Mathematisches Forschungsinstitut Oberwolfach

Report No. 33/2018

DOI: 10.4171/OWR/2018/33

Classical Algebraic Geometry

Organised by
Olivier Debarre, Paris
David Eisenbud, Berkeley
Gavril Farkas, Berlin
Ravi Vakil, Stanford

15 July – 21 July 2018

Abstract. Progress in algebraic geometry often comes through the introduction of new tools and ideas to tackle the classical problems in the development of the field. Examples include new invariants that capture some aspect of geometry in a novel way, such as the derived category, and the extension of the class of geometric objects considered to allow constructions not previously possible, such as the transition from varieties to schemes or from schemes to stacks. Many famous old problems and outstanding conjectures have been resolved in this way over the last 50 years. While the new theories are sometimes studied for their own sake, they are in the end best understood in the context of the classical questions they illuminate. The goal of the workshop was to study new developments in algebraic geometry, with a view toward their application to the classical problems.

Mathematics Subject Classification (2010): 14-XX.

Introduction by the Organisers

The workshop *Classical Algebraic Geometry* held July 16–20, 2018 at the "Mathematisches Forschungsinstitut Oberwolfach" was organized by Olivier Debarre (ENS), David Eisenbud (Berkeley), Gavril Farkas (Berlin), and Ravi Vakil (Stanford). There were 17 one-hour talks with a maximum of four talks a day, and an evening session of short presentations allowing young participants to introduce their current work (and themselves). The schedule deliberately left plenty of room for informal discussion and work in smaller groups.

The extended abstracts give a detailed account of the broad variety of topics of the meeting, often classical questions in algebraic geometry approached with modern methods. (It should be noted that five of the lectures were given by recent Ph.D.'s.) We focus on a representative sample here:

- *A real period index theorem* (Olivier Benoist)
  One of the younger participants, Olivier Benoist, gave an inspiring and beautifully presented talk on the "period index" problem with an application to families of real algebraic varieties. The original problem is to determine when the period (or order) of an element in a Brauer group is equal to it index, defined as the degree of the smallest field extension over which the element splits. The first example where these numbers are different was given by Adrian Albert in the first half of the 20th century. An important result of Johan de Jong from 2004 asserts that over the function field of a complex surface the period and index of any element of the Brauer group coincide. This is false for real surfaces, but conjectures of Lang suggest that it should be true over the function field of a surface with no real points. Benoist uses modern ideas about the Hodge theory of rationally connected varieties and the density of Noether–Lefschetz loci to prove even more: it suffices that the element restrict to 0 at every real point of the surface.

- *Syzygies of canonical curves via Koszul modules* (Claudiu Raicu)
  Formulated in 1984, Mark Green's conjecture predicts that the intrinsic complexity of an algebraic curve of genus g, encoded in its Clifford index, can be read off from the resolution (syzygies) of its canonical embedding. The conjecture led to a remarkable amount of activity and has been proved in 2002–05 by Claire Voisin (in characteristic zero) for *general* curves of arbitrary genus. The conjecture is known in many other cases, but a solution for *arbitrary* curves remains elusive. Claudiu Raicu gave the inaugural talk of our workshop, presenting a dramatically simpler proof of Green's conjecture for generic curves of genus g, which applies not only to characteristic zero but also to characteristic $p \geq \frac{g+3}{2}$ (which is very near to the sharp bound $p \geq \frac{g-1}{2}$ conjectured by Eisenbud and Schreyer). The new proof relies on specializing the curve to sections of the tangential variety of a rational normal curve of degree g and uses, in a subtle way, representation theory and the geometry of Grassmannians of lines in projective space. This new approach to syzygies is expected to lead to further progress on topological invariants of groups.

- *Stable cohomology of complements of discriminants* (Orsola Tommasi)
  Orsola Tommasi gave an impressive talk on her recent result proving a very general stabilization theorem for the cohomology of complements of discriminants on algebraic varieties. There is a long history to deriving non-trivial cohomological information about varieties lying at the heart of algebraic geometry, like moduli spaces of curves, by stratifying them with strata being given by complements of discriminants. Tommasi, using a version of a method pioneered by Vassiliev, manages to prove that on an

arbitrary smooth variety X, the kth cohomology of the complement of the discriminant with respect to a sufficiently high multiple of any ample line bundle L on X stabilizes.

- *Gonality and zero-cycles of general abelian varieties* (Claire Voisin)
  A few years ago, Ein and Lazarsfeld, inspired by earlier work of Pirola, introduced several numerical invariants which measure how far from being rational a given variety is. These included the measure of irrationality, that is, the smallest degree of a generically finite dominant map from the variety to a projective space, and the covering gonality, that is, the minimal gonality of a curve passing though a general point of the variety. It remained a challenge to compute these invariants in the case of interesting varieties. An important progress in this direction is the recent work of Claire Voisin, on which she reported during the first day of our workshop. She showed that for every fixed gonality k, there exists an integer d_k such that any curve passing through a general point of a general abelian variety of dimension at least d_k has gonality at least k. In particular, the covering gonality of a sufficiently general abelian variety is unbounded (in terms of dimension). This is in stark contrast with the case of polarized $K3$ surfaces, which are covered by elliptic curves, thus irrespective of the degree of the polarization have covering gonality 2.

The young participants' presentations, listed below, covered a similarly wide range of topics. As with previous years' young participants, we expect these researchers to quickly establish themselves as leaders in their areas.

- Daniele Agostini (Leipzig)
  Asymptotic syzygies and higher order embeddings
- Jonathan Montaño (New Mexico State University)
  Asymptotic vanishing behavior of local cohomology
- Johannes Schmitt (Ph.D. student, ETH Zürich)
  Tautological zero-cycles on moduli spaces of curves
- Ulrike Riess (ETH Zürich)
  Base divisors for big and nef line bundles on irreducible symplectic varieties
- René Mboro (Vienna)
  Remarks on varieties of essential CH_0*-dimension at most 2*
- Ignacio Barros (Berlin)
  Uniruledness of strata of differentials in small genus
- Susanna Zimmermann (Angers)
  The higher rank Cremona groups are not simple
- Botong Wang (Wisconsin-Madison)
  A Hard Lefschetz theorem in combinatorics

Acknowledgement: The MFO and the workshop organizers would like to thank the US National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, “US Junior Oberwolfach Fellows”. Moreover, the MFO and the workshop organizers would like to thank the Simons Foundation for supporting Joe Harris in the “Simons Visiting Professors” program at the MFO.

Workshop: Classical Algebraic Geometry

Table of Contents

Abstracts

Koszul modules and the Green conjecture

Claudiu Raicu
(joint work with Marian Aprodu, Gavril Farkas, Ştefan Papadima, Jerzy Weyman)

Let V be a vector space of dimension $n \geq 3$ over some field $\boldsymbol{k}$, and let $K \subseteq \bigwedge^2 V$ be a subspace of dimension $m \leq \binom{n}{2}$. We let $S = \mathrm{Sym}(V)$ denote the symmetric algebra on V, and consider the 3-term complex

$$K \otimes S \xrightarrow{\delta_2|_{K\otimes S}} V \otimes S \xrightarrow{\delta_1} S \tag{1}$$

where $\delta_1 : V \otimes S \longrightarrow S$ is the natural multiplication map, and

$$\delta_2 : \bigwedge^2 V \otimes S \longrightarrow V \otimes S, \quad (v \wedge v') \otimes f \xrightarrow{\delta_2} v \otimes (v'f) - v' \otimes (vf) \text{ for } v, v' \in V, f \in S,$$

is the second differential of the Koszul complex on V. The *Koszul module* $W(V,K)$ is defined to be the middle homology of the complex (1), and it is a graded module generated in degree 0 if we make the convention that K is placed in degree 0.

Papadima and Suciu have shown in [4, Lemma 2.4] that the set-theoretic support of $W(V,K)$ is given by the *resonance variety*

$$\mathcal{R}(V,K) := \Big\{ a \in V^\vee \mid \text{ there exists } b \in V^\vee \text{ such that } a \wedge b \in K^\perp \setminus \{0\} \Big\} \cup \{0\}, \tag{2}$$

where $K^\perp$ is the vector space of 2-forms in $\bigwedge^2 V^\vee$ vanishing identically on K. In particular $\mathcal{R}(V,K) = \{0\}$ if and only if $W_q(V,K) = 0$ for $q \gg 0$, and [4] suggests to look for an effective bound for when the vanishing $W_q(V,K) = 0$ starts. We provide such an effective bound in almost all characteristics, as follows.

Theorem 1. *If* $\mathrm{char}(\boldsymbol{k}) = 0$ *or* $\mathrm{char}(\boldsymbol{k}) \geq n-2$ *then we have the equivalence*

$$\mathcal{R}(V,K) = \{0\} \Longleftrightarrow W_q(V,K) = 0 \textit{ for } q \geq n-3. \tag{3}$$

Experiments in small characteristic suggest that the assumption $\mathrm{char}(\boldsymbol{k}) \geq n-2$ is the best possible in order for (3) to hold. It would be interesting to generate examples in arbitrary characteristic that satisfy $\mathrm{char}(\boldsymbol{k}) \leq n-3$ but fail (3). We expect this to be correlated with the failure of the Borel–Weil–Bott theorem in positive characteristic. As the next theorem shows, the vanishing range $q \geq n-3$ in Theorem 1 is optimal, since $W_{n-4}(V,K) \neq 0$ when $\dim(K) = 2n-3$ and $n \geq 4$.

Theorem 2. *If* $\mathrm{char}(\boldsymbol{k}) = 0$ *or* $\mathrm{char}(\boldsymbol{k}) \geq n-2$, *and if* $\mathcal{R}(V,K) = \{0\}$, *then*

$$\dim W_q(V,K) \leq \binom{n+q-1}{q} \frac{(n-2)(n-q-3)}{q+2} \qquad \textit{for } q = 0, \cdots, n-4.$$

Moreover, equality holds for all q *if* $\dim(K) = 2n-3$.

The key role played by the case $\dim(K) = 2n-3$ should not be surprising, given the following simple geometric observation. Using (2), the vanishing of the resonance is the condition that $\mathbb{P}K^{\perp}$ is disjoint from $\mathrm{Gr}_2(V^{\vee})$ inside the Plücker space $\mathbb{P}\left(\bigwedge^2 V^{\vee}\right)$, which can happen only when $m = \mathrm{codim}(\mathbb{P}K^{\perp})$ is larger than $\dim(\mathrm{Gr}_2(V^{\vee})) = 2n-4$. If $m = 2n-3$ then the condition $\mathbb{P}K^{\perp} \cap \mathrm{Gr}_2(V^{\vee}) = \emptyset$ is divisorial on $\mathrm{Gr}_{2n-3}\left(\bigwedge^2 V\right)$, given by the Cayley–Chow divisor of $\mathrm{Gr}_2(V^{\vee})$ in its Plücker embedding. Theorem 1 provides an alternative description of this divisor as the degeneracy locus of a map of vector bundles of equal rank

$$K \otimes \mathrm{Sym}^{n-3} V \xrightarrow{\delta_2} \ker(\delta_{1,n-2}),$$

where the kernel of $\delta_{1,n-2} : V \otimes \mathrm{Sym}^{n-2} V \to \mathrm{Sym}^{n-1} V$ is a trivial bundle, and K is thought of as the tautological subbundle on $\mathrm{Gr}_{2n-3}\left(\bigwedge^2 V\right)$.

Green's Conjecture for cuspidal curves. Formulated in 1984, Green's Conjecture [3, Conjecture 5.1] predicts that one can recognize the intrinsic complexity of a smooth algebraic curve from the syzygies of its canonical embedding, and has been one of the most intensely studied questions in the theory of curves. If $C \hookrightarrow \mathbb{P}^{g-1}$ is a non-hyperelliptic canonically embedded curve of genus g, and if we denote by $K_{i,j}(C, \omega_C)$ the Koszul cohomology group of i-th syzygies of weight j, then Green's Conjecture predicts the equivalence

$$K_{i,1}(C, \omega_C) = 0 \Longleftrightarrow i \geq g - \mathrm{Cliff}(C) - 1,$$

where $\mathrm{Cliff}(C)$ denotes the Clifford index of C. Although for arbitrary curves the conjecture remains open, for general curves Green's Conjecture has been resolved using geometric methods in two landmark papers by Voisin [5, 6]. In this case, the statement of the conjecture reduces to

$$K_{\lfloor g/2 \rfloor,1}(C, \omega_C) = 0. \tag{4}$$

More elementary, algebraic approaches have been proposed over the years to solve the generic Green conjecture (even prior to Voisin's papers), but none has been brought to fruition. One of them is described in [1, Section 3.I] and relies on computing the syzygies of a general canonically embedded g-cuspidal rational curve, which is known to arise as a generic hyperplane section of the tangent developable $\mathcal{T}$ to a rational normal curve of degree g. We prove the following.

Theorem 3. *Let $n \geq 3$ and suppose that $g = 2n-3$ or $g = 2n-4$. If $\mathrm{char}(\boldsymbol{k}) = 0$ or $\mathrm{char}(\boldsymbol{k}) \geq n$ then $K_{n-2,1}(\mathcal{T}, \mathcal{O}_{\mathcal{T}}(1)) = 0$.*

The idea of the proof is to construct a 3-term complex which is exact on the right, has middle homology given by a graded component of a Koszul module, and whose homology on the left is the Koszul cohomology group $K_{n-2,1}(\mathcal{T}, \mathcal{O}_{\mathcal{T}}(1))$. A direct computation of the Euler characteristic of the complex, combined with a good understanding of the Hilbert function of the relevant Koszul module as in Theorems 1 and 2, implies the desired vanishing. By passing to a general hyperplane section of $\mathcal{T}$ in Theorem 3 we find that (4) holds for general g-cuspidal rational curves, and by semicontinuity of Koszul cohomology groups we obtain:

Corollary 4. *Green's Conjecture holds for generic canonical curves of genus g, provided that* $\mathrm{char}(\boldsymbol{k}) = 0$ *or* $\mathrm{char}(\boldsymbol{k}) \geq (g+3)/2$.

In fact, Eisenbud and Schreyer predict in [2, Conjecture 0.1] that the conclusion of Corollary 4 should hold whenever $\mathrm{char}(\boldsymbol{k}) \geq (g-1)/2$. However, if $\mathrm{char}(\boldsymbol{k}) = p \leq (g+1)/2$ then it can be shown that $\mathcal{T}$ is contained in a rational normal scroll of codimension $g-p \geq \lfloor g/2 \rfloor$, and consequently $K_{\lfloor g/2 \rfloor,1}(\mathcal{T}, \mathcal{O}_{\mathcal{T}}(1)) \neq 0$. This implies as before that g-cuspidal rational curves do not satisfy (4), and therefore can't be used to prove the Eisenbud–Schreyer Conjecture. The case when $p = (g+2)/2$ is still undecided, but experiments suggest that $K_{\lfloor g/2 \rfloor,1}(\mathcal{T}, \mathcal{O}_{\mathcal{T}}(1)) = 0$, so the conclusion of Corollary 4 should extend at least to this case.

References

[1] D. Eisenbud, *Green's conjecture: An orientation for algebraists*, in: Free resolutions in commutative algebra and algebraic geometry (Sundance 1990), Res. Notes Math. **2**, Jones and Bartlett, Boston (1992), 51–78.

[2] D. Eisenbud, F. Schreyer, *Equations and syzygies of K3 carpets and unions of scrolls*, preprint, arXiv:1804.08011.

[3] M. Green, *Koszul cohomology and the geometry of projective verieties* J. Differential Geometry **19** (1984), 125–171.

[4] Ş. Papadima, A. Suciu, *Vanishing resonance and representations of Lie algebras*, J. Reine Angewandte Math. **706** (2015), 83–101.

[5] C. Voisin, *Green's generic syzygy conjecture for curves of even genus lying on a K3 surface*, Journal of the European Math. Society **4** (2002), 363–404.

[6] C. Voisin, *Green's canonical syzygy conjecture for generic curves of odd genus*, Compositio Math. **141** (2005), 1163–1190.

Gonality and zero-cycles of general abelian varieties

Claire Voisin

The gonality of a projective algebraic variety X is the minimal gonality of an irreducible projective curve C mapping nontrivially to X. In a recent paper, Bastianelli–de Poi, Ein, Lazarsfeld and Ullery (BDELU) considered the covering gonality, which is the minimum of the gonalities of curves C which are general members of a family of curves covering X. For an abelian variety, the covering gonality and the gonality coincide. The gonality of a very general abelian variety of dimension g is a well-defined number depending only on g. This is indeed an invariant which depends only on the isogeny class of the abelian variety, and it is constant on the complement of a countable union of proper closed algebraic subsets of any moduli space of abelian varieties. We answer affirmatively a question asked in [1].

Theorem 1. *When g tends to infinity, the gonality of a very general abelian variety of dimension g tends to infinity.*

We give precise estimates for that: If $g \geq 2^k(k+1)$, then a very general abelian variety of dimension g has gonality at least $k+1$.

A different version, with a better estimate, of that result is the following:

Theorem 2. *If $g \geq 2k-1$, then for a very general abelian variety A, there is no non constant morphism $j : C \to A$ from a curve C admitting a degree k morphism $f : C \to \mathbb{P}^1$ which is totally ramified over 0.*

This result generalizes a result due to Pirola [2]:

Theorem 3 (Pirola)**.** *A very general abelian variety of dimension at least 3 does not contain any hyperelliptic curve.*

The theorems above admit versions involving orbits of zero-cycles for rational equivalence: For example, we have the following results which imply the theorem above:

Theorem 4. *If A is a very general abelian variety of dimension $g \geq 2k-1$, then*
(i) the orbit of $k\{0_A\}$ is countable.
(ii) The set of divisors $D \in Pic^0(A)$ such that $D^k = 0$ in $CH^k(A)$ is at most countable.

The proof ultimately relies to a general fact about "naturally defined subsets of abelian varieties". By this, we mean the data of a subset $\Sigma_A \subset A$ for any abelian variety A, that satisfy some axioms:

(1) The sets Σ_A should be countable unions of closed algebraic subsets of A,
(2) They should be defined in family.
(3) They are stable under morphisms of abelian varieties.

Our abelian varieties are actually groups (they have a zero), and in the last item, the morphisms are morphisms preserving the group structure. By using a method of Pirola in [2], we show that naturally defined subsets of abelian varieties have dimension decreasing with g once we know they do not fill the whole general abelian variety.

References

[1] F. Bastianelli, P. De Poi, L. Ein, R. Lazarsfeld, B. Ullery *Measures of irrationality for hypersurfaces of large degree.* Compos. Math. 153 (2017), no. 11, 2368–2393.
[2] G.-P. Pirola *Curves on generic Kummer varieties*, Duke Math. J. 59 (1989), 701–708.
[3] C. Voisin *Chow rings and gonality of general abelian varieties*, preprint 2018.

Some curious properties of combinatorial intersection cohomology

Karim Adiprasito

I presented two result: Firstly, considering a d-polytope P, define P_W is the induced subcomplex of P on vertex set W, and $\alpha_{k-1}(P_W)$ denotes the dimension of the image of $H_{k-1}(P_W;\mathbb{R}) \to H_{k-1}(\partial P \setminus P_{V\setminus W};\mathbb{R})$. Moreover, let $g_k(P)$ denote the dimension of the primitive intersection cohomology in degree k, as defined by Barthel–Brasselet–Fieseler–Kaup and Karu for general (possibly non-rational) polytopes. Then we have the following inequality.

Theorem 1 (arxiv:1805.03267)**.** *Let P be a d-polytope, and W any subset of its vertices $V = V(P)$. Let $k \leq \frac{d}{2}$. Then the induced simplicial subcomplex P_W satisfies*

$$\alpha_{k-1}(P_W) \leq g_k(P).$$

This inequality implies the solution to several conjectures in polytope theory, in particular showing that polytopes with small primitive intersection cohomology cannot approximate smooth convex bodies well.

The second curious observation addresses a question of Adin: For cubical polytopes P, there exists a graded commutative ring $C(P)$ such that

(1) $C(P)$ satisfies the Hard Lefschetz theorem, i.e. for a d-polytope P, $k \leq \frac{d}{2}$ and

$$C^k(P) \xrightarrow{\cdot \ell^{d-2k}} C^{d-k}(P)$$

is an isomorphism.

(2) Every non-negative linear combination of face numbers of P is a non-negative linear combination of the primitive Betti numbers of $C(P)$.

In particular, the primitive Betti numbers for $C(P)$ are a finer invariant than the primitive Betti numbers of intersection cohomology of P.

Connections betweens some conjectures on subvarieties of abelian varieties

Mihnea Popa

The talk was devoted to a series of conjectures on subvarieties of principally polarized abelian varieties and on singularities of theta divisors, and especially to pointing out some surprising connections between them that have emerged in recent work.

All throughout, we denote by (A, Θ) an indecomposable principally polarized abelian variety (ppav) of dimension g. The starting point is the main conjecture in this subject, sometimes known as the "minimal class conjecture".

Conjecture 1 (Debarre)**.** *Let X be a closed subscheme of A of dimension $1 \leq d \leq g-2$. The following are equivalent:*

(1) *X is reduced of pure dimension and has minimal cohomology class, i.e. $[X] = \frac{\theta^{g-d}}{(g-d)!}$.*

(2) *One of the following holds:*
   (a) *There is a smooth genus g curve C and an isomorphism $(A, \Theta) \cong (JC, \Theta_C)$ that identifies X with $W_d(C)$.*
   (b) *$g = 5$, $d = 2$, and there is a smooth cubic threefold Y and an isomorphism $(A, \Theta) \cong (JY, \Theta_Y)$ that identifies X with F, the Fano surface of lines on Y.*

When X is a curve this is the celebrated Matsusaka–Ran criterion, while otherwise the conjecture is known to hold in dimension up to four. In [1] the author has proposed, together with Pareschi, an analogy with the classification of subvarieties of minimal degree in projective space, which consists just as in the conjecture above of (cones over) a general class, namely rational normal scrolls (the analogues of the W_d's), and an isolated example, namely the Veronese surface in $\mathbf{P}^5$ (the analogue of the Fano surface in the 5-dimensional intermediate Jacobian).

Given the difficulty of Conjecture 1, we have also proposed in [2] an intermediate conjecture which bridges the gap between its two parts, inspired via the analogy above by the characterization of subvarieties of minimal degree in $\mathbf{P}^n$ in terms of the 2-regularity of their ideal sheaf, in the sense of Castelnuovo–Mumford.

Conjecture 2 ([2])**.** *The two items in Conjecture 1 are also equivalent to the fact that X is a geometrically nondegenerate GV-subscheme, i.e. X is geometrically nondegenerate and $\mathcal{I}_X(\Theta)$ is a GV-sheaf on A.*

The property in the conjecture is the natural analogue of Castelnuovo–Mumford 2-regularity, as discovered in the author's work with Pareschi on M-regularity. It means that the twisted ideal sheaf $\mathcal{I}_X(\Theta)$ satisfies the generic vanishing property, i.e.

$$\operatorname{codim}_{\operatorname{Pic}^0(A)}\{\alpha \in \operatorname{Pic}^0(A) \mid h^i(A, \mathcal{I}_X(\Theta) \otimes \alpha) \neq 0\} \geq i$$

for all $i \geq 0$. This property is known to hold for the special subvarieties in (2) in Conjecture 1, by joint work with Pareschi, as well as work of Höring. The main result of [2] is that it implies (1) in Conjecture 1 as well.

In the direction of Conjecture 2, the most advanced classification result to date is the following:

Theorem 3 ([3])**.** *The regularity condition in Conjecture 2 implies the classification (2) in Conjecture 1 in dimension up to five.*

In fact everything besides the case when X is a surface and A is a fivefold follows already from [2]; this last case however is the most difficult, and it includes both types of special ppavs appearing in Conjecture 1. Recognizing the two different types requires different techniques. Jacobians of curves are recognized via the condition that the theta divisor have a curve summand, a criterion proved recently by Schreieder. On the other hand, intermediate Jacobians of smooth cubic threefolds are recognized via the fact that the theta divisor has unusually large multiplicity (in this case 3) at the origin. The method is based on the analysis of the difference map

$$d \colon X \times X \longrightarrow X - X \subset A,$$

especially in the case when $X - X = \Theta$. It suggests a general procedure where even in higher dimension a better understanding of the singularities of Θ could help towards proving parts of Conjectures 1 and 2. This provides one more reason for addressing the well-known problem of giving effective bounds for the multiplicities of points on theta divisors.

In this latter direction, there is the following folklore:

Conjecture 4. *For every* $x \in \Theta$ *we have* $\mathrm{mult}_x \Theta \leq \frac{g+1}{2}$.

Equality holds for certain points on intermediate Jacobians of smooth cubic threefolds, as observed above, and on Jacobians of hyperelliptic curves of odd genus. In general it is only known that $\mathrm{mult}_x \Theta \leq g$, due to a result of Kollár. Together with Mustaţă, we have recently obtained:

Theorem 5 ([4])**.** *Conjecture 4 holds when* Θ *has isolated singularities. Moreover, in this case there can be at most one point where equality is attained. When* $g \gg 0$*, the bound can be improved to roughly* g/e *(i.e. the transcendental number* e*).*

The same result has been obtained by Codogni–Grushevsky–Sernesi, using a different method. The method we use in the work with Mustaţă is based on the study of Hodge ideals [4], a generalization of multiplier ideals which comes from the theory of mixed Hodge modules. In particular, there is an ideal sheaf $I_1(\Theta) \subseteq \mathcal{O}_A$, which is non-trivial at points where the multiplicity goes beyond what is predicted by the conjecture, and has a Nadel-type vanishing theorem. These two facts, combined with precise knowledge about the Kummer map of A, lead to the statement.

What is interesting here is that even when Θ does not have isolated singularities, if $I_1(\Theta)$ is the ideal sheaf of a closed subscheme Z, a (hopefully small) strengthening of the vanishing theorem for Hodge ideals implies that Z is a GV-subscheme. Conjecture 2 then predicts that A and Z should be of the special type in Conjecture 1, in which case the singularities of Θ are completely understood. Thus in fact Conjecture 4 should be improved to the statement that equality can hold only on hyperelliptic Jacobians and intermediate Jacobians of cubic threefolds. Moreover, we obtain the following somewhat vaguely formulated connections:

- Conjecture 2 plus an ϵ improvement implies Conjecture 4.

- Conjecture 4 can help with proving important special cases of Conjecture 2.

References

[1] G. Pareschi and M. Popa, *Castelnuovo theory and the geometric Schottky problem*, J. Reine Angew. Math. **615** (2008), 25–44.

[2] G. Pareschi and M. Popa, *Generic vanishing and minimal cohomology classes on abelian varieties*, Math. Ann. **340** (2008), 209–222.

[3] S. Casalaina–Martin, M. Popa and S. Schreieder, *Generic vanishing and minimal cohomology classes on abelian fivefolds*, J. Alg. Geom. **27** (2018), 553–581.

[4] M. Mustaţă and M. Popa, *Hodge ideals*, preprint arXiv:1605.08088, to appear in Mem. Amer. Math. Soc.

Moduli spaces of unstable curves

Frances Kirwan

This is a report on the work of my student Joshua Jackson [6] on the construction of moduli spaces of unstable projective curves of given 'Rosenlicht–Serre type'. This construction uses an extension to non-reductive group actions of geometric invariant theory (GIT), and is based on joint work with Gergely Bérczi, Brent Doran, Tom Hawes and Vicky Hoskins [1, 2, 3, 4, 5]

Let C be a projective curve (over $\mathbb{C}$, or more generally an algebraically closed field of characteristic zero) of genus $g \geqslant 2$. Recall that C is stable iff its singularities are at worst nodes and its automorphism group is finite, and that the moduli space $\overline{\mathcal{M}}_g$ of stable curves of genus g is a projective scheme. Moreover $\overline{\mathcal{M}}_g = \mathcal{K}^s/SL(r+1)$ is a quotient by the special linear group $SL(r+1)$ of an open subset $\mathcal{K}^s$ (the locus of stable curves) of the closure

$$\mathcal{K} = \overline{\{k-\text{canonically embedded nonsing curves of genus } g\}}$$

in the Hilbert scheme of curves in $\mathbb{P}^r$ with Hilbert polynomial $P(m) = dm + 1 - g$ for $r = (2k-1)(g-1) - 1$ and $d = 2k(g-1)$ for $k >> 1$. Here $\mathcal{K}^s$ coincides with the stable locus in the sense of GIT for a suitable linearisation of the action of $SL(r+1)$ on $\mathcal{K}$, and so $\overline{\mathcal{M}}_g$ can be identified with the GIT quotient $\mathcal{K}//SL(r+1)$.

There is a description due to Rosenlicht and Serre [7] of a singular curve C in terms of its normalisation $\tilde{C}$. The normalisation map $p : \tilde{C} \to C$ defines an equivalence relation $\sim$ on $\tilde{C}$ with associated homeomorphism $\tilde{C}/\!\sim \to C$ where $\tilde{C}/\!\sim$ has the co-finite topology. For $z \in \tilde{C}/\!\sim$ let $\mathcal{O}_z$ be the semi-local ring

$$\bigcap_{p(y)=z} \mathcal{O}_{y,\tilde{C}}$$

with radical $\mathfrak{r}_z$. Let $\mathcal{O}'_z$ be $\mathcal{O}_z$ if $z \notin \mathrm{Sing}(C)$ and otherwise a subring such that

$$\mathbb{C} + \mathfrak{r}_z \supseteq \mathcal{O}'_z \supseteq \mathbb{C} + \mathfrak{r}_z^n$$

for some $n = n_C >> 1$. Then the ringed space $(\tilde{C}/\!\sim, \mathcal{O}')$ is always a projective curve, and is isomorphic to the original curve C for appropriate choices of $n_C >> 1$ and of $\mathcal{O}'_z$ when $z \in \mathrm{Sing}(C)$. In 1964 Ebey used this description to classify unibranched singularities, by using slices for a non-reductive linear algebraic group action.

Let G be a complex reductive group and X a complex projective scheme acted on linearly by G. The best situation for GIT is when $X^{ss} = X^s \neq \emptyset$. Then

$$X^s/G = X//G = \mathrm{Proj}(\bigoplus_{k=0}^{\infty} H^0(X, L^{\otimes k})^G)$$

is a projective scheme and is a geometric quotient of X^s by the action of G. More generally, if the stable locus X^s is non-empty, then there is a 'partial desingularisation' $\tilde{X}//G = \tilde{X}^{ss}/G$ of the GIT quotient $X//G$. This partial desingularisation is a geometric quotient by G of the open subscheme $\tilde{X}^{ss} = \tilde{X}^s$ of a G-equivariant blow-up $\tilde{X}$ of X. Here $\tilde{X}^{ss}$ is obtained from X^{ss} by successively blowing up along the subschemes of semistable points stabilised by (reductive) subgroups of G of maximal dimension and then removing the unstable points in the resulting blow-up. When X is nonsingular then $\tilde{X}//G$ is an orbifold.

We would like to use GIT quotient constructions of $\overline{\mathcal{M}}_g$, and the related moduli spaces $\overline{\mathcal{M}}_{g,n}$ of stable curves with marked points, to stratify the moduli stack of projective curves, finding discrete invariants of unstable curves such that moduli spaces of curves with these invariants fixed can also be constructed by GIT methods. Similarly we would like to construct other moduli spaces of unstable objects, such as moduli spaces of unstable sheaves of fixed Harder–Narasimhan type over a fixed nonsingular projective scheme, or moduli spaces of unstable maps from curves (perhaps with marked points) into a projective variety X. To do this, we are led to use a non-reductive version of GIT, rather than classical GIT, even though initially we might have an action of a reductive group such as $G = SL(r+1)$. Just as for moduli spaces of stable objects, in practice we need to guess the answer in advance in order to set up the GIT constructions appropriately.

Why does non-reductive GIT appear here? Classical GIT tells us that when a reductive group G acts linearly on a projective scheme X, then X has a stratification $X = \bigsqcup_{\beta \in \mathcal{B}} S_\beta$ indexed by a finite subset $\mathcal{B}$ of a positive Weyl chamber for G, with

(i) $S_0 = X^{ss}$, and for each $\beta \in \mathcal{B}$

(ii) the closure of S_β is contained in $\bigcup_{\gamma \geqslant \beta} S_\gamma$,

(iii) $S_\beta \cong (G \times Y_\beta^{ss})/P_\beta$ where P_β is a parabolic subgroup of G and Y_β^{ss} is an open subset of a projective subscheme $\overline{Y}_\beta$ of X, determined by the action of the Levi subgroup of P_β with respect to a twisted linearisation.

To try to construct a quotient of (an open subset of) an unstable stratum S_β by G, we can study the linear action on $\overline{Y}_\beta$ of the parabolic subgroup P_β, twisted by a suitable (rational) character. In general G itself will not have suitable characters.

If G is a linear algebraic group which is not reductive, the graded algebra $\bigoplus_{k=0}^\infty H^0(X, L^{\otimes k})^G$ is not necessarily finitely generated. Nonetheless it is possible to define open subschemes X^s ('the stable locus') and X^{ss} ('the semistable locus') with a geometric quotient $X^s \to X^s/G$ and an 'enveloping quotient' $X^{ss} \to X//G$. However in general $X//G$ is not necessarily projective and $X^{ss} \to X//G$ is not necessarily onto. Also the Hilbert–Mumford criteria for (semi)stability in classical GIT do not generalise, at least not in a very obvious way.

Let us call a unipotent linear algebraic group U graded unipotent if there is a homomorphism $\lambda : \mathbb{C}^* \to Aut(U)$ with the weights of the $\mathbb{C}^*$ action on $Lie(U)$ all strictly positive. Then let $\hat{U} = U \rtimes \mathbb{C}^*$ be the associated semi-direct product. Suppose that $\hat{U}$ acts linearly (with respect to an ample line bundle L) on a projective scheme X. We can twist the action of $\hat{U}$ by any character (or any rational character, after replacing L with $L^{\otimes m}$ for sufficiently divisible positive m). If we are willing to twist by an appropriate rational character, then GIT for the $\hat{U}$ action is nearly as well behaved as in the classical case for reductive groups.

Theorem 1 ([2, 3]). *Let U be graded unipotent acting linearly on a projective scheme X. Suppose that the linear action extends to $\hat{U} = U \rtimes \mathbb{C}^*$, and*

$$(*) \quad x \in Z_{\min} \Rightarrow \dim \mathrm{Stab}_U(x) = 0;$$

here $Z_{\min}$ is the union of connected components of $X^{\mathbb{C}^*}$ where the $\mathbb{C}^*$-action on L^* has minimum weight. We can twist the action of $\hat{U}$ by a (rational) character so that 0 lies just above the minimum weight for the $\mathbb{C}^*$ action on X, and

(i) *the algebra of $\hat{U}$-invariants is finitely generated, and* $X/\!/\hat{U} = Proj(\bigoplus_{k=0}^{\infty} H^0(X, L^{\otimes k})^{\hat{U}})$ *is projective;*

(ii) $X/\!/\hat{U}$ *is a geometric quotient of $X^{ss,\hat{U}} = X^{s,\hat{U}}$ by $\hat{U}$ and $X^{ss,\hat{U}}$ has a Hilbert–Mumford description.*

Moreover, even without condition (*) there is a projective completion of $X^{s,\hat{U}}/\hat{U}$ which is a geometric quotient by $\hat{U}$ of an open subset $\tilde{X}^{ss}$ of a $\hat{U}$-equivariant blow-up $\tilde{X}$ of X.

A parabolic subgroup P_β of a reductive group G has the form $P_\beta = U_\beta \rtimes L_\beta$, with its unipotent radical U_β 'internally graded' by a central 1-parameter subgroup of its Levi subgroup L_β. Thus to construct a quotient of (an open subscheme of) an unstable stratum S_β by G, we can study the linear action, appropriately twisted, of P_β on the closure of Y_β^{ss}, and quotient first by $\hat{U}_\beta$ and then by the residual action of the reductive group $P_\beta/\hat{U}_\beta = L_\beta/\mathbb{C}^*$. We can then refine the stratification so that all the strata have geometric quotients [5]. A similar stratification exists when G is non-reductive with 'internally graded unipotent radical'.

This suggests that non-reductive GIT for linear actions, twisted appropriately, can be used to construct moduli spaces of unstable curves of fixed 'type' [6]. One nice aspect is that we can ensure that the condition (*), that 'semistability coincides with stability for the unipotent radical', is satisfied for moduli of (unstable) curves, except in very special cases which can be dealt with separately. The 'Rosenlicht–Serre type' of a projective curve C is made up of the genus of (each connected component of) its normalisation (or resolution of singularities) $\tilde{C}$, and some additional data involving the singularities of C.

References

[1] G. Bérczi, B. Doran, T. Hawes, F. Kirwan, *Constructing quotients of algebraic varieties by linear algebraic group actions*, arXiv:1512.02997, to appear in Handbook of Group Actions.

[2] G. Bérczi, B. Doran, T. Hawes, F. Kirwan, *Geometric invariant theory for graded unipotent groups and applications*, arXiv: 1601.00340, to appear in J. Topology.

[3] G. Bérczi, B. Doran, T. Hawes, F. Kirwan, *Projective completions of graded unipotent quotients*, arXiv:1607.04181.

[4] G. Bérczi, B. Doran, F. Kirwan, *Graded linearisations*, arXiv:1703.05226, to appear in 'Modern Geometry: a celebration of the work of Simon Donaldson', AMS Proc Symp Pure Math.

[5] G. Bérczi, V. Hoskins, F. Kirwan, *Stratifying quotient stacks and moduli stacks*, arXiv:1711.10215, to appear in 'Geometry of moduli', Abel Symposium 2017.
[6] J. J. Jackson, Oxford D.Phil. thesis, 2018.
[7] V. L. Popov, *Algebraic curves with an infinite automorphism group*, Mathematical Notes of the Academy of Sciences of the USSR, **23 (2)** 102-108, 1978.

Big polynomial rings and Stillman's conjecture

Steven V. Sam
(joint work with Daniel Erman and Andrew Snowden)

Throughout, $\mathbf{k}$ will refer to an algebraically closed field (not fixed). Stillman's conjecture (now a theorem of Ananyan–Hochster [AH]) is the following statement:

Theorem 1 (Ananyan–Hochster)**.** *Fix integers $d_1, \dots, d_r \geq 1$. There is a constant C such that any ideal in $\mathbf{k}[x_1, \dots, x_n]$ generated by homogeneous polynomials of degrees $d_1, \dots, d_r$ has projective dimension $\leq C$ (independent of n and $\mathbf{k}$).*

The motivation for the work in this talk was to show this as a consequence of a more basic structural result. Our first idea involved "**GL**-noetherianity":

- Remove independence on n by working in $\mathbf{k}[x_1, x_2, \dots] = \mathrm{Sym}(\mathbf{k}^\infty)$.
- The relevant parameter space for choices of polynomials is
$$X = \mathrm{Sym}^{d_1}(\mathbf{k}^\infty) \times \cdots \times \mathrm{Sym}^{d_r}(\mathbf{k}^\infty).$$
- For each d, we define a $\mathbf{GL}_\infty(\mathbf{k})$-equivariant subset
$$X_{\geq d} = \{(f_1, \dots, f_r) \in X \mid \mathrm{pdim}(f_1, \dots, f_r) \geq d\},$$
which gives a decreasing chain $X_{\geq 1} \supseteq X_{\geq 2} \supseteq \cdots$.
- If each $X_{\geq d}$ is closed and X is $\mathbf{GL}_\infty(\mathbf{k})$-noetherian (i.e., decreasing chains of closed $\mathbf{GL}_\infty(\mathbf{k})$-invariant subsets always stabilize), then we would have $X_{\geq d} = X_{\geq d+1}$ for $d \gg 0$ and get an upper bound for projective dimension.
- Noetherianity follows from work of Draisma [Dr]; closedness of $X_{\geq d}$ is less clear. A priori, it is only an infinite union of closed sets given by vanishing conditions on graded Betti numbers: $\beta_{d,d} = \beta_{d,d+1} = \cdots = 0$.

This idea can be developed to give a proof (see [ESS1, §5]), but we ended up finding a simpler proof which we explain.

A key step to proving this is the notion of strength [AH]: a homogeneous element f in a graded ring has **strength** $\leq s$ if we can write $f = g_1 h_1 + \cdots + g_s h_s$ with g_i, h_i homogeneous and $0 < \deg g_i < \deg f$ for all i. The strength is s if it has strength $\leq s$ but not strength $\leq s - 1$. The strength is ∞ if there is no such decomposition. The strength of a linear space of elements is the minimal strength of a nonzero homogeneous element in it. Then:

Theorem 2 (Ananyan–Hochster)**.** *Fix integers $d_1, \dots, d_r \geq 1$. Given polynomials $f_1, \dots, f_r$ with $\deg(f_i) = d_i$, if the strength of $\langle f_1, \dots, f_r \rangle$ is sufficiently large (with respect to $d_1, \dots, d_r$), then $f_1, \dots, f_r$ is a regular sequence.*

In particular, there is a constant C such that any $f_1, \ldots, f_r \in \mathbf{k}[x_1, \ldots, x_n]$ with $\deg(f_i) = d_i$ belong to a subalgebra generated by a regular sequence with $\leq C$ homogeneous elements.

The C in the theorem gives a bound for the original statement: a minimal free resolution for $(f_1, \ldots, f_r)$ can first be computed in the subalgebra generated by the regular sequence; by flatness, its base change to $\mathbf{k}[x_1, \ldots, x_n]$ remains a resolution.

Ultraproducts give a context for working with the notion of "sufficiently large" without having to explicitly identify bounds.

Let $\mathcal{I}$ be an infinite set (typically the positive integers). We fix a **non-principal ultrafilter** $\mathcal{F}$ on $\mathcal{I}$, which is a collection of subsets of $\mathcal{I}$ satisfying the following properties:

(1) $\mathcal{F}$ contains no finite sets,
(2) if $A \in \mathcal{F}$ and $B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$,
(3) if $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$,
(4) for all $A \subseteq \mathcal{I}$, either $A \in \mathcal{F}$ or $\mathcal{I} \setminus A \in \mathcal{F}$ (but not both).

Intuition: the sets in $\mathcal{F}$ are neighborhoods of some hypothetical (and non-existent) point $*$ of $\mathcal{I}$. We say that some condition holds near $*$ if it holds in some neighborhood of $*$.

Given a family of sets $\{X_i\}_{i \in \mathcal{I}}$, their ultraproduct $\operatorname{ulim}_{i \in \mathcal{I}} X_i$ is the quotient of the usual product $\prod_{i \in \mathcal{I}} X_i$ in which two sequences (x_i) and (y_i) are identified if the equality $x_i = y_i$ holds near $*$.

Suppose that each X_i is a graded abelian group. We define the **graded ultraproduct** of the X_i's to be the subgroup of the usual ultraproduct consisting of elements x such that $\deg(x_i)$ is bounded near $*$. The graded ultraproduct is a graded abelian group; the degree d piece of the graded ultraproduct is the usual ultraproduct of the degree d pieces of the X_i's. We again denote this by $\operatorname{ulim}_i X_i$.

The following 3 statements about ultraproducts and regular sequences give a proof of Stillman's conjecture:

Lemma 3. *For $i \geq 1$, let V_i be linear subspaces of polynomial rings R_i with strength tending to ∞. Then $\operatorname{ulim}_i V_i \subset \operatorname{ulim}_i R_i$ has infinite strength.*

This follows from the definitions of ultraproducts.

Theorem 4. *For $i \geq 1$, let $R_i = \mathbf{k}_i[x_1, x_2, \ldots]$ with $\deg(x_j) = 1$. Let $\mathbf{S} = \operatorname{ulim}_i R_i$ and let $\mathfrak{m}$ be its homogeneous maximal ideal. Let $\mathcal{E} \subset \mathfrak{m}$ be a subset of homogeneous elements whose image in $\mathfrak{m}/\mathfrak{m}^2$ is a basis (over $\mathbf{K} = \operatorname{ulim}_i \mathbf{k}_i$). Then $\mathbf{S}$ is a polynomial ring over $\mathbf{K}$ with generators $\mathcal{E}$.*

Note that $\mathfrak{m}^2$ is precisely the set of polynomials of finite strength, so a linear subspace of $\mathfrak{m}$ has infinite strength if and only if its image in $\mathfrak{m}/\mathfrak{m}^2$ is linearly independent.

Lemma 5. *For $i \geq 1$, let $f_{i,1}, \ldots, f_{i,r} \in R_i$ be homogeneous polynomials of degrees $d_1, \ldots, d_r$. Then $\operatorname{ulim}_i f_{i,1}, \ldots, \operatorname{ulim}_i f_{i,r} \in \operatorname{ulim}_i R_i$ is a regular sequence if and only if $f_{i,1}, \ldots, f_{i,r}$ is a regular sequence for i near $*$.*

Here is the proof that sufficiently large strength (relative to degrees) implies regular sequence: if not, then we can find a sequence of polynomials $(f_{i,1}, \ldots, f_{i,r})$ whose strength goes to ∞ for $i \gg 0$ but which do not form a regular sequence for any i. The ultralimit has infinite strength in $\mathbf{S}$ (Lemma 3), which is then part of an algebraically independent generating set for $\mathbf{S}$ (Theorem 4), and hence form a regular sequence. But this contradicts Lemma 5.

Lemma 3 follows from the definitions.

Theorem 4 is proven with the following criterion for polynomiality:

Theorem 6. *Let R be a $\mathbf{Z}_{\geq 0}$-graded ring over $\mathbf{k}$ (characteristic 0 and not necessarily algebraically closed) with $R_0 = \mathbf{k}$. Assume that R has "enough derivations", i.e., for every positive degree element f, there exists a negative degree derivation ∂ of R such that $\partial(f) \neq 0$. Then R is a polynomial ring over $\mathbf{k}$, and a generating set can be obtained by taking any lift of a $\mathbf{k}$-basis for $R_{>0}/R_{>0}^2$.*

For fields of positive characteristic, this doesn't work (pth powers are killed by any derivation and $\mathbf{F}_p[x]/(x^p)$ has enough derivations) but we can give slightly different criteria using Hasse derivatives ($\partial_k x^n = \binom{n}{k} x^{n-k}$) when $\mathbf{k}$ is perfect (the imperfect case is handled in [ESS2]).

The idea is that any lift of a basis for $R_{>0}/R_{>0}^2$ generates R, and the derivations are used to show that these elements don't satisfy any nontrivial algebraic relations since any relation can always be used to produce one of lower degree. To get the derivations on $\mathbf{S}$, we can take ultraproducts of usual partial derivatives.

Finally, here is a sketch of the proof of Lemma 5:

- Let $\mathbf{S} = \operatorname{ulim}_i R_i$ and let $I = \operatorname{ulim}_i I_i$ with $I_i \subseteq R_i$ ideals generated in the same degrees. We will show that $\operatorname{codim} I = \operatorname{codim} I_i$ for i near $*$ and apply this to the ideals generated by $f_{i,1}, \ldots, f_{i,r}$.
  Also, set $R'_i = \mathbf{k}_i[x_2, x_3, \ldots]$ and $\mathbf{S}' = \operatorname{ulim}_i R'_i$. Then $\mathbf{S}'[x_1] \cong \mathbf{S}$.
- First, $\operatorname{codim}(I) < \infty$ since I is finitely generated and defined by finitely many of the variables in $\mathcal{E}$. Let $c = \operatorname{codim}(I)$.
- Now do induction on c. If $c = 0$, then $I = 0$ and $I_i = 0$, so there is nothing to show. Otherwise, pick nonzero $f \in I$.
- Pick change of variables γ so that $\gamma(f)$ is monic as a polynomial in x_1. Then $\operatorname{codim}_{\mathbf{S}}(I) - 1 = \operatorname{codim}_{\mathbf{S}'}(\mathbf{S}' \cap I)$ since $\mathbf{S}' \to \mathbf{S}/(f)$ is finite and flat. Now use the induction hypothesis.

References

[AH] Tigran Ananyan and Melvin Hochster, Small subalgebras of polynomial rings and Stillman's conjecture, arXiv:1610.09268v1.

[Dr] Jan Draisma, Topological noetherianity for polynomial functors, arXiv:1705.01419v1.

[ESS1] Daniel Erman, Steven V Sam, Andrew Snowden, Big polynomial rings and Stillman's conjecture, arXiv:1801.09852v3.

[ESS2] Daniel Erman, Steven V Sam, Andrew Snowden, Big polynomial rings with imperfect coefficient fields, arXiv:1806.04208v1.

Stable rationality and root stacks

Brendan Hassett

(joint work with Andrew Kresch and Yuri Tschinkel)

We exhibit smooth projective stably rational threefolds that deform to varieties that are not stably rational. Thus stable rationality is not a deformation invariant in dimension three [4].

A complex variety V is *stably rational* if the product $V \times \mathbb{P}^r$ is rational for some r. Voisin's technique of decomposition of the diagonal [9] is a powerful tool for proving that varieties are not stably rational. It is the key to showing that stable rationality is not a deformation invariant of smooth projective complex varieties of dimension at least four [5]. The case of dimension three was left open.

The first stably rational non-rational varieties were found in by Beauville, Colliot-Thélène, Sansuc, and Swinnerton-Dyer [2]. They offered two related classes of examples. The first is Châtelet surfaces, defined over a field k by

$$\{y^2 - az^2 = P(x)\} \subset \mathbb{A}^3,$$

where $P(x) \in k[x]$ is a cubic polynomial with Galois group $\mathfrak{S}_3$ and discriminant a. These are stably rational but non-rational over k. Geometrically, they admit conic bundle fibrations

$$\varphi : \widetilde{V} \to \mathbb{P}^1_x,$$

with four degenerate fibers corresponding to the roots of $P(x)$ and $x = \infty$. Passing to $k = \mathbb{C}(t)$, we obtain smooth projective threefolds with fibrations

$$\mathcal{V}_0 \stackrel{\phi_0}{\to} \mathcal{S}_0 \stackrel{\rho}{\to} \mathbb{P}^1_t. \tag{1}$$

Here $\mathcal{S}_0$ is a smooth projective surface birationally ruled over $\mathbb{P}^1_t$, and $\mathcal{V}_0 \to \mathcal{S}_0$ is a conic fibration degenerate over a curve

$$D_0 = C \cup R \subset \mathcal{S}_0, \tag{2}$$

where C is a trisection and R a section of ρ. (These correspond to $P(x) = 0$ and $x = \infty$ respectively.) The variety $\mathcal{V}_0$ is stably rational over $\mathbb{C}(t)$ and thus over $\mathbb{C}$. The Clemens–Griffiths theory of intermediate Jacobians shows it is often non-rational.

Here are the elements of the construction of a specialization of smooth projective threefolds $\mathcal{X} \rightsquigarrow \mathcal{X}_0$ with $\mathcal{X}_0$ stably rational but $\mathcal{X}$ not stably rational.

First, we use an extension of the class of Châtelet surfaces analyzed in [6]. We consider all degree four del Pezzo surfaces with conic fibrations

$$\widetilde{W} \to \mathbb{P}^1$$

over k, admitting the same Galois structure as above. Precisely, the Galois actions on the Picard groups of $\widetilde{V}$ and $\widetilde{W}$ are equivalent. These depend on *two* parameters rather than the one parameter governing Châtelet surfaces. Nevertheless, the new surfaces remain birational over k to Châtelet surfaces and thus are stably rational.

From now on, take $k = \mathbb{C}(t)$ and seek towers (1) associated with the generalizations of Châtelet surfaces discussed above.

We analyze the possible degeneracy data (2) for our generalized Châtelet surfaces over $\mathbb{C}(t)$, in terms of branched coverings. We take $f : C \to \mathbb{P}^1$ to be an arbitrary simply branched triple cover of genus g, and $p_1, \dots, p_{2\mathsf{g}+4} \in C$ the points residual to ramification points. Consider

$$D_0 = C \cup_{p_i = f(p_i)} R, \quad R \simeq \mathbb{P}^1,$$

where we glue the residual points and their images in $\mathbb{P}^1$. Note the induced degree four morphism $g_0 : D_0 \to \mathbb{P}^1$. The discriminant double cover induces an admissible cover

$$\widetilde{D_0} \to D_0$$

which will encode the irreducible components of the reducible conic fibers.

The third step is to construct embeddings

$$D_0 \hookrightarrow \mathcal{S}_0 \to \mathbb{P}^1$$

of D_0 into a birationally ruled surface that induces g_0. We take $\mathsf{g} = 1$ and $\mathcal{S}_0$ to be the blow up of $\mathbb{P}^2$ at four points, three of whom are collinear. Here $D_0 \in |-2K_{\mathcal{S}_0}|$, i.e., is bi-anticanonical.

The next step is to construct conic bundles $\mathcal{W}_0 \to \mathcal{S}_0$ with the degeneracy (ramification) data $(D_0 \subset \mathcal{S}_0, \widetilde{D_0} \to D_0)$. The existence of such a conic bundle goes back to work of Artin–Mumford [1] and Sarkisov [8]. Indeed, there are many such conic bundles related by explicit birational modifications.

The main technical challenge is to construct these in such a way that everything deforms in families. Indeed, consider pairs $(\mathcal{S}, D)$ where $\mathcal{S}$ is a quintic del Pezzo surface and $D \in |-2K_{\mathcal{S}}|$ is a general bi-anticanoncial divisor. We can clearly specialize

$$(D \subset \mathcal{S}, \widetilde{D} \to D) \rightsquigarrow (D_0 \subset \mathcal{S}_0, \widetilde{D_0} \to D_0)$$

but we would also like conic bundles $\mathcal{X} \to \mathcal{S}$ and $\mathcal{X}_0 \to \mathcal{S}_0$, with the prescribed degeneracy data, such that

$$\mathcal{X} \rightsquigarrow \mathcal{X}_0 \overset{\sim}{\dashrightarrow} \mathcal{W}_0.$$

This is the point where stack-theoretic techniques come into play. These allow us to show that, given ramification data $(D \subset \mathcal{S}, \widetilde{D} \to D)$, we can produce conic bundles lifting these data. We reinterpret a conic bundle over a surface as a rank two vector bundle over a μ_2-gerbe over a root stack. Elementary transformations can be used to modify a rank-two bundle on a surface to one with the same first Chern class and very large second Chern class, but with good deformation properties [7]. We emulate this to construct $\mathcal{X} \rightsquigarrow \mathcal{X}_0$.

The final step is to prove that a very general such $\mathcal{X}$ is not stably rational. The decomposition of the diagonal technique has been implemented for conic bundles over rational surface [3]. For our application, we specialize

$$D \rightsquigarrow D_1 \cup D_2, \quad D_1, D_2 \in |-K_{\mathcal{S}}|$$

to a union of two smooth elliptic curves. It follows that $\mathcal{X}$ fails to admit a decomposition of the diagonal and thus is not stably rational.

References

[1] M. Artin, D. Mumford, *Some elementary examples of unirational varieties which are not rational*, Proceedings of the London Mathematical Society **25** (1972), no. 3, 75–95.

[2] A. Beauville, J.L. Colliot-Thélène, J.J. Sansuc, P. Swinnerton-Dyer, *Variétés stablement rationnelles non rationnelles*, Annals of Mathematics (2) **121** (1985), no. 2, 283-318.

[3] B. Hassett, A. Kresch, Y. Tschinkel, *Stable rationality and conic bundles*, Mathematische Annalen **365** (2016), no. 3-4, 1201–1217.

[4] B. Hassett, A. Kresch, Y. Tschinkel, *Stable rationality in smooth families of threefolds*, arXiv:1802.06107.

[5] B. Hassett, A. Pirutka, Y. Tschinkel, *Stable rationality of quadric surface bundles over surfaces*, Acta Mathematica, to appear, arXiv:1603.09262.

[6] B.É. Kunyavskiĭ, A.N. Skorobogatov, M.A. Tsfasman, *del Pezzo surfaces of degree four*, Mémoires de la Société Mathématique de France **37** (1989), 113 pp.

[7] K.G. O'Grady, *Moduli of vector-bundles on surfaces*, Algebraic geometry–Santa Cruz 1995, 101–126, Proceedings of Symposia in Pure Mathematics **62**, Part 1, American Mathematical Society, Providence, RI, 1997.

[8] V.G. Sarkisov, *On conic bundle structures*, Izv. Akad. Nauk SSSR Ser. Mat. Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya **46** (1982), 371–408.

[9] C. Voisin, *Unirational threefolds with no universal codimension 2 cycle*, Inventiones Mathematicae **201** (2015), no. 1, 207-237.

Extremal syzygies of canonical curves

Michael Kemeny

The equations of curves embedded in projective space have long been studied in algebraic geometry, starting with the work of Noether, Petri, Babbage and others. Since the 80s, particular attention has focused on a conjecture of Mark Green concerning canonically embedded curves $C \subseteq \mathbb{P}^{g-1}$. In modern language, Green's Conjecture predicts that the length of the linear part of the minimal free resolution of the canonical ring

$$\bigoplus_{n\in\mathbb{Z}} H^0(C, \omega_C^{\otimes n})$$

is the difference $g - \mathrm{Cliff}(C) - 2$, where $\mathrm{Cliff}(C)$ is the Clifford index of the curve. In my talk, I discussed a perspective of Schreyer which both explains Green's conjecture and goes much further in predicting the values of the Betti numbers of a canonical curve. The idea is that if a curve C has gonality k and a unique pencil $A \in W^1_k(C)$, then all linear Betti number $b_{p,1}(C, \omega_C)$ should coincide with the Betti numbers of the scroll $X := \bigcup_{D\in|A|} \mathrm{Span}(D) \subseteq \mathbb{P}^{g-1}$, provided p is close enough to the extremal value $g-k$. In particular, one expects $b_{g-k,1}(C,\omega_C) = b_{g-k,1}(X, \mathcal{O}_{\mathbb{P}^{g-1}}(1)) = g-k$, which is a statement strictly stronger than Green's Conjecture for such a curve C.

We further explained a connection between Green's conjecture, Schreyer's conjecture and projections. Let C be a curve of gonality k, and let x, y be points such that there exists $A \in W^1_k(C)$ with $h^0(A(-x-y)) = 1$. A very useful result of Aprodu tells us that if Green's conjecture holds for C, then it holds for the 1-nodal curve D obtained by identifying x and y. We explained how to go in the other direction: if one assumes that Schreyer's conjecture $b_{g(D)-k,1}(D,\omega_D) = g(D)-k$ holds for D, then one obtains that Green's conjecture holds for C. The value of this is that Schreyer's conjecture may hold for D even when $\dim W^1_k(C) \neq 0$, and so one can use this to prove cases of Green's conjecture which have previously been out of reach. In particular, one may use this to prove Green's conjecture for many new cases of curves of even genus and maximum gonality or for covers of elliptic curves.

Hodge ideals for Q-divisors

Mircea Mustaţă
(joint work with Mihnea Popa)

Given a reduced hypersurface Z in a smooth complex algebraic variety X, the *Hodge ideals* of Z form a sequence of ideals $I_p(Z)$, for $p \geq 0$, that are defined using Saito's theory of mixed Hodge modules [5]. Namely, the Hodge filtration on the $\mathcal{D}_X$-module

$$\mathcal{O}_X(*Z) := \bigcup_{m\geq 0} \mathcal{O}_X(mZ)$$

can be written as

$$F_p\mathcal{O}_X(*Z) = I_p(Z) \otimes \mathcal{O}_X\big((p+1)Z\big).$$

These ideals have been studied systematically in [1] and [2]. The first ideal $I_0(Z)$ is a multiplier ideal: it is equal to $\mathcal{I}\big(X, (1-\epsilon)Z\big)$ for $0 < \epsilon \ll 1$. The higher Hodge ideals can be considered as a "higher-level" version of multiplier ideals; in fact, many general properties of multiplier ideals extend to all Hodge ideals. On the other hand, multiplier ideals can be defined for arbitrary effective **Q**-divisors. We explain below, following [3], an extension of the notion of Hodge ideals to effective **Q**-divisors. We also state, following [4], a result describing the Hodge ideals in terms of the V-filtration of Malgrange and Kashiwara.
Let X be a smooth n-dimensional complex algebraic variety and D an effective **Q**-divisor on X. Let us write $D = \alpha H$, for a divisor H and $\alpha \in \mathbf{Q}_{>0}$. Working locally, we may assume that we have a global equation $h \in \mathcal{O}_X(X)$ defining H. We denote by Z the support of H. The key object is the $\mathcal{D}_X$-module

$$\mathcal{M}(h^{-\alpha}) := \mathcal{O}_X(*Z)h^{-\alpha}.$$

This is a free $\mathcal{O}_X(*Z)$-module of rank 1, with the $\mathcal{D}_X$-module structure given by

$$u \cdot h^{-\alpha} = -\frac{\alpha \cdot u(h)}{h} h^{-\alpha} \quad \text{for every} \quad u \in \mathrm{Der}_{\mathbf{C}}(\mathcal{O}_X).$$

It turns out that $\mathcal{M}(h^{-\alpha})$ carries a canonical Hodge filtration. In order to see this, let us fix an integer $\ell \geq 2$ such that $\ell\alpha \in \mathbf{Z}$, and consider the inclusion map $j\colon U = X \setminus Z \hookrightarrow X$ and the finite étale map $p\colon V = \mathcal{S}pec\, \mathcal{O}_U[y]/(y^\ell - h^{-\ell\alpha}) \to U$. It is easy to see that we have an isomorphism of $\mathcal{D}_X$-modules

$$j_+ p_+ \mathcal{O}_V = \bigoplus_{0 \leq i \leq \ell-1} \mathcal{M}(h^{-i\alpha}). \tag{1}$$

We make a parenthesis to recall that a mixed Hodge module M on X in the sense of [5] consists of the following data:

1) A $\mathcal{D}_X$-module $\mathcal{M}$ on X (which is holonomic, with regular singularities).
2) A good filtration $F_\bullet \mathcal{M}$ on $\mathcal{M}$, with respect to the order filtration on $\mathcal{D}_X$ (this is the Hodge filtration on $\mathcal{M}$).
3) A $\mathbf{Q}$-structure on M (given by a perverse sheaf $\mathcal{P}$ of $\mathbf{Q}$-vector spaces on X, together with an isomorphism $\alpha\colon \mathcal{P}_{\mathbf{C}} \simeq \mathrm{DR}_X(\mathcal{M})$).
4) A weight filtration on M.

This data is supposed to satisfy a complicated set of conditions of an inductive nature. When X is a point, a mixed Hodge module on X is the same as a mixed Hodge structure.
A fundamental example of a mixed Hodge on X is $\mathbf{Q}_X^H[n]$. In this case the $\mathcal{D}_X$-module is $\mathcal{O}_X$, with the filtration given by $F_p\mathcal{O}_X = \mathcal{O}_X$ for $p \geq 0$ and $F_p\mathcal{O}_X = 0$ for $p < 0$. The $\mathbf{Q}$-structure is provided by the perverse sheaf $\mathbf{Q}_X[n]$.
Saito defined a push-forward functor for mixed Hodge modules which, at the level of the underlying $\mathcal{D}_X$-modules, is the usual $\mathcal{D}$-module push-forward. In particular, the left-hand side of (1) underlies the mixed Hodge module $j_+p_+\mathbf{Q}_V^H[n]$. This implies that we have a canonical filtration on the right-hand side of (1). One can show that this is the direct sum of the filtrations induced on each of the summands. We thus obtain a canonical good filtration on $\mathcal{M}(h^{-\alpha})$ (it is easy to see that this is independent of ℓ).
We compute explicitly this filtration when Z is smooth. By choosing an open subset W of X such that $\mathrm{codim}_X(X \setminus W) \geq 2$ and $W \cap Z$ is smooth, we see that for every $p \geq 0$, we have ideals $I_p(D)$, the *Hodge ideals* of D, such that

$$F_p\mathcal{M}(h^{-\alpha}) = I_p(D) \otimes \mathcal{O}_X(pZ)h^{-\alpha}.$$

We show that the Hodge ideals are independent of the choice of H and of the local equation for H.

Given a log resolution of (X, D) that is an isomorphism over U, we compute the ideals $I_p(D)$ as the derived push-forward of a suitable twisted De Rham complex with log poles on the resolution. A first consequence of this computation is that the first Hodge ideal is the multiplier ideal of a small perturbation of D:

$$I_0(D) = \mathcal{I}\big(X, (1-\epsilon)D\big) \quad \text{for} \quad 0 < \epsilon \ll 1.$$

Several properties proved in [1] and [2] for reduced divisors extend to the case of $\mathbf{Q}$-divisors. For example, we have a necessary and sufficient criterion for the Hodge filtration on $\mathcal{M}(h^{-\alpha})$ to be generated at level q. In particular, we see that it is always generated at level $n-1$. We also prove extensions of the restriction and semicontinuity theorems to this setting. Finally, we prove a vanishing result. Here are some peculiar features of Hodge ideals for $\mathbf{Q}$-divisors:

1) In general, the ideals $I_k(\alpha H)$, for various $\alpha \in \mathbf{Q}_{>0}$, are not comparable (this is in contrast with the case of multiplier ideals). For example, if H is the divisor in $\mathbf{A}^2$ defined by x^2+y^3, then for $\frac{5}{6} < \alpha \leq 1$, we have
$$I_2(\alpha H) = \big(x^3, x^2y^2, xy^3, y^4 - (2\alpha+1)x^2y\big).$$
2) For $p \geq 1$, the support Z plays an important role in the behavior of the Hodge ideals. For example, if $\mathrm{mult}_x(Z) \geq 2 + \frac{n}{p}$ for some $x \in X$, then $I_p(\alpha H)_x \neq \mathcal{O}_{X,x}$ for all $\alpha > 0$ (again, this is in contrast with the behavior for multiplier ideals).

We end by stating a complete description of Hodge ideals in terms of the V-filtration. For the sake of simplicity, we assume that $D = \alpha Z$, where Z is a reduced divisor. Suppose that $h \in \mathcal{O}_X(X)$ is a global equation for Z and let $i\colon X \hookrightarrow X \times \mathbf{A}^1$ be the graph embedding $i(x) = \big(x, h(x)\big)$. Recall that the V-filtration is a filtration on
$$B_h := i_+\mathcal{O}_X = \mathcal{O}_X[t]_{h-t}/\mathcal{O}_X[t].$$
Note that B_h is a $\mathcal{D}_{X\times \mathbf{A}^1}$-module, that can be written as
$$B_h = \bigoplus_{j\geq 0} \mathcal{O}_X \cdot \partial_t^j \delta,$$
where δ is the class of $\frac{1}{h-t}$. The V-filtration $(V^\alpha B_h)_{\alpha\in\mathbf{Q}}$ was constructed by Malgrange, and then extended by Kashiwara, in order to describe the nearby cycles of h. Its construction makes use of the existence of the Bernstein–Sato polynomial of h and the rationality of its roots.

Theorem 1. For every non-negative integer p and every $\alpha \in \mathbf{Q}_{>0}$, we have
$$I_p(\alpha Z) = \left\{ \sum_{j=0}^{p} Q_j(\alpha) h^{p-j} v_j \,\Big|\, \sum_{j=0}^{p} v_j \partial_t^j \delta \in V^\alpha B_h \right\},$$
where $Q_j(x) = x(x+1)\cdots(x+j-1)$. In particular, we have
$$I_p(\alpha Z) + (h) = \widetilde{I}_p(\alpha Z) + (h),$$
where $\widetilde{I}_p(\alpha Z)$ consists of those sections v of $\mathcal{O}_X$ with the property that there are $v_0, \ldots, v_{p-1}, v_p = v$ such that $\sum_{j=0}^{p} v_j \partial_t^j \delta \in V^\alpha B_h$.

For $\alpha = 1$, the last assertion in the theorem was proved by Saito in [6], by making use of the compatibility between the Hodge filtration and the V-filtration on B_f. We rely on similar arguments to prove the above statement. Another key ingredient is a result relating the V-filtration on B_f and that on $i_+\mathcal{M}(h^{-\alpha})$.

The theorem has some interesting consequences concerning the behavior of Hodge ideals. For example, we obtain a necessary and sufficient criterion for having $I_p(\alpha Z) = \mathcal{O}_X$ in terms of the roots of the Bernstein–Sato polynomial of h (see [4]).

References

[1] M.Mustaţă and M. Popa, *Hodge ideals*, Memoirs of the AMS, to appear.

[2] M.Mustaţă and M. Popa, *Restriction, subadditivity, and semicontinuity theorems for Hodge ideals*, Int. Math. Res. Not. IMRN 2018, no. 11, 3587–3605.

[3] M.Mustaţă and M. Popa, *Hodge ideals for* **Q**-*divisors I: birational approach*, preprint, arXiv:1807.01932.

[4] M.Mustaţă and M. Popa, *Hodge ideals for* **Q**-*divisors II: V-filtration*, preprint, arXiv:1807.01935.

[5] M. Saito, *Mixed Hodge modules*, Publ. Res. Inst. Math. Sci. **26** (1990), 221–333.

[6] M. Saito, *Hodge ideals and microlocal V-filtration*, preprint, arXiv:1612.08667.

The "top-heavy" conjecture and the Kazhdan–Lusztig theory of matroids

Botong Wang

(joint work with Tom Braden, June Huh, Jacob Matherne, Nick Proudfoot)

The "top-heavy" conjecture ([2]) in matroid theory predicts the following.

Conjecture 1. *Let M be a rank d matroid, and let W_k be the number of rank k flats of M. Then $W_k \leq W_{d-k}$, for any $k \leq \frac{d}{2}$.*

For realizable matroids, the conjecture is proved in [3]. The key step in the proof is to use the hard Lefschetz theorem of the intersection cohomology groups of a singular variety Y_M. An open set of the same variety Y_M (called the reciprocal plane) was studied in [1], where the authors introduced the notion of Kazhdan–Lusztig polynomial $P_M(t)$ of a matroid M. It was proved that for a realizable matroid M, the Kazhdan–Lusztig polynomial is equal to the Poincaré polynomial of certain (local) intersection complex of Y_M. More precisely, there exists a point $x \in Y_M$ such that

$$P_M(t) = \sum_i \dim IH^i_x(Y, \overline{\mathbb{Q}}_l) \cdot t^i.$$

Corollary 2. *For any realizable matroid M, the coefficients of the Kazhdan–Lusztig polynomial $P_M(t)$ are nonnegative.*

The above consequence was conjectured to be true for arbitrary matroids.

Conjecture 3. *For any matroid M, the coefficients of $P_M(t)$ are nonnegative.*

In an on-going project, we are working on the proof of the above two conjectures for non-realizable matroids. The problem is similar to understanding the Soergel bimodules in the Kazhdan–Lusztig theory of Coxeter groups. There exists analogous Bott–Samelson resolution both in the realizable and non-realizable case. This allows us to combinatorially define the intersection cohomology groups as subspaces of the cohomology groups of the analogous Bott–Samelson resolution. The key problem is to prove the Kähler package, i.e., Poincaré duality, hard Lefschetz property and Hodge Riemann relation, of the combinatorially defined intersection cohomology groups.

We would also like to compare the variety Y_M with the classical Schubert varieties, and compare the Kazhdan–Lusztig polynomial of matroid with the classical one. We will refer to the survey paper [4] for more known results and conjectures.

References

[1] B. Elias, N. Proudfoot, M. Wakefield, *The Kazhdan–Lusztig polynomial of a matroid*, Adv. Math. **299** (2016), 36–70.

[2] T. Dowling, R. Wilson, *Whitney number inequalities for geometric lattices*, Proc. Amer. Math. Soc. **47** (1975), 504–512.

[3] J. Huh, B. Wang, *Enumeration of points, lines, planes, etc.* Acta Math. 218 (2017), no. **2**, 297–317.

[4] K. Gedeon, N. Proudfoot, B. Young, *Kazhdan–Lusztig polynomials of matroids: a survey of results and conjectures*, Sém. Lothar. Combin. **78B** (2017), Art. 80

On 5-dimensional minifolds and their Hilbert schemes of lines

Alexander Kuznetsov

Definition 1. A minifold is a smooth Fano variety X which is "cohomologically minimal", i.e., one of the following conditions hold:

(1) $\dim H^\bullet(X,\mathbb{Q}) = n+1$, or
(2) $K_0(X) \cong \mathbb{Z}^{n+1}$, or
(3) $\mathrm{D}^{\mathrm{b}}(X) = \langle E_0, E_1, \dots, E_n\rangle$, an exceptional collection of length $n+1$,

where $n = \dim X$.

This definition was introduced in [2]. Note that (3) $\Rightarrow$ (2) $\Rightarrow$ (1), so the last condition is the strongest, while the first is the easiest to check.

So far, only two series and five sporadic examples of minifolds are known. All of them are listed in the next table:

	n	index	exceptional collection
$\mathbb{P}^n$	any	$n+1$	$\langle \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n)\rangle$
Q^n	odd	n	$\langle \mathcal{S}, \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n-1)\rangle$
$V_5^3 = \mathrm{Gr}(2,5)\cap\mathbb{P}^6$	3	2	$\langle \mathcal{O}, \mathcal{U}_2^\vee, \mathcal{O}(1), \mathcal{U}_2^\vee(1)\rangle$
$V_{22}^3 = \mathrm{IGr}_{\omega_1,\omega_2,\omega_3}(3,7)$	3	1	$\langle \mathcal{O}, \mathcal{U}_3^\vee, \mathcal{U}_2^\vee, \wedge^2\mathcal{U}_3^\vee\rangle$
$V_{18}^5 = \mathrm{G}_2^{\mathrm{ad}} \subset \mathrm{Gr}(2,7)$	5	3	$\langle \mathcal{O}, \mathcal{U}_2^\vee, \mathcal{O}(1), \mathcal{U}_2^\vee(1), \mathcal{O}(2), \mathcal{U}_2^\vee(2)\rangle$
$V_{16}^5 = \mathrm{LGr}(3,6)\cap H$	5	3	$\langle \mathcal{O}, \mathcal{U}_3^\vee, \mathcal{O}(1), \mathcal{U}_3^\vee(1), \mathcal{O}(2), \mathcal{U}_3^\vee(2)\rangle$
$V_{12}^5 = \mathrm{OGr}_+(5,10)\cap\mathbb{P}^{10}$	5	3	$\langle \mathcal{O}, \mathcal{U}_5^\vee, \mathcal{O}(1), \mathcal{U}_5^\vee(1), \mathcal{O}(2), \mathcal{U}_5^\vee(2)\rangle$

Here $\mathrm{Gr}(k,m)$ denotes the Grassmannian of linear k-subspaces in a vector space of dimension m, $\mathrm{LGr}(m,2m)$ denotes the Lagrangian Grassmannians of a symplectic vector space, and $\mathrm{OGr}_+(m,2m)$ denotes (one of the two connected components of) the isotropic Grassmannian of a vector space with a non-degenerate quadratic form. Finally, $\mathrm{IGr}_{\omega_1,\omega_2,\omega_3}$ stands for the subvariety of the Grassmannian, parameterizing 3-subspaces isotropic with respect to a general triple of skew-forms, and $\mathrm{G}_2^{\mathrm{ad}}$ is the adjoint Grassmannian of the group G_2. Furthermore, $\mathcal{S}$ stands for the spinor bundle on the quadric, while $\mathcal{U}_k$ stands for the tautological bundle (of rank k) on the Grassmannian.

The varieties V_{22}^3 and V_{12}^5 have moduli spaces of dimensions 6 and 10 respectively, while all the other varieties are rigid.

It is an interesting question to find more examples of minifolds. Note that in the dimensions up to 3 the above table is complete, since classification of Fano varieties in these dimensions is known and easy to check. Furthermore, in dimension 4 the only possible minifold satisfying (3) is $\mathbb{P}^4$ (see [2, Section 2]). But starting from dimension 5 nothing is known except the examples listed in the table.

A simple computation shows that for a minifold X of dimension n and index 3 the Hilbert scheme of lines $F(X)$ on X also has dimension n. Among those in the table, these are $\mathbb{P}^2$, Q^3, and three 5-folds V_{18}^5, V_{16}^5, and V_{12}^5. The first three of these minifolds are homogeneous, and it is easy to see that

$$F(\mathbb{P}^2) \cong \mathbb{P}^2, \qquad F(Q^3) \cong \mathbb{P}^3, \qquad F(V_{18}^5) \cong Q^5.$$

In particular, all these Hilbert schemes are also minifolds! Furthermore, this phenomenon persists on the next step.

Theorem 2 ([3, Corollary 6.7]). *We have $F(V_{16}^5) \cong V_{18}^5$.*

So, it is intriguing to get a description of $F(V_{12}^5)$. The following result is known:

Theorem 3 ([6]). *For a general variety V_{12}^5 there is a cubic threefold $Y \subset \mathbb{P}^4$ such that the Hilbert scheme of lines $F(V_{12}^5)$ is the variety of sums of powers for Y:*

$$F(V_{12}^5) \cong \mathrm{VSP}(Y, 8)$$

Moreover, $F(V_{12}^5)$ is a Fano variety of index 1 and degree 660.

Recall the notion of Gushel–Mukai varieties from [1]. The main result discussed in this talk is the following

Theorem 4. *For a general variety V_{12}^5 there is a smooth Gushel–Mukai 5-fold $R = \mathrm{Gr}(2,5) \cap Q$ and a diagram*

$$\begin{array}{ccc} \widetilde{F}(V_{12}^5) & \dashleftarrow\dashrightarrow & \widetilde{R} \\ \downarrow & & \downarrow \\ F(V_{12}^5) & & R \end{array}$$

where the vertical arrows are $\mathbb{P}^1$-bundles and the horizontal arrow is a flop.

The proof of the theorem is based on the study of geometry of linear sections of $\mathrm{OGr}_+(5,10)$ that was performed in [4]. In particular, the crucial role is played by the spinor quadratic line complex introduced there — the Gushel–Mukai 5-fold R in the theorem is a section of this complex.

Corollary 5. *The Hilbert scheme of lines $F(V_{12}^5)$ is not a minifold.*

In fact, the top half of the Hodge diamond of $F(V_{12}^5)$ looks like

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 0 & & 0 & & & & \\ & & & 0 & & 1 & & 0 & & & \\ & & 0 & & 0 & & 0 & & 0 & & \\ & 0 & & 0 & & 2 & & 0 & & 0 & \\ 0 & & 0 & & 10 & & 10 & & 0 & & 0 \end{array}$$

and the results of [5] suggest that $\mathrm{D^b}(F(V_{12}^5))$ has a semiorthogonal decomposition with 6 exceptional objects and a Gushel–Mukai category of R as components.

References

[1] Debarre, Olivier; Kuznetsov, Alexander, *Gushel–Mukai varieties: classification and birationalities*, Algebr. Geom. 5 (2018), no. 1, 15–76.

[2] S. Galkin, L. Katzarkov, A. Mellit, E. Shinder, *Minifolds and phantoms*, arXiv preprint arXiv:1305.4549.

[3] A. Kuznetsov, *Küchle fivefolds of type c5*, Math. Z. 284 (2016), no. 3-4, 1245–1278.

[4] A. Kuznetsov, *On linear sections of the spinor tenfold, I*, Izv. RAN. Ser. Mat., 82:4 (2018), arXiv preprint arXiv:1801.00037.

[5] A. Kuznetsov, A. Perry, *Derived categories of Gushel–Mukai varieties*, Compos. Math. 154 (2018), no. 7, 1362–1406.

[6] K. Ranestad, F.-O. Schreyer, *Varieties of sums of powers*, J. Reine Angew. Math. 525 (2000), 147–181.

Special subvarieties in holomorphic symplectic varieties

Qizheng Yin

(joint work with Georg Oberdieck, Junliang Shen, Xiaolei Zhao)

Holomorphic symplectic varieties form one of the three building blocks of varieties with trivial canonical bundle (the other two being abelian varieties and Calabi–Yau varieties). The study of these varieties has attracted much attention within recent years.

A smooth projective complex variety X is called *holomorphic symplectic* (equivalently, *hyper-Kähler*) if

- $\pi_1(X) = 0$;
- $H^2(X, \Omega_X^2) = \mathbb{C}\sigma$, where σ is a nowhere degenerate holomorphic 2-form.

In dimension 2, this defines a $K3$ surface. Higher-dimensional examples include the Hilbert schemes of points on a $K3$ surface, the generalized Kummer varieties associated to an abelian surface (both due to Beauville), and two sporadic families constructed by O'Grady, one in dimension 10 and the other in dimension 6. Moreover, deformations of holomorphic symplectic varieties remain holomorphic symplectic.

In this report, we discuss the search of subvarieties in a holomorphic symplectic variety which are special with respect to the holomorphic 2-form. Let X, σ be as above with $\dim X = 2n$. The following notion was introduced by Voisin in [15]. A subvariety $Z \subset X$ of codimension i is called *algebraically coisotropic* if Z admits a dominant rational map

$$p : Z \dashrightarrow B$$

with general fibers of dimension i and such that

$$\sigma|_Z = p^*\sigma_B \tag{1}$$

for some holomorphic 2-form σ_B on B. Roughly speaking, this means that the tangent space $T_{Z,z}$ at each smooth point $z \in Z$ is a coisotropic subspace of $T_{X,z}$ with respect to σ, and that the induced foliation on Z is algebraically integrable. When $i = n$, this defines a *Lagrangian* subvariety.

A even more algebraic notion replaces (1) by the requirement that the general fibers of $p : Z \dashrightarrow B$ are *constant cycle* subvarieties of X, *i.e.*, subvarieties whose points share the same class in the Chow group $\mathrm{CH}_0(X)$. By Mumford's theorem, this is indeed a stronger condition. We shall refer to such $Z \subset X$ as algebraically coisotropic subvarieties with constant cycle fibers, or simply, *special* subvarieties. For example, every curve in a $K3$ surface is Lagrangian, yet the constant cycle curves are rare and studied in detail in [3].

The main question concerns the existence of special subvarieties.

Conjecture 1 (Voisin [15])**.** *Let X be holomorphic symplectic of dimension $2n$. For all $0 \leq i \leq n$, there exist special subvarieties $Z \subset X$ in codimension i.*

Conjecture 1 is motivated by the Beauville–Voisin study of Chow rings of holomorphic symplectic varieties; see [1, 14]. Roughly speaking, the Beauville–Voisin conjectures predict a filtration on $\mathrm{CH}^*(X)$ which is opposite to the conjectural Bloch–Beilinson filtration, resulting in a multiplicative decomposition of $\mathrm{CH}^*(X)_{\mathbb{Q}}$. In the case of 0-cycles, a candidate of the Beauville–Voisin filtration

$$S_0\mathrm{CH}_0(X) \subset S_1\mathrm{CH}_0(X) \subset \cdots \subset S_n\mathrm{CH}_0(X) = \mathrm{CH}_0(X)$$

is given by

$$S_i\mathrm{CH}_0(X) = \{\, \text{0-cycle classes supported on special subvarieties } Z \subset X \text{ of codimension} \geq n - i \,\}.$$

The existence and abundance of special subvarieties is needed to justify this proposal; *c.f.* [15].

Conjecture 1 remains wide open. Previous results were obtained in [2, 4, 5, 6, 7, 9, 10, 15]. We state two new results in this direction, one via sheaf theory and the other via the study of rational curves.

Theorem 2 ([13], Marian–Zhao [8])**.** *Let X be a smooth projective moduli space of stable sheaves (or complexes) on a $K3$ surface. Then Conjecture 1 holds for X in all codimensions.*

By a result of Mukai, such a moduli space is holomorphic symplectic of $K3^{[n]}$ type. Theorem 2 thus verifies Conjecture 1 for a large class of 19-dimensional families of polarized holomorphic symplectic varieties. In [12], we also indicated how to establish Conjecture 1 for the moduli spaces of stable objects in the Kuznetsov category of a cubic 4-fold (conditionally on a conjecture).

We turn to the rational curve approach. Let X be holomorphic symplectic of dimension $2n$, which we assume to be *very general*. Consider the moduli space M of rational curves in the *primitive* curve class of X. Below are some useful facts about M; *c.f.* [11].

- The space M is pure of the expected dimension $2n - 2$.
- Let $\mathcal{C} \to M$ be the universal curve with $\phi : \mathcal{C} \to X$ the natural map. Then there is a decomposition

$$M = M^1 \cup M^2 \cup \cdots \cup M^n$$

into components such that $Z^i = \phi(\mathcal{C}|_{M^i}) \subset X$ is of codimension i.
- If nonempty, each irreducible component of $Z^i \subset X$ is special via the MRC fibration.

With Conjecture 1 in mind, it is tempting to ask the following question.

Question 3 (Mongardi–Pacienza [10])**.** *It is true that $M^i \neq \emptyset$ for all $1 \leq i \leq n$?*

Theorem 4 ([11])**.**

- *For a very general holomorphic symplectic 4-fold of $K3^{[2]}$ type of polarization degree 6 and divisibility 2, we have $M^2 = \emptyset$.*
- *There exist very general holomorphic symplectic varieties of $K3^{[8]}$ type satisfying $M^1 = \emptyset$.*

The proof of Theorem 4 uses a combination of classical geometry and the (reduced) Gromov–Witten theory of holomorphic symplectic varieties. In conclusion, the structure of M remains mysterious.

References

[1] A. Beauville, *On the splitting of the Bloch–Beilinson filtration.* Algebraic cycles and motives. Vol. 2, 38–53, London Math. Soc. Lecture Note Ser., **344**, Cambridge Univ. Press, Cambridge, 2007.

[2] F. Charles, G. Pacienza, *Families of rational curves on holomorphic symplectic varieties and applications to 0-cycles.* arXiv:1401.4071v2.

[3] D. Huybrechts, *Curves and cycles on K3 surfaces.* With an appendix by C. Voisin. Algebr. Geom. **1** (2014), no. 1, 69–106.

[4] A. L. Knutsen, M. Lelli-Chiesa, G. Mongardi, *Wall divisors and algebraically coisotropic subvarieties of irreducible holomorphic symplectic manifolds.* Trans. Amer. Math. Soc., to appear.

[5] Ch. Lehn, G. Pacienza, *Stability of coisotropic fibrations on holomorphic symplectic manifolds.* Ann. Sc. Norm. Super. Pisa Cl. Sci., to appear.

[6] H.-Y. Lin, *On the Chow group of zero-cycles of a generalized Kummer variety.* Adv. Math. **298** (2016), 448–472.

[7] H.-Y. Lin, *Lagrangian constant cycle subvarieties in Lagrangian fibrations.* Int. Math. Res. Not. IMRN, to appear.

[8] A. Marian, X. Zhao, *On the group of zero-cycles of holomorphic symplectic varieties.* arXiv:1711.10045v2.

[9] G. Mongardi, G. Pacienza, *Polarized parallel transport and uniruled divisors on deformations of generalized Kummer varieties.* Int. Math. Res. Not. IMRN (2018), no. 11, 3606–3620.
[10] G. Mongardi, G. Pacienza, *Density of Noether–Lefschetz loci of polarized irreducible holomorphic symplectic varieties and applications.* arXiv:1804.09440v1.
[11] G. Oberdieck, J. Shen, Q. Yin, *Rational curves in the Fano varieties of cubic 4-folds and Gromov–Witten invariants.* arXiv:1805.07001v2.
[12] J. Shen, Q. Yin, *$K3$ categories, one-cycles on cubic fourfolds, and the Beauville–Voisin filtration.* arXiv:1712.07170v1.
[13] J. Shen, Q. Yin, X. Zhao, *Derived categories of $K3$ surfaces, O'Grady's filtration, and zero-cycles on holomorphic symplectic varieties.* arXiv:1705.06953v2.
[14] C. Voisin, *On the Chow ring of certain algebraic hyper-Kähler manifolds.* Pure Appl. Math. Q. **4** (2008), no. 3, part 2, 613–649.
[15] C. Voisin, *Remarks and questions on coisotropic subvarieties and 0-cycles of hyper-Kähler varieties.* $K3$ surfaces and their moduli, 365–399, Progr. Math., **315**, Birkhäuser/Springer, Cham, 2016.

Stable cohomology of complements of discriminants

Orsola Tommasi

The term *discriminant* refers in general to the locus of degenerate elements in a family of varieties. An easy example is given by the vector space $V = \mathbf{C}[x_0, \ldots, x_n]_d$ of homogeneous polynomials of degree d in a fixed number of variables. If we regard it as a parameter space for degree d hypersurfaces in $\mathbf{P}^n$, we can define the discriminant as the Zariski closed subset

$$\Sigma = \{f \in V \mid f \text{ is singular}\}.$$

Then the complement $X = V \setminus \Sigma$ parametrizes smooth degree d hypersurfaces in $\mathbf{P}^n$ (up to scaling).

A natural generalization of this situation is the following.

Definition 1. Let M be a non-singular projective variety and L a very ample line bundle on M. Let us denote by $V_{(M,L),d}$ the space $\Gamma(M, L^{\otimes d})$ of global sections of $L^{\otimes d}$. The *discriminant* $\Sigma_{(M,L),d}$ is the Zariski closed subset of $V_{(M,L),d}$ consisting of singular sections, i.e. of the sections whose zero locus is a singular divisor of M together with the zero section. The complement of the discriminant will be denoted by $X_{(M,L),d} := V_{(M,L),d} \setminus \Sigma_{(M,L),d}$.

From a geometric point of view, the complement $X_{(M,L),d}$ is the locus parametrizing the non-singular sections of $L^{\otimes d}$. It is natural to wonder about the dependency of the geometry of $X_{(M,L),d}$ on d. More precisely, we are interested in one of its main topological invariants, the cohomology with rational coefficients.

Question 2. *Is the kth cohomology group $H^k(X_{(M,L),d}; \mathbf{Q})$ independent of d if k is sufficiently large with respect to k?*

This question can be rephrased as asking whether the rational cohomology of $X_{(M,L),d}$ stabilizes with respect to d. An affirmative answer to the question is a special case of [VW15, Conjecture B]. One of the main results of [VW15] also provides powerful evidence for this conjecture, since it proves that the class of the complement of the discriminant in the Grothendieck ring of varieties stabilizes in a suitable sense. Moreover, [VW15] also provides an explicit formula for the limit class in a certain completion of a localization of the Grothendieck ring.

Our main result is an affirmative answer to the question above.

Theorem 3. *For every (M, L) as above and $k \geq 1$, there exists an index d_k with $1 \leq d_k \leq k(\dim M) + 1$ such that for every two integers $d \geq d' \geq d_k$ there exists a natural isomorphism $H^k(X_{(M,L),d}; \mathbf{Q}) \to H^k(X_{(M,L),d'}; \mathbf{Q})$.*

One of the reasons why the question above is tricky is that there is no natural map from $X_{(M,L),d}$ to $X_{(M,L),d'}$ for $d' > d$. Therefore, it is not too clear how an isomorphism between their cohomology groups should arise. On the other hand, tensoring with a fixed non-zero element of $V_{(M.L),d'-d}$ will define a map $\Sigma_{(M,L),d} \to \Sigma_{(M,L),d'}$. However, the topology of the discriminant and that of its complement are strictly related and working with the discriminant is often more suitable for explicit results. This is an approach that goes back to Arnold ([Arn70]) and which is also the basis for Vassiliev's method for computing the cohomology of complements of discriminants ([Vas92, Vas99, Gor05]).

The proof of Theorem 3 relies on an adaptation of Vassiliev's method which is inspired by the construction of mixed Hodge structures on singular spaces given in [GNAPGP88] and [PS08, §5]. A similar approach was already used by the author to prove stabilization in the easier case of degree d hypersurfaces in projective space.

Theorem 4 ([Tom14])**.** *Let $(M, L) = (\mathbf{P}^n, \mathcal{O}(1))$. Then the cohomology with rational coefficients of $X_{(\mathbf{P}^n,\mathcal{O}(1)),d}$ stabilizes in degree $k < \frac{d+1}{2}$. In this range, it is isomorphic to the cohomology of $\mathrm{GL}_{n+1}(\mathbf{C})$ considered as a topological space.*

Let us remark that in view of the Leray spectral sequence of the quotient map, Theorem 4 implies that the the rational cohomology of the moduli space of degree d hypersurfaces $X_{(\mathbf{P}^n,\mathcal{O}(1)),d}/\mathrm{GL}_{n+1}(\mathbf{C})$ vanishes in degree k for $0 < k < \frac{d+1}{2}$, provided d is at least 3.

We would like to finish with a remark on the boundedness of stable cohomology. In the case $M = \mathbf{P}^n$, the previous theorem implies that the stable cohomology of $X_{(M,L),d}$ is 0 if the degree is larger than $(n+1)^2$. The proof of the main theorem also implies that stable cohomology is going to be bounded if M is a union of cells isomorphic to affine spaces $\mathbf{A}^m$. However, in most cases the stable cohomology of $X_{(M,L),d}$ is going to be non-trivial in arbitrarily high degree. In particular, the proof of Theorem 3 suggests that this happens when M has non-trivial cohomology in odd degree. The easiest example of this is the case of curves.

Theorem 5. *Let us assume M is a curve and denote by $\mathbf{V}_k := \mathrm{Sym}^k H^1(M; \mathbf{Q})$ the symmetric products of the middle cohomology of M. Then in the stable range (i.e. for all k such that $d_{2k+1} > d$) the cohomology of $X_{(M,L),d}$ together with its* *mixed Hodge structures is given by*

$$H^{2k}(X_{(M,L),d};\mathbf{Q}) = \mathbf{V}_k(-k) \oplus \mathbf{V}_{k-2}(-k-1)$$
$$H^{2k+1}(X_{(M,L),d};\mathbf{Q}) = \mathbf{V}_{k-1}(-k-1) + \mathbf{V}_k(-k-1),$$

where the notation $(-k)$ denotes the twist by the Tate Hodge structure $\mathbf{Q}(-k)$.

References

[Arn70] V. I. Arnold. On some topological invariants of algebraic functions. *Trans. Moscow Math. Soc.*, 21:27–46, 1970.

[GNAPGP88] F. Guillén, V. Navarro Aznar, P. Pascual Gainza, and F. Puerta. *Hyperrésolutions cubiques et descente cohomologique*, volume 1335 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1988. Papers from the Seminar on Hodge–Deligne Theory held in Barcelona, 1982.

[Gor05] A. G. Gorinov. Real cohomology groups of the space of nonsingular curves of degree 5 in $\mathbf{C}P^2$. *Ann. Fac. Sci. Toulouse Math. (6)*, 14(3):395–434, 2005.

[PS08] C. A. M. Peters and J. H. M. Steenbrink. *Mixed Hodge structures*, volume 52 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer–Verlag, Berlin, 2008.

[Tom14] O. Tommasi. Stable cohomology of spaces of non-singular hypersurfaces. *Adv. Math.*, 265:428–440, 2014.

[Vas92] V. A. Vassiliev. *Complements of discriminants of smooth maps: topology and applications*, volume 98 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1992. Translated from the Russian by B. Goldfarb.

[Vas99] V. A. Vasil′ev. How to calculate the homology of spaces of nonsingular algebraic projective hypersurfaces. *Tr. Mat. Inst. Steklova*, 225(Solitony Geom. Topol. na Perekrest.):132–152, 1999.

[VW15] R. Vakil and M. M. Wood. Discriminants in the Grothendieck ring. *Duke Math. J.*, 164(6):1139–1185, 2015.

A real period–index theorem

Olivier Benoist

A field K is said to be C_i if every degree d hypersurface $X \subset \mathbb{P}^N_K$ with $d^i \leq N$ has a K-point. A key example of C_i fields is given by the Tsen–Lang theorem [5].

Theorem 1 (Tsen–Lang)**.** *If B is an integral complex variety of dimension i, the field $\mathbf{C}(B)$ is C_i.*

Geometrically, this means that hypersurface fibrations over complex varieties have a rational section, if the degree of the hypersurfaces is low enough. When the base B is a curve, the inequality $d \leq N$ exactly means that the hypersurfaces are in the Fano range. This suggests that one might expect a more general statement, for rationally connected fibrations: this is the Graber–Harris–Starr theorem [4].

Theorem 2 (Graber–Harris–Starr)**.** *If C is an integral complex curve, every rationally connected variety X over $\mathbf{C}(C)$ has a $\mathbf{C}(C)$-point.*

Both theorems fail badly if **C** is replaced by the field $\mathbb{R}$ of real numbers, as some real points of the base might not lift to real points of the total space of the fibration. For instance, the hypersurface $X := \{X_0^2 + \cdots + X_N^2 = 0\} \subset \mathbb{P}^N_{\mathbb{R}}$ over $B = \mathrm{Spec}(\mathbb{R})$ has no $\mathbb{R}$-point. Lang suggested in [6, p. 379] that one might still obtain correct statements if the base has no real points (thus expressing the hope that real varieties with no real points behave as complex varieties).

Conjecture 3 (Lang)**.** *If B is an integral real variety of dimension i such that $B(\mathbb{R}) = \emptyset$, the field $\mathbb{R}(B)$ is C_i.*

By analogy with the complex situation, when B is a curve, it is natural to formulate a real variant of the Graber–Harris–Starr theorem:

Conjecture 4 (Manin, Kollár)**.** *If C is an integral real curve with $C(\mathbb{R}) = \emptyset$, every rationally connected variety X over $\mathbb{R}(C)$ has a $\mathbb{R}(C)$-point.*

Applying Conjecture 4 when the base C is the real conic with no real points, and when X is defined over $\mathbb{R}$ (i.e. when the fibration is trivial) would answer positively the following question of Kollár:

Conjecture 5 (Kollár)**.** *Every rationally connected variety over $\mathbb{R}$ contains a geometrically integral curve of geometric genus 0.*

Very little is known concerning these conjectures. Lang had answered positively Conjecture 3 for odd degree hypersurfaces [6, p. 390], mimicking the proof of Theorem 1 and taking advantage of the fact real polynomials of odd degree have a real root. Conjecture 4 has been answered positively by Steinberg [8] for compactifications of varieties that are homogeneous under the action of a connected linear algebraic group. The case of conics (that is Conjecture 3 for $i = 1$ and $d = 2$) was already known to Witt [9, Satz 22]. Steinberg's theorem has nothing to do with real algebraic geometry: it remains valid if one replaces the function field of a real curve with no real points with any field of cohomological dimension 1.

Our goal is to solve new cases of Conjectures 3 and 4, providing evidence for their validity beyond Lang's and Steinberg's results.

Theorem 6. *Let S be a real surface such that $S(\mathbb{R}) = \emptyset$. Then every quadric of dimension ≥ 3 over $\mathbb{R}(S)$ has a $\mathbb{R}(S)$-point.*

Theorem 7. *Let C be a real curve such that $C(\mathbb{R}) = \emptyset$. Then every degree 4 del Pezzo surface over $\mathbb{R}(C)$ has a $\mathbb{R}(C)$-point.*

Theorem 6 is Conjecture 3 for $i = d = 2$. Theorem 7 follows at once from Theorem 6, by applying the Amer–Brumer theorem [1, Théorème 1]: a degree 4 del Pezzo surface over K has a rational point if and only if the pencil of quadrics that defines it, viewed as a quadric over $K(t)$, has a rational point.

It has been understood by Elman, Lam and Pfister (see [7, Proposition 9]) that Theorem 6 would be a consequence of the following real period–index theorem:

Theorem 8. *Let S be a smooth integral surface over $\mathbb{R}$, and let $\alpha \in \mathrm{Br}(S) \subset \mathrm{Br}(\mathbb{R}(S))$ be such that $\alpha|_x = 0 \in \mathrm{Br}(\mathbb{R})$ for every $x \in S(\mathbb{R})$. Then* $\mathrm{ind}(\alpha) = \mathrm{per}(\alpha)$.

Over the complex numbers, Theorem 8 is the celebrated period–index theorem of de Jong [3]. Only the particular case where $S(\mathbb{R}) = \emptyset$ is needed to prove Theorem 6. The finer hypothesis that α vanish in restriction to real points was put forward by Pfister in [7].

De Jong's proof of the period–index theorem does not adapt over $\mathbb{R}$. The argument given in [2] to prove Theorem 8 uses a different strategy, relying on Hodge theory. The talk was devoted to explaining the principle of this strategy.

Let us just mention here how Hodge theory enters the picture. One has a short exact sequence $0 \to \mathrm{Pic}(S)/n \to H^2_{\text{ét}}(S, \mu_n) \to \mathrm{Br}(S)[n] \to 0$. To show that the Brauer class associated to $\beta \in H^2_{\text{ét}}(S, \mu_n)$ has index dividing n, one has to find a degree n ramified cover $p : T \to S$ such that $p^*\beta \in H^2_{\text{ét}}(T, \mu_n)$ is algebraic. This is only possible if T carries enough algebraic cycles. To ensure this, we choose T in an appropriate Noether–Lefschetz locus, using Green's infinitesimal criterion.

References

[1] A. Brumer, *Remarques sur les couples de formes quadratiques*, C. R. Acad. Sci. Paris Sr. A-B **286** (1978), A679–A681.

[2] O. Benoist, *The period–index problem for real surfaces*, eprint arXiv 2018.

[3] A. J. de Jong, *The period–index problem for the Brauer group of an algebraic surface*, Duke Math. J. **123** (2004), 71–94.

[4] T. Graber, J. Harris, J. Starr *Families of rationally connected varieties*, JAMS **16** (2003), 57–67.

[5] S. Lang, *On quasi algebraic closure*, Ann. of Math. **55** (1952), 373–390.

[6] S. Lang, *The theory of real places*, Ann. of Math. **57** (1953), 378–391.

[7] A. Pfister, *On quadratic forms and abelian varieties over function fields*, In Ordered fields and real algebraic geometry (San Francisco, Calif., 1981), Contemp. Math. **8**, 249–264, Amer. Math. Soc., Providence, R.I., 1982.

[8] R. Steinberg, *Regular elements of semisimple algebraic groups*, Publ. Math. I.H.E.S. **25** (1965), 281–312.

[9] E. Witt, *Theorie der quadratischen Formen in beliebigen Körpern.*, J. Reine Angew. Math. **176** (1937), 31–44.

Differential forms on singular spaces

Stefan Kebekus
(joint work with Christian Schnell)

1. Extension Theorems for Differential Forms

This talk was concerned with the following "extension problem" for holomorphic differential forms on complex spaces. Let X be a complex space and let $r\colon \widetilde{X} \to X$ be a resolution of singularities. Under what conditions on the singularities of X is it true that every holomorphic p-form defined on the smooth locus U_{reg} of an open subset $U \subseteq X$ extends to a holomorphic p-form on the complex manifold $r^{-1}(U)$? Equivalently, what conditions on the singularities would guarantee that the natural morphism $r_*\Omega^p_{\widetilde{X}} \hookrightarrow \iota_*\Omega^p_{X_{\mathrm{reg}}}$ is isomorphic, where $\iota : X_{\mathrm{reg}} \to X$ is the inclusion map? If X is normal, this is equivalent to asking whether $r_*\Omega^p_{\widetilde{X}}$ is reflexive.

The best existing result concerning extension of differential forms is [GKKP11, Thm. 1.4], which says that if X underlies a normal algebraic variety with Kawamata log terminal (=klt) singularities, then $r_*\Omega^p_{\widetilde{X}}$ is reflexive for every $0 \le p \le \dim X$.

Note 1. The definition of klt includes normality, and implies that $r_*\omega_{\widetilde{X}}$ is reflexive for any resolution of singularities.

Our main result is the following theorem, which works in substantially higher generality. In particular, it does not assume that X is algebraic, normal, or $\mathbb{Q}$-Gorenstein. In a nutshell, it asserts that if top-forms extend from X_{reg} to a resolution of singularities, then so will p-forms, for all values of p.

Theorem 2 (Extension theorem for p-forms)**.** *Let X be a locally irreducible complex space and write $\iota : X_{\mathrm{reg}} \to X$ for the inclusion map. Let $r\colon \widetilde{X} \to X$ be a resolution of singularities. If the natural map $r_*\omega_{\widetilde{X}} \hookrightarrow \iota_*\omega_{X_{\mathrm{reg}}}$ is isomorphic, then the maps $r_*\Omega^p_{\widetilde{X}} \hookrightarrow \iota_*\Omega^p_{X_{\mathrm{reg}}}$ are isomorphic for every $0 \le p \le \dim X$.*

Note 3. If $\operatorname{codim}_X X_{\mathrm{sing}} \le 1$, then $r_*\omega_{\widetilde{X}}$ and $\iota_*\omega_{X_{\mathrm{reg}}}$ will never agree, so that the statement of Theorem 2 is empty in this case.

Note 4. Theorem 2 discusses subsheaves $r_*\Omega^p_{\widetilde{X}} \hookrightarrow \iota_*\Omega^p_{X_{\mathrm{reg}}}$. Since any two resolutions of X are dominated by a common third, these subsheaves are independent of the choice of the resolution r. Assumption and conclusion of Theorem 2 are therefore independent of r. Because of its role in the Grauert–Riemenschneider vanishing theorem, we refer to $r_*\omega_{\widetilde{X}}$ as the *Grauert–Riemenschneider sheaf* and write ω_X^{GR}.

The talk discussed the proof of Theorem 2 at length. The key idea is to use the Decomposition Theorem [BBD82, Sai88], in order to relate the $\mathcal{O}_X$-module $r_*\Omega^p_{\widetilde{X}}$ to the intersection complex of X, viewed as a polarisable Hodge module.

Rational singularities. Theorem 2 applies to normal complex spaces with rational singularities. If X has rational singularities, it follows almost directly from the definition that the natural map of Theorem 2, $r_*\omega_{\widetilde{X}} \hookrightarrow \iota_*\omega_{X_{\mathrm{reg}}}$, is isomorphic. We refer to [KM98, Sect. 5.1] for details. The following corollary is then immediate.

Corollary 5 (Extension theorem for p-forms on spaces with rational singularities)**.** *Let X be a normal complex space with only rational singularities, and let $r\colon \widetilde{X} \to X$ be a resolution of singularities. Then, every holomorphic differential form defined on X_{reg} extends uniquely to a holomorphic form on $\widetilde{X}$.* □

1.1. **Extension of differential forms with logarithmic poles.** We also obtain a version of Theorem 2 with log poles, by adapting the techniques in the proof to a certain class of graded-polarisable mixed Hodge modules. Recall that a resolution of singularities $r\colon \widetilde{X} \to X$ of a complex space is called a *log resolution* if the r-exceptional set is a divisor with normal crossings on $\widetilde{X}$.

Theorem 6 (Extension theorem for logarithmic p-forms)**.** *Let X be a locally irreducible complex space and write $\iota : X_{\mathrm{reg}} \to X$ for the inclusion map. Let $r\colon \widetilde{X} \to X$ be a log resolution with exceptional divisor $E \subset \widetilde{X}$. If the natural map $r_*\omega_{\widetilde{X}}(E) \hookrightarrow \iota_*\omega_{X_{\mathrm{reg}}}$ is isomorphic, then the maps $r_*\Omega^p_{\widetilde{X}}(\log E) \hookrightarrow \iota_*\Omega^p_{X_{\mathrm{reg}}}$ are isomorphic for every $0 \le p \le \dim X$.*

Note 7. By a result of Kovács, Schwede, and Smith [KSS10, Thm. 1], a complex algebraic variety X that is normal and Cohen–Macaulay has Du Bois singularities if and only if $r_*\omega_{\widetilde{X}}(E)$ is a reflexive $\mathcal{O}_X$-module for some log resolution $r\colon \widetilde{X} \to X$.

1.2. **Local vanishing conjecture.** The methods developed in this paper also settle the "local vanishing conjecture" proposed by Mustaţă, Olano, and Popa [MOP17, Conj. A]. The original conjecture contained the assumption that X is a normal algebraic variety with rational singularities. Our proof shows that the weaker assumption $R^{n-1}r_*\mathcal{O}_{\widetilde{X}} = 0$ is sufficient.

Theorem 8 (Local vanishing)**.** *Let X be a locally irreducible complex space of dimension n. Let $r\colon \widetilde{X} \to X$ be a log resolution with exceptional divisor $E \subset \widetilde{X}$. If $R^{n-1}r_*\mathcal{O}_{\widetilde{X}} = 0$, then $R^{n-1}r_*\Omega_{\widetilde{X}}(\log E) = 0$.*

1.3. **Functorial pull-back.** Theorem 2 can be seen as saying that any differential form $\sigma \in H\big(X_{\mathrm{reg}}, \Omega_{X_{\mathrm{reg}}}\big) = H\big(X, \Omega^{[1]}_X\big)$ induces a *pull-back form* $\widetilde{\sigma} \in H\big(\widetilde{X}, \Omega_{\widetilde{X}}\big)$. More generally, we show that pull-back exists for reflexive differentials and arbitrary morphisms between varieties with rational singularities. The paper [Keb13b] discusses these matters in detail.

Theorem 9 (Functorial pull-back for reflexive differentials)**.** *Let $f\colon X \to Y$ be any morphism between normal complex spaces with only rational singularities. Then, there exists a pull-back morphism*

$$\mathrm{d}_{\mathrm{refl}}\, f\colon f^*\Omega^{[p]}_Y \to \Omega^{[p]}_X,$$

uniquely determined by natural universal properties.

The "natural universal properties" mentioned in Theorem 9 require in essence that the pull-back morphisms agree with the pull-back of Kähler differentials wherever this makes sense, and that they satisfy the composition law.

Note 10. Theorem 9 applies to morphisms $X \to Y$ whose images are entirely contained in the singular locus of Y. Taking the inclusion of the singular set for a morphism, Theorem 9 implies that every differential form on Y_{reg} induces a differential form on every stratum on the singularity stratification.

1.4. **First applications.** Theorem 9 can be formulated in terms of h-differentials; these are obtained as the sheafification of Kähler differential forms with respect to the h-topology on the category of complex spaces, as introduced by Voevodsky. We refer the reader to [HJ14] and to the survey [Hub16] for a gentle introduction to these matters. Using the description of h-differentials found in [HJ14, Thm. 1], the following is an immediate consequence of Theorem 9.

Corollary 11 (h-differentials on spaces with rational singularities)**.** *Let X be a normal complex space with only rational singularities. Then,* h*-differentials and reflexive differentials agree:* $\Omega^p_{\mathrm{h}}(X) = \Omega^{[p]}_X(X)$. □

The extension theorem for klt spaces has had a number of applications, pertaining to integral Hodge classes [HV11], hyperbolicity of moduli [Keb13a], the structure of minimal varieties with trivial canonical class [GKP16, GGK17], the nonabelian Hodge correspondence for singular spaces [GKPT17], and quasi-étale uniformisation [LT14, GKPT15]. Here, we mention only one immediate application of Theorem 2.

Theorem 12 (Closedness of forms and Bogomolov–Sommese vanishing)**.** *Let X be a complex, projective variety. If ω_X^{GR} is reflexive, then any differential form on X_{reg} is closed. If $\mathcal{A} \subseteq \Omega^{[p]}_X$ is locally free, then $\kappa(\mathcal{A}) \le p$.* □

References

[BBD82] Alexander Beĭlinson, Joseph Bernstein, and Pierre Deligne. Faisceaux pervers. In *Analysis and topology on singular spaces, I (Luminy, 1981)*, volume 100 of *Astérisque*, pages 5–171. Soc. Math. France, Paris, 1982. Available online at `https://publications.ias.edu/deligne/paper/396`.

[GGK17] Daniel Greb, Henri Guenancia, and Stefan Kebekus. Klt varieties with trivial canonical class — holonomy, differential forms, and fundamental groups. Preprint arXiv:1704.01408, April 2017.

[GKKP11] Daniel Greb, Stefan Kebekus, Sándor J. Kovács, and Thomas Peternell. Differential forms on log canonical spaces. *Inst. Hautes Études Sci. Publ. Math.*, 114(1):87–169, November 2011. DOI:10.1007/s10240-011-0036-0 An extended version with additional graphics is available as arXiv:1003.2913.

[GKP16] Daniel Greb, Stefan Kebekus, and Thomas Peternell. Singular spaces with trivial canonical class. In *Minimal Models and Extremal Rays, Kyoto, 2011*, volume 70 of *Adv. Stud. Pure Math.*, pages 67–113. Mathematical Society of Japan, Tokyo, 2016. Preprint arXiv:1110.5250.

[GKPT15] Daniel Greb, Stefan Kebekus, Thomas Peternell, and Behrouz Taji. The Miyaoka-Yau inequality and uniformisation of canonical models. Preprint arXiv:1511.08822. To appear in the Annales scientifiques de l'École normale supérieure, November 2015.

[GKPT17] Daniel Greb, Stefan Kebekus, Thomas Peternell, and Behrouz Taji. Nonabelian Hodge theory for klt spaces and descent theorems for vector bundles. Preprint arXiv:1711.08159, version 2 or greater, October 2017.

[HJ14] Annette Huber and Clemens Jörder. Differential forms in the h-topology. *Algebr. Geom.*, 1(4):449–478, 2014. DOI:10.14231/AG-2014-020. Preprint arXiv:1305.7361.

[Hub16] Annette Huber. Differential forms in algebraic geometry—a new perspective in the singular case. *Port. Math.*, 73(4):337–367, 2016. DOI:10.4171/PM/1990.

[HV11] Andreas Höring and Claire Voisin. Anticanonical divisors and curve classes on Fano manifolds. *Pure Appl. Math. Q.*, 7(4, Special Issue: In memory of Eckart Viehweg):1371–1393, 2011. DOI:10.4310/PAMQ.2011.v7.n4.a13.

[Keb13a] Stefan Kebekus. Differential forms on singular spaces, the minimal model program, and hyperbolicity of moduli stacks. In Gavril Farkas and Ian Morrison, editors, *Handbook of Moduli, in honour of David Mumford, Vol. II*, volume 25 of *Advanced Lectures in Mathematics*, pages 71–114. International Press, March 2013. ISBN 9781571462589. Preprint arXiv:1107.4239.

[Keb13b] Stefan Kebekus. Pull-back morphisms for reflexive differential forms. *Adv. Math.*, 245:78–112, 2013. DOI:10.1016/j.aim.2013.06.013. Preprint arXiv:1210.3255.

[KM98] János Kollár and Shigefumi Mori. *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original. DOI:10.1017/CBO9780511662560.

[KSS10] Sándor J. Kovács, Karl Schwede, and Karen E. Smith. The canonical sheaf of Du Bois singularities. *Adv. Math.*, 224(4):1618–1640, 2010. DOI:10.1016/j.aim.2010.01.020.

[LT14] Steven S.Y. Lu and Behrouz Taji. A characterization of finite quotients of Abelian varieties. *IMRN*, 2018(1):292–319, October 2014. DOI:10.1093/imrn/rnw251. Preprint arXiv:1410.0063.

[MOP17] Mircea Mustaţă, Sebastián Olano, and Mihnea Popa. Local vanishing and Hodge filtration for rational singularities, 2017. Preprint arXiv:1703.06704.

[Sai88] Morihiko Saito. Modules de Hodge polarisables. *Publ. Res. Inst. Math. Sci.*, 24(6):849–995 (1989), 1988. DOI:10.2977/prims/1195173930.

The *d*-primary Brauer–Manin obstruction

Bianca Viray

(joint work with Brendan Creutz and José Felipe Voloch)

A fundamental question in the study of the arithmetic of algebraic varieties is how to determine whether a "nice" variety over $\mathbb{Q}$ has a $\mathbb{Q}$-point. Manin observed that for any nice variety X there is a pairing of the Brauer group of X with the set of adelic points of X and that the rational points are always contained in the so-called Brauer–Manin set $X(\mathbb{A})^{\mathrm{Br}}$, i.e., the set of adelic points orthogonal to *every element* of the Brauer group of X. For all but the simplest varieties (e.g. Severi–Brauer varieties), the Brauer–Manin set can be strictly smaller than the set of adelic points, so understanding the rational points on X often necessitates understanding this set.

The pairing between a Brauer element and an adelic point is a powerful theoretical tool. However, in practice it can often be difficult to compute the adelic points orthogonal to a single Brauer class, let alone the full Brauer group. This makes computation of the Brauer–Manin set particularly daunting in cases when the Brauer group is large.

For proper varieties, the continuity of the Brauer–Manin pairing and the compactness of $X(\mathbb{A})$ guarantee that if the Brauer–Manin set is empty, this emptiness can be deduced from a finite subgroup $B \subset \mathrm{Br}X$. Thus one can ask whether such a finite subgroup B can be detected a priori, or, for X in a suitable family, whether the prime divisors of $\#B$ is constant across a family and, if so, what can be said about this set of primes. These questions, and closely related questions, lead us to consider the following properties.

Definition 1. Fix an integer d and a smooth projective variety X.

(1) We say that BM_d holds when we have $X(\mathbb{A})^{\mathrm{Br}} = \emptyset \Leftrightarrow X(\mathbb{A})^{\mathrm{Br}[d^\infty]} = \emptyset$. In this case we say that the d-primary subgroup *captures* the Brauer–Manin obstruction.

(2) We say that $\mathrm{BM}_d^\perp$ holds when we have $X(\mathbb{A}) \neq \emptyset \Leftrightarrow X(\mathbb{A})^{\mathrm{Br}[d^\perp]} \neq \emptyset$, where $\mathrm{Br}[d^\perp]$ denotes the subgroup of elements with order coprime to d.

(3) We say that $\widehat{\mathrm{BM}}_d$ holds when for all subgroups $B \subset \mathrm{Br}X$, we have $X(\mathbb{A})^B = \emptyset \iff X(\mathbb{A})^{B[d^\infty]}$. In this case we say that the d-primary subgroup *completely captures* the Brauer–Manin obstruction.

Our broad motivating question is to determine how geometry controls the sets

$$\{d : \mathrm{BM}_d \text{ holds}\}, \quad \left\{d : \mathrm{BM}_d^\perp \text{ holds}\right\}, \quad \text{and} \quad \left\{d : \widehat{\mathrm{BM}}_d \text{ holds}\right\}.$$

We attack this problem from two directions. First, in joint work with B. Creutz, we determine families of varieties and positive integers d where BM_d or $\widehat{\mathrm{BM}}_d$ hold for every variety in the family.

Theorem 2 ([1]).

(1) *Let V be a torsor under an abelian variety and let P be the period of V. Then $\widehat{\mathrm{BM}}_P$ holds for V.*
(2) *Let $V \to A$ be a 2-covering of an abelian variety and let Y be the associated Kummer variety, i.e., the minimal desingularization of the quotient $V/\pm 1$. Then BM_2 holds for Y.*
(3) *Let X be a bielliptic surface, let n be the order of K_X, and let P be the period of the Albanese torsor Alb^1_X. If $3|n$ then assume that Alb^1_X is not a nontrivial divisible element in the Tate–Shafarevich group of Alb^0_X. Then BM_{nP} holds for X.*

In these cases, the proofs of our original results and subsequent work of Skorobogatov show that the set $\{d : \mathrm{BM}_d \text{ holds}\}$ is a finitely generated multiplicative subset.

The second direction of attack, in joint work with B. Creutz and J.F. Voloch, is to construct examples that demonstrate possible "extreme" behavior (extreme in comparison to the above results) for the above sets.

Theorem 3 ([2]).

(1) *There exists a curve C such that $\{d : \mathrm{BM}_d \text{ holds}\}$ contains infinitely many coprime d, so in particular is not finitely generated.*
(2) *For any prime ℓ, there exists a genus 2 curve over a global field such that BM_ℓ fails. So in particular, the set $\{d : \mathrm{BM}_d \text{ holds}\}$ is not constant on the family of genus 2 curves.*
(3) *There exists a genus 3 curve C with index 1 such that $C(\mathbb{A})^{\mathrm{Br}[2]} = \emptyset$ and $C(\mathbb{A})^{\mathrm{Br}[2^\perp]} \neq \emptyset$.*

References

[1] B. Creutz and B. Viray, *Degree and the Brauer–Manin obstruction*, preprint `arXiv:1703.02187`.
[2] B. Creutz, B. Viray, and J.F. Voloch, *The d-primary Brauer–Manin obstruction for curves*, Research in Number Theory **4** (2018), no. 2, 4:26.

Automorphism of $K3$ surfaces and decomposition groups of rational sextics

Shigeru Mukai
(joint work with Hisanori Ohashi)

Let $\bar{B} \subset \mathbb{P}^2$ be a sextic curve with only simple singularity and assume that all components are rational. After blowing up suitably we obtain a Coble surface S with anti-bicanonical boundary $B = \sum_1^m B_i \in |-2K_S|$ of disjoint union of $(-4)\mathbb{P}^1$s. The automorphism group of S coincides with the decomposition group $\mathrm{Dec}\,\bar{B} \subset \mathrm{Cr}_2$ of $\bar{B} \subset \mathbb{P}^2$ by [2, §4]. I discussed the following:

Conjecture 1. *Let S be a Coble, Enriques or elliptic $K3$ surface. Then*

$$vcd(\mathrm{Aut}\, S) = \max_f MW\text{-}rk(f), \tag{1}$$

where $f : S \to \mathbb{P}^1$ runs over all genus-1 fibration of S. The left hand side is the virtual cohomological dimension, see [5] *and* [3]. *MW-rk in the right hand side denotes the rank of Mordell–Weil group of the Jacobian fibration* $\mathrm{Jac}\, f : \mathrm{Jac}\, S \to \mathbb{P}^1$ *of f.*

By the Shioda–Tate formula,

$$\text{MW-rk}(f) = 8 - \sum_{fibers} \{(\#\,\text{of irred. comp.}) - 1\} \tag{2}$$

holds for a Coble and Enriques surface.

1. Coble (and Enriques) surfaces

In my talk I gave three examples for which the conjecture holds true. Let R_5 be a *quintic del Pezzo surface*, that is, the blow-up of the projective plane $\mathbb{P}^2$ at four points $p_1, \dots, p_4$ in general position. The strict transforms of lines joining two points and four exceptional curves are called *lines* on R_5 since their anti-canonical degree are one. The dual graph of their configuration is the Petersen graph. There are 15 intersection points in total among the ten lines. It is well known that the automorphism group $\mathrm{Aut}\, R_5$ is isomorphic to the symmetric group $\mathfrak{S}_5$ of degree five. Let R_{-10} be the blow-up of R_5 at the 15 intersection points. R_{-10} is a Coble surface with boundary $B = \sum_{0 \le i < j \le 4} B_{ij}$, where B_{ij}'s are the strict transforms of 10 lines.

1.1. **Case where characteristic is not 2.** We take a system of homogeneous coordinates $(x_0 : x_1 : x_2)$ such that three diagonal point A, B, C are the coordinate points $(1:0:0), (0:1:0)$ and $(0:0:1)$. Then the four points $p_1, \dots, p_4$ are the fixed points of the quadratic Cremona transformation $(x_0 : x_1 : x_2) \mapsto (a/x_0 : b/x_1 : c/x_2)$ with center A, B, C for suitable constants a, b, c. The Cremona transformation induces an involution of R_{-10}, which we denote by σ. Taking conjugate by the action of $\mathrm{Aut}\, R_5 \simeq \mathfrak{S}_5$, we obtain five involutions $\sigma = \sigma_1, \dots, \sigma_5$ of R_{-10}. The automorphism group of S, or equivalently, the decomposition group of 6 lines $\bar{B} = \sum_{1 \le i < j \le 4} \overline{p_i p_j}$, is described as follows:

Theorem 2. *The automorphism group of* R_{-10} *is generated by* $\operatorname{Aut} R_5$ *and* σ*. Moreover, it is isomorphic to the semi-direct product of the amalgam of five involutions* $\langle\sigma_1\rangle * \cdots * \langle\sigma_5\rangle$ *by* $\mathfrak{S}_5$*.*

By the theorem $\operatorname{Aut} R_{-10}$ contains a free group $F_4 = \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$ as a subgroup of finite index. Hence the virtual cohomological dimension is equal to 1. There are four types of genus-1 fibrations, $1)A_8, 2)A_7 + A_1, 3)A_4 + A_4$ and $4)A_5 + A_2$. The last fibrations have Mordell–Weil rank 1 and others 0 by the formula (2). Therefore, the conjecture holds since both sides of (1) are equal to 1.

1.2. **Case where characteristic is 2.** Three diagonal points A, B, C of the complete quadrilateral $\bar{B} = \sum_{1\le i<j\le 4} \overline{p_i p_j}$ are colinear (Fano plane). Hence the Coble surface S has five $(-2)\mathbb{P}^1$s (instead of five involutions). All genus-1 fibration of S has Mordell–Weil rank 0 by virtue of these curves. More precisely, there are 51 such fibrations of the following types and their Mordell–Weil group are as follows:

	ADE type	MW group	cardinality
1)	A_8	$\mathbb{Z}/3\mathbb{Z}$	20
2)	$A_7 + A_1$	$\mathbb{Z}/4\mathbb{Z}$	15
3)	$A_4 + A_4$	$\mathbb{Z}/5\mathbb{Z}$	6
4)	$A_5 + A_2 + A_1$	$\mathbb{Z}/3\mathbb{Z}$	10

By Vinbergs criterion, $\operatorname{Aut} R_{-10}$ is finite. Moreover, we have

Theorem 3. *The automorphism group* $\operatorname{Aut} R_{-10}$ *of* R_{-10} *is the same as* $\operatorname{Aut} R_5 \simeq \mathfrak{S}_5$*.*

Hence the conjecture holds with both sides of (1) being 0.

1.3. **Very general case.** When S is a very general Enriques surface or a very general Coble surface with $m = 1, 2$, the conjecture holds since both sides of (1) are equal to 8. See [4, Remark 5] for Enriques case. (The conjecture holds true for a certain 1-dimensional family of Enriques surfaces by the main result of [4].)

2. Singular K3 surfaces

We restrict ourselves to K3 surfaces X of Picard number 20 over the complex number field $\mathbf{C}$. Such a K3 surface is called *singular* and always has an elliptic pencil. By [6, §5], the automorphism group of X is infinite. In fact, X has an elliptic fibration with positive Mordell–Weil rank.

By the main result of Vinberg[8], AutX is of cohomological dimension 1 when the discriminant is 3 or 4. It is easy to see that the conjecture holds in this case. The following characterizes these two singular K3 surfaces by the virtual cohomological dimension ($vcd = 1$). We call the right hand side of (1) the MW-rank of X for the sake of convenience.

Theorem 4. *A singular K3 surface other than* [8] *has MW-rank* ≥ 2*. Moreover, the equality holds if and only if the discriminant is 7 or 8.*

The following is a special case of the conjecture (1).

Subconjecture 5 (Next two most algebraic K3 surfaces)**.** *The automorphism group of a singular $K3$ surface of discriminant 7 or 8 has cohomological dimension 2.*

A system of generators of $\operatorname{Aut} X$ is determined by Ujikawa [7] in the former case. ADE-type of all elliptic fibrations are determined by Bertin–Lecacheux [1] in the latter case.

References

[1] M.J. Bertin and O. Lecacheux, *Elliptic fibrations on the modular surface associated to* $\Gamma_1(8)$, in Arithmetic geometry of $K3$ surfaces and Calabi–Yau threefolds, Fields Inst. Comm. **67**(2013), 153265.
[2] J. Blanc, I. Pan and T. Vust, *On birational transfromations of pairs in the complex plane*, Geom. Dedicata, **139**(2009), 5773.
[3] K. S. Brown, Cohomology of groups, Springer–Verlag, 1982.
[4] S. Mukai and H. Ohashi.: *The automorphism groups of Enriques surfaces covered by symmetric quartic surfaces*, in "Recent Advances in Algebraic Geometry", A volume in honor of Rob Lazarsfeld's 60th birthday, eds. Hacon, Mustata and Popa, Cambridge Univ. Press, 2015, 307–320.
[5] J.P. Serre, *Cohomologie des groupes discrets*, in "Prospects in Math.", Ann. of Math. Studies **70**(1971), 77–169.
[6] T. Shioda, and H. Inose, *On singular surfaces*, in "Complex analysis and algebraic geometry", Iwanami-Shoten, 1977.
[7] M. Ujilkawa, *The automorphism group of the singular $K3$ surface of discriminant 7*, Comment. Math. Univ. St. Pauli, **62** (2013), 11-29.
[8] E.B. Vinberg, *The two most algebraic K3 surfaces*, Math. Ann. **265** (1983), 1–21.

Reporter: Asher Auel