) G-extensions E_{t_0}/k, with $t_0 \in k$, solving the given Grunwald problem.

Stronger versions of this result are contained in [2], including counting results for specializations with local conditions and bounded discriminant. The motivation for [8] is to extend the investigation to specialization with arbitrary (tamely) ramified local extensions i.e. to investigate the solvability of tamely ramified Grunwald problems via specialization of Galois covers. The main underlying tool is a precise statement about the decomposition groups at ramified primes in specializations of function field extensions ([8], Thm. 4.1), improving in particular on Beckmann's theorem on inertia groups in specializations ([1], Prop. 4.2). As a consequence, one obtains solutions to a certain class of ramified Grunwald problems via specialization of a k-regular extension $E/k(t)$ (depending only on the local behaviour at branch points of $E/k(t)$).

On the other hand, our results show that (under mild assumptions on the group G; cf. [8], Thm. 6.2), the Hilbert-Grunwald property can never carry over *in full* to ramified Grunwald problems, for any k-regular G-extension $E/k(t)$. As an application, we obtain a new method for disproving the existence of parametric Galois extensions for certain groups G. Here, following e.g. [9] or [7], a regular Galois extension $E/k(t)$ with group G is called parametric if every G-extension of K occurs as a specialization of $E/k(t)$.
We obtain that a group fulfilling a certain very weak "Grunwald-type" property cannot have a parametric extension. In particular, this property is fulfilled by the alternating groups A_n with $n \geq 4$, due to a famous construction of Mestre.We therefore obtain that the alternating groups do not possess parametric Galois extensions over any number field (cf. [8], Cor. 7.3). This is the first result of this kind for finite simple groups. Our results also suggest (using heuristics on the distribution of Galois extensions with given Galois group) that in general, the set of all specializations of a single Galois cover, when counted by discriminant, is

"small" compared to the set of all G-extensions of k, which would considerably strengthen previous non-parametricity results.

The next step logical step is therefore to investigate the behaviour of specialization sets of infinite families of G-covers. This is the content of [6], which shows that already for specializations of a *one-parameter family* of regular G-extensions, i.e., a $k(s)$-regular G-extension $E/k(s)(t)$, the obstructions to solvability of all Grunwald problems by specializing a single regular extension can potentially vanish for all finite groups. Namely, assuming the existence of a one-parameter family $E/k(s)(t)$ with certain conditions on the local extensions at the ramified places, one obtains a tamely ramified analog of the Hilbert-Grunwald property (HG), with the single regular extension $E/k(t)$ replaced by a family $E/k(s)(t)$ (cf. Theorem 1.1 in [6]). While these conditions will be difficult to guarantee in practice, solvability of the weak Grunwald problem is reduced to a geometric problem in the form of a strong version of the regular inverse Galois problem. In particular, the notion of one-parameter families translates to existence of rational curves on suitable Hurwitz spaces.

Application: Admissibility of finite groups. As an application of our investigation of local behaviour of specializations, we obtain new results on $\mathbb{Q}$-admissibility, i.e., existence of G-crossed product division algebras over $\mathbb{Q}$. Recall that a group G is called k-admissible, if there exists a division algebra A over k with $Z(A) = k$ and with a maximal subfield F/k of A such that F/k is Galois with group G. The admissibility of finite groups over number fields was investigated extensively by Schacher ([11]), Sonn (e.g.,[12]) and others, leading to the $\mathbb{Q}$-admissibility conjecture, which states that a finite group is $\mathbb{Q}$-admissible if and only if it is Sylow-metacyclic. While [12] established the conjecture for solvable groups, only a few isolated cases had previously been known for non-solvable groups.

Applying our results on decomposition groups in specializations, we obtain for the first time a $\mathbb{Q}$-admissibility result for infinitely many non-abelian simple groups $PSL_2(p)$ (see [8], Section 5).

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Algebraic Cycles in Generalized Cohomology Theories

Gereon Quick
(joint work with Michael J. Hopkins)

Let X be a smooth projective complex variety. It is an important goal in algebraic geometry to understand the cycle map which associates to an algebraic subvariety $Y \subset X$ a class in the singular cohomology of the space of complex points of X. In particular, one would like to understand the kernel and the image of the cycle map.

In this talk we study the following analogue of this problem. Instead of considering (possibly singular) subvarieties of X, we assume that $Y \to X$ is any proper algebraic map for a smooth proper complex variety Y. We would like to answer the following questions:

(a) Given a complex compact manifold Y over X, how can we decide whether Y is an algebraic variety?
(b) Given a smooth compact algebraic variety Y over X, how can we decide whether Y is a boundary of a manifold without being an "algebraic boundary"?

The goal of the talk is to discuss topological invariants which help to give at least partial answers to the above questions. As a first step, we explain how to consider them from the point of view of cobordism theories in the classical and the motivic homotopy category of Morel and Voevodsky [4].

For question (a), we show how cohomology operations allow to construct examples of non-algebraic classes in Brown-Peterson cohomology theories which form a family of cohomologies which interpolate between singular cohomology and cobordism [6]. Geometrically, these theories correspond to the types of singularities which are allowed for the cycles $Y \to X$. The construction of these examples is based on the work of Atiyah-Hirzebruch [1] and Totaro [8].

There are similar examples for varieties defined over the algebraic closure of finite fields. The key is that there is a suitable stable étale realization functor on the stable motivic homotopy category of smooth varieties over the algebraic closure of finite fields.

For question (b), we present joint work with Michael J. Hopkins in which we construct an invariant which plays the role of the classical Abel-Jacobi invariant for algebraic cobordism of Voevodsky, Levine and Morel [3]. This new invariant is defined for any smooth complex variety over X which is the boundary of a compact almost complex manifold. It takes values in a bordism version of Griffiths' intermediate Jacobian [5].

In fact, there is a Hodge filtered cobordism theory which is an analogue of Deligne cohomology for smooth complex varieties together with a natural map from algebraic cobordism [2]. This map restricts to the Abel-Jacobi invariant for topologically trivial cycles.

For schemes defined over number rings, the pushforward of this theory receives a natural map of spectra from the motivic Thom spectrum. The homotopy fiber of this map represents Arakelov algebraic cobordism. This is related to recent work of Rodriguez on weak arithmetic cobordism [7]. To give a geometric interpretation of the Chern classes in Arakelov algebraic cobordism is an interesting open problem.

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Operad Structure on π_1-sections of Conf_n

Kirsten Wickelgren
(joint work with Craig Westerland)

This describes joint work in progress with Craig Westerland. Benjamin Collas has informed us that he has independent work in progress which overlaps with these ideas – see his report in this volume.

Let k be a number field, and let $G_k = \mathrm{Gal}(\overline{k}/k)$ denote the absolute Galois group of k for an algebraic closure $k \subset \overline{k}$, which we will embed in $\mathbb{C}$. For a geometrically connected variety X over k, a π_1-section of X over k refers to a conjugacy class of sections of the homotopy exact sequence of étale fundamental groups

$$1 \to \pi_1(X_{\overline{k}}) \to \pi_1(X) \to G_k \to 1,$$

where a section s is conjugate to the section $g \mapsto \gamma^{-1}s(g)\gamma$ for any γ in $\pi_1(X_{\overline{k}})$. Let $\mathcal{S}(\pi_1(X/k))$ denote the set of π_1-sections of X over k.

By functoriality of π_1, k-points of X give rise to π_1-sections, as well as k-rational tangential basepoints in the sense of Deligne [1] and Nakamura [5]. See also [6]. For example, maps $\operatorname{Spec} \cup_{n=1}^{\infty} k((z^{1/n})) \to X$. By abuse of notation, $a \in k^*$ will also refer to the associated element of $\mathcal{S}(\pi_1(\mathbb{G}_m/k))$, and a similar convection applies for $X = \mathbb{P}^1 - \{0, 1, \infty\}$. The map $k[x, x^{-1}] \to \overline{k}((z))$ sending x to az is the tangential basepoint associated to the tangent vector at 0 pointing towards a and will be denoted $\overrightarrow{0a}$. There is similar notation for the elements of $\mathcal{S}(\pi_1(\mathbb{P}^1 - \{0, 1, \infty\}/k))$ associated to tangent vectors at $0, 1$ or ∞. Please see [7, 12.2.2] for more information.

Let Conf_n denote the scheme parameterizing configurations of n-ordered points of $\mathbb{A}^1$ up to translation

$$\mathrm{Conf}_n \cong (\mathbb{A}^n - \Delta)/\mathbb{A}^1 \cong \operatorname{Spec} \mathbb{Q}[u_{12}, \ldots, u_{1n}, (u_{1i} - u_{1j})^{-1} : i \neq j],$$

where Δ denotes the union of the closed subspaces where at least two of the standard coordinate projections of $\mathbb{A}^n$ are equal. The notation u_{ij} is borrowed from [4] and indicates $u_{ij} = u_i - u_j$ where u_i denotes the ith coordinate projection. The spaces of the little 2-disks operad are homotopy equivalent to the analytic space of $\mathbb{C}$-points of Conf_n. However, the operad maps do not come from scheme maps. Instead, we wish to construct operad composition maps on π_1-sections of Conf_n.

We were inspired to look for this by work of Horel [3] and Fresse [2]. Let GT denote the Grothendieck-Teichmüller group, PaB the parenthesized braid operad, and $\mathrm{PaB}^\wedge$ its profinite completion. There is an injection $G_{\mathbb{Q}} \to \mathrm{GT}$ and

Theorem (Drinfel'd)**.** GT *is the group of automorphism of* $\mathrm{PaB}^\wedge$ *inducing the identity on objects.*

Conjecture/Theorem in progress (Westerland – W.) *For each n, there is a set P_n of $\mathbb{Q}$-rational tangential basepoints of* Conf_n *and a $G_{\mathbb{Q}}$-equivariant isomorphism*

$$\pi_1(\mathrm{Conf}_{n,\overline{\mathbb{Q}}}, P_n) \cong \mathrm{PaB}_n^\wedge$$

Our set of basepoints P_n are constructed following Ihara-Matsumoto [4].

Corollary. $\pi_1(\mathrm{Conf}_{n,\overline{\mathbb{Q}}}, P_n)$ *is an (explicit) operad in groupoids with an action of* $G_{\mathbb{Q}}$*.*

Corollary. *There are operad composition maps*

$$\begin{aligned} \mathcal{S}(\pi_1(\mathrm{Conf}_n/\mathbb{Q})) \times (\mathcal{S}(\pi_1(\mathrm{Conf}_{m_1}/\mathbb{Q})) \times \ldots \times \mathcal{S}(\pi_1(\mathrm{Conf}_{m_n}/\mathbb{Q}))) \\ \to \mathcal{S}(\pi_1(\mathrm{Conf}_{\sum_{i=1}^n m_i}/\mathbb{Q})) \end{aligned} \tag{1}$$

We expect to compute the maps (1) explicitly on rational points as follows. Let $(a_1, \ldots a_n)$ be a rational point of Conf_n, and for $i = 1, \ldots, n$, let $(b_{i1}, b_{i2}, \ldots, b_{im_i})$ be a rational point of Conf_{m_i}. Except for finitely many values of t, the $(\sum_{i=1}^n m_i)$-tuple

$$(a_1+tb_{11}, a_1+tb_{12}, \ldots, a_1+tb_{1m_1}, \ldots, a_i+tb_{i1}, a_i+tb_{i2}, \ldots, a_i+tb_{im_i}, \ldots, a_n+tb_{nm_n})$$

determines a rational point of $\mathrm{Conf}_{\sum_{i=1}^n m_i}$. In other words, there is an open set U of $\mathbb{A}^1$ and a map $f : U \to \mathrm{Conf}_{\sum_{i=1}^n m_i}$. Under (1), the tuples of sections corresponding to the tuple of points

$$(a_1, \dots a_n) \times ((b_{11}, b_{12}, \dots, b_{1m_1}) \times \dots \times (b_{n1}, b_{n2}, \dots, b_{nm_n}))$$

is sent to $f_*\overline{01}$.

Let's be more explicit for small values of n and the m_i. Note the following.

$\mathrm{Conf}_1 \cong \operatorname{Spec} k$.

$\mathrm{Conf}_2 \cong \operatorname{Spec} k[u_{21}, u_{21}^{-1}] \cong \mathbb{G}_m$.

$\mathrm{Conf}_3 \cong \operatorname{Spec} k[u_{21}, u_{31}, u_{21}^{-1}, u_{31}^{-1}, (u_{21} - u_{31})^{-1}] \cong \mathbb{G}_m \times \mathbb{G}_m - \Delta$.

It is convenient to use the isomorphism

$$\mathbb{G}_m \times \mathbb{G}_m - \Delta \cong \mathbb{G}_m \times \mathbb{P}^1 - \{0, 1, \infty\} \cong \operatorname{Spec} k[x, x^{-1}] \times \operatorname{Spec} k[y, y^{-1}, (y-1)^{-1}]$$

given by $(u_{21}, u_{31}) \mapsto (u_{21}, [u_{21}, u_{31}])$ and $(x, y) \mapsto (x, yx)$, and then to use the above notation for tangential basepoints of $\mathbb{G}_m$ and $\mathbb{P}^1 - \{0, 1, \infty\}$ to denote tangential base points of $\mathbb{G}_m \times \mathbb{G}_m - \Delta$.

We have the map

$$\mathcal{S}(\pi_1(\mathrm{Conf}_2 /\mathbb{Q})) \times (\mathcal{S}(\pi_1(\mathrm{Conf}_2 /\mathbb{Q})) \times \mathcal{S}(\pi_1(\mathrm{Conf}_1 /\mathbb{Q}))) \to \mathcal{S}(\pi_1(\mathrm{Conf}_3 /\mathbb{Q}))$$

$$a \times b \times * \mapsto b \times \overline{\infty \frac{b}{a}}. \tag{2}$$

As well as

$$\mathcal{S}(\pi_1(\mathrm{Conf}_2 /\mathbb{Q})) \times (\mathcal{S}(\pi_1(\mathrm{Conf}_1 /\mathbb{Q})) \times \mathcal{S}(\pi_1(\mathrm{Conf}_2 /\mathbb{Q}))) \to \mathcal{S}(\pi_1(\mathrm{Conf}_3 /\mathbb{Q}))$$

$$a \times * \times b \mapsto a \times \overline{1 \frac{b}{a}}. \tag{3}$$

To see that the above computations are reasonable, first compute the image of tangential basepoints under the composition

$$\mathcal{S}(\pi_1(\mathbb{G}_m \times \mathbb{P}^1 - \{0,1,\infty\}/\mathbb{Q})) \to \mathcal{S}(\pi_1(\mathbb{G}_m \times \mathbb{G}_m - \Delta/\mathbb{Q})) \overset{(u_{12}, u_{13}) \mapsto u_{ij}}{\to} \mathcal{S}(\pi_1(\mathbb{G}_m/\mathbb{Q})).$$

$$\overline{0a} \times \overline{0b} \mapsto \begin{cases} a & \text{if } (i,j) = (2,1) \\ (0 + az)(0 + bz) = abz = ab & \text{if } (i,j) = (3,1) \end{cases}$$

Then note that in case (2), the three points of the configuration have the first two clustered around 0, the second with tangent direction b. The third point is at a. Thus the u_{ij} coordinates of the image are $u_{21} = b$, $u_{31} = u_{32} = a$, whence $x = u_{21} = b$ and $[1 : y] = [u_{21}, u_{31}] = [0 + bz, a] = [0 + \frac{b}{a}z, 1]$. Similarly, in case (3), the u_{ij} coordinates of the image are $u_{21} = u_{31} = a$, and $u_{32} = b$, whence $x = u_{21} = b$ and $[1 : y] = [u_{21}, u_{21} + u_{32}] = [a, a + bz] = [1, 1 + \frac{b}{a}z]$.

The motivation for studying operad composition maps on π_1-sections of configuration spaces is to lift this structure to Hurwitz spaces, thereby producing operadic operations on Galois covers.

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