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Interactions between Algebraic Geometry and Noncommutative Algebra

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Abstract. The workshop presented current developments exploring the boundaries and intersections between the fields of noncommutative algebra and representation theory on the one hand, and algebraic geometry on the other hand, with particular emphasis on topics where such interactions have led to substantial recent progress.

Mathematics Subject Classification (2010): 14A22.

Introduction by the Organisers

The workshop brought together 49 participants from ten countries (Australia, Belgium, Canada, France, Germany, Japan, Netherlands, Russia, UK, US). 24 talks were presented in the five-day period, a considerable part of them given by younger participants. To advance interaction between the participants working at the different strands of the topic during the workshop, three introductory/overview talks were given on the first two days by B. Davison, K. McGerty and M. Wemyss.

The workshop explored the application of ideas and techniques from algebraic geometry to noncommutative algebra and vice versa. There has been a considerable amount of activities at the interface between these different areas, with several very recent exciting developments.

A broad range of these interfaces was reflected in the participants' research activities and the talks, including

- **Hall algebras** in relation to quantum groups and Donaldson-Thomas theory
- **Symplectic and Poisson geometry** in relation to quantization, quiver varieties, and Cherednik algebras
- **Categorification** and (triangulated) categories in relation to homological algebra and Lie-theoretic representation theory
- **Noncommutative resolutions of singularities** in relation to decompositions of derived categories of sheaves and to representation theory
- **Noncommutative algebraic geometry** in relation to classical projective geometry and ring theory

The broad perspective of the workshop, the interactions between many different areas, and the range of substantial advances is illustrated by the abstracts.

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Workshop: Interactions between Algebraic Geometry and Noncommutative Algebra

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Abstracts

Symplectic resolutions of Hamiltonian reduction

Gwyn Bellamy and Travis Schedler

Hamiltonian reduction is an extremely powerful technique, in both physics and differential geometry, for producing rich new symplectic manifolds from a manifold with Hamiltonian G-action. The same technique also works well in the algebraic setting, except that the resulting spaces are often singular, and hence cannot be (algebraic) symplectic manifolds. Thanks to Beauville, there is an effective generalization of algebraic symplectic manifold to the singular setting, appropriately called "symplectic singularities". Thus, it is natural to ask if algebraic Hamiltonian reduction gives rise to spaces with symplectic singularities? In general, examples make it clear that the answer is sometimes yes, sometimes no. For instance, examples show that even beginning with a symplectic G-module, the resulting Hamiltonian reduction can be non-reduced or reducible; even if is reduced and irreducible, it is often not normal. On the other hand, we show in [2] that large classes of examples such as quiver varieties, as defined by Nakajima in [6], give rise to symplectic singularities .

Recently, Herbig–Schwarz–Seaton [5] have shown that 2-large G-modules give rise to a very large class of examples whose Hamiltonian reductions do admit symplectic singularities (in some sense, this is the generic situation). In this case, the obvious question is: do these symplectic singularities admit symplectic resolutions? For G semi-simple we show that the Hamiltonian reduction never admits a symplectic resolution.

Quiver varieties. In this section we describe recent joint work contained in the preprint [2]. We begin by fixing notation. Let Q be a finite quiver with vertices Q_0 and arrows Q_1. Choose $\alpha \in \mathbb{N}^{Q_0}$ a dimension vector, $\lambda \in \mathbb{R}^{Q_0}$ a deformation parameter and $\theta \in \mathbb{Z}^{Q_0}$ a stability parameter such that $\lambda \cdot \alpha = \theta \cdot \alpha = 0$. We write $\mathrm{GL}_\alpha = \prod_{i \in Q_0} \mathrm{GL}(\alpha_i, \mathbb{C})$ and $\mathfrak{gl}_\alpha = \mathrm{Lie}\ \mathrm{GL}_\alpha$. If $\overline{Q}$ denotes the double of Q then the action of GL_α on $T^*\mathrm{Rep}(Q, \alpha) = \mathrm{Rep}(\overline{Q}, \alpha)$ is Hamiltonian with moment map $\mu : \mathrm{Rep}(\overline{Q}, \alpha) \to \mathfrak{gl}_\alpha^* \simeq \mathfrak{gl}_\alpha$. Recall that a representation M with $\theta \cdot \dim M = 0$ is said to be θ-semistable if $\theta \cdot \dim N \leq 0$ for all subrepresentations $N \subset M$. If $\theta \cdot \dim N < 0$ for all proper subrepresentations then M is said to be θ-stable. Let $\mu^{-1}(\lambda)^\theta \subset \mu^{-1}(\lambda)$ be the θ-semistable locus, and let $\mathbb{C}[\mu^{-1}(\lambda)]^{k\theta} \subset \mathbb{C}[\mu^{-1}(\lambda)]$ be the subset of functions on which GL_α acts by the character $(g_i) \mapsto \prod_{i \in Q_0} \det(g_i)^{-\theta_i}$.

The quiver variety is defined to be

$$\mathfrak{M}_{\alpha,\lambda}(\theta) := \mu^{-1}(\lambda)^\theta /\!/ GL_\alpha = \mathrm{Proj} \bigoplus_{k \geq 0} \mathbb{C}[\mu^{-1}(\lambda)]^{k\theta}.$$

We first address the obvious question: Does there exist a stable representation of moment λ? In general, the answer to this question is no. More precisely, the following theorem was proven by Crawley-Boevey for $\theta = 0$, and extend to the general case by the authors.

Theorem 1. There is a combinatorially defined set $\Sigma_{\lambda,\theta} \subset \mathbb{N}^{Q_0}$ such that there is a stable representation of dimension α if and only $\alpha \in \Sigma_{\lambda,\theta}$. For general α there is a canonical decomposition $\alpha = k_1\alpha^{(1)} + \cdots + k_m\alpha^{(m)}$ such that $\alpha^{(i)} \in \Sigma_{\lambda,\theta}$; moreover

$$\mathfrak{M}_{\alpha,\lambda}(\theta) \simeq \mathrm{Sym}^{k_1}\mathfrak{M}_{\alpha^{(1)},\lambda}(\theta) \times \cdots \times \mathrm{Sym}^{k_m}\mathfrak{M}_{\alpha^{(m)},\lambda}(\theta).$$

Note that $\Sigma_{\lambda,\theta}$ is contained in the set of positive roots of the Kac-Moody Lie algebra corresponding to the quiver Q. We recall that these roots fit into one of three families, using the Cartan pairing $(-,-)$. Namely, if $(\alpha,\alpha) = 2$ we say that α is real; if $(\alpha,\alpha) = 0$ we say it is isotropic (imaginary); and if $(\alpha,\alpha) \leq -2$ then we say α is anisotropic (imaginary).

As an example, a real root α belongs to $\Sigma_{\lambda,\theta}$ if and only if $\alpha = e_i$ for some $i \in Q_0$ and $\lambda_i = \theta_i = 0$. In this case, $\mathfrak{M}_{\alpha,\lambda}(\theta)$ is a point. If $\alpha \in \Sigma_{\lambda,\theta}$ is isotropic then $\mathfrak{M}_{\alpha,\lambda}(\theta)$ is a partial resolution/deformation of a du Val singularity.

In the canonical decomposition described in the above theorem, if $\alpha^{(i)}$ is anisotropic then necessarily $k_i = 0$. Thus, the symmetric powers only occur non-trivially for isotropic roots.

We recall that a normal variety X is said to have *symplectic singularities* if there is a symplectic two form ω_{reg} defined on the regular locus X^{reg} and for some/every resolution of singularities $\rho : Y \to X$, the section $\rho^*(\omega_{\mathrm{reg}})$ extends to a *regular* section $\omega_Y \in \Gamma(Y, \Omega_Y)$ on the whole of Y. Moreover, we say that ρ is a *symplectic resolution* if ω_Y is non-degenerate (equivalently, symplectic) on Y.

Theorem 2 (BS)**.** Every quiver variety $\mathfrak{M}_{\alpha,\lambda}(\theta)$ has symplectic singularities. The variety admits a symplectic resolution if and only if, for each $\alpha^{(i)}$, exactly one of the following holds:

(a) $\alpha^{(i)}$ is real.
(b) $\alpha^{(i)}$ is isotropic.
(c) $\alpha^{(i)}$ is anisotropic and Σ-indivisible.
(d) $\alpha^{(i)} = 2\beta^{(i)}$, where $\beta^{(i)} \in \Sigma_{\lambda,\theta}$ is Σ-indivisible and $(\beta^{(i)}, \beta^{(i)}) = -2$.

Here a root $\beta \in \Sigma_{\lambda,\theta}$ is said to be Σ-indivisible if there is no integer $m > 1$ such that $\frac{1}{m}\beta$ also belongs to $\Sigma_{\lambda,\theta}$. In case (d) the symplectic resolution is obtained in two steps. First one chooses a generic stability parameter θ' such that there is a projective map $f : \mathfrak{M}_{\alpha,\lambda}(\theta') \to \mathfrak{M}_{\alpha,\lambda}(\theta)$. Then, a resolution of $\mathfrak{M}_{\alpha,\lambda}(\theta)$ is given by blowing up the reduced subscheme of $\mathfrak{M}_{\alpha,\lambda}(\theta')$ defining its singular locus.

Analogous results. Let S_g be a compact Riemann surface of genus g. Then the space

$$X(g,n) := \mathrm{Hom}(\pi_1(S_g), \mathrm{GL}_n)//\mathrm{GL}_n$$

is the associated character variety. We show:

Theorem 3 (BS)**.** The character variety $X(g,n)$ has symplectic singularities. It admits a symplectic resolution if and only if $g = 1$, $n = 1$ or $(g,n) = (2,2)$.

Using Simpson's Isosingularity Theorem, A. Tirelli has also shown in [9] that if $\mathcal{M}_{\text{Higgs}}(S_g, \text{GL}_n)$ is the moduli space of rank n and degree 0 Higgs bundles on the Riemann surface S_g then:

Theorem 4. The moduli space $\mathcal{M}_{\text{Higgs}}(S_g, \text{GL}_n)$ has symplectic singularities. It admits a symplectic resolution if and only if $g = 1$, $n = 1$ or $(g, n) = (2, 2)$.

In joint work in progress with A. Tirelli, the second author has extended the theorem on character varieties to character varieties of certain surfaces with punctures and monodromy conditions at the punctures. These varieties are all examples of *multiplicative* quiver varieties [4]. The precise statement which has been proved is that, for multiplicative quiver varieties associated to parameters $q \in (\mathbb{C}^\times)^{Q_0}$ and $\theta \in \mathbb{Z}^{Q_0}$, and of dimension vector in $\Sigma_{q,\theta}$, then after taking normalization, the variety is a symplectic singularity. Moreover, we proved that, if we are not in the case (d) before ($\alpha = 2\beta$ for $(\beta, \beta) = -2$), then a symplectic resolution exists if and only if we are in the cases (a)–(c) as before. To extend to the case (d), remove the condition of taking normalization, and extend beyond $\Sigma_{q,\theta}$, a local structure theorem is needed. Such a result should follow by proving that the multiplicative preprojective algebra of a non-Dynkin quiver is 2-Calabi–Yau (many ingredients needed to prove this already appeared in [4, §3], but a certain bimodule map appearing there was not proved to be injective). In fact, by [3, Theorem 6.3, Theorem 6.6], all representation spaces of 2-Calabi–Yau algebras are formally locally quiver varieties; this can be viewed as a local singular analogue of recent work announced by Brav and Dyckerhoff, explaining that the derived moduli of representations of an n-Calabi–Yau category form a $(2 - n)$–shifted symplectic stack.

2-large representations. This part of our report is based partly on recent work of Herbig–Schwarz–Seaton [5]. Their work allows us to apply our arguments and show that a generic Hamiltonian reduction by a semi-simple group does not admit a symplectic resolution. Let G be reductive and connected, V a G-module. Then G acts on $T^*V = V \times V^*$ via Hamiltonian automorphisms, with moment map $\mu : T^*V \to \mathfrak{g}^*$ given by

$$\mu(v, \lambda)(x) = \lambda(x \cdot v), \quad v \in V, \lambda \in V^*, x \in \mathfrak{g}.$$

As mentioned in the introduction, the variety $\mu^{-1}(0)/\!/G$ is badly behaved in general—it need not be reduced, or irreducible, and even when it has both of these properties, it need not be normal. However, by imposing certain natural conditions on V, it is possible to ensure that the Hamiltonian reduction is always well-behaved.

Let $V_{(r)} = \{v \in V \mid \dim G_v = r\}$.

We say that the representation V is 2-large if:

(a) A generic $v \in V$ has finite stabilizer group and the property that $G \cdot v$ is closed in V.
   Under this hypothesis, we can let $V_{\text{pri}} \subseteq V$ be the subset of all v with $G \cdot v$ closed having minimal stabilizer group. The stabilizer groups of all

$v \in V_{\mathrm{pri}}$ are conjugate by Luna's slice theorem. There are now two more conditions:

(b) $\mathrm{codim} V \backslash V_{\mathrm{pri}} \geq 2$.

(c) $\mathrm{codim} V_{(r)} \geq r+2$ for $1 \leq r \leq \dim G$.

Note that in this definition, we clearly have $V_{\mathrm{pri}} \subset V_{(0)}$, but this is a proper inclusion in general. It was shown by G. Schwarz that if G is simple then all but finitely many V (with $V^G = \{0\}$) are 2-large. The main result of [5] shows that:

Theorem 5. Assume that (V, G) is 2-large. Then $\mu^{-1}(0)/\!/G$ is a symplectic singularity.

In particular, $\mu^{-1}(0)$ is reduced, irreducible and normal. In [1], we show that:

Theorem 6 (BS)**.** Assume that G is semi-simple and V is 2-large. Then $\mu^{-1}(0)/\!/G$ is $\mathbb{Q}$-factorial terminal.

More precisely, we show that $\mu^{-1}(0)/\!/G$ is $\mathbb{Q}$-factorial if and only if the reductive group G is semi-simple. If it is not semi-simple, then locally one can take a (partial) resolution by applying GIT.

Note that, for general reductive G, the Hamiltonian reduction is still terminal, since it has symplectic singularities in codimension at least four (then apply [7]). Thus Van der Waerden purity implies:

Corollary 7 (BS)**.** For G semi-simple, the variety $\mu^{-1}(0)/\!/G$ does not admit a symplectic resolution.

Applying Namikawa's results [8] one can also deduce that, in this case, the variety $\mu^{-1}(0)/\!/G$ does not admit any non-trivial (flat, graded) Poisson deformations. In particular, the second Poisson cohomology group $\mathrm{HP}^2(\mu^{-1}(0)/\!/G)$ is zero.

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Semiorthogonal decompositions for moduli of sheaves on curves

Pieter Belmans
(joint work with Swarnava Mukhopadhyay, Sergey Galkin)

Let C be a smooth projective curve[1]. The curve C is often studied through the properties its various associated moduli spaces of sheaves. Because a sheaf on C splits as the direct sum of a vector bundle and a sheaf with finite support, these moduli spaces come in 2 (or 3) families:

1. the symmetric powers $\mathrm{Sym}^i C$, which are isomorphic to the Hilbert scheme of i points $\mathrm{Hilb}^i C$ on C;

2a. the Picard varieties $\mathrm{Pic}^d C$ parametrising line bundles of degree d;

2b. the moduli spaces $\mathrm{M}_C(r,d)$ of (semi)stable vector bundles of fixed rank $r \geq 2$ and degree d, and we will assume $\gcd(r,d)=1$ to ensure that there are no strictly semistable bundles and that the moduli spaces are smooth.

The study of the moduli space $\mathrm{M}_C(r,d)$ is often reduced to that of $\mathrm{M}_C(r,\mathcal{L})$, which is the fiber of the determinant map $\det\colon \mathrm{M}_C(r,d) \to \mathrm{Pic}^d C$ over a line bundle $\mathcal{L}$ of degree d. The moduli space $\mathrm{M}_C(r,\mathcal{L})$ is a smooth projective Fano variety of dimension $(r^2-1)(g-1)$, with $\mathrm{Pic}\,\mathrm{M}_C(r,\mathcal{L}) \cong \mathbb{Z}\cdot\Theta$ where Θ is the ample generator, and index 2, i.e. we have that $\omega^\vee_{\mathrm{M}_C(r,\mathcal{L})} \cong \Theta^{\otimes 2}$.

I will explain how the derived categories of these moduli spaces of sheaves (are expected to) behave, depending on the parameters, by describing natural semiorthogonal decompositions. For background one is referred to Kuznetsov's ICM address [6]. These decompositions give rise to new connections between the different families of moduli spaces, and it is moreover possible to recover various known results as corollaries (not discussed here).

Under our assumptions on k we have that $\mathrm{Pic}^d C \cong \mathrm{Jac}\, C$, and it is well-known that the derived category $\mathbf{D}^{\mathrm{b}}(\mathrm{Jac}\, C)$ is indecomposable as $\mathrm{Jac}\, C$ is an abelian variety. On the other hand one expects interesting semiorthogonal decomposition for $\mathrm{M}_C(r,\mathcal{L})$, as one does for every Fano variety. And it is well-known that $\mathrm{Sym}^i C$ when $i \geq 2g-1$ is a projective bundle over $\mathrm{Pic}^i C$, so one again obtains a semiorthogonal decomposition. Hence one desires to understand decompositions of the derived categories

1. $\mathbf{D}^{\mathrm{b}}(\mathrm{Sym}^i C)$ for all $i \geq 1$,

2b'. $\mathbf{D}^{\mathrm{b}}(\mathrm{M}_C(r,\mathcal{L}))$ for $r \geq 2$ and $\mathcal{L}$ of degree d coprime to r

into indecomposable pieces. I will summarise the state of the art on decompositions for $\mathrm{Sym}^i C$ and $\mathrm{M}_C(r,\mathcal{L})$, and state some natural conjectures in this context.

Besides proving these conjectures in the stated settings it could be interesting to study these moduli problems in the context of noncommutative algebraic geometry for sheaves of hereditary orders on smooth projective curves, or equivalently, orbifold curves, where parabolic sheaves might play a role.

[1] We will work over an algebraically closed field k of characteristic 0, suitable adaptations of the statements are expected to hold over more general fields.

1. Symmetric powers

First we consider symmetric powers, as they will also make an appearance when describing $\mathbf{D}^{\mathrm{b}}(\mathrm{M}_C(r,\mathcal{L}))$. Recently, Toda showed the following semiorthogonal decomposition [9, corollary 5.11].

Theorem 1 (Toda)**.** *With C as above, there exists a semiorthogonal decomposition*

$$\mathbf{D}^{\mathrm{b}}(\mathrm{Sym}^{g-1+n} C) = \left\langle \overbrace{\mathbf{D}^{\mathrm{b}}(\mathrm{Jac}\, C), \ldots, \mathbf{D}^{\mathrm{b}}(\mathrm{Jac}\, C)}^{n \text{ copies}}, \mathbf{D}^{\mathrm{b}}(\mathrm{Sym}^{g-1-n} C) \right\rangle \tag{1}$$

for all $n \geq 0$.

When $n \geq g$ the Abel–Jacobi morphism $\mathrm{Sym}^{g-1+n} C \to \mathrm{Pic}^{g-1+n} C$ is a $\mathbb{P}^{n-1}$-bundle, and $\mathrm{Sym}^{g-1-n} C = \emptyset$, so the description reduces to Orlov's blow-up formula. For $n \leq g-1$ the semiorthogonal decomposition is obtained by studying wall-crossing for moduli spaces on Calabi–Yau 3-folds, and it would be interesting to give a more self-contained proof.

The following natural conjecture then provides the final ingredient for the study of semiorthogonal decompositions on symmetric powers.

Conjecture 2. *With C as above, the derived category $\mathbf{D}^{\mathrm{b}}(\mathrm{Sym}^i C)$ is indecomposable for $i = 1, \ldots, g-1$.*

If $i = 1$, then $\mathrm{Sym}^1 C = C$ and indecomposability was shown in [8]. For $i = 2, \ldots, g-1$ the symmetric power is an i-dimensional variety of general type, where the study of the base locus of the complete linear system $|\omega_{\mathrm{Sym}^i C}|$ should shed light on the indecomposability.

For a smooth projective surface Krug–Sosna showed that the Fourier–Mukai functor associated to the universal ideal sheaf $\mathcal{I}$ on $S \times \mathrm{Hilb}^n S$ is fully faithful [5], provided that $\mathrm{H}^1(S, \mathcal{O}_S) = \mathrm{H}^2(S, \mathcal{O}_S) = 0$. When S is a K3 surface the functor is a $\mathbb{P}^n$-functor towards an indecomposable category. This leads us to the following question, which would be interesting for understanding the relationship between the Hochschild cohomologies and (noncommutative) deformation theories of these varieties.

Question 3. *What are the special properties of the Fourier–Mukai functor from C (resp. S) to $\mathrm{Sym}^i C$ (resp. $\mathrm{Hilb}^n S$)?*

2. Moduli of vector bundles

As $\mathrm{M}_C(r,\mathcal{L})$ is a fine moduli space we have a universal (Poincaré) bundle $\mathcal{W}$ on $C \times \mathrm{M}_C(r,\mathcal{L})$. We can use it to construct the Fourier–Mukai functor

$$\Phi_{\mathcal{W}} \colon \mathbf{D}^{\mathrm{b}}(C) \to \mathbf{D}^{\mathrm{b}}(\mathrm{M}_C(r,\mathcal{L})). \tag{2}$$

Recently the following result was shown by Fonarev–Kuznetsov [4] and indepently by Narasimhan [7].

Theorem 4 (Fonarev–Kuznetsov, Narasimhan)**.** *With C as above, when $r = 2$ and $\deg \mathcal{L} = 1$, the functor $\Phi_{\mathcal{W}}$ is fully faithful.*

In the approach by Fonarev–Kuznetsov one needs to take C generic, as the fully faithfulness is based on an explicit check for hyperelliptic curves. The approach by Narasimhan uses the Hecke correspondence, and one needs to impose $g \geq 4$. In a joint work with Swarnava Mukhopadhyay [1] we have generalised the approach using the Hecke correspondence to arbitrary rank, and degree 1. The proof for other degrees will require a new ingredient.

Theorem 5 (Belmans–Mukhopadhyay)**.** *With C as above, when $r \geq 2$ and $\deg \mathcal{L} = 1$, the functor $\Phi_{\mathcal{W}}$ is fully faithful for $g \geq g_0(r)$.*

Having found a non-trivial component in the derived category, one would like to know how it relates to the exceptional line bundles $\Theta^\vee$ and $\mathcal{O}_{\mathrm{M}_C(r,\mathcal{L})}$. A special case of the following result when $r = 2$ and involving only a single copy of the curve was shown by Narasimhan.

Theorem 6 (Belmans–Mukhopadhyay)**.** *With C as above and $\mathcal{L}$ as above, we have that the sequence*

$$\Theta^\vee, \Phi_{\mathcal{W}}(\mathbf{D}^{\mathrm{b}}(C)) \otimes \Theta^\vee, \mathcal{O}_{\mathrm{M}_C(r,\mathcal{L})}, \Phi_{\mathcal{W}}(\mathbf{D}^{\mathrm{b}}(C) \tag{3}$$

is the start of a semiorthogonal decomposition for $\mathbf{D}^{\mathrm{b}}(\mathrm{M}_C(r,\mathcal{L}))$, for $g \geq g_0(r)$.

To describe the complement of these 4 admissible subcategories I propose the following conjecture.

Conjecture 7. *With C as above and $\mathcal{L}$ as above, the derived category $\mathbf{D}^{\mathrm{b}}(\mathrm{M}_C(2,\mathcal{L}))$ has a semiorthogonal decomposition of the form*

$$\begin{aligned}\mathbf{D}^{\mathrm{b}}(\mathrm{M}_C(2,\mathcal{L})) = \big\langle\, & \mathbf{D}^{\mathrm{b}}(k), \mathbf{D}^{\mathrm{b}}(k), \\ & \mathbf{D}^{\mathrm{b}}(C), \mathbf{D}^{\mathrm{b}}(C), \\ & \mathbf{D}^{\mathrm{b}}(\operatorname{Sym}^2 C), \mathbf{D}^{\mathrm{b}}(\operatorname{Sym}^2 C), \\ & \ldots, \\ & \mathbf{D}^{\mathrm{b}}(\operatorname{Sym}^{g-2} C), \mathbf{D}^{\mathrm{b}}(\operatorname{Sym}^{g-2} C), \\ & \mathbf{D}^{\mathrm{b}}(\operatorname{Sym}^{g-1} C)\big\rangle\end{aligned} \tag{4}$$

where there are 2 copies of $\mathbf{D}^{\mathrm{b}}(\operatorname{Sym}^i C)$ for $i = 0, \ldots, g-2$ and 1 copy of $\mathbf{D}^{\mathrm{b}}(\operatorname{Sym}^{g-1} C)$.

For $g = 2$ we have that $\mathrm{M}_C(2,\mathcal{L})$ is the intersection of two quadrics in $\mathbb{P}^5$, and the proposed semiorthogonal decomposition is known by [3].

In a joint work in progress with Sergey Galkin and Swarnava Mukhopadhyay [2] we have collected some evidence for this conjecture, from two angles:

1. an identity in the Bondal–Larsen–Lunts Grothendieck ring of categories;
2. a (partial) description of the Landau–Ginzburg mirror, and a study of the eigenvalues of the endomorphism $\mathrm{c}_1(\mathrm{M}_C(2,\mathcal{L})) \star -$ on the quantum cohomology of $\mathrm{M}_C(2,\mathcal{L})$.

It is not clear at the moment what the precise statement for the analogous conjecture for $r \geq 3$ would have to be, but it will probably involve components like $\mathbf{D}^{\mathrm{b}}(C^i)$ and $\mathbf{D}^{\mathrm{b}}(C^i \times \operatorname{Sym}^j C)$. A better understanding of the Chow motive or the class in the Grothendieck ring of varieties would be helpful.

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Perverse sheaves and knot contact homology

Yuri Berest

(joint work with Wai-Kit Yeung, Alim Eshmatov)

In this talk, we give a new algebraic construction of knot contact homology in the sense of L. Ng [6, 7]. For a link L in $\mathbb{R}^3$, we define a differential graded (DG) k-category $\mathscr{A}_L$ with finitely many objects, whose quasi-equivalence class is a topological invariant of L. In the case when L is a knot, the endomorphism algebra of a distinguished object of $\mathscr{A}_L$ coincides with the fully noncommutative knot DGA defined by Ekholm, Etnyre, Ng and Sullivan in [4]. Our construction of knot contact homology is a special case of a general categorical construction (which we call the *homotopy braid closure*) that produces invariants of links in $\mathbb{R}^3$ starting with a braid group action on objects of a given (model) category $\mathcal{C}$. The DG k-category $\mathscr{A}_L$ is obtained by taking the homotopy braid closure of a natural action of the braid group B_n on the category of perverse sheaves on a two-dimensional disk with singularities at n marked points, studied by Gelfand, MacPherson and Vilonen in [5]. This places knot contact homology in one row with other classical link invariants, such as link groups and Alexander modules.

As an application, we show that the category of finite-dimensional representations of the link k-category $A_L := H_0(\mathscr{A}_L)$ defined as the 0th homology of our DG category $\mathscr{A}_L$ is equivalent to the category $\operatorname{Perv}(\mathbb{R}^3, L)$ of perverse sheaves on

$\mathbb{R}^3$ which are singular along the link L. This gives a description of the category $\mathrm{Perv}(\mathbb{R}^3, L)$ in terms of linear algebra data similar, in spirit, to the quiver description of the category of perverse sheaves on the disk given in [5]. We also construct a natural generalization of the category $\mathscr{A}_L$ by extending the Gelfand-MacPherson-Vilonen braid group action.

The main results of this talk have been announced in [1]; detailed proofs, examples, applications and further generalizations will appear in [2] and [3].

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Tautological moduli problems and tilting theory

Daniel Chan

(joint work with Tarig Abdelgadir, Boris Lerner)

In recent years, it has been increasingly appreciated that moduli stacks of modules are a fruitful way to study noncommutative algebra, since their universal module can be used to produce functors between the algebra and the moduli stack. The classic example of this is King's moduli interpretation of Beilinson's derived equivalence for $\mathbb{P}^1$ where, starting from the Kronecker quiver, one can reproduce $\mathbb{P}^1$ as a moduli stack of representations of dimension vector $(1, 1)$ and the universal object is the classic tilting bundle $\mathcal{O} \oplus \mathcal{O}(1)$. Unfortunately, this approach as is, cannot reproduce Geigle-Lenzing's derived equivalence between weighted projective lines and Ringel's canonical algebras.

In this talk, we introduce a new moduli stack, the tensor stable moduli stack, which allows one to rectify this problem. To understand the problem, it is natural to "tilt" the moduli problem on the module category back to a "tautological moduli problem" on the stack, namely: How can you recover a stack $\mathbb{X}$ as a moduli problem on $\mathrm{Coh}(\mathbb{X})$. We look at this problem too and what it has to say about Geigle-Lenzing spaces.

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An introduction to Hall algebras

Ben Davison

This is an overview talk on Hall algebras, where I will try to offer a chronological sketch starting with Hall's original algebra and ending with more recent work on cohomological Hall algebras and Yangians. In the interests of time and space I have had to exclude whole areas of the subject, and have somewhat arbitrarily focused only on those parts of the theory that involve quivers and quantum groups. But I can't give the talk without pointing out that for much of what follows one can replace the category of representations of a quiver with the category of coherent sheaves on a projective smooth curve, arriving at, for instance the Hall algebra of an elliptic curve [1], which has turned out to be of central importance in geometric representation theory.

So to start with, let $\mathcal{A}$ be an exact subcategory of a finitary hereditary Abelian category over a finite field $\mathbb{F}_q$, and let

$$(M,N) = \dim_{\mathbb{F}_q}\mathrm{Hom}(M,N) - \dim_{\mathbb{F}_q}\mathrm{Ext}^1(M,N)$$

be the Euler form (which plays a vital role in all incarnations of Hall algebras). We first consider the case where $\mathcal{A}$ is the category of finite-dimensional nilpotent $\mathbb{F}_q[t]$-modules, where isomorphism classes of modules are indexed in a natural way by partitions, and it was proved by Hall that for three partitions λ, μ, ν the number

$$\#\{M \subset M_\nu | M \cong M_\lambda, M_\nu/M \cong M_\mu\}$$

is a polynomial expressed in q, the so-called Hall polynomial. These are the structure constants for the "original" Hall algebra, with $\mathbb{Z}[q]$-basis symbols δ_λ, and structure constants given by the above count. This algebra plays a key role in the theory of symmetric functions [8]. It is $\mathbb{Z}$-graded by letting the degree of δ_λ equal $|\lambda|$. Different Hall algebras are graded by considering classes in $K(\mathcal{A})$, which we pick to be some manageable quotient of the Grothendieck group of $\mathcal{A}$, such that the Euler form descends to a form on $K(\mathcal{A})$.

More generally, for classes $r = r' + r''$ in $K(\mathcal{A})$, (a graded piece of) the Hall algebra multiplication is defined via the following diagram

$$Ob(\mathcal{A}_{r'}) \times Ob(\mathcal{A}_{r''}) \xleftarrow{\pi} Ex(\mathcal{A}_{r',r''}) \xrightarrow{p} Ob(\mathcal{A}_r)$$

where the middle term is the stack/set of short exact sequences with first and last term of class r' and r'' respectively. Hall's multiplication can be written rather succinctly as $p_* \circ \pi^*$, as an operator on the set of functions on the set of isomorphism classes, and this remains true more generally — the way to get different Hall algebras is to answer differently the following questions:

(1) What to choose for $\mathcal{A}$
(2) How to interpret $Ob(\mathcal{A})$ (e.g. set, or stack...)
(3) How to interpret the operators p_*, π^*.

In order to define the Ringel-Hall algebra associated to a finite quiver Q, we let $Ob(\mathcal{A})$ be the set of functions from the set of isomorphism classes of finite-dimensional Q-representations over $\mathbb{F}_q$, and p_* and π^* be the pushforward and pullback of these functions. In order to define the Hall multiplication we twist, by letting u be a square root of q, and setting

$$\alpha \star \beta = p_* \circ \pi^*(\alpha \boxtimes \beta) \cdot u^{(|\alpha|,|\beta|)}.$$

This twist by a power of u is a recurring feature of the story. It has to do with shift of perverse degrees when we pull back along π. Another common feature of the story is that, reading the above diagram the other way, there is a coproduct (Green's coproduct), the (r', r'')-dimensional piece of which is defined by

$$\Delta(\alpha) = \pi_* \circ p^*(\alpha) \cdot u^{(r',r'')}.$$

Here are the key results regarding this algebra. The theorems of Ringel and Green state that there is an embedding of the quantum group $U_q(\mathfrak{g}_Q^+)$, the deformed universal enveloping algebra of the Kac-Moody algebra associated to the underlying graph of Q, sending the standard generators E_i to the indicator functions for the simple 1-dimensional nilpotent representations S_i associated to the vertices of Q, and this embedding is an isomorphism if and only if Q is of ADE type. An excellent account of this theory is provided by Schiffmann's notes [10]. There is an extension of this result to deal with $U_q(\mathfrak{b}_Q^+)$, but for some time, the question of how to embed the whole of $U_q(\mathfrak{g}_Q)$ inside a Hall algebra remained mysterious. This mystery was solved by Tom Bridgeland, using a Hall algebra of $\mathbb{Z}_2$-graded Q-representations [2]. Somewhat intriguingly, Van den Bergh and Sevenhant showed that even in general (i.e. not ADE, or even affine) type, the Ringel-Hall algebra is the quantized enveloping algebra of the positive part of a generalized Kac–Moody algebra, or Borcherds algebra. This suggested that on top of producing interesting realizations of quantum groups for known Lie algebras, Hall algebras may be a way to construct new such algebras from scratch. This idea will be touched on at the end of the talk, and provides a hint of some of the most interesting developments when we come to cohomological Hall algebras.

Revisiting the three questions from above, we may instead consider $Ob(\mathcal{A})$ as the stack of finite-dimensional Q-representations over the algebraic closure $\overline{\mathbb{F}}_q$, but now interpret p_* and π^* as operations on ℓ-adic sheaves, or Weil sheaves (equivariant sheaves for the action of Frobenius). If we start with the constant sheaf on each of the simples S_i as above, and take the closure under the operations of Hall algebra convolution and taking direct summands, we arrive at the *Lusztig category*. Passing to the Grothendieck group of the Lusztig category, we obtain a Hall algebra that is isomorphic to the Ringel-Hall algebra, which now has a *canonical basis* given by classes of simple perverse sheaves. This aspect of the theory is beautifully explained in Schiffmann's notes [11].

An important subsequent development in the theory of Hall algebras, starting with Reineke's paper [9] has to do with *integration maps*. An integration map is usually a ring homomorphism from a Hall algebra (in this case Ringel's Hall algebra) to a "quantum torus" $R[[x_1, \ldots, x_n]]$, where $1, \ldots, n$ are the vertices of Q, and R here stands for ring and also realization, and is a ring in which counts of objects can take place. The idea is that the coefficient of x^r in $\int \alpha$ should be the sum/integral of the function/object α across all objects of class r. So in the case of Ringel's Hall algebra, we set $R = \mathbb{Z}((q^{1/2}))$, the ring of rational functions in $q^{1/2}$, since taking into account stabilizers, the counts of objects are typically rational functions in q, with a square root added to accommodate the $q^{1/2}$-powers mentioned above. The multiplication in the target ring of the integration map is R-linear, but mildly noncommutative: we set

$$x^{r'} \cdot x^{r''} = q^{(r',r'')/2} x^{r'+r''}.$$

Reineke used such maps to deduce recursion results on the count (over $\mathbb{F}_q$) and cohomology (over $\mathbb{C}$) of semistable representations, and relate these objects to the quantum group $U_q(\mathfrak{g}^+)$. Generalised integration maps have proved to be tremendously fruitful in the theory of motivic Hall algebras and Donaldson–Thomas invariants for 3-Calabi–Yau categories (e.g. the category of coherent sheaves on a 3CY variety), and foundations were developed extensively in the subsequent work by Joyce and Song [4], and separately by Kontsevich and Soibelman [5], before being used in the work of Bridgeland, Toda and many others studying moduli of coherent sheaves on CY 3-folds.

More recently, a theory of cohomological Hall algebras has emerged as a new way of understanding quantum groups. For this theory we let $\mathcal{A}$ be the category of finite dimensional representations of a quiver over $\mathbb{C}$, we interpret $Ob(\mathcal{A})$ as an Artin stack over $\mathbb{C}$, and we interpret p_* and π^* as operations in some equivariant cohomology theory which we'll take to be Borel–Moore homology for this talk. One gets different quantum groups for different cohomology theories, e.g. K theory and elliptic cohomology, which we won't discuss. The cohomological algebra was first introduced in [6]. There is again a coproduct, defined exactly as above (the twist by u now becomes a shift in cohomological degree). Alternatively we can let $\mathcal{A}$ be the category of representations of the preprojective algebra associated to Q and interpret p_* and π^* as morphisms in BM homology, to arrive at the Hall algebra defined by Schiffmann and Vasserot in [12]. Additionally, there is a modification of the Kontsevich–Soibelman construction involving potentials and vanishing cycles, for an arbitrary quiver with potential, such that for a particular quiver with potential $(\tilde{Q}, W)$ built out of the original quiver Q, we get a cohomological Hall algebra that turns out to be isomorphic to the Schiffmann–Vasserot algebra. As a kind of cohomological version of the Van den Bergh–Sevenhat result above, Sven Meinhardt and I proved [3] that the cohomological Hall algebra associated to an *arbitrary* quiver with potential is a quantum deformation of a Lie algebra $\tilde{\mathfrak{g}}^+_{Q,W} \otimes \mathbb{Q}[t]$, where t is a variable of cohomological degree 2, and $\mathfrak{g}_{Q,W}$ is a new Lie algebra. In the case of the quiver with potential $(\tilde{Q}, W)$, this turns out to be a Borcherds algebra again, and it also seems that this algebra is not *brand* new, and is in fact isomorphic to the Lie algebra constructed by Maulik and Okounkov in their book on equivariant quantum cohomology of Nakajima quiver varieties and Yangian algebras [7].

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Donaldson–Thomas Invariants of Quivers as Chow Groups

Hans Franzen

(joint work with Markus Reineke)

The Cohomological Hall algebra of a quiver Q was introduced by Kontsevich and Soibelman. It is defined as

$$\mathrm{CoHA}(Q) = \bigoplus_{d \in \mathbb{Z}_{\geq 0}^{Q_0}} H^*_{G_d}(R_d;\mathbb{Q})$$

where $R_d = \bigoplus_{a\in Q_1} \mathrm{Hom}(\mathbb{C}^{d_{s(a)}}, \mathbb{C}^{d_{t(a)}})$ which is acted upon by the group $G_d = \prod_{i\in Q_0} \mathrm{GL}_{d_i}(\mathbb{C})$. The multiplication is given by the maps

$$R_d \times R_e \leftarrow \left(\begin{smallmatrix} R_d & * \\ & R_e \end{smallmatrix}\right) \to R_{d+e}$$

called Hecke correspondences. Let $\theta : \mathbb{Z}^{Q_0} \to \mathbb{Z}$ be a $\mathbb{Z}$-linear map. Let $\mu : \mathbb{Z}_{\geq 0}^{Q_0} - \{0\} \to \mathbb{Q}$ be the associated slope function $\mu(d) = \theta(d)/(\sum d_i)$. This slope function defines open subsets $R_d^{\theta-\mathrm{st}} \subseteq R_d^{\theta-\mathrm{sst}} \subseteq R_d$ of θ-stable and θ-semi-stable representations. For a fixed rational number m define $\Lambda_m = \{d \in \mathbb{Z}_{\geq 0}^{Q_0} \mid d =$

0 or $\mu(d) = m\}$. The above Hecke correspondences restrict to the semi-stable loci to give a multiplication on

$$\mathrm{CoHA}^{\theta,m}(Q) = \bigoplus_{d\in\Lambda_m} H^*_{G_d}(R_d^{\theta-\mathrm{sst}};\mathbb{Q}).$$

The induced algebra is called the semi-stable Cohomological Hall algebra. We show $H^{2i+1}_{G_d}(R_d^{\theta-\mathrm{sst}}) = 0$ and that the equvariant cycle map $A^i_{G_d}(R_d^{\theta-\mathrm{sst}}) \to H^{2i}_{G_d}(R_d^{\theta-\mathrm{sst}})$ is an isomorphism. We use this result to deduce that for $d \in \Lambda_m$ the maps $\mathrm{CoHA}_d \to \mathrm{CoHA}_d^{\theta,m}$ and $\mathrm{CoHA}_d^{\theta,m} \to A^*_{G_d}(R_d^{\theta-\mathrm{st}})_{\mathbb{Q}}$ are surjective and the kernels are given as

$$\ker(\mathrm{CoHA}_d \to \mathrm{CoHA}_d^{\theta,m}) = \sum_{\substack{e+f=d\\ \mu(e)>\mu(f)}} \mathrm{CoHA}_e * \mathrm{CoHA}_f$$

$$\ker(\mathrm{CoHA}_d^{\theta,m} \to A^*_{G_d}(R_d^{\theta-\mathrm{st}})_{\mathbb{Q}}) = \sum_{\substack{e+f=d\\ e,f\in\Lambda_m-\{0\}}} \mathrm{CoHA}_e^{\theta,m} * \mathrm{CoHA}_f^{\theta,m}.$$

Using Efimov's result that the CoHA of a symmetric quiver is—up to a slight modification—a free super-commutative algebra, we deduce that if Q is symmetric the Donaldson–Thomas invariants of Q can be realized as Chow groups of stable moduli spaces.

Mukai flop from point of view of noncommutative crepant resolutions

Wahei Hara

Let $Y = T^*\mathbb{P}(V)$ be the cotangent bundle of the n-dimensional projective space $\mathbb{P}(V)$ where $V = \mathbb{C}^{n+1}$, and let $Y' = T^*\mathbb{P}(V^*)$ be the cotangent bundle of the dual projective space. Then there exists a canonical isomorphism of $\mathbb{C}$-algebras

$$R := H^0(Y, \mathcal{O}_Y) \simeq H^0(Y', \mathcal{O}_{Y'}),$$

and if we put $X := \mathrm{Spec}(R)$, two canonical morphisms $\phi : Y \to X$ and $\phi' : Y' \to X$ give symplectic (and hence crepant) resolutions of X. The diagram

$$Y \xrightarrow{\phi} X \xleftarrow{\phi'} Y'$$

is a local model of Mukai flop. For two crepant resolutions, it is widely expected that they are derived equivalent. Kawamata [Kaw02] and Namikawa [Nam03] proved this expectation for Mukai flops. They proved a Fourier-Mukai transform whose kernel is the structure sheaf of $Y \times_X Y'$. We call this functor Kawamata-Namikawa equivalence.

If we put $\pi : Y \to \mathbb{P}(V)$ and $\pi' : Y' \to \mathbb{P}(V^*)$ be the projections and put $\mathcal{O}_Y(a) := \pi^*\mathcal{O}_{\mathbb{P}(V)}(a)$ and $\mathcal{O}_{Y'}(a) := \pi'^*\mathcal{O}_{\mathbb{P}(V^*)}(a)$, then

$$T_0 = \mathcal{O}_Y \oplus \mathcal{O}_Y(-1) \oplus \cdots \oplus \mathcal{O}_Y(-n),$$
$$T_0' = \mathcal{O}_{Y'} \oplus \mathcal{O}_{Y'}(-1) \oplus \cdots \oplus \mathcal{O}_{Y'}(-n)$$

are tilting bundles on Y and Y' respectively, and R-modules $\phi_*(T_0)$ and $\phi'_*(T'_0)$ gives non-commutative crepant resolutions of R. Although vector bundles

$$T_k := T_0 \otimes \mathcal{O}_Y(k),\ T'_k := T'_0 \otimes \mathcal{O}_{Y'}(k)$$

are also tilting bundles, the bundles T_0 and T'_0 are canonical in the following sense. Put

$$W_0 = \mathcal{O}_Y \oplus \mathcal{O}_Y(-1) \oplus \cdots \oplus \mathcal{O}_Y(-n+1),$$
$$W'_0 = \mathcal{O}_{Y'} \oplus \mathcal{O}_{Y'}(-1) \oplus \cdots \oplus \mathcal{O}_{Y'}(-n+1)$$

and let

$$A_{n-1} := \mathrm{End}_Y(W_0) \text{ and } A'_{n-1} := \mathrm{End}_{Y'}(W'_0)$$

be their endomorphism rings. Then Toda and Uehara [TU10] constructed perverse hearts

$${}^0\mathrm{Per}(Y/A_{n-1}) \subset D^b(\mathrm{coh}(Y)) \text{ and } {}^0\mathrm{Per}(Y'/A'_{n-1}) \subset D^b(\mathrm{coh}(Y'))$$

that are natural generalizations of perverse hearts in dimension three (see, for example, [VdB04a]), and showed that T_0 and T'_0 are projective generators of ${}^0\mathrm{Per}(Y/A_{n-1})$ and ${}^0\mathrm{Per}(Y'/A'_{n-1})$.

However the R-modules $\phi_*(T_0)$ and $\phi'_*(T'_0)$ that give non-commutative crepant resolutions are not isomorphic to each other if $n \geq 2$. Thus it is natural to ask what is the relation between these two modules. The answer of this question is the following: they are connected by applying the operation *mutation* which is introduced by Iyama and Wemyss.

Let us recall the definition of the mutation. Let M be a reflexive module that gives a non-commutative crepant resolution and N be a direct summand of M. Then we can define a (left) mutation $\mu_N(M)$ of M at N in the following way. First take a morphism

$$a : L \to M^*$$

such that $L \in \mathrm{add}(N^*)$ and the induced morphism

$$\mathrm{Hom}(N^*, L) \xrightarrow{a\circ -} \mathrm{Hom}(N^*, M^*)$$

is surjective. Such morphism is called a right $\mathrm{add}(N^*)$-approximation of M^*. Let $K := \mathrm{Ker}(a)$ and define

$$\mu_N(M) := N \oplus K^*.$$

Iyama and Wemyss [IW14] proved that the new module $\mu_N(M)$ also gives a non-commutative crepant resolution and that there exists a canonical derived equivalence

$$\Phi_N : D^b(\mathrm{mod}(\mathrm{End}(M))) \xrightarrow{\sim} D^b(\mathrm{mod}(\mathrm{End}(\mu_N(M)))).$$

Note that the mutated module $\mu_N(M)$ is well-defined up to additive closures.

If the module that gives a non-commutative crepant resolution comes from a tilting bundle, then we can compute its mutation using the following lemma.

Lemma 1 ([H17b])**.** Let $\phi : Y \to X = \mathrm{Spec}(R)$ be a crepant resolution. Let W be a vector bundle on Y and

$$0 \to E_0 \xrightarrow{f_0} E_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{l-2}} E_{l-1} \xrightarrow{f_{l-1}} E_l \to 0$$

an exact sequence of vector bundles on Y. Assume that

(i) $T_0 = E_0 \oplus W$ and $T_{l-1} = E_l \oplus W$ are tilting bundles on Y.
(ii) W contains $\mathcal{O}_Y$ as a direct summand.
(iii) E_i is in $\mathrm{add}(W)$ for $1 \le i \le l-1$.

Then we have

(1) $T_i := W \oplus \mathrm{Im}(f_i)$ is a tilting bundle for all $0 \le i \le l-1$.
(2) If we put $N := \phi_*(W)$ and $M_i := \phi_*(T_i)$, then

$$\mu_N(M_i) \simeq M_{i+1}$$

for $0 \le l-2$.
(3) There is a functor isomorphism

$$\Phi_N \simeq \mathrm{RHom}_{\mathrm{End}(M_i)}(\mathrm{Hom}_Y(T_i, T_{i+1}), -) : D^b(\mathrm{mod}(\mathrm{End}(M_i))) \xrightarrow{\sim} D^b(\mathrm{mod}(\mathrm{End}(M_{i+1}))).$$

Using this lemma, we can compute mutations of non-commutative crepant resolutions in the case of the Mukai flop. Extending the long Euler sequence on $\mathbb{P}(V)$, we have a long exact sequence

$$0 \to \mathcal{O}_Y(-n) \to V \otimes_{\mathbb{C}} \mathcal{O}_Y(-n+1) \to \cdots \to V^* \otimes_{\mathbb{C}} \mathcal{O}_Y \to \mathcal{O}_Y(1) \to 0$$

on $Y = T^*\mathbb{P}(V)$. Then we can apply the above lemma to this sequence and we have

$$\mu^n_{\phi_*(W_0)}(\phi_*(T_0)) \simeq \phi_*(W_0 \oplus \mathcal{O}_Y(1)) = \phi_*(T_1).$$

where $\mu^n_{\phi_*(W_0)} := \mu_{\phi_*(W_0)} \circ \cdots \circ \mu_{\phi_*(W_0)}$ is a composition of $n-1$ mutations at $\phi_*(W_0)$. Using similar argument repeatedly, we have

$$(\mu^n_{\phi_*(W_{n-1})} \circ \cdots \circ \mu^n_{\phi_*(W_1)} \circ \mu^n_{\phi_*(W_0)})(\phi_*(T_0)) \simeq \phi_*(T_n)$$

where $W_k := W_0 \otimes \mathcal{O}_Y(k)$. Since there is an isomorphism $\phi_*\mathcal{O}_Y(a) \simeq \phi'_*\mathcal{O}_{Y'}(-a)$, we have an isomorphism

$$\phi_*(T_n) \simeq \phi'_*(T'_0),$$

and hence we have

$$(\mu^n_{\phi_*(W_{n-1})} \circ \cdots \circ \mu^n_{\phi_*(W_1)} \circ \mu^n_{\phi_*(W_0)})(\phi_*(T_0)) \simeq \phi'_*(T'_0).$$

Note that the Euler sequence we used also defines the mutation of the exceptional object $\mathcal{O}_{\mathbb{P}(V)}(-n)$ over a partial exceptional collection

$$\langle \mathcal{O}_{\mathbb{P}(V)}(-n+1), \mathcal{O}_{\mathbb{P}(V)}(-n+2), \ldots, \mathcal{O}_{\mathbb{P}(V)} \rangle.$$

This means that mutations of non-commutative crepant resolutions are parallel with mutations of exceptional collections in some good situations.

In the following, we discuss derived equivalences coming from mutations. First, applying derived equivalence for mutations, we have a derived equivalence

$$\begin{aligned} D^b(\mathrm{coh}(Y)) &\xrightarrow{\mathrm{RHom}_Y(T_0,-)} D^b(\mathrm{mod}(\mathrm{End}(\phi_*(T_0)))) \\ &\xrightarrow{\Phi^n_{\phi_*(W_{n-1})}\circ\cdots\circ\Phi^n_{\phi_*(W_1)}\circ\Phi^n_{\phi_*(W_0)}} D^b(\mathrm{mod}(\mathrm{End}(\phi'_*(T'_0)))) \\ &\xrightarrow{T'_0\otimes_{\mathrm{End}(\phi'_*T'_0)}-} D^b(\mathrm{coh}(Y')). \end{aligned}$$

Theorem 2 ([H17a])**.** This equivalence is naturally isomorphic to the inverse of Kawamata-Namikawa equivalence.

This is an analog of the result of Wemyss for three dimensional flops [Wem17, Theorem 4.2]. Next we discuss autoequivalence on $D^b(\mathrm{coh}(Y))$ coming from mutations. Recall that we have

$$\mu^n_{\phi_*(W_0)}(\phi_*(T_0)) \simeq \phi_*(W_0 \oplus \mathcal{O}_Y(1)) = \phi_*(T_1).$$

We can continue to apply mutations at $\phi_*(W_0)$ using the geometry on the side of $Y' = T^*\mathbb{P}(V^*)$. Let us consider a long Euler sequence

$$0 \to \mathcal{O}_{Y'}(-1) \to V \otimes_{\mathbb{C}} \mathcal{O}_{Y'} \to \cdots \to V^* \otimes_{\mathbb{C}} \mathcal{O}_{Y'} \to \mathcal{O}_{Y'}(n-1) \to 0$$

on Y'. Applying the lemma again, we have

$$\mu^n_{\phi'_*(W'_{n-1})}(\phi'_*(T'_{n-1})) \simeq \phi'_*(W'_{n-1} \oplus \mathcal{O}_{Y'}(n)) = \phi'_*(T'_n).$$

Since there are isomorphisms $\phi_*(W_0) \simeq \phi'_*(W'_{n-1})$ and $\phi_*(T_0) \simeq \phi'_*(T'_n)$, we have

$$\mu^{2n}_{\phi_*(W_0)}(\phi_*(T_0)) \simeq \phi_*(T_0).$$

Thus we have an autoequivalence

$$\Phi^{2n}_{\phi_*(W_0)} \in \mathrm{Auteq}(D^b(\mathrm{mod}(\mathrm{End}(\phi_*(T_0))))).$$

Theorem 3 ([H17a])**.** Under the identification

$$\mathrm{RHom}(T_0,-) : D^b(\mathrm{coh}(Y)) \xrightarrow{\sim} D^b(\mathrm{mod}(\mathrm{End}(\phi_*T_0))),$$

the autoequivalence $\Phi^{2n}_{\phi_*(W_0)}$ corresponds to a P-twist on $D^b(\mathrm{coh}(Y))$ around P-object $\mathcal{O}_{Y_0}(-n-1)$ where $Y_0 \subset Y$ is the zero-section.

P-twists are important autoequivalences appearing in Mathematical Physics. This results means we can understand P-twists as a certain "monodromy" of mutations of non-commutative crepant resolutions. This is an analog of the result of Donovan and Wemyss [DW16]. Using this result, we can also recover "flop-flop=twist" results proved by Cautis [Cau12] and Addington-Donovan-Meachan [ADM15].

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On the notion of an enhanced category

Dmitry Kaledin

The idea of considering mathematical objects "up to an equivalence" of some sort is conveniently formalised by the categorical notion of *localization*. Localization of a category $\mathcal{C}$ with respect to a class of morphisms W is a category $h(\mathcal{C}, W)$ equipped a functor $\mathcal{C} \to h(\mathcal{C}, W)$ that inverts morphisms in W and is universal among such functors. Localization is unique by universality, exists modulo relatively minor set-theoretical issues, and is quite ubiquitous in mathematics. The simplest example is probably the category Cat of small categories, with W the class of equivalences; in this case, localization exists, and $h(\mathrm{Cat}, W)$ is the category $\overline{\mathrm{Cat}}$ of small categories and isomorphism classes of functors between them. There is also a relative version $\overline{\mathrm{Cat}}/I$ for the category Cat/I of small categories equipped with a functor to some fixed category I. Other examples are derived categories, homotopy categories of various origin, and so on and so forth.

Unfortunately, while passing from a category $\mathcal{C}$ to its localization $h(\mathcal{C}, W)$ by design loses some information, it seems that some of the information lost is essential. As a result of this, one cannot do a number of quite natural construction on the level of the localized category $h(\mathcal{C}, W)$. For example, for a small category I, one can never recover the localization $h(\mathcal{C}^I, W^I)$ of the functor category $\mathcal{C}^I$ purely in terms of $h(\mathcal{C}, W)$ and I. What one needs is to equip the category $h(\mathcal{C}, W)$ with some "enhancement".

This hypothetical enhancement should at least contain a homotopy type of maps $\mathcal{H}(C,W)(c,c')$ for any two objects $c,c' \in \mathcal{C}$ such that $h(\mathcal{C},W)(c,c') = \pi_0(\mathcal{H}(C,W)(c,c')$, but this is not enough. What exactly one needs to add constitutes "a theory of enhancement", and several such theories exist at present. For example, there are complete Segal spaces of Rezk, quasicategories of Joyal and Lurie, relative categories of Barwick and Kan, and more. In all these theories, the category Enh of enhanced categories is itself obtained by localization: for example, in Rezk's approach, we have $\mathrm{Enh} = h(\mathrm{CSS}, W)$, where CSS is a category of complete Segal spaces, and W is a natural class of weak equivalences between them. Since Enh is obtained by localization, it itself has an enhanced version $\mathcal{E}nh$. The localization one uses is not an easy one, so to control it, Rezk equips CSS with a model structure in the sense of Quillen. The shape of the other approaches is exactly the same, the only thing that differs is a model category one constructs. They are all "Quillen-equivalent" which in particular means that Enh is the same in all approaches, but the proofs of these facts are not particularly illuminating. If one assumes that Enh must come from a model category, then it has been proved by Toën that CSS is a universal theory, modulo some conditions. The conditions are quite natural; however, the model category assumption itself is not. It does not seem to be justified by anything other than pure belief.

The goal of the present research, then, is to suggest a theory of enhancements that is free from all the arbitrary choices hardcoded into the existing definitions, and does not rely on any *a priori* assumptions or beliefs (we do not even need to know beforehand that homotopy types play any role in the theory). Neither do we need to assume that the result is obtained by localizing a model category — rather, we aim for a much simpler type of localization: in the end, we simply embed Enh into the category $\overline{\mathrm{Cat}}/I$, for an appropriate choice of I.

The idea of our approach actually predates all the existing approaches and goes back to Grothendieck: it is his idea of a "derivator" suggested in the late 70-ies in a rather long letter to Quillen. We just need to assume the following general principles:

(1) No matter what else enhanced categories form, they at least form a category Enh.
(2) A category is tautologically an enhanced category, so we have an embedding $\sigma : \overline{\mathrm{Cat}} \to \mathrm{Enh}$, and it has a left-adjoint truncation functor $\tau : \mathrm{Enh} \to \overline{\mathrm{Cat}}$ (this corresponds to replacing homotopy types of maps with their π_0).
(3) The category Enh is cartesian closed, in that for any $I, \mathcal{C} \in \mathrm{Enh}$ we have the functor enhanced category $\mathcal{C}^I \in \mathrm{Enh}$ defined as the inner Hom with respect to the cartesian product.
(4) In addition to (i), enhanced categories themselves also form an enhanced category $\mathcal{E}nh$.

Starting from these principles, one observes that any $\mathcal{C} \in \mathrm{Enh}$ defines a functor $\mathcal{D}(\mathcal{C}) : \overline{\mathrm{Cat}} \to \overline{\mathrm{Cat}}$, $I \mapsto \tau(\mathcal{C}^{\sigma(I)})$. Roughly speaking, this $\mathcal{D}(\mathcal{C})$ is a "derivator", and the hope is that it remembers all the relevant information about $\mathcal{C}$.

We have found that this idea works, with a slight refinement (instead of functors from $\overline{\mathrm{Cat}}$ to $\overline{\mathrm{Cat}}$ one needs to consider "pseudofunctors" Cat $\to$ Cat conveniently encoded by Grothendieck fibrations $\mathcal{C} \to \mathrm{Cat}$). Moreover, we do not need to index our derivators on all small categories, it suffices to consider the category Pos^f of partially ordered sets of finite chain dimension. Furthermore, it suffices to remember the isomorphism groupoids $\mathrm{Iso}(\tau(\mathcal{C}^\sigma(I)))$ of the categories $\tau(\mathcal{C}^{\sigma(I)})$. Our result, then, is roughly the following:

- Let Enh be the homotopy category of complete Segal spaces (or any other of the known models, since they are all equivalent). For any $\mathcal{C} \in \mathrm{Enh}$, let $\mathcal{D}(\mathcal{C}) \to \mathrm{Pos}^f$ be the fibration with fibers $\mathcal{D}(\mathcal{C})_I = \mathrm{Iso}(\tau(\mathcal{C}^{\sigma(I)}))$, $I \in \mathrm{Pos}^f$. Then the correpondence $\mathcal{C} \mapsto \mathcal{D}(\mathcal{C})$ provides a fully faithful embedding $\mathrm{Enh} \to \overline{\mathrm{Cat}}/\,\mathrm{Pos}^f$.

We also characterize the image of this embedding, and prove some further results (in particular, we give a very simple direct contruction of the derivator enhancement of the category Enh).

2-representations of fiat 2-categories and applications to Soergel bimodules

Vanessa Miemietz
(joint work with Marco MacKaay, Volodymyr Mazorchuk, Xiaoting Zhang)

Finitary 2-categories are desigend as 2-analogues of finite-dimensional algebras, and *fiat* 2-categories as such of finite-dimensional algebras with a simple-preserving involution. We explain the basics of their 2-representation theory, in particular, how this is governed by the left/right and two-sided cell structure of the 2-category. The analogues of simple modules are *simple transitive* 2-representations and their classification is a major current problem in the theory. We describe how this classification can be reduced to studying simple transitive 2-representations for fiat 2-categories with only one left and one right (non-identity) cell. Furthermore, we give applications to 2-categories of Soergel bimodules.

References

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Introduction to symplectic varieties and their quantizations

Kevin McGerty

Symplectic varieties have been the centre of considerable activity in the last few years. In this talk I will survey a range of results in this subject, focusing on those which extend classical results from geometric representation theory. We will begin by reviewing the definition of symplectic singularities and symplectic resolutions and describe some important classes of examples, such as quiver varieties. We will then discuss the theory of quantization for conic symplectic resolutions and results on localization, which provides a bridge to the representation theory of important classes of algebras. We will also discuss categorical decompositions of the category of sheaves of modules for quantizations of symplectic varieties and the important subcategories, geometric versions of the classical category $\mathcal{O}$ representations of a semisimple Lie algebra, which arise in the presence of Hamiltonian $\mathbb{C}^\times$ actions.

Quantum groups via Bridgeland's Hall Algebra of Complexes

Eoin Murphy

In this talk we give an overview of Bridgeland's realization of the whole quantized enveloping algebra as a Hall algebra.

A foundational result in the theory of Hall algebras, due to Ringel [2], is that the positive part of the quantized enveloping algebra can be realized as a Hall algebra of categories of quiver representations. It was an open question whether or not one could extend this result to recover the whole quantized enveloping algebra by taking the Hall algebra of some other natural category. Bridgeland answered this affirmative in [1] by taking the Hall algebra of the category of $\mathbb{Z}_2$-graded complexes in projective quiver representations.

Let's set up some background notation. Take $\vec{Q}$ to be a quiver without oriented cycles. We denote by $\mathrm{Rep}_{\mathbb{F}_q}(\vec{Q})$ its category of finite dimensional representations over a finite field $\mathbb{F}_q$ with q elements. Let $\mathcal{C}_{\mathbb{Z}_2}$ be the category of $\mathbb{Z}_2$-graded complexes in projective objects in $\mathrm{Rep}_{\mathbb{F}_q}(\vec{Q})$. The objects in $\mathcal{C}_{\mathbb{Z}_2}$ are of the following form where L_1 and L_0 are projective objects in $\mathrm{Rep}_{\mathbb{F}_q}(\vec{Q})$.

$$L_\bullet = \; L_1 \underset{g}{\overset{f}{\rightleftarrows}} L_0 \;, \qquad f \circ g = g \circ f = 0$$

Morphisms are given by usual morphisms of complexes. A feature of the $\mathbb{Z}_2$-grading is that the usual shift functor for complexes induces an involution $[1] : \mathcal{C}_{\mathbb{Z}_2} \to \mathcal{C}_{\mathbb{Z}_2}$.

To the quiver $\vec{Q}$ one can associate a Kac-Moody Lie algebra $\mathfrak{g}$ in a natural way. Moreover one can define a $\mathbb{C}$-algebra called the quantized enveloping algebra $\mathrm{U}_{q^{1/2}}(\mathfrak{g})$. The algebra $\mathrm{U}_{q^{1/2}}(\mathfrak{g})$ can be defined via generators E_i, F_i and $K_i^{\pm 1}$ along with certain relations. Here i runs over the set of vertices of $\vec{Q}$. See for example

Appendix A.4 [3].

We will now outline Bridgeland's construction. The following definition is the usual definition of a Hall algebra applied to the category $\mathcal{C}_{\mathbb{Z}_2}$.

Definition 1. Define the Hall algebra of $\mathcal{C}_{\mathbb{Z}_2}$ to the the $\mathbb{C}$-vector space $\mathrm{H}(\mathcal{C}_{\mathbb{Z}_2})$ with basis indexed by the set of isomorphism class of complexes $\mathrm{Iso}(\mathcal{C}_{\mathbb{Z}_2})$ and multiplication defined as follows[2]

$$[M_\bullet][N_\bullet] = \sum_{L_\bullet \in \mathrm{Iso}(\mathcal{C}_{\mathbb{Z}_2})} \frac{|\mathrm{Ext}^1_{\mathcal{C}_{\mathbb{Z}_2}}(M_\bullet, N_\bullet)_{L_\bullet}|}{|\mathrm{Hom}_{\mathcal{C}_{\mathbb{Z}_2}}(M_\bullet, N_\bullet)|}[L_\bullet] \tag{1}$$

Here $\mathrm{Ext}^1_{\mathcal{C}_{\mathbb{Z}_2}}(M_\bullet, N_\bullet)_{L_\bullet}$ is the set of extensions whose middle term is isomorphic to $L_\bullet$.

One can check that (1) gives an associative product. However as observed by Bridgeland in [1] the Hall algebra $\mathrm{H}(\mathcal{C}_{\mathbb{Z}_2})$ is almost, but not quite the correct one to recover the whole quantized enveloping algebra. In particular the analogous relations to those of $K_iK_i^{-1} = 1$ in $\mathrm{U}_{q^{1/2}}(\mathfrak{g})$ do not hold in $\mathrm{H}(\mathcal{C}_{\mathbb{Z}_2})$. To remedy the situation one must impose these by hand.

Definition 2. Define the Bridgeland-Hall algebra DH to be the quotient of $\mathrm{H}(\mathcal{C}_{\mathbb{Z}_2})$ by the following ideal:

$$([L_\bullet] - 1 \mid L_\bullet \text{ is acyclic and } L_\bullet \cong L_\bullet[1])$$

Armed with this modified Hall algebra, Bridgeland proved the following extension of Ringel's theorem.

Theorem 3 (Bridgeland)**.** *There is an embedding of $\mathbb{C}$-algebras $U_{q^{1/2}}(\mathfrak{g}) \to DH$ which is an isomorphism in the case that $\vec{Q}$ is of type ADE.*

References

[1] T. Bridgeland, *Quantum groups via Hall algebras of complexes*, Ann. Math. **177** (2013), 739-759.
[2] C. Ringel, *Hall algebras and quantum groups*, Invent. Math. **101** (1990), 583591.
[3] O. Schiffman, *Lectures on Hall algebras*, Cambridge University Press (2009).

[2]For the sake of clarity in this product we omit a slight factor depending on the Euler form of $\mathrm{Rep}_{\mathbb{F}_q}(\vec{Q})$

Poisson geometry and moduli spaces

Thomas Nevins
(joint work with Kevin McGerty, Emily Cliff, Shiyu Shen)

Many symplectic algebraic varieties that arise naturally in geometric representation theory and supersymmetric quantum field theories—for example, Nakajima quiver varieties, and moduli spaces of Higgs bundles on a curve—can be interpreted as open symplectic leaves of moduli spaces of "sheaves on a (possibly noncommutative) Poisson surface." An important problem for representation theory and mathematical physics is to compute the cohomology of such symplectic varieties. The talk explained that there are general principles, first appearing in commutative algebraic geometry in work of Ellingsrud-Strømme, Beauville, and deepened in work of Markman and others, which provide generators of the cohomology of a symplectic algebraic variety whenever it can be interpreted as an appropriate open set in a moduli space of sheaves on a proper noncommutative Poisson surface.

In particular, applying these principles in the setting of Nakajima quiver varieties proves (joint work with McGerty) that the cohomology (and bounded coherent derived category) of a quiver variety is generated by Chern classes of tautological vector bundles. Markman had earlier used a similar technique to show that the cohomology of the moduli space of $GL(r)$-Higgs bundles of degree d on a curve Σ is generated by tautological classes when d and r are coprime. By contrast, in joint work with Cliff and Shen, I have shown that the cohomology of the moduli space (and stack) of semistable G-Higgs bundles on Σ is never generated by tautological classes if the center $Z(G)$ of G is disconnected.

Defining noncommutative del Pezzo surfaces as AS-regular I-algebras

Shinnosuke Okawa
(joint work with Tarig Abdelgadir, Kazushi Ueda)

We assume that the base field $\mathbf{k}$ is algebraically closed of characteristic zero.

In the seminal paper [2], a class of noncommutative graded algebras (nowadays called Artin-Schelter regular algebra) was introduced and those which are of dimension 3 and generated in degree one were partially classified. In particular, they proved that such an algebra is either, what they called, a quadratic algebra or a cubic algebra. The classification was completed later in [3], based on the quite important discovery that such an algebra is in one-to-one correspondence with a commutative algebro-geometric data, which generically is a pair of an elliptic curve and its translation.

The notion of flat deformation of abelian categories is introduced in [11] (see also [10]). For an algebraic variety X, we call the flat deformations of $\operatorname{coh} X$ *noncommutative deformations* of X. The previous paragraph is related to this notion in the following way: for a graded algebra R one can define the category $\operatorname{qgr} R$, the Serre quotient of the category $\operatorname{gr} R$ of finitely generated graded right R-modules by the subcategory $\operatorname{tors} R$ of finite dimensional (over $\mathbf{k}$) modules. The categories $\operatorname{qgr} R$ for 3-dimensional quadratic (resp. cubic) AS-regular algebras R form flat deformations of $\operatorname{coh} \mathbb{P}^2$ (resp. $\operatorname{coh} \mathbb{P}^1 \times \mathbb{P}^1$).

The notion of 3-dimensional quadratic AS-regular algebra is satisfactory also from this point of view, in the sense that any formal deformation of $\operatorname{coh} \mathbb{P}^2$ is obtained from such an algebra; i.e., for any flat deformation of $\operatorname{coh} \mathbb{P}^2$ over a commutative complete local $\mathbf{k}$-algebra $(B, \mathfrak{m})$ whose base change by $B \to B/\mathfrak{m}$ ("the central fiber") is equivalent to $\operatorname{coh} \mathbb{P}^2$, there exists a 3-dimensional quadratic AS-regular algebra $\mathcal{R}$ over B such that the category is equivalent to $\operatorname{qgr} \mathcal{R}$ as abelian B-linear categories. On the other hand, this is not the case for $\mathbb{P}^1 \times \mathbb{P}^1$. The moduli of 3-dimensional cubic AS-regular algebras is 2, which is one less than that of $\operatorname{coh} \mathbb{P}^1 \times \mathbb{P}^1$ $(= \dim \mathrm{HH}^2 - \dim \mathrm{HH}^1 = 3)$.

A generalization of the notion of graded algebra is that of I-*algebra* for a set I, which is just a $\mathbf{k}$-linear category $\mathcal{C}$ equipped with a bijection $I \xrightarrow{\sim} \operatorname{Obj} \mathcal{C}$. This is equivalently described as an associative $\mathbf{k}$-algebra $A = \operatorname{Alg}(\mathcal{C}) = \bigoplus_{i,j \in I} A_{ij}$, where $A_{ij} = \operatorname{Hom}_{\mathcal{C}}(j, i)$, equipped with the local unit $e_i = \mathrm{id}_i \in A_{ii}$. The category of right modules over $\mathcal{C}$ is defined as $\operatorname{Gr} \mathcal{C} = \operatorname{Gr} A = \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \operatorname{Vect} \mathbf{k})$. Namely, a module over A is a contravariant $\mathbf{k}$-linear functor from $\mathcal{C}$ to the category of vector spaces $\operatorname{Vect} \mathbf{k}$ and a homomorphism between modules is nothing but a natural transformation. A typical examples of A-modules are the i-th projective module $P_i := e_i A$ and the i-th simple module S_i which admits an epimorphism $P_i \twoheadrightarrow S_i$ and $\dim_{\mathbf{k}} S_i = 1$. A graded algebra $R = \bigoplus_{i \in \mathbb{Z}} R_i$ yields an $I = \mathbb{Z}$-algebra $\check{R} = \bigoplus_{i,j \in \mathbb{Z}} R_{i-j}$, and there is a canonical isomorphism of categories $\operatorname{Gr} R \simeq \operatorname{Gr} \check{R}$ (see [14] for details).

In [14], the notion of 3-dimensional quadratic and cubic AS-regular $\mathbb{Z}$-*algebras* is introduced. The former is defined as a $\mathbb{Z}$-algebra $\mathcal{C}$ ($\iff A = \operatorname{Alg}(\mathcal{C})$) with the following properties.

- $A_{ij} = 0$ if $i > j$, and $A_{ii} = \mathbf{k} e_i$.
- For each $i \in I$ there exists an exact sequence in $\operatorname{Gr} A$ of the following form.

$$0 \to P_{i+3} \to P_{i+2}^{\oplus 3} \to P_{i+1}^{\oplus 3} \to P_i \to S_i \to 0. \tag{1}$$

- For each $i \in I$, the following AS-Gorenstein condition is satisfied.

$$\sum_{\substack{\ell \in \mathbb{Z} \\ j \in I}} \dim_{\mathbf{k}} \operatorname{Ext}^{\ell}_{\operatorname{Gr} A}(S_i, P_j) = 1. \tag{2}$$

The definition for the cubic version is obtained just by replacing the resolution (1) with the following one.

$$0 \to P_{i+4} \to P_{i+3}^{\oplus 2} \to P_{i+1}^{\oplus 2} \to P_i \to S_i \to 0. \tag{3}$$

In fact, a 3-dimensional quadratic AS-regular $\mathbb{Z}$-algebra is nothing but the class of $\mathbb{Z}$-algebras considered in the yet another seminal paper [5], in which the authors considered (derived) noncommutative deformations of $\mathbb{P}^2$ from the point of view of $\mathbb{Z}$-algebras. They in particular proved that the isomorphism classes of 3-dimensional quadratic AS-regular $\mathbb{Z}$-algebras are in one-to-one correspondence with the commutative algebro-geometric data (Y, L_0, L_1), where

- $(Y, L_0, L_1) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(1))$ or
- Y is a curve of genus 1, L_i is a very ample line bundle on Y which embeds Y into $\mathbb{P}^2$ as a cubic divisor $(i = 0, 1)$ such that $\deg L_0|_C = \deg L_1|_C$ for each irreducible component $C \subset Y$ and $L_0 \not\simeq L_1$.

The similar correspondence for cubic $\mathbb{Z}$-algebras is established in [14]. Moreover it is shown that any flat formal deformation of $\operatorname{coh} \mathbb{P}^1 \times \mathbb{P}^1$ is obtained as qgr of a cubic $\mathbb{Z}$-algebra in the similar sense as above. Based on this, one *defines* noncommutative $\mathbb{P}^2$ (resp. $\mathbb{P}^1 \times \mathbb{P}^1$) as an abelian **k**-linear category which is equivalent to $\operatorname{qgr} A$ for a 3-dimensional quadratic (resp. cubic) AS-regular $\mathbb{Z}$-algebra.

A graded algebra R is 3-dimensional quadratic (resp. cubic) AS-regular if and only if the $\mathbb{Z}$-algebra $\check{R}$ is. It turns out that any (3-dimensional AS-regular) quadratic $\mathbb{Z}$-algebra is obtained from a graded algebra, which is not the case for cubic $\mathbb{Z}$-algebras.

There are 10 deformation types for (commutative) del Pezzo surfaces, so it is natural to ask if one can generalize the notion of 3-dimensional quadratic/cubic AS-regular $\mathbb{Z}$-algebras so that all noncommutative del Pezzo surfaces are defined similarly as above. This question is answered in the following theorem from our joint work in progress, in such a way that

- 3-dimensional quadratic/cubic AS-regular $\mathbb{Z}$-algebras appear as two special cases of the general construction, and
- All commutative del Pezzo surfaces are qgr of such algebras.

Theorem 1. *Choose a commutative del Pezzo surface X and a geometric helix $\mathcal{E} = (\mathcal{E}_i)_{i\in\mathbb{Z}}$ on X which consist of vector bundles. Then the $\mathbb{Z}$-algebra $\mathcal{C} = (\mathcal{E}_{-i})_{i\in\mathbb{Z}}$ ($\Longleftrightarrow A = \operatorname{Alg}(\mathcal{C})$)[3] satisfies the following properties.*

- *$A_{ij} = 0$ for $i > j$ and $A_{ii} = \mathbf{k}\,\mathrm{id}_{\mathcal{E}_{-i}}$.*
- *For each $i \in I$ there exists a projective resolution of S_i of the form* (6) *below.*
- *A is AS-Gorenstein.*

A helix $\mathcal{E} = (\mathcal{E}_i)_{i\in\mathbb{Z}}$ is said to be *geometric* if $\operatorname{Ext}^{\ell}_X(\mathcal{E}_i, \mathcal{E}_j) = 0$ for all $i < j$ and $\ell \neq 0$. This definition is taken from [6], and slightly differs from the usage in [5]. Under this assumption, any full exceptional collection (a.k.a. foundation) $\mathcal{F} \subset \mathcal{E}$ is strong. Since $\mathcal{E}_i \otimes \omega_X \simeq \mathcal{E}_{i-r}$ for all $i \in \mathbb{Z}$, where $r = \#\mathcal{F} = \operatorname{rank} K_0(X)$, one has a free (in the sense of [7]) action of the group $\mathbb{Z}$ on the $\mathbb{Z}$-algebra $\mathcal{C}$. $\mathcal{F}$ is a "fundamental domain" of this action.

Consider the total space of the canonical line bundle $\pi\colon \omega_X \to X$. The geometricity of $\mathcal{E}$ implies that $\mathcal{T} = \bigoplus_{\mathcal{E}_i\in\mathcal{F}} \pi^*\mathcal{E}_i$ is a tilting object of $D^b \operatorname{coh} \omega_X$, so that one obtains a derived equivalence $D^b_0 \operatorname{coh} \omega_X \simeq D^b \operatorname{mod} B$ to $B := \operatorname{End}_{\omega_X}(\mathcal{T})$. The left hand side is the full subcategory of objects supported on the 0-section, and the right hand side is the derived cateogry of finite dimensional right B-modules.

[3] $\mathcal{C}$ is the full subcategory of $\operatorname{coh} X$ consisting of the objects in $\mathcal{E}$ with the indexing $i \mapsto \mathcal{E}_{-i}$.

By a standard geometric argument, one can check that the canonical line bundle of ω_X is trivial. This implies that the triangulated categories above are 3 Calabi-Yau. Algebra B with this property is thoroughly studied in [4], and as its corollary we obtain the following conclusions.

- There exists a quiver with superpotential (Q, W) such that $Q_0 = \{1, 2, \ldots, r\} \simeq \mathcal{F}$ and $B \simeq \mathbf{k}(Q, W)$. Moreover,
- For each $\bar{\imath} \in Q_0$ there exists an exact sequence of the following form, where the middle map is given by the "Hessian" of W.

$$0 \to P_{\bar{\imath}} \to \bigoplus_{s(a)=\bar{\imath}} P_{t(a)} \to \bigoplus_{t(b)=\bar{\imath}} P_{s(b)} \to P_{\bar{\imath}} \to S_{\bar{\imath}} \to 0 \tag{4}$$

- For each $\bar{\imath} \in Q_0$,

$$\sum_{\substack{\ell \in \mathbb{Z} \\ \bar{\jmath} \in Q_0}} \dim_{\mathbf{k}} \operatorname{Ext}^{\ell}_{\operatorname{Gr} B} (S_{\bar{\imath}}, P_{\bar{\jmath}}) = 1. \tag{5}$$

In [4], an explicit bimodule resolution of B is described. (4) is simply obtained by applying $S_{\bar{\imath}} \otimes_B -$ to the bimodule resolution, and (5) follows from its self-duality.

If one regards B as a $\mathbf{k}$-linear category whose set of objects is $Q_0 (\simeq \mathcal{F} \simeq \mathcal{C}/\mathbb{Z})$, then it is equivalent to the skew category of $\mathcal{C}$ in the sense of [7] under the action of $\mathbb{Z}$ which we discussed above. With this in mind, for each $i \in I$ one obtains the projective resolution of S_i

$$0 \to P_{i+r} \to \bigoplus_{s(a)=i} P_{t(a)} \to \bigoplus_{t(b)=i} P_{s(b)} \to P_i \to S_i \to 0 \tag{6}$$

by simply "lifting" the projective resolution (4) of $S_{\bar{\imath}}$, where $\bar{\imath} \in Q_0$ corresponds to the orbit of i. The "arrows" a and b should be interpreted appropriately as morphisms between the objects in $\mathcal{C}$. They uniquely correspond to the arrows a and b in (4) once we fix the lift i of $\bar{\imath}$. This is analogous to the fact in topology that the lift of a path to a covering space is uniquely determined by the lift of its endpoint. In general, the form of the resolution (6) varies depending on the congruence of i modulo r. Finally, (5) immediately implies the AS-Gorenstein property of A.

Now we define an *AS-regular $\mathbb{Z}$-algebra of type $\mathcal{E}$* to be a $\mathbb{Z}$-algebra $\mathcal{C}(\iff A = \operatorname{Alg}(\mathcal{C}))$ which is AS-Gorenstein and each simple S_i admits a projective resolution of the form (6). The definition of 3-dimensional quadratic (resp. cubic) AS-regular $\mathbb{Z}$-algebra is recovered as a special example of this general definition. In fact, it corresponds to the geometric helix on $\mathbb{P}^2$ which is generated by the full exceptional collection $\mathcal{F} = (\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(2))$ (resp. $\mathcal{F} = (\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,0), \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1), \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2,1))$ on $\mathbb{P}^1 \times \mathbb{P}^1$).

Actually it is natural to take a more general index set I than $\mathbb{Z}$ depending on the nature of the helix $\mathcal{E}$. For example, it is known by [9] that each del Pezzo surface other than $\mathbb{F}_1$ or two point blowup of $\mathbb{P}^2$ admits a three block exceptional collection which generates a geometric helix. Such an exceptional collection splits into three blocks in each of which objects are orthogonal to each other, and it is natural to leave those objects unordered. We illustrate this by describing the definition of the AS-regular $I = \mathbb{Z} \times \mathbb{Z}/3$-algebra arising from the geometric helix $\mathcal{E}_{(6.1),(1,1,1)}$[4] on cubic surfaces, which is generated by the three block collection $\tau_{(6.1)}$ considered in [9, p. 453]. We write $i = (i_1, i_2) \in I$ and so on.

Definition 2 (Sample)**.** An AS-regular $\mathbb{Z} \times \mathbb{Z}/3(= I)$ algebra of type $\mathcal{E}_{(6.1),(1,1,1)}$ is an I-algebra $\mathcal{C}$ ($\iff$ $A = \mathrm{Alg}\,(\mathcal{C})$) such that

- $A_{ij} = 0$ if $i_1 > j_1$ or $[i_1 = j_1$ and $i_2 \neq j_2]$, and $A_{ii} = \mathbf{k}e_i$.
- For each $i \in I$ there exists an exact sequence in $\mathrm{Gr}\,A$ of the following form.

$$0 \to P_{i+(3,0)} \to \bigoplus_{a \in \mathbb{Z}/3} P_{(i_1+2,a)} \to \bigoplus_{b \in \mathbb{Z}/3} P_{(i_1+1,b)} \to P_i \to S_i \to 0. \tag{7}$$

- For each $i \in I$, the following AS-Gorenstein condition is satisfied.

$$\sum_{\substack{\ell \in \mathbb{Z} \\ j \in I}} \dim_{\mathbf{k}} \mathrm{Ext}^{\ell}_{\mathrm{Gr}\,A}\,(S_i, P_j) = 1. \tag{8}$$

Note that, even for a fixed X, there is the dependence of the definition of AS-regular I-algebra on the choice of the geometric helix, which remains to be understood. The modular interpretation, from the point of view of AS-regular I-algebra, of the compactified moduli space of noncommutative del Pezzo surfaces introduced by the authors in [1] [12] is also to be investigated. We also have to compare the blowup construction, i.e., the graded algebra which appears in [13], with the Veronese subalgebra of the AS-regular I-algebra corresponding to a $\mathbb{Z}$-orbit (noncommutative deformations of the anti-canonical rings).

Also we hope to establish in general the correspondence with AS-regular I-algebras to the commutative algebro-geometric data obtained as follows. Starting with such an I-algebra A, fix a foundation $\mathcal{F} \subset \mathcal{E} = (\pi(P_i))_{i \in I}$, where $\pi\colon\ \mathrm{gr}\,A \to \mathrm{qgr}\,A$ is the quotient functor. Consider the finite dimensional algebra $C = \mathrm{End}\,\mathcal{F}$. Let Y be the moduli space of right modules M over C which satisfy $\dim_{\mathbf{k}} M \cdot \mathrm{id}_{\mathcal{E}_i} = \mathrm{rank}\,\mathcal{E}_i$ for each $\mathcal{E}_i \in \mathcal{F}$. Y comes with the universal C-module, i.e., a collection of vector bundles $(\mathcal{M}_{\mathcal{E}_i})_{\mathcal{E}_i \in \mathcal{F}}$. One expects to recover A from the geometric data $\big(Y, (\mathcal{M}_{\mathcal{E}_i})_{\mathcal{E}_i \in \mathcal{F}}\big)$. The geometric data for 3-dimensional quadratic/cubic $\mathbb{Z}$-algebras described above are, actually, examples of this construction. For the general case there are some details to be clarified, notably the appropriate choice of a stability condition for modules in the definition of the moduli space Y.

Finally, we remark that there is a closely related paper [8] in which the equivalence of deformation theories for $\mathrm{Qcoh}\,X$, A, and C is shown (the author is indebted to Michel Van den Bergh for this remark).

[4]The subscript (6.1) comes from the Diophantine equation (6.1) of [9, p. 446, Table]. $(1,1,1)(=$ "$\mathrm{rank}\,\tau_{(6.1)}$"$)$ is its minimal solution. See [9] for the detail.

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Rational Cherednik algebras, Hilbert schemes and combinatorics

Tomasz Przezdziecki

Rational Cherednik algebras were introduced by Etingof and Ginzburg [4]. They depend on a complex reflection group as well as two parameters $\mathbf{h}$ and t. In the case of Weyl groups they coincide with certain degenerations of double affine Hecke algebras introduced earlier by Cherednik [2]. When $t = 0$, rational Cherednik algebras have large centres and the corresponding affine varieties are known as generalized Calogero-Moser spaces. In this talk we consider rational Cherednik algebras $\mathbb{H}_\mathbf{h} := \mathbb{H}_{t=0,\mathbf{h}}$ associated to complex reflection groups of type $G(l,1,n)$, i.e., the wreath products $\Gamma_n := (\mathbb{Z}/l\mathbb{Z}) \wr S_n$. Calogero-Moser spaces of type $G(l,1,n)$ are especially interesting because they admit a Nakajima quiver variety interpretation. This fact was exploited by Gordon [7], who used reflection functors to relate Calogero-Moser spaces to Hilbert schemes of points in the plane. His results were later used by Bezrukavnikov and Finkelberg [1] to prove Haiman's wreath Macdonald positivity conjecture.

Let us recall the framework of [7] in more detail. Assume that the variety $\mathcal{Y}_\mathbf{h} := \operatorname{Spec} Z(\mathbb{H}_\mathbf{h})$ is smooth. Etingof and Ginzburg showed in [4] that $\mathcal{Y}_\mathbf{h}$ is isomorphic to a cyclic quiver variety $\mathcal{X}_\theta(n\delta)$ generalizing Wilson's construction of the Calogero-Moser space in [13]. Considering $\mathcal{X}_\theta(n\delta)$ as a hyper-Kähler manifold, one can use reflection functors, defined by Nakajima in [11], to construct a hyper-Kähler isometry $\mathcal{X}_\theta(n\delta) \to \mathcal{X}_{-\frac{\mathbf{1}}{\mathbf{2}}}(\gamma)$ between quiver varieties associated to different parameters. Furthermore, rotation of complex structure yields a diffeomorphism between $\mathcal{X}_{-\frac{\mathbf{1}}{\mathbf{2}}}(\gamma)$ and a certain GIT quotient $\mathcal{M}_{-\mathbf{1}}(\gamma)$. The latter is isomorphic to an irreducible component Hilb_K^ν of $\mathrm{Hilb}_K^{\mathbb{Z}/l\mathbb{Z}}$, where Hilb_K denotes the Hilbert scheme of K points in $\mathbb{C}^2$. The following diagram summarizes all the maps involved:

$$\mathcal{Y}_{\mathbf{h}} \xrightarrow{\mathsf{EG}} \mathcal{X}_\theta(n\delta) \xrightarrow{\mathsf{Refl.Fun.}} \mathcal{X}_{-\frac{\mathbf{1}}{\mathbf{2}}}(\gamma) \xrightarrow{\mathsf{Rotation}} \mathcal{M}_{-\mathbf{1}}(\gamma) \hookrightarrow \mathrm{Hilb}_K^{\mathbb{Z}/l\mathbb{Z}}. \tag{1}$$

Let us explain the parameters. The affine symmetric group $\tilde{S}_l$ acts on dimension vectors and the parameter space associated to the cyclic quiver with l vertices. We apply this action to the dimension vector $n\delta$ and the parameter $-\frac{\mathbf{1}}{\mathbf{2}} := -\frac{1}{2l}(1,\dots,1)$. Fix $w \in \tilde{S}_l$ and set $\theta := w^{-1} \cdot (-\frac{\mathbf{1}}{\mathbf{2}})$ and $\gamma := w * n\delta$. Then $\gamma = n\delta + \gamma_0$, where γ_0 is the l-residue of a uniquely determined l-core partition ν. Set $K := nl + |\nu|$.

Both $\mathcal{Y}_{\mathbf{h}}$ and Hilb_K carry natural $\mathbb{C}^*$-actions with respect to which (1) is equivariant. It follows from [6] that the closed $\mathbb{C}^*$-fixed points in $\mathcal{Y}_{\mathbf{h}}$ are labelled by l-multipartitions of n. On the other hand, the $\mathbb{C}^*$-fixed points in Hilb_K correspond to monomial ideals in $\mathbb{C}[x,y]$ of colength K and are therefore labelled by partitions of K. In particular, the $\mathbb{C}^*$-fixed points in Hilb_K^ν are labelled by partitions of K with l-core ν. Since (1) is equivariant, it induces a bijection

$$\mathcal{P}(l,n) \longleftrightarrow \mathcal{P}_\nu(K), \tag{2}$$

where $\mathcal{P}_\nu(K)$ denotes the set of partitions of K with l-core ν and $\mathcal{P}(l,n)$ the set of l-multipartitions of n.

One of our aims is to give an explicit combinatorial description of this bijection. With this goal in mind, we first classify $\mathbb{C}^*$-fixed points in cyclic quiver varieties and calculate the $\mathbb{C}^*$-characters of the fibres of their tautological bundles.

Theorem 1. *Let $u \in \tilde{S}_l$, $\xi := u * n\delta = n\delta + \xi_0$ and let ω be the transpose of the l-core corresponding to ξ_0. Set $L := nl + |\omega|$. Let $\alpha \in \mathbb{Q}^l$ be any parameter such that $\mathcal{X}_\alpha(\xi)$ is smooth. Let $\mathcal{V}_\alpha(\xi)$ denote the tautological bundle on $\mathcal{X}_\alpha(\xi)$. Then:*

a) *The $\mathbb{C}^*$-fixed points in $\mathcal{X}_\alpha(\xi)$ are naturally labelled by $\mathcal{P}_\omega(L)$.*
b) *Let $\mu \in \mathcal{P}_\omega(L)$. Then the $\mathbb{C}^*$-character of the fibre of $\mathcal{V}_\alpha(\xi)$ at μ is given by*

$$\mathrm{ch}_t \mathcal{V}_\alpha(\xi)_\mu = \mathrm{Res}_\mu(t) := \sum_{\square \in \mu} t^{c(\square)}.$$

Our second result describes the bijection between the $\mathbb{C}^*$-fixed points induced by the Etingof-Ginzburg isomorphism.

Theorem 2. *The map $\mathcal{Y}_{\mathbf{h}} \xrightarrow{\mathsf{EG}} \mathcal{X}_\theta(n\delta)$ induces a bijection*

$$\mathcal{P}(l,n) \to \mathcal{P}_\varnothing(nl), \quad \underline{\mathsf{Quot}}(\mu)^\flat \mapsto \mu,$$

where $\underline{\mathsf{Quot}}(\mu)^\flat$ *denotes the multipartition obtained from the* l*-quotient of* μ *by reversing the order of the constituent partitions and* $\varnothing$ *is the empty partition.*

We next consider Nakajima's reflection functors. To each simple reflection $\sigma_i \in \tilde{S}_l$, we associate a hyper-Kähler isometry $\mathfrak{R}_i : \mathcal{X}_\alpha(\xi) \to \mathcal{X}_{\sigma_i \cdot \alpha}(\sigma_i * \xi)$. One can show that $\sigma_i * \xi = n\delta + \sigma_i * \xi_0$, where $\sigma_i * \xi_0$ is the l-residue of a uniquely determined l-core ω'. The reflection functor $\mathfrak{R}_i$ induces a bijection between the labelling sets of $\mathbb{C}^*$-fixed points

$$\mathbf{R}_i : \mathcal{P}_\omega(L) \to \mathcal{P}_{(\omega')^t}(L'), \tag{3}$$

where $L' := nl + |\omega'|$. Our third result gives a combinatorial description of this bijection. We use the $\tilde{S}_l$-action on the set of all partitions defined by Van Leeuwen in [9]. This action involves combinatorial ideas reminiscent of those describing the $\hat{\mathfrak{sl}}_l$-action on the Fock space. More precisely, if μ is a partition then $\sigma_i * \mu$ is the partition obtained by simultaneously removing and adding all the removable (resp. addable) cells of content i mod l from (to) the Young diagram of μ.

Theorem 3. *Let* $\mu \in \mathcal{P}_\omega(L)$. *Then* $\mathbf{R}_i(\mu) = (\sigma_i * \mu^t)^t$.

Combining Theorem 2 with (iterated applications of) Theorem 3 allows us to deduce an explicit combinatorial description of (2). Given $w \in \tilde{S}_l$, we define the *w-twisted l-quotient bijection* to be the map

$$\tau_w : \mathcal{P}(l, n) \to \mathcal{P}_\nu(K), \quad \underline{\mathsf{Quot}}(\mu) \mapsto w * \mu.$$

Theorem 4. *Bijection* (2) *is given by the formula:* $\underline{\lambda} \mapsto \tau_w(\underline{\lambda}^t)$.

0.1. **The higher level** q**-hook formula.** We apply our results to give a new proof as well as a generalization of the q-hook formula:

$$\sum_{\square \in \mu} t^{c(\square)} = [n]_t \sum_{\lambda \uparrow \mu} \frac{f_\lambda(t)}{f_\mu(t)}, \tag{4}$$

where μ is a partition of n and $f_\mu(t)$ is the fake degree polynomial associated to μ. The q-hook formula has been proven by Kerov [8], Garsia and Haiman [5] and Chen and Stanley [3] using probabilistic, combinatorial and algebraic methods. We prove the following generalization.

Theorem 5. *Let* $\mu \in \mathcal{P}_\varnothing(nl)$. *Then:*

$$\sum_{\square \in \mu} t^{c(\square)} = [nl]_t \sum_{\underline{\lambda} \uparrow \underline{\mathsf{Quot}}(\mu)^\flat} \frac{f_{\underline{\lambda}}(t)}{f_{\underline{\mathsf{Quot}}(\mu)^\flat}(t)}. \tag{5}$$

We call (5) the *higher level* q-hook formula. Setting $l = 1$ we recover the classical q-hook formula. Our proof of Theorem 5 is geometric. Let e_n denote the trivial idempotent in Γ_n. The right $e_n \mathbb{H}_{\mathbf{h}} e_n$-module $\mathbb{H}_{\mathbf{h}} e_n$ defines a coherent sheaf on $\mathcal{Y}_{\mathbf{h}}$. Since $\mathcal{Y}_{\mathbf{h}}$ is smooth, this sheaf is also locally free. Let $\mathcal{R}_{\mathbf{h}}$ denote the corresponding vector bundle. It was shown in [4] that there exists an isomorphism of vector bundles $\mathcal{R}_{\mathbf{h}}^{\Gamma_{n-1}} \xrightarrow{\sim} \mathcal{V}_\theta(n\delta)$ lifting the Etingof-Ginzburg isomorphism.

Fix $\mu \in \mathcal{P}_{\varnothing}(nl)$. By Theorem 2, the Etingof-Ginzburg map sends the fixed point labelled by $\underline{\mathsf{Quot}}(\mu)^\flat$ to the fixed point labelled by μ. We obtain the higher level q-hook formula (5) by comparing the $\mathbb{C}^*$-characters of the corresponding fibres $(\mathcal{R}_{\mathbf{h}}^{\Gamma_{n-1}})_{\underline{\mathsf{Quot}}(\mu)^\flat}$ and $\mathcal{V}_\theta(n\delta)_\mu$.

0.2. **Wreath Macdonald polynomials.** In [7] Gordon defined a geometric ordering $\prec_{\mathbf{h}}^{\mathsf{geo}}$ on $\mathcal{P}(l,n)$ using the closure relations between the attracting sets of $\mathbb{C}^*$-fixed points in $\mathcal{M}_{2\theta}(n\delta)$. There is also a combinatorial ordering $\prec_{\mathbf{h}}^{\mathsf{com}}$ on $\mathcal{P}(l,n)$ defined by

$$\underline{\mu} \preceq_{\mathbf{h}}^{\mathsf{com}} \underline{\lambda} \iff \tau_w(\underline{\lambda}^t) \trianglelefteq \tau_w(\underline{\mu}^t),$$

where $\trianglelefteq$ denotes the dominance ordering on partitions. Using Corollary 4 and the results of Nakajima from [10] we deduce the following.

Corollary 6. *Let* $\underline{\mu}, \underline{\lambda} \in \mathcal{P}(l,n)$. *Then* $\underline{\mu} \preceq_{\mathbf{h}}^{\mathsf{geo}} \underline{\lambda} \Rightarrow \underline{\mu} \preceq_{\mathbf{h}}^{\mathsf{com}} \underline{\lambda}$.

Corollary 6 has an important application - it is a key ingredient in the proof of Haiman's wreath Macdonald positivity conjecture by Bezrukavnikov and Finkelberg (see §2.3 and Lemma 3.8 in [1]).

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Hilbert squares for varieties with exceptional structure sheaf

Theo Raedschelders
(joint work with Pieter Belmans, Lie Fu)

For a smooth projective surface S with $H^1(S, \mathcal{O}_S) = 0$, Hitchin showed in [1] that there is an intimate connection between the deformation theory of S, and that of the Hilbert scheme $S^{[n]}$ of n points on S. On the side of the surface, these deformations are possibly noncommutative. Infinitesimally, this connection is solidified by a natural short exact sequence

$$0 \to H^1(S, T_S) \to H^1(S^{[n]}, T_{S^{[n]}}) \to H^0(S, \wedge^2 T_S) \to 0. \tag{1}$$

In this talk we provide an alternative explanation for this sequence using results of Krug and Sosna [2] on the Fourier-Mukai functor induced by the ideal sheaf of the universal family on $S \times S^{[n]}$. We then go on and explore generalizations to Hilbert schemes of 2 points on varieties of arbitrary dimension $d \geq 2$ which have an exceptional structure sheaf.

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What goes wrong when enhancing triangulated categories (over a field)

Alice Rizzardo

Derived and triangulated categories are a fundamental object of study for many mathematicians, both in geometry and in topology. Their structure is however in many ways insufficient, and usually an enhancement is needed to carry out many important constructions on them. In this talk we will provide counterexamples to existence and uniqueness of enhancements of triangulated categories over a field of characteristic zero. The basic idea is that by replacing A_∞ categories with A_n-categories we are still able to carry on many constructions for which the triangulated category structure is not sufficient, and at the same time an A_n category that does not lift to an A_∞ category will not admit an enhancement. This is joint work with Michel Van den Bergh.

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The categorified Chern character

Sarah Scherotzke
(joint work with Marc Hoyois, Pavel Safronov, Nicolo Sibilla)

The Chern character is a central construction which appears in topology, representation theory and algebraic geometry. In algebraic topology it is for instance used to probe K-theory which is notoriously hard to compute, in representation theory it takes the form of classical character theory. Recently, Toën and Vezzosi suggested a new construction of the Chern character, using derived algebraic geometry. Let X be a derived stack. The Chern character is a map

$$\mathrm{ch}: \iota_0 \mathcal{P}\mathrm{erf}(X) \to HH_*(X)$$

from the ∞-groupoid of perfect complexes on X to the Hochschild homology of X. It has a refinement given by the Dennis trace map

$$K(X) \to NC^-(X).$$

We categorify this Chern character in [2]. The categorified counterparts of these invariants can be tabulated as follows:

Classical	Categorified
The stable category of perfect complexes $\mathcal{P}\mathrm{erf}(X)$, X a derived stack	$\mathrm{Mod}_X^{\mathrm{dual}}$, the $(\infty, 2)$-category of fully dualizable stable categories tensored over $\mathcal{P}\mathrm{erf}(X)$
Hochschild homology, $HH_*(X)$	$\mathcal{P}\mathrm{erf}(\mathcal{L}X)$, where $\mathcal{L}X$ is the loop stack of X
Algebraic K-theory, $K(X)$	$\mathrm{NMot}(X)$ the category of noncommutative motives
Negative cyclic homology, $NC^-(X)$	$\mathcal{P}\mathrm{erf}(\mathcal{L}X)^{S^1}$ the S^1-equivariant perfect complexes.

It is a categorification in the following sense: taking the endomorphism ring of the unit object with respect to the symmetric monoidal structure on the right hand side yields the left hand side. So for instance

$$HH_*(X) \simeq Hom_{\mathcal{P}\mathrm{erf}(\mathcal{L}X)}(\mathcal{O}_{\mathcal{L}X}, \mathcal{O}_{\mathcal{L}X})$$

The categorified Chern character is a symmetric monoidal functor

$$\mathrm{Ch}\colon \mathrm{Mod}_X^{\mathrm{dual}} \to \mathcal{P}\mathrm{erf}\mathcal{L}X. \tag{1}$$

It factors uniquely through NMot(X). Note that restricting to endomorphism rings of the unit objects recovers the classical Chern character.

In classical homological algebra, the Chern character factors through the fixed locus for the canonical S^1-action on the Hochschild complex. This is a manifestation of general rotation invariance properties of trace maps, best understood in the context of TQFT, see [1], [2] and [4]. This feature persists at the categorified level: we have an S^1-equivariant refinement of the categorified Chern character

$$\mathrm{Ch}^{S^1} : \mathrm{Mod}_X^{\mathrm{dual}} \longrightarrow (\mathcal{P}\mathrm{erf}\mathcal{L}X)^{S^1}. \tag{2}$$

0.1. **The Grothendieck-Riemann-Roch.** The classical GRR Theorem states functoriality with respect to the pushforward functor. In the categorified setting the GRR takes the following form: let $f\colon X \to Y$ be a map between derived stacks. Under appropriate conditions on f (see [3], the map f induces a functor between perfect complexes on loop spaces

$$\mathcal{L}f_*\colon \mathcal{P}\mathrm{erf}\mathcal{L}X \to \mathcal{P}\mathrm{erf}\mathcal{L}Y.$$

Passability implies that there is a well-defined push-forward of dualizable modules

$$f_*\colon \mathrm{Mod}_X^{\mathrm{dual}} \longrightarrow \mathrm{Mod}_Y^{\mathrm{dual}}.$$

The categorified Grothendiek-Riemann-Roch Theorem states that the following square of ∞-categories commutes:

$$\begin{array}{ccc} \mathrm{Mod}_X^{\mathrm{dual}} & \xrightarrow{\mathrm{Ch}^{S^1}} & (\mathcal{P}\mathrm{erf}\mathcal{L}X)^{S^1} \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle \mathcal{L}f_*} \\ \mathrm{Mod}_Y^{\mathrm{dual}} & \xrightarrow{\mathrm{Ch}^{S^1}} & (\mathcal{P}\mathrm{erf}\mathcal{L}Y)^{S^1}. \end{array} \tag{3}$$

The Grothendieck-Riemann-Roch Theorem has interesting consequences:

- *The ordinary Chern character.* Toën and Vezzosi give a new construction of the Chern character. Using the GRR, we can show that their definition agrees with the classical Chern character definitions of McCarthy, Ben-Zvi–Nadler, and Keller.
- *The secondary Chern character.* The GRR yields a comparison between the secondary Chern character and motivic character maps that had already appeared in the literature.
- *The de Rham realization.* In the geometric setting Ch^{S^1} matches the de Rham realization. This shows in particular that the Gauss–Manin connection is of non-commutative origin.
- *The classical GRR.* The categorified GRR statement implies the classical GRR.

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Chain conditions in the enveloping algebra of the Witt algebra

Susan J. Sierra
(joint work with Alexey Petukhov)

Let W_+ be the *positive Witt algebra*, which has a $\mathbb{C}$-basis

$$\{e_n : n \in \mathbb{Z}_{\geq 1}\},$$

with Lie bracket

$$[e_i, e_j] = (j-i)e_{i+j}. \tag{1}$$

We study the two-sided ideal structure of $U(W_+)$.

In 2013, the second author and Walton proved [SW1] that $U(W_+)$ is neither left nor right noetherian, by establishing the analogous properties for the quotient ring $B = U(W_+)/(e_1e_5 - 4e_2e_4 + 3e_3^2 + 2e_6)$. However, by [SW2, Proposition 6.6], two-sided ideals of B satisfy the ascending chain condition, and B has Gelfand-Kirillov dimension (GK-dimension) 3. The main question we investigate is how far these properties generalise to arbitrary quotients of $U(W_+)$.

The enveloping algebra $U(W_+)$ is highly noncommutative — it is well-known, for example, that the Weyl algebra $A_n(\mathbb{C})$ is a quotient of $U(W_+)$ for any n. One thus expects that two-sided ideals of $U(W_+)$ are large, and computer experiments have supported this. In fact, all known proper quotients of $U(W_+)$ have finite GK-dimension, even though $U(W_+)$ has subexponential growth and thus infinite GK-dimension. We conjecture:

Conjecture 1. *The enveloping algebra $U(W_+)$ has* just infinite GK-dimension *in the sense that if I is a nonzero ideal of $U(W_+)$, then the GK-dimension of $U(W_+)/I$ is finite.*

If nontrivial ideals in $U(W_+)$ are large, it is natural to expect that the lattice of two-sided ideals is well-behaved. In fact, we conjecture:

Conjecture 2. *Two-sided ideals of $U(W_+)$ satisfy the ascending chain condition: all strictly ascending chains of ideals are finite.*

The second conjecture, asked in [SW2, Question 0.11], was first brought to the second author's attention by Lance Small.

The first author and Penkov have shown that the ideal structure of enveloping algebras of infinite-dimensional Lie algebras can be extremely sparse; for example, for the majority of locally simple Lie algebras $\mathfrak{g}_\infty$, the universal enveloping algebra $U(\mathfrak{g}_\infty)$ has only finitely many two-sided ideals by [PP1, Corollary 3.2 and Section 6]. Further, the analogue of Conjecture 2 holds for $U(\mathfrak{sl}(\infty))$ by [PP2, Corollary 5.4]. In general, two-sided ideals of enveloping algebras of infinite-dimensional Lie algebras form an interesting area of research with many unexpected phenomena.

Although we do not prove either conjecture, we make progress towards both, establishing several partial results that support the conjectures. Our key method is to work with the symmetric algebra $S(W_+)$ under the natural Poisson structure induced from $U(W_+)$. It is well-known that ideals of $U(W_+)$ give rise, via the associated graded construction, to Poisson ideals of $S(W_+)$. We show that if I is a nontrivial radical Poisson ideal of $S(W_+)$ then $S(W_+)/I$ embeds in a finitely generated commutative algebra. As a consequence, we obtain:

Theorem 3. *Let K be a nontrivial Poisson ideal of $S(W_+)$. Then K has finitely many minimal primes, and $S(W_+)/K$ has finite GK-dimension.*

Using this result, we show:

Theorem 4. *The algebra $S(W_+)$ satisfies the ascending chain condition on radical Poisson ideals. Thus $U(W_+)$ satisfies the ascending chain condition on ideals whose associated graded ideal is radical.*

We then turn to studying the GK-dimension of quotients of $U(W_+)$ more directly. Since $S(W_+)$ is not finitely generated as an algebra, there is no clear reason for the GK-dimension of the associated quotient of $S(W_+)$ to give a bound on the GK-dimension of R in general. However, $S(W_+)$ is finitely generated as a Poisson algebra. Thus suggests defining the *Poisson Gelfand-Kirillov dimension* $\operatorname{PGKdim} A$, which measures the growth of a Poisson algebra A as a Poisson algebra. We show that the GK-dimension of a quotient of $U(W_+)$ is equal to the Poisson GK-dimension of the associated quotient of $S(W_+)$. We further show:

Theorem 5. *If K is a nontrivial radical Poisson ideal of $S(W_+)$, then*

$$\operatorname{PGKdim} S(W_+)/K = \operatorname{GKdim} S(W_+)/K,$$

which we have seen previously is finite.

Therefore, if I is an ideal of $U(W_+)$ whose associated graded ideal is radical, then $\operatorname{GKdim} U(W_+) < \infty$, and thus Conjectures 1 and 2 both hold for ideals whose associated graded ideal is radical.

We then turn our attention to quadratic elements in the symmetric algebra, i.e. elements of $S^2(W_+)$. Through explicit computations, we show that $S^2(W_+)$ is a noetherian W_+-module, and further it is GK 2-critical, and as a consequence that $S(W_+)$ satisfies the ascending chain condition on Poisson ideals generated by quadratic elements. Finally, we show:

Theorem 6. *If I is an ideal of $U(W_+)$ that contains a quadratic expression in the e_i, then $U(W_+)/I$ has finite GK-dimension.*

Recall that W_+ is a subalgebra of the (full) *Witt algebra* W, which has a $\mathbb{C}$-basis $\{e_n : n \in \mathbb{Z}\}$ and Lie bracket defined by (1). Recall also that W is obtained from the *Virasoro algebra* V (which we do not define) by setting the central charge equal to zero. We conjecture that analogues of Conjectures 1 and 2 and Theorem 6 hold for $U(W)$ and $U(V)$. These questions will be the subject of future work.

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Tilting bundles on hypertoric varieties

Špela Špenko

(joint work with M. Van den Bergh)

McBreen and Webster have recently [1] constructed a tilting bundle on a smooth hypertoric variety and showed that its endomorphism ring is Koszul (using the Bezrukavnikov-Kaledin method based on reduction to finite characteristics). We show an alternative method for proving these results.

Our proof is based on the simple observation that the tilting bundle constructed by Halpern-Leistner and Sam [2] on a generic open GIT substack of the ambient linear space restricts to a tilting bundle on the hypertoric variety. The Koszulity then easily follows from the fact that the hypertoric variety is defined by a quadratic regular sequence.

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Representation theory of elliptic algebras that are module finite over their center, with use of Poisson geometry

Chelsea Walton

(joint work with Xingting Wang, Milen Yakimov)

Take S to be a 3-dimensional or 4-dimensional Sklyanin (elliptic) algebra that is module-finite over its center Z; thus, S is PI. Our first result is the construction of a Poisson Z-order structure on S such that the induced Poisson bracket on Z is non-vanishing. We also provide the explicit Jacobian structure of this bracket, leading to a description of the symplectic core decomposition of the maximal spectrum Y of Z. We then classify the irreducible representations of S by combining (1) the geometry of the Poisson order structures, with (2) algebro-geometric methods for the elliptic curve attached to S, along with (3) representation-theoretic methods using line and fat point modules of S. Along the way, we improve results of Smith and Tate obtaining a description the singular locus of Y for such S.

Talk notes are available here:

https://faculty.math.illinois.edu/ notlaw/PoissonSkly4-MFO.pdf,

and the preprints for the talk are [1] and [2].

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Coulomb branches and Khovanov-Lauda-Rouquier algebras

Ben Webster

In this talk, we discuss the definition of Coulomb branches for quiver gauge theories. Since these are gauge theories in 3-dimensions, their Coulomb branch is a conic variety with a Poisson bracket of degree -2; the incorporation of the rotation action of S^1 on S^2 (in physics terms, of an Ω-background) also induces a non-commutative deformation of this coordinate ring.

Braverman, Finkelberg and Nakajima [BFNb] have recently given a geometric construction of this non-commutative deformation, using sheaves on the affine Grassmannian of the gauge group. In this paper, we discuss an algebraic way of constructing this coordinate ring using a diagrammatic approach, related to Khovanov-Lauda-Rouquier algebras. Special cases of this construction include many well-known and interesting varieties, including slices between Schubert varieties in affine Grassmannians, finite and affine type A quiver varieties, and type A Slodowy slices [NT17, BFNa].

This approach allows us to analyze the representation theory of these Coulomb branches, and relate them to graded combinatorial algebras. In particular, in ongoing work [KTWWYb] with Kamnitzer, Tingley, Weekes and Yacobi, we show that the category $\mathcal{O}$ of truncated shifted Yangians is (typically) a tensor product categorification in the sense of earlier joint worth with Losev [LW15]. This proves the conjecture from [KTWWYa] relating highest weights over these shifted Yangians to the monomial product crystal. This also establishes a Koszul dual duality between category $\mathcal{O}$'s of these Coulomb branches and Nakajima quiver varieties, which are the corresponding Higgs branches [Web].

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Introduction to Noncommutative Resolutions

Michael Wemyss

This is an overview talk, which will motivate and introduce noncommutative resolutions, then survey some of their main uses.

The definition of a noncommutative crepant resolution (NCCR) is due to Van den Bergh [5], motivated in part due to his pioneering work on flops [4] from the viewpoint of noncommutative rings, and in part through his generalisation of Bridgeland–King–Reid's McKay correspondence as an equivalence of derived categories [1].

Consider a reasonable singularity $\operatorname{Spec} R$, which for this abstract will mean a Gorenstein d-fold with only rational singularities, over $\mathbb{C}$. An NCCR of R is defined to be a noncommutative ring Λ of the form $\Lambda \cong \operatorname{End}_R(M)$ for some nonzero reflexive R-module M, such that

(1) $\Lambda \in \operatorname{CM} R$,
(2) $\operatorname{gldim} \Lambda_{\mathfrak{p}} = \dim R_{\mathfrak{p}}$ for all primes $\mathfrak{p}$ of R.

The first condition corresponds to crepancy, the second to smoothness. Although the first condition is the less intuitive, it is usually the easier to check. It turns out [2] that NCCRs are the only possible algebras derived equivalent to crepant resolutions of $\operatorname{Spec} R$, which is why they arise naturally in various contexts. In fact, this observation more-or-less forces the axioms of NCCRs onto us.

The remainder of the survey talk splits into sections, describing some of the main uses of NCCRs:

- Relationship to geometry. Main conjectures in the area, and the current state-of-the-art.
- Relationship to cluster theory. Finite type phenomenon.

- Relationship to derived equivalences, through mutation.
- Relationship to GIT and wall crossing.

There are already various surveys in the literature on noncommutative resolutions, describing in detail some of the above: see for example [3] or [6].

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Prime spectra of abelian and triangulated monoidal categories

Milen Yakimov

(joint work with Kent Vashaw)

We construct a general framework for prime, completely prime, semiprime, and primitive ideals of an abelian 2-category. This provides a noncommutative version of Balmer's prime spectrum [1, 3] of a tensor triangulated category. These notions are based on containment conditions in terms of thick subcategories of an abelian or triangulated category and thick ideals of a monoidal abelian and triangulated category.

A *monoidal abelian category* is an abelian category $\mathcal{M}$ which is equipped with a monoidal structure such that the bifunctor $- \otimes - : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is biexact. A *thick* (*two-sided*) *ideal* of $\mathcal{M}$ is a *thick* (or *wide*) subcatgory $\mathcal{I}$ of $\mathcal{M}$ which is closed under left and right tensoring with $\mathcal{M}$.

For a proper thick ideal $\mathcal{P}$ of $\mathcal{M}$, the following are equivalent:

(1) For all objects X, Y of $\mathcal{M}$, $X \otimes \mathcal{M} \otimes Y \subseteq \mathcal{P} \Rightarrow X \in \mathcal{P}$ or $Y \in \mathcal{P}$.

(2) For all thick ideals $\mathcal{I}, \mathcal{J}$ of $\mathcal{M}$, $\mathcal{I} \otimes \mathcal{J} \subseteq \mathcal{P} \Rightarrow \mathcal{I} \subseteq \mathcal{M}$ or $\mathcal{J} \subseteq \mathcal{P}$.

The proper thick ideals of $\mathcal{M}$ with the above property are called *prime*. The set of those is denoted by Spec$\mathcal{M}$ and equipped with a Zariski type topology. We show that maximal elements in various sets of thick ideals of $\mathcal{M}$ are prime and derive from this that Spec$\mathcal{M}$ is always *nonempty*.

We prove categorical analogs of the main properties of noncommutative prime spectra [4]. For instance we show that if $\mathcal{M}$ has ACC on two-sided thick ideals, then for every proper thick ideal $\mathcal{I}$ of $\mathcal{M}$, there are finitely many primes over it.

Similar notions, starting with Serre subcategories of an abelian category and Serre ideals of a monoidal abelian category $\mathcal{M}$, are defined. Denote by

Serre-Spec$\mathcal{M}$ the topological subspace of Spec$\mathcal{M}$ consisting of Serre prime ideals. If every object of $\mathcal{M}$ has finite length, then $K_0(\mathcal{M})$ is $\mathbb{Z}_+$-ring in the sense of Etingof et al [2]. There is a natural notion of Serre prime ideals of such rings; this gives rise to a topological space Serre-Spec$K_0(\mathcal{M})$. We prove that *the canonical map*

$$K_0 : \text{Serre-Spec}\mathcal{M} \to \text{Serre-Spec}K_0(\mathcal{M}) \tag{1}$$

is a homeomorphism.

As an application, we construct a categorification of the quantized coordinate rings of open Richardson varieties for symmetric Kac–Moody groups, by constructing Serre completely prime ideals of monoidal categories of modules of the KLR algebras, and by taking Serre quotients with respect to them.

Fix a symmetrizable Kac–Moody algebra $\mathfrak{g}$ and the corresponding group G (with a pair of opposite Borel subgroups $B_\pm$ and Weyl group W). The Schubert cell decompositions

$$G/B_+ = \bigsqcup_{w\in W} B_+wB_+/B_+ = \bigsqcup_{u\in W} B_-uB_+/B_+.$$

give rise to the *open Richardson varieties*

$$R_{u,w} = (B_+wB_+ \cap B_-uB_+)/B_+ \neq \emptyset \text{ iff } u \leq w.$$

The quantum Schubert cell algebras of De Concini–Kac–Procesi and Lusztig are

$$U_q^-[w] = U_q(\mathfrak{n}_- \cap w(\mathfrak{n}_-)) \text{ for } w \in W.$$

The maximal torus T of G acts on $U_q^-[w]$ and the T-prime ideals of $U_q^-[w]$ are parametrized by $W^{\leq w}$, [8]. Let $I_w(u)$ denote the corresponding ideal of $U_q^-[w]$ for $u \in W^{\leq w}$. The algebra $U_q^-[w]/I_w(u)$ quantizes $\mathbb{C}[R_{u,w}]$ (more precisely, the coordinate ring of the closure of $R_{u,w}$ in the Schubert cell B_+wB_+/B_+). For $\mathcal{A} := \mathbb{Z}[q^{\pm 1}]$, the *dual integral form* $U_{\mathcal{A}}^-[w]^\vee$ of $U_q^-[w]$ has an $\mathcal{A}$-basis given by the *dual canonical basis* $\mathcal{B}_w^\vee$.

Khovanov–Lauda [6] and Rouquier [7] constructed a monoidal category $\mathcal{C}$ of finite dim modules of the *KLR algebras* satisfying $K_0(\mathcal{C}) \cong U_{\mathcal{A}}^-(\mathfrak{g})^\vee$. For every $w \in W$, it has a monoidal subcategory

$$\mathcal{C}_w \subset \mathcal{C} \quad \text{such that} \quad K_0(\mathcal{C}_w) \cong U_{\mathcal{A}}^-[w]^\vee.$$

We prove that:

For a symmetric Kac–Moody algebra $\mathfrak{g}$ and $u \leq w \in W$, the following hold:

(1) *The integral form $I_w(u)_{\mathcal{A}} := I_w(u) \cap U_{\mathcal{A}}^-[w]$ of the prime ideal $I_w(u)$ has an $\mathcal{A}$-basis given by the elements of the dual canonical basis $\mathcal{B}_w^\vee$ that belong to it, i.e., $I_w(u)_{\mathcal{A}} \in$ Serre-Spec$(U_{\mathcal{A}}^-[w])$.*

(2) *There exists a Serre completely prime ideal $\mathcal{P}_w(u)$ of the category $\mathcal{C}_w$ such that*

$$K_0(\mathcal{P}_w(u)) \cong I_w(u)_{\mathcal{A}},$$

so $K_0(\mathcal{C}_w/\mathcal{P}_w(u)) \cong U_{\mathcal{A}}^-[w]/I_w(u)_{\mathcal{A}}$.

The proof of the first part of the theorem uses Kashiwara's theorems on Deamuze crystals, and that of the second part the relation between Serre spectra (1) and the theorem of Varagnolo–Vasserot and Rouquier linking the dual canonical basis and the KLR categorification. Recently Kashiwara, Kim, Oh, Park (Aug 2017) constructed a categorification of $\mathbb{C}[R_{u,w}]$ using different ideas via monoidal subcategories [5].

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Frobenius-Perron theory of endofunctors

J.J. Zhang

(joint work with J.M. Chen, Z.B. Gao, E. Wicks, X.-H. Zhang, H. Zhu)

We introduce Frobenius-Perron dimension of endofunctors of k-linear categories, study basic properties and provide many examples. The talk is based on [1].

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