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## Flat Surfaces and Algebraic Curves

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ABSTRACT. This workshop brought together two distinct communities: “flat” geometers, studying the moduli of flat surfaces, and Teichmüller dynamics, and algebraic geometers studying the moduli space of curves. While both communities study similar or often the same objects, very different viewpoints and toolboxes lead to different questions being addressed, and different progress being made. The workshop sought to educate each community about the techniques of the other, and to foster communication between the two groups. One particular focus was enumerative geometry.

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### Introduction by the Organisers

The workshop “Flat surfaces and algebraic curves” aimed at creating a bridge between two communities working with the moduli space of curves and various related moduli spaces. A *flat* surface is a complex algebraic curve, with the additional datum of a holomorphic one-form. The moduli space of flat surfaces is the total space of the Hodge vector bundle over the moduli space of curves. It is stratified according to the multiplicities of zeroes of the one-form. Each stratum admits an action of the group  $SL_2(\mathbb{R})$ , and the study of the properties action is a vibrant field of mathematics called Teichmüller dynamics. Many of the leaders of Teichmüller dynamics were present at the workshop.

The moduli space of (complex algebraic compact genus  $g$ ) curves is one of the classical central objects of study in algebraic geometry. While the beginnings of the subject go back to the 19th century school, tremendous progress has been made

in the last 50 years by utilizing modern techniques, starting with the Deligne-Mumford construction of a natural compactification of the moduli space. Many of the experts working on the birational geometry of the moduli space of curves, on its homology, and on related counting/enumerative problems participated in the workshop.

In recent years, it has become obvious in the flat surface community that the strata of flat surfaces are amenable to a study from the viewpoint of algebraic geometry, and that the techniques of algebraic geometry may provide key insights to questions of dynamics. The breakthrough results of Eskin-Mirzakhani-Mohammadi and Filip showed in particular that the topological closure of any  $SL_2(\mathbb{R})$  orbit is an algebraic variety — and can be studied as such. For algebraic geometers, flat surfaces have been providing new geometric constructions, and a source of questions, while the enumerative geometry is in some ways paralleled by the actively studied double ramification loci in moduli spaces.

The aim of the workshop was to bridge the gaps in background, and to update both communities on the developments and questions arising in both.

One tool to achieve these goals was a series of overview talks from both viewpoints. These were given on the Kodaira dimension of moduli spaces (Farkas), affine invariant submanifolds (Apisa), enumerative aspects of moduli of curves (Pixton), higher Teichmüller theory (Wienhard), Lyapunov exponents (Filip) and tropical curves (Ulirsch). These talks allowed all the participants to understand the language of each other, and to discuss the problems of mutual interest. The wealth of the resulting common background then allowed faster communication, and allowed us to have shorter (half-hour) research talks, and more of these.

Perhaps the main focus of research interface between the two communities during the workshop was around the enumerative aspects of flat surfaces and of moduli curves. On the flat surfaces side, these were discussed as Masur-Veech volumes, and related properties of strata, in the talks of Aggarwal, Goujard, Masur, while on the algebraic geometry side, Buryak, Norbury and Rossi, discussed the geometry of the double ramification cycle; the talk of Sauvaget brought techniques of algebraic geometry proper to bear on a counting problem in flat surfaces.

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## Workshop: Flat Surfaces and Algebraic Curves

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## Abstracts

### Large Genus Asymptotics for Volumes of Strata of Abelian Differentials

AMOL AGGARWAL

#### 1. BACKGROUND AND RESULTS

Fix a positive integer  $g > 1$ , and let  $\mathcal{H} = \mathcal{H}_g$  denote the moduli space of Abelian differentials of genus  $g$ . For any partition  $m = (m_1, m_2, \dots, m_n)$  of  $2g - 2$ , let  $\mathcal{H}(m) \subset \mathcal{H}$  denote the associated stratum; the period map  $\Phi : \mathcal{H}(m) \rightarrow \mathbb{C}^{2g+n-1}$  defines a local coordinate chart for  $\mathcal{H}(m)$ .

Pulling back the Lebesgue measure on  $\mathbb{C}^{2g+n-1}$  yields a measure  $\nu$  on  $\mathcal{H}(m)$ . To ensure finiteness of this measure, let  $\mathcal{H}_1(m) \subset \mathcal{H}(m)$  denote the moduli space of pairs  $(X, \omega) \in \mathcal{H}(m)$  with  $\omega$  of area one. Let  $\nu_1$  denote the measure induced by  $\nu$  on  $\mathcal{H}_1(m)$ . It was established independently by Masur [5] and Veech [7] that the volume  $\nu_1(\mathcal{H}_1(m))$  is finite for each  $m$ . This volume  $\nu_1(\mathcal{H}_1(m))$  is called the *Masur-Veech volume* of the stratum indexed by  $m$ .

Based on the representation theory of the symmetric group and asymptotic Hurwitz theory, Eskin-Okounkov [3] proposed a general algorithm that, given an integer  $g > 1$  and partition  $m = (m_1, m_2, \dots, m_n)$  of  $2g - 2$ , determines the volume of the stratum  $\nu_1(\mathcal{H}_1(m))$ . This algorithm was reasonably intricate, in the sense that the time taken to implement it grows exponentially in  $g$ .

Once it is known that these volumes are finite and can in principle be determined, a question of interest is to understand how they behave as the genus  $g$  tends to  $\infty$ . To that end, the algorithm of Eskin-Okounkov enabled Eskin to write a computer program to evaluate the volumes  $\nu_1(\mathcal{H}_1(m))$  with  $g \leq 10$ . Based on the numerical data provided by this program, Eskin and Zorich predicted (see Conjecture 1 and equations (1) and (2) of [4]) that

$$(1) \quad \nu_1(\mathcal{H}_1(m)) = \frac{4 + o(1)}{\prod_{i=1}^n (m_i + 1)},$$

uniformly in  $g$  and  $m$ . Before this work, the asymptotic (1) had been verified in two cases, namely for the principal stratum (when  $m = 1^{2g-2}$ ) by Chen-Möller-Zagier [2] and the minimal stratum (when  $m = (2g - 2)$ ) by Sauvaget [6].

We establish the asymptotic (1) for all strata, as indicated by the following theorem.

**Theorem 1.** *Let  $g > 1$  be a positive integer, and let  $m = (m_1, m_2, \dots, m_n)$  denote a partition of size  $2g - 2$ . Then,*

$$(2) \quad \frac{4}{\prod_{i=1}^n (m_i + 1)} \left(1 - \frac{2^{2^{200}}}{g}\right) \leq \nu_1(\mathcal{H}_1(m)) \leq \frac{4}{\prod_{i=1}^n (m_i + 1)} \left(1 + \frac{2^{2^{200}}}{g}\right).$$

The proof of Theorem 1 is based on a combinatorial analysis of the original algorithm proposed by Eskin and Okounkov for evaluating  $\nu_1(\mathcal{H}_1(m))$  in [3]. However, as mentioned previously, this algorithm is intricate; it expresses the Masur-Veech volume through the composition of three identities, each of which involves a sum whose number of terms increases exponentially in the genus  $g$ . To establish (2), one shows that each of these sums is dominated by a single term, and the remaining (non-dominant) terms in the sum decay rapidly and can be viewed as negligible. However, instead of explaining this method in full generality, let us provide a heuristic in a special case.

## 2. A HEURISTIC FOR THE PRINCIPAL STRATUM

The proof of Theorem 1 simplifies considerably in the case when  $m = 1^{2g-2}$  corresponds to the principal stratum, due to a result of Eskin-Okounkov [3] that provides an explicit expression for the volume  $\nu_1(\mathcal{H}_1(1^{2g-2}))$ . To state it, we define the quantity

$$\mathfrak{z}(k) = (2 - 2^{2-k})\zeta(k)\mathbf{1}_{k \in 2\mathbb{Z}_{\geq 0}},$$

where  $\zeta(k)$  denotes the Riemann zeta function and  $\mathbf{1}_E$  denotes the indicator for any event  $E$ . Furthermore, for any partition  $\mu$ , let  $\ell(\mu)$  denote the number of parts of  $\mu$ ; moreover, for any integer  $j \geq 1$ , let  $M_j(\mu)$  denote the number of indices  $1 \leq i \leq \ell(\mu)$  such that  $\mu_i = j$ .

**Lemma 2** ([3, Theorem 7.1]). *For any even positive integer  $n$ , we have that*

$$(3) \quad \nu(\mathcal{H}(1^n)) = 2n! \sum_{\mu} \frac{(-1)^{\ell(\mu)-1}}{(2n - \ell(\mu) + 2)! \prod_{i=2}^{\infty} M_i(\mu)!} \prod_{i=1}^{\ell(\mu)} (2\mu_i - 3)!! \mathfrak{z}(\mu_i),$$

where  $\mu$  is summed over all partitions of  $n+2$  with only even parts, and  $a!!$  denotes the product of all odd positive integers at most equal to  $a$ .

Now let us explain how one might use (3) to provide a heuristic for the asymptotic (2) when  $m = 1^{2g-2}$ . In this case, (2) states that  $\nu_1(\mathcal{H}(1^{2g-2})) \approx 2^{4-2g}$ .

To that end, observe that the product  $\prod_{i=1}^{\ell(\mu)} (2\mu_i - 3)!!$  is maximized (over all partitions  $\mu$  of  $n+2$  with all even parts) when  $\mu = (n+2)$  consists of one part. In this case, we have that  $\ell(\mu) = 1$ , and  $M_i(\mu) \in \{0, 1\}$  for all  $i$ . Thus, the contribution of the  $\mu = (n+2)$  term to the right side of (3) is given by

$$2n! \frac{(2n+1)!! \mathfrak{z}(n+2)}{(2n+1)!} = 2^{1-n} \mathfrak{z}(n+2),$$

where we have used the fact that  $(2n+1)! = \prod_{i=1}^{n+1} (2n-1)(2i) = 2^n n! (2n+1)!!$ . Since  $\mathfrak{z}(n+2) = (2 - 2^{-n})\zeta(n+2) \approx 2$  for  $n$  large, it follows that the contribution to (3) from the  $\mu = (n+2)$  term is approximately  $2^{2-n} = 2^{4-2g}$ . This is the leading order term predicted by (2), which suggests that the right side of (3) is in fact dominated by the  $\mu = (n+2)$  term.

This can indeed be shown to be the case but, instead of detailing such a proof here, let us explain how the above heuristic can be extended to guess the second

order correction to the approximation  $\nu_1(\mathcal{H}(1^{2g-2})) \approx 2^{4-2g}$ . To that end, observe that the maximum of the product  $\prod_{i=1}^{\ell(\mu)} (2\mu_i - 3)!!$  over all even partitions  $\mu$  of  $n + 2$  with at least two parts (equivalently, when  $\mu \neq (n + 2)$ ) is obtained when  $\mu = (n, 2)$ . Then,  $\ell(\nu) = 2$  and  $M_i(\nu) \in \{0, 1\}$  for all  $i$  (if  $n \geq 4$ ). Thus, the contribution to (3) resulting from this term is

$$\begin{aligned} -2n! \frac{(2n-3)!! \mathfrak{z}(2) \mathfrak{z}(n)}{(2n)!} &= -\frac{4n!(1-2^{1-n})(2n-3)! \zeta(2) \zeta(n)}{2^{n-2}(n-2)!(2n)!} \approx -2^{2-n} \frac{\zeta(2)}{2n} \\ &\approx -2^{4-2g} \frac{\pi^2}{24g}, \end{aligned}$$

where we used the fact that  $\zeta(2) = \frac{\pi^2}{6}$  and  $n = 2g - 2$ . This leads to the guess

$$\nu_1(\mathcal{H}_1(1^{2g-2})) = 2^{4-2g} \left( 1 - \frac{\pi^2}{24g} + \mathcal{O}\left(\frac{1}{g^2}\right) \right),$$

which was in fact shown to be correct in earlier work of Chen-Möller-Zagier (see the proof of Theorem 19.3 of [2]) through an entirely different method.

By following the above sort of reasoning, one might predict that the contribution of an individual summand on the right side of (3) corresponding to some partition  $\mu$  decays exponentially in the length  $\ell(\mu)$  of  $\mu$ . Such a statement can be proven and used to bound the contribution from the (large number of) remaining terms in (3), leading to a justification of the above volume asymptotic heuristic.

#### REFERENCES

- [1] A. Aggarwal, Large Genus Asymptotics for Volumes of Strata of Abelian Differentials, preprint, arXiv:1804.05431.
- [2] D. Chen, M. Möller, and D. Zagier, Quasimodularity and Large Genus Limits of Siegel-Veech Constants, *J. Amer. Math. Soc.* **31**, 1059–1163, 2018.
- [3] A. Eskin and A. Okounkov, Asymptotics of Numbers of Branched Coverings of a Torus and Volumes of Moduli Spaces of Holomorphic Differentials, *Invent. Math.* **145**, 59–103, 2001.
- [4] A. Eskin and A. Zorich, Volumes of Strata of Abelian Differentials and Siegel-Veech Constants in Large Genera, *Arnold Math. J.* **1**, 481–488, 2015.
- [5] H. Masur, Interval Exchange Transformations and Measured Foliations, *Ann. Math.* **115**, 169–200, 1982.
- [6] A. Sauvaget, Volumes and Siegel-Veech Constants of  $\mathcal{H}(2g-2)$  and Hodge Integrals, preprint, arXiv:1801.01744.
- [7] W. A. Veech, Gauss Measures for Transformations on the Space of Interval Exchange Maps, *Ann. Math.* **115**, 201–242, 1982.

### Around the Classification of Affine Invariant Submanifolds

PAUL APISA

The moduli space  $\Omega\mathcal{M}_g$  of Abelian differentials on genus  $g$  Riemann surfaces admits actions of complex scalar multiplication and Teichmüller geodesic flow, which generate a  $\mathrm{GL}(2, \mathbb{R})$  action. The space  $\Omega\mathcal{M}_g$  admits a  $\mathrm{GL}(2, \mathbb{R})$ -invariant stratification by specifying the number and order of zeros of the Abelian differentials. By

work of Eskin-Mirzakhani [6] and Eskin-Mirzakhani-Mohammadi [7], the closure in a stratum of any  $GL(2, \mathbb{R})$  orbit is a linear manifold that is locally defined by real homogeneous linear equations in period coordinates. Such a manifold is called an affine invariant submanifold.

The goal of the talk was to describe recent work that develops tools for classifying affine invariant submanifolds. The first progress on this problem was made by McMullen [9], who classified affine invariant submanifolds in genus two.

In the first part of the talk, examples of affine invariant submanifolds were presented and used to illustrate the natural identification of the tangent space at a point of an affine invariant submanifold with a subspace of relative cohomology. Results of Avila-Eskin-Möller [5] and Wright [12] were described in order to define the notion of the rank of an affine invariant submanifold. Finally, results of Filip [8] were described which provided a completely algebraic characterization of affine invariant submanifolds.

In the second part of the talk, the Mirzakhani-Wright partial compactification of an affine invariant submanifold was described (see [10]). Using the cylinder deformation theorem of Wright (see [12]), an example of a cylinder degeneration was given and used to illustrate how studying the boundary objects produced by such degenerations greatly constrain the original affine invariant submanifolds.

In the third and final part of the talk, work of Apisa [1] and Mirzakhani-Wright [11] that classifies affine invariant submanifolds of complex-dimension at least three in hyperelliptic components and of full rank respectively were presented. The proof sketches were presented in a united way that highlighted the similarity of the two strategies. It was remarked that a similar strategy might potentially be employed to prove a conjecture of Mirzakhani characterizing the affine invariant submanifolds of rank at least  $\frac{g}{2}$ . The approach would use the classification of rank two affine invariant submanifolds achieved by Aulicino, Nguyen, and Wright [2], [3], [4].

## REFERENCES

- [1] Apisa, Paul.  *$GL_2\mathbb{R}$  orbit closures in hyperelliptic components of strata*, Duke Math J. (2018) Vol. 167 No. 4, 679-742.
- [2] Aulicino, David and Nguyen, Duc-Manh. *Rank two affine submanifolds in  $\mathcal{H}(2, 2)$  and  $\mathcal{H}(3, 1)$* , Geom. Topol. (2016). Vol 20 No. 5, 2837-2904.
- [3] Aulicino, David and Nguyen, Duc-Manh. *Rank Two Affine Manifolds in Genus 3*, arXiv:1612.06970 (2016).
- [4] Aulicino, David and Nguyen, Duc-Manh and Wright, Alex. *Classification of higher rank orbit closures in  $\mathcal{H}^{\text{odd}}(4)$* , J. Eur. Math. Soc. (JEMS) (2016) Vol. 18 No. 8, 1855-1872.
- [5] Avila, Artur and Eskin, Alex and Möller, Martin. *Symplectic and isometric  $SL(2, \mathbb{R})$ -invariant subbundles of the Hodge bundle*, J. Reine Angew. Math. (2017) Vol. 732, 1-20.
- [6] Eskin, Alex and Mirzakhani, Maryam. *Invariant and stationary measures for the  $SL(2, \mathbb{R})$  action on moduli space* Publ. Math. Inst. Hautes Études Sci. (2018) Vol. 127, 95-324.
- [7] Eskin, Alex and Mirzakhani, Maryam and Mohammadi, Amir. *Isolation, equidistribution, and orbit closures for the  $SL(2, \mathbb{R})$  action on moduli space*, Ann. of Math (2) (2015) Vol. 182 No. 2, 673-721.
- [8] Filip, Simion. *Splitting mixed Hodge structures over affine invariant manifolds*, Ann. of Math. (2) (2016) Vol. 183 No. 2, 681-713.

- [9] McMullen, Curtis T. *Dynamics of  $SL_2(\mathbb{R})$  over moduli space in genus two*, Ann. of Math (2) (2007) No. 2, 397-456.
- [10] Mirzakhani, Maryam and Wright, Alex. *The boundary of an affine invariant submanifold*, Invent. Math. (2017) Vol. 209 No. 3, 927-984.
- [11] Mirzakhani, Maryam and Wright, Alex. *Full-rank affine invariant submanifolds*, Duke Math J. (2018) Vol. 167 No. 1, 1-40.
- [12] Wright, Alex. *Cylinder deformations in orbit closures of translation surfaces*, Geom. Topol. (2015) Vol. 19 No. 1, 413-438.

## Geodesics on Platonic solids and flat surfaces

JAYADEV ATHREYA

(joint work with David Auricino, Pat Hooper)

Platonic solids are among the oldest objects of recorded mathematical study. In 1906-07, Rodenberg [7] and Stäckel [8] (in discussions with each other, but separate papers) initiated the study of the *surfaces* of these solids (and more generally convex polyhedra) as geometric objects, by considering the natural (singular) flat metric coming from the flat metric on faces. The singularities occur at vertices where the total angle is not  $2\pi$ . They discussed how geodesics could be unambiguously extended over sides but not through vertices. This leads naturally to the question of understanding *saddle connections*, geodesic trajectories connecting two vertices (with none in the interior)- for example, edges of the polyhedra are saddle connections.

The next important development was the 1936 paper [4] of Ralph Fox and Richard Kershner (who were 23-year-old graduate students at the time the paper was written) which described an *unfolding* procedure which turned rational polyhedra (polyhedra with angles which are rational multiples of  $\pi$ ) into singular flat surfaces described by polygons and identifications of sides by translations, what we know in modern language as *translation surfaces*. In fact, in complex analytic terms what they describe is a method to take a  $k$ -differential on  $\mathbb{C}P^1$  (here  $k$  is the smallest integer so that  $k\theta$  is an integer multiple of  $2\pi$  for all angles  $\theta$  of the polyhedron) and take the canonical  $k$ -cover to obtain an abelian differential on a higher-genus surface.

For example, the process they describe would take the tetrahedron, view the net for the tetrahedron as a triangle with each side cut in two and identified via rotation by  $\pi$ , and double it along one edge to obtain a rhombus with parallel sides identified by translation, that is, a flat torus (with one marked point). This is a covering map of a quadratic differential on  $\mathbb{C}P^1$  with four simple poles by a flat torus, branched over the singular points.

A natural question is to understand the *closed saddle connections* on Platonic solids, namely, the saddle connections that start and end at the same vertex. Using the basic theory of Weierstrass points (fixed points of involutions) and the symmetries of polyhedra and covers, it is not hard to show:

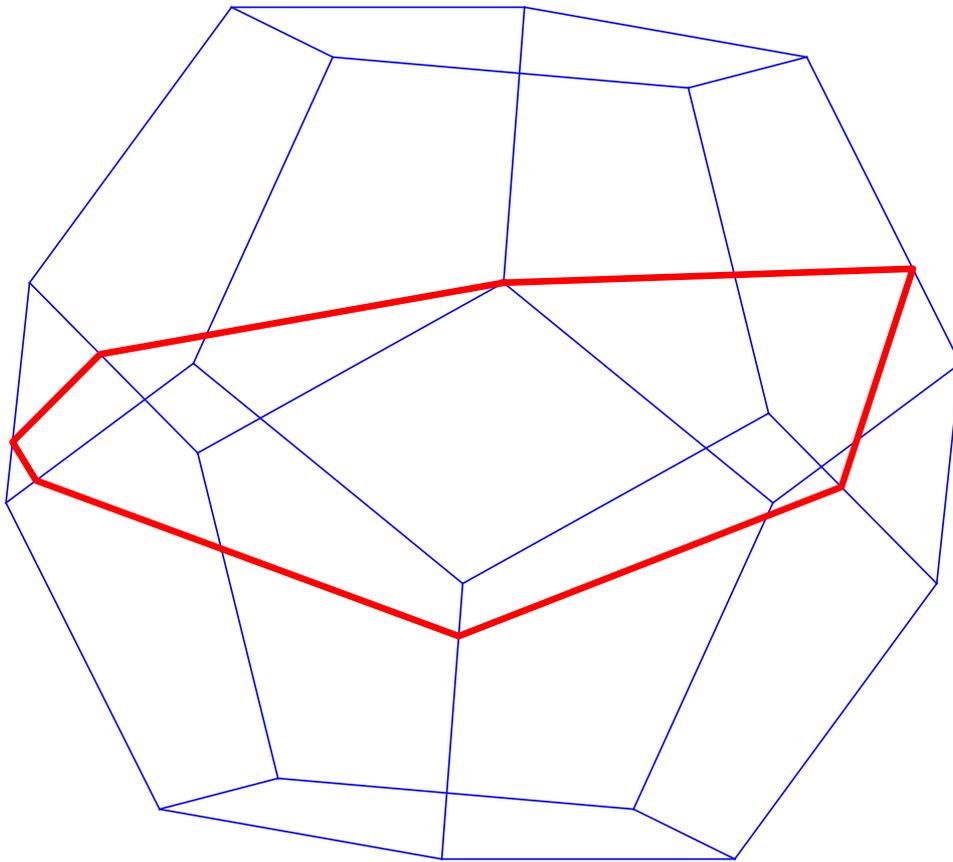
*There are no closed saddle connections on the tetrahedron, octahedron, cube, or icosahedron.*

This result was known, and in fact, Fox-Kershner observed that the study of geodesics on these solids was closely related to the fact that they are tiled by polygons that tile the plane.

Proofs of the above result are given in Fuchs [6], Fuchs-Fuchs [5], and Davis-Dodds-Traub-Yang [2], and in these and other related papers, they posed the natural question: *What about the dodecahedron?* Fuchs conjectured that there *should* exist such a trajectory based on numerical evidence.

In joint work with D. Auricino [1], we found such a closed saddle connection, and in subsequent work with D. Auricino and P. Hooper, we have classified all such closed saddle connections up to the equivalence relation of affine automorphisms on a natural cover.

FIGURE 1. A closed saddle connection on the dodecahedron



Namely, associated to each of the platonic solids is a natural cover which is an abelian differential on a higher-genus surface given by unfolding. The covers associated to the solids tiled by squares or triangles yield *arithmetic* or square-tiled surfaces. The cover associated to the dodecahedron instead covers the *double pentagon* surface. Symmetries of translation surfaces have natural affine automorphism groups, and if surfaces are square-tiled these are finite index subgroups of

Polyhedron	Stratum of $k$ -differentials	Stratum of unfolding	Genus of unfolding
Tetrahedron	$\mathcal{H}_2(-1^4)$	$\mathcal{H}_1(0^4)$	1
Octahedron	$\mathcal{H}_3(-1^6)$	$\mathcal{H}_1(1^6)$	4
Cube	$\mathcal{H}_4(-1^8)$	$\mathcal{H}_1(2^8)$	9
Icosahedron	$\mathcal{H}_6(-1^{12})$	$\mathcal{H}_1(4^{12})$	25
Dodecahedron	$\mathcal{H}_{10}(-1^{20})$	$\mathcal{H}_1(8^{20})$	81

TABLE 1. Strata of the Platonic solids and their unfoldings.

$SL(2, \mathbb{Z})$ , the affine automorphism group of the torus. For the platonic solids, these indices are in fact relatively small, with the largest being 10 (for the icosahedron).

In contrast, the double pentagon has affine symmetry group given by Hecke triangle group  $\Delta(2, 5, \infty)$ , and the unfolded dodecahedron, which is a degree 60 cover of the double pentagon, has affine symmetry group  $\Gamma$  which is index 2106 in this group! The associated hyperbolic surface  $\mathbb{H}^2/\Gamma$ , known as the Teichmüller curve, has genus 131.

We use Sage, and in particular the packages `flat_surf` and `surf_dynamics` [3] to compute  $\Gamma$ , and then to analyze equivalence classes of saddle connections under the action of  $\Gamma$ . We say two objects on the dodecahedron are *unfolding equivalent* if they have lifts to the cover which are  $\Gamma$ -equivalent. We conclude:

*Up to unfolding equivalence, there are 31 equivalence classes of closed saddle connections on the dodecahedron.*

## REFERENCES

- [1] J. S. Athreya and D. Auricino, *A Trajectory from a Vertex to Itself on the Dodecahedron*, arXiv:1802.00811, to appear in Amer. Math. Monthly (2018).
- [2] Diana Davis, Victor Dods, Cynthia Traub, and Jed Yang, *Geodesics on the regular tetrahedron and the cube*, Discrete Math. **340** (2017), no. 1, 3183–3196.
- [3] V. Delecroix and W. Patrick Hooper, *Sage-FlatSurf*
- [4] R. Fox and R. Kershner, *Concerning the transitive properties of geodesics on a rational polyhedron*. Duke Mathematical Journal, volume 2, 147-150, 1936.
- [5] Dmitry Fuchs and Ekaterina Fuchs, *Closed geodesics on regular polyhedra*, Mosc. Math. J. **7** (2007), no. 2, 265–279, 350.
- [6] Dmitry Fuchs, *Geodesics on regular polyhedra with endpoints at the vertices*, Arnold Math. J. **2** (2016), no. 2, 201–211.
- [7] C. Rodenberg, *Geodätische Linien auf Polyederflächen*, Rendiconti del Circolo Matematico di Palermo, volume 23, 107-125, 1907.
- [8] P. Stäckel, *Geodätische Linien auf Polyederflächen*, Rendiconti del Circolo Matematico di Palermo, volume 22, 141-151, 1906.

## Rank 3 Affine Submanifolds of $\mathcal{H}(10)$ Are Arithmetic

DAVID AULICINO

The work of Eskin, Mirzakhani, and Mohammadi [EM18, EMM15] established that all  $\mathrm{GL}_2(\mathbb{R})$  orbit closures have a very nice structure in the moduli space of translation surfaces. They are affine immersed suborbifolds (which will be referred to as submanifolds here) with local coordinates given by period coordinates, which is modeled on first relative cohomology. Locally, the affine submanifold is a linear subspace of first relative cohomology. The *rank* of an affine submanifold  $\mathcal{M}$  is half the total degrees of freedom to vary absolute periods of  $M \in \mathcal{M}$  such that the resulting surface remains in  $\mathcal{M}$ . The *degree* of  $\mathcal{M}$  is the degree of the smallest (real) extension over  $\mathbb{Q}$  needed to contain the coefficients of the linear equations in periodic coordinates defining the charts of  $\mathcal{M}$ .

Veech [Vee89] found the first examples of rank one affine submanifolds (Teichmüller curves in his case) with degree two in the moduli space  $\mathcal{H}(2)$  of genus two translation surfaces with a double order zero. Infinitely many more were discovered in  $\mathcal{H}(2)$  by McMullen and Calta. More recently, Wright discovered evidence for a rank two quadratic affine submanifold in  $\mathcal{H}(6)$ . Its existence was established by [EMMW18] and its uniqueness in  $\mathcal{H}(6)$  by [Api17]. This was discovered in spite of a belief that higher rank should preclude such exceptional behavior, i.e. higher rank affine submanifolds should only arise from branched coverings of strata.

This left the door open to determine if a quadratic rank three affine submanifold could exist. By work of [Wri15], genus six is the smallest genus in which it could exist, and  $\mathcal{H}(10)$  is the smallest stratum in genus six.

**Theorem 1.** *Let  $\mathcal{M}$  be a rank  $k$  affine submanifold in  $\mathcal{H}(4k - 2)$ . If  $k > 2$ , then  $\mathcal{M}$  is arithmetic.*

This is the first indication that exceptional behavior might not occur for rank three affine submanifolds.

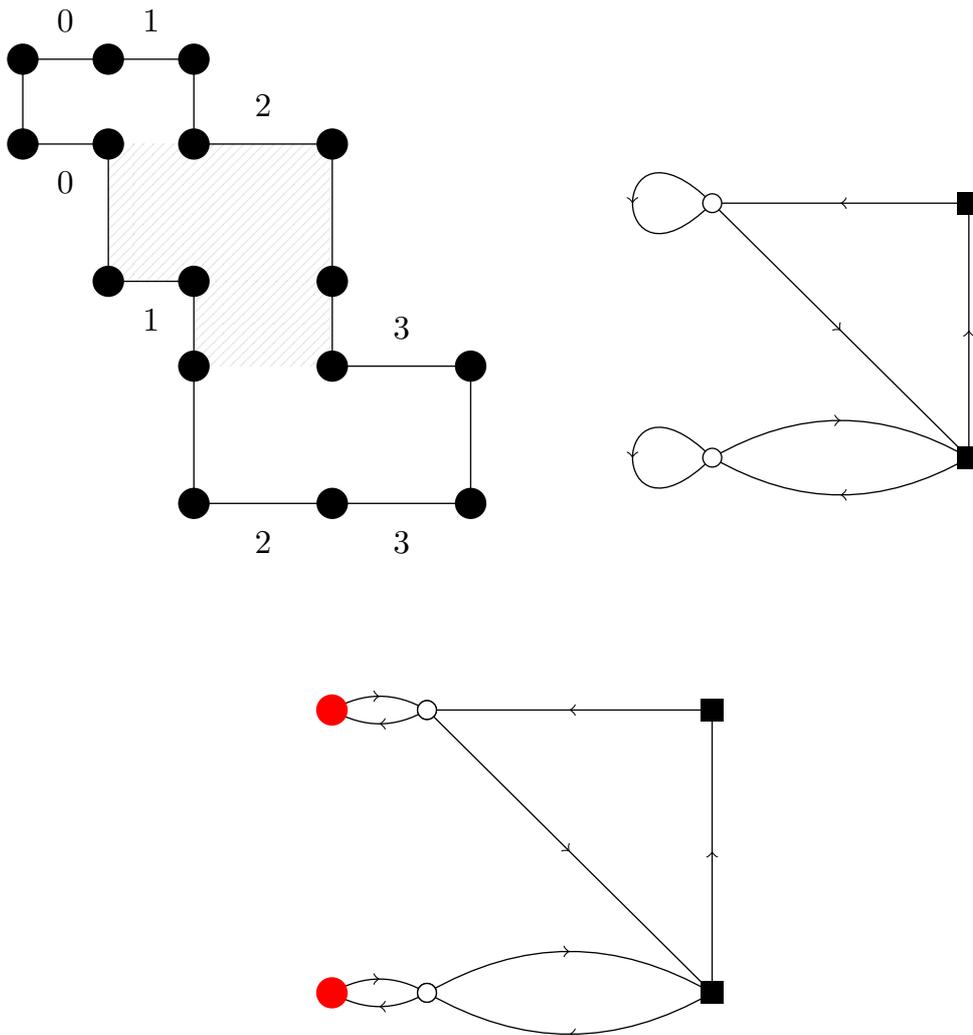
**Proof Sketch and Ideas.** The proof primarily uses degeneration techniques to leverage the uniqueness results of [Api17]. Several new techniques are introduced to address the distinct challenges of this problem. The first key result of [BM12] is that the quadratic assumption implies that the existence of one hyperbolically short curve in a non-trivial absolute homology class implies the existence of a second one. This allows the degeneration from genus six directly to genus four.

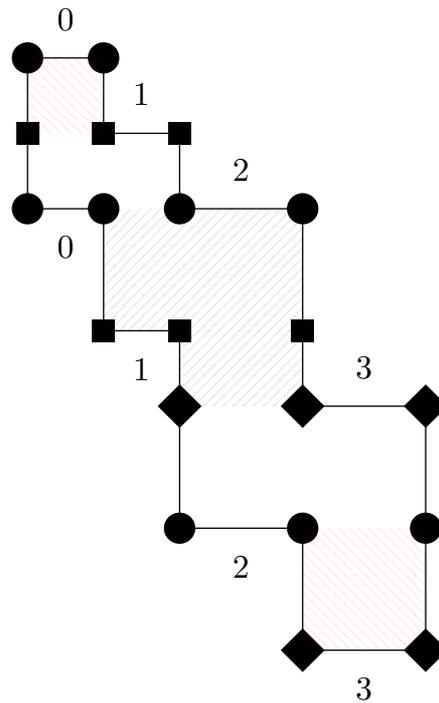
Secondly, we prove that it is always possible to degenerate from a rank  $k$  affine submanifold in  $\mathcal{H}(4k - 2)$  to rank three in  $\mathcal{H}(10)$  and to rank two in  $\mathcal{H}(6)$ . Moreover, we must always be able to degenerate from rank three in  $\mathcal{H}(10)$  to rank two in  $\mathcal{H}(n, m, \ell)$ , where  $n + m + \ell = 6$  is a non-negative partition. Then we argue that no such affine submanifold in the boundary of a rank three affine submanifold  $\mathcal{H}(10)$  could exist.

To do this we classify 4-cylinder diagrams in the quadratic rank two affine submanifold  $\mathcal{N} \subset \mathcal{H}(6)$ . Then we consider all possible ways of splitting the zero of

order six into three zeros and exclude each case to show that no such intermediate affine submanifold could exist.

To address the combinatorics of splitting the zero, we use a classical concept from graph theory called a subdivision of a graph. A *cylinder diagram* is the topological data consisting of cylinders with marked saddle connections in their boundaries and identifications specified among them. The *dual graph* of a cylinder diagram is the directed graph where each vertex represents a cylinder and each directed edge corresponds to a saddle connection connecting the bottom cylinder of the saddle connections to the top one. A *simple cylinder* is a cylinder with exactly one saddle connection in each boundary. Adding a simple cylinder to a saddle connection corresponds to subdividing an edge of a dual graph. We prove that a hypothetical  $\mathcal{M}' \subset \mathcal{H}(\ell, m, n)$  must have a cylinder diagram with two simple cylinders. By considering all admissible subdivisions of the dual graphs, we deduce all necessary cases to analyze. A depiction of this procedure is provided in the figure below.





## REFERENCES

- [Api17] Paul Apisa, *Periodic points in genus two: Holomorphic sections over Hilbert modular varieties, Teichmüller dynamics, and billiards*, Preprint arXiv:1710.05505 (2017), 1–49.
- [BM12] Matt Bainbridge and Martin Möller, *The Deligne-Mumford compactification of the real multiplication locus and Teichmüller curves in genus 3*, Acta Math. **208** (2012), no. 1, 1–92. MR 2910796
- [EM18] Alex Eskin and Maryam Mirzakhani, *Invariant and stationary measures for the  $SL(2, \mathbb{R})$  action on moduli space*, Publ. Math. Inst. Hautes Études Sci. **127** (2018), 95–324. MR 3814652
- [EMM15] Alex Eskin, Maryam Mirzakhani, and Amir Mohammadi, *Isolation, equidistribution, and orbit closures for the  $SL(2, \mathbb{R})$  action on moduli space*, Ann. of Math. (2) **182** (2015), no. 2, 673–721. MR 3418528
- [EMMW18] Alex Eskin, Curtis T. McMullen, Ronen E. Mukamel, and Alex Wright, *Billiards, quadrilaterals and moduli spaces*, Preprint (2018), 1–63.
- [Vee89] W. A. Veech, *Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards*, Invent. Math. **97** (1989), no. 3, 553–583. MR 1005006 (91h:58083a)
- [Wri15] ———, *Cylinder deformations in orbit closures of translation surfaces*, Geom. Topol. **19** (2015), no. 1, 413–438. MR 3318755

## Enumerative geometry of the moduli space of curves and integrable systems

ALEXANDR BURYAK

We consider the moduli space  $\overline{\mathcal{M}}_{g,n}$  of stable algebraic curves of genus  $g$  with  $n$  marked points. In order to describe the class of enumerative invariants on the moduli space  $\overline{\mathcal{M}}_{g,n}$ , we have to introduce the notion of a *cohomological field theory*, first appeared in the work of M. Kontsevich and Yu. Manin [9].

Let  $V$  be a finite dimensional vector space over  $\mathbb{C}$  with a fixed basis  $e_1, \dots, e_N \in V$  and  $N := \dim V$ . Let us also fix a non-degenerate symmetric matrix  $\eta = (\eta_{\alpha\beta})_{1 \leq \alpha, \beta \leq N}$  with complex coefficients. Denote by  $H^{\text{even}}(\overline{\mathcal{M}}_{g,n}, \mathbb{C})$  the even part in the cohomology of  $\overline{\mathcal{M}}_{g,n}$ . A cohomological field theory (CohFT) is a collection of linear homomorphisms  $c_{g,n}: V^{\otimes n} \rightarrow H^{\text{even}}(\overline{\mathcal{M}}_{g,n}, \mathbb{C})$ , defined for all  $g$  and  $n$ , and satisfying the following properties:

1. The map  $c_{g,n}$  is  $S_n$ -equivariant.
2. We have  $\eta_{\alpha\beta} = c_{0,3}(e_\alpha \otimes e_\beta \otimes e_1) \in H^*(\overline{\mathcal{M}}_{0,3}, \mathbb{C}) = \mathbb{C}$ .
3. For the forgetful map  $\pi: \overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$ , which forgets the last marked point, we have  $\pi^* c_{g,n}(e_{\alpha_1} \otimes \dots \otimes e_{\alpha_n}) = c_{g,n+1}(e_{\alpha_1} \otimes \dots \otimes e_{\alpha_n} \otimes e_1)$ .
4. a) For the gluing map  $\text{gl}: \overline{\mathcal{M}}_{g_1, n_1+1} \times \overline{\mathcal{M}}_{g_2, n_2+1} \rightarrow \overline{\mathcal{M}}_{g,n}$ ,  $n = n_1 + n_2$ ,  $g = g_1 + g_2$ , we have

$$\text{gl}^* c_{g,n}(\otimes_{i=1}^n e_{\alpha_i}) = \sum_{\alpha, \beta} c_{g_1, n_1+1}(\otimes_{i=1}^{n_1} e_{\alpha_i} \otimes e_\alpha) \times c_{g_2, n_2+1}(\otimes_{j=1}^{n_2} e_{\alpha_{n_1+j}} \otimes e_\beta) \eta^{\alpha\beta},$$

where the coefficients  $\eta^{\alpha\beta}$  are defined by  $(\eta^{\alpha\beta}) := \eta^{-1}$ .

- b) For the gluing map  $\text{gl}: \overline{\mathcal{M}}_{g-1, n+2} \rightarrow \overline{\mathcal{M}}_{g,n}$  we have

$$\text{gl}^* c_{g,n}(\otimes_{i=1}^n e_{\alpha_i}) = \sum_{\alpha, \beta} c_{g-1, n+2}(\otimes_{i=1}^n e_{\alpha_i} \otimes e_\alpha \otimes e_\beta) \eta^{\alpha\beta}.$$

The class  $\psi_i \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{C})$  is defined as the first Chern class of the line bundle over  $\overline{\mathcal{M}}_{g,n}$  formed by the cotangent lines at the  $i$ -th marked point. The *correlators* of our CohFT are defined by

$$\langle \tau_{\alpha_1, d_1} \tau_{\alpha_2, d_2} \dots \tau_{\alpha_n, d_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} c_{g,n}(e_{\alpha_1} \otimes \dots \otimes e_{\alpha_n}) \prod_{i=1}^n \psi_i^{d_i}.$$

The Gromov–Witten invariants of algebraic varieties are particular examples of such correlators [9]. The so-called Fan–Jarvis–Ruan–Witten invariants [7, 11] are also contained in the class of correlators of CohFTs.

Among the variety of techniques developed for the description of the correlators of CohFTs, we would like to mention the following:

- Mirror symmetry of different kinds.
- Modular forms: the Gromov–Witten invariants of some classes of target varieties can be effectively described using modular forms.
- Integrable systems.

Here we would like to focus on the approach, based on the theory of integrable systems. The starting point for it was the conjecture of E. Witten [10]. Consider the trivial cohomological field theory, given by

$$V = \mathbb{C} = \langle e_1 \rangle, \quad \eta_{1,1} = 1, \quad c_{g,n}(e_1^{\otimes n}) = 1 \in H^0(\overline{\mathcal{M}}_{g,n}, \mathbb{C}).$$

Let  $\varepsilon$  and  $t_0, t_1, t_2, \dots$  be formal variables and consider the generating series

$$F(t_0, t_1, \dots, \varepsilon) := \sum_{g,n} \frac{\varepsilon^{2g}}{n!} \sum_{d_1, \dots, d_n \geq 0} \langle \tau_{1,d_1} \tau_{1,d_2} \cdots \tau_{1,d_n} \rangle_g \prod_{i=1}^n t_{d_i} \in \mathbb{C}[[t_0, t_1, \dots, \varepsilon]].$$

E. Witten conjectured that the second derivative  $w := \frac{\partial^2 F}{\partial t_0^2}$  satisfies the *Korteweg – de Vries (KdV) hierarchy*

$$\begin{aligned} \frac{\partial w}{\partial t_1} &= w w_x + \frac{\varepsilon^2}{12} w_{xxx}, \\ \frac{\partial w}{\partial t_2} &= \frac{w^2 w_x}{2} + \varepsilon^2 \left( \frac{w w_{xxx}}{12} + \frac{w_x w_{xx}}{6} \right) + \varepsilon^4 \frac{w_{xxxx}}{240}, \\ \frac{\partial w}{\partial t_3} &= \frac{w^3 w_x}{6} + \varepsilon^2 \left( \frac{w^2 w_{xxx}}{24} + \frac{w_x^3}{24} + \frac{w w_x w_{xx}}{6} \right) + \\ &\quad + \varepsilon^4 \left( \frac{w w_{xxxx}}{240} + \frac{w_{xxxx} w_x}{80} + \frac{w_{xxx} w_{xx}}{48} \right) + \varepsilon^6 \frac{w_{xxxxxx}}{6720}, \\ &\vdots \end{aligned}$$

where we identify  $x = t_0$ . The famous KdV equation is the first equation of the hierarchy. Witten's conjecture was proved by M. Kontsevich [8]. This result allows to determine the generating function  $F$  uniquely.

Consider now an arbitrary cohomological field theory. Let  $w_d^\alpha$ ,  $1 \leq \alpha \leq N$ ,  $d \geq 0$ , be formal variables; we also use the notations  $w_0^\alpha = w^\alpha$ ,  $w_1^\alpha = w_x^\alpha$ ,  $w_2^\alpha = w_{xx}^\alpha, \dots$ . A *differential polynomial* is a polynomial in the variables  $w_d^\alpha$ ,  $d \geq 1$ , with coefficients in the ring of formal power series  $\mathbb{C}[[w^1, \dots, w^N]]$ . The set of differential polynomials forms a ring, which we denote by  $\mathcal{A}_N$ . Let  $t_d^\alpha$ ,  $1 \leq \alpha \leq N$ ,  $d \geq 0$ , be formal variables and consider the generating series

$$F(t_*^*, \varepsilon) := \sum_{g,n} \frac{\varepsilon^{2g}}{n!} \sum_{\substack{1 \leq \alpha_1, \dots, \alpha_n \leq N \\ d_1, \dots, d_n \geq 0}} \langle \tau_{\alpha_1, d_1} \tau_{\alpha_2, d_2} \cdots \tau_{\alpha_n, d_n} \rangle_g \prod_{i=1}^n t_{d_i}^{\alpha_i} \in \mathbb{C}[[t_*^*, \varepsilon]].$$

**Theorem 1.** [4] *If the CohFT satisfies an additional assumption of semisimplicity, then there exist unique elements  $K_{\beta,d}^\alpha \in \mathcal{A}_N[[\varepsilon]]$ , such that the functions  $w^\alpha := \eta^{\alpha\mu} \frac{\partial^2 F}{\partial t_0^1 \partial t_0^\mu}$  satisfy the system of PDEs*

$$(1) \quad \frac{\partial w^\alpha}{\partial t_d^\beta} = K_{\beta,d}^\alpha, \quad 1 \leq \alpha, \beta \leq N, \quad d \geq 0,$$

where we identify  $x = t_0^1$ .

The semisimplicity condition is satisfied in a large class of interesting examples. In [4] we also constructed a Hamiltonian structure for the system (1). B. Dubrovin and Y. Zhang conjectured that in the case of a homogeneous cohomological field theory the system (1) possesses a bi-Hamiltonian structure [6]. As it is well-known, such a structure gives an efficient reconstruction procedure for the functions  $K_{\beta,d}^{\alpha}$ . This conjecture is one of the most important open problems in the area.

Trying to find new approaches to the systems (1) and, in particular, to the Dubrovin-Zhang conjecture, in [1], I introduced a new construction of a Hamiltonian system of PDEs associated to an arbitrary CohFT. The key role in the construction is played by the double ramification (DR) cycle. I also conjectured that the new system, called the *DR hierarchy*, is related to the system (1) by a polynomial change of variables.

During the last several years, together with Paolo Rossi and also with B. Dubrovin and J. Guéré, we were studying the structure of the DR hierarchy and its relation to the system (1) (see e.g. [2, 3, 5]). Parts of this project are presented in the companion talk of Paolo Rossi.

#### REFERENCES

- [1] A. Buryak, *Double ramification cycles and integrable hierarchies*, Communications in Mathematical Physics **336** (2015), no. 3, 1085–1107.
- [2] A. Buryak, B. Dubrovin, J. Guéré, P. Rossi, *Tau-structure for the Double Ramification Hierarchies*, Communications in Mathematical Physics **363** (2018), no. 1, 191–260.
- [3] A. Buryak, J. Guéré, P. Rossi, *DR/DZ equivalence conjecture and tautological relations*, arXiv:1705.03287.
- [4] A. Buryak, H. Posthuma, S. Shadrin, *A polynomial bracket for the Dubrovin-Zhang hierarchies*, Journal of Differential Geometry **92** (2012), no. 1, 153–185.
- [5] A. Buryak, P. Rossi, *Recursion Relations for Double Ramification Hierarchies*, Communications in Mathematical Physics **342** (2016), no. 2, 533–568.
- [6] B. Dubrovin, Y. Zhang, *Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov-Witten invariants*, arXiv:math/0108160.
- [7] H. Fan, T. Jarvis, Y. Ruan, *The Witten equation, mirror symmetry, and quantum singularity theory*, Annals of Mathematics **178** (2013), no. 1, 1–106.
- [8] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*, Communications in Mathematical Physics **147** (1992), no. 1, 1–23.
- [9] M. Kontsevich, Yu. Manin, *Gromov-Witten classes, quantum cohomology, and enumerative geometry*, Communications in Mathematical Physics **164** (1994), no. 3, 525–562.
- [10] E. Witten, *Two-dimensional gravity and intersection theory on moduli space*, Surveys in differential geometry (Cambridge, MA, 1990), 243–310, Lehigh Univ., Bethlehem, PA, 1991.
- [11] E. Witten, *Algebraic geometry associated with matrix models of two-dimensional gravity*, Topological methods in modern mathematics (Stony Brook, NY, 1991), 235–269, Publish or Perish, Houston, TX, 1993.

## Moduli of Differentials

DAWEI CHEN

Let  $\mu = (m_1, \dots, m_n)$  be a collection of integers such that  $\sum_{i=1}^n m_i = 2g - 2$ . Let  $\mathbb{P}\mathcal{H}(\mu)$  be the projectivized stratum parameterizing differentials  $\omega$  (up to scale) on genus  $g$  Riemann surfaces such that the associated divisor of  $\omega$  is of zero and pole type  $\mu$ , i.e.  $(\omega) = \sum_{i=1}^n m_i p_i$  for distinct points  $p_1, \dots, p_n$ . In this talk we study some aspects of  $\mathbb{P}\mathcal{H}(\mu)$  from the viewpoint of algebraic geometry and compare them to the moduli space of pointed genus  $g$  curves  $\mathcal{M}_{g,n}$ .

First we consider complete subvarieties. Diaz ([7]) first proved that any complete subvariety in  $\mathcal{M}_g$  has dimension bounded above by  $g - 2$ . The bound is not known to be sharp in general. In particular we do not know whether  $\mathcal{M}_4$  contains a complete surface. For the case of differentials, the author ([3]) proved that the strata of meromorphic differentials (i.e. some  $m_i < 0$ ) do not contain any complete curve. It remains to be an open question whether the strata of holomorphic differentials can contain a complete curve.

Next we consider tautological rings. The tautological ring of  $\mathcal{M}_{g,n}$  is generated by Mumford's  $\kappa_j$  classes and the  $\psi_i$  classes associated to each marked point. In general the tautological ring structure of  $\mathcal{M}_{g,n}$  can be complicated. We define the tautological ring of the strata  $\mathbb{P}\mathcal{H}(\mu)$  to be generated by the tautological classes pulled back from  $\mathcal{M}_{g,n}$  together with the tautological line bundle class  $\eta = c_1(\mathcal{O}(-1))$  of the projective Hodge bundle. The author ([4]) proved that the tautological ring of the strata of differentials without simple poles (i.e. all  $m_i \neq -1$ ) is generated by  $\eta$  only. It remains to be an open question which powers of  $\eta$  vanish on the strata.

Finally we consider positivity of divisor classes. Cornalba and Harris ([6]) proved that  $a\lambda - b\delta$  is ample on the Deligne-Mumford compactification  $\overline{\mathcal{M}}_g$  if and only if  $a > 11b > 0$ , where  $\lambda$  is the first Chern class of the Hodge bundle and  $\delta$  is the total boundary divisor class. The author ([5]) proved that  $a(\kappa_1 + \psi) - b\eta$  is ample on the compactified stratum  $\overline{\mathbb{P}\mathcal{H}}_{g,n}(\mu)$  (in the sense of [1]) if and only if  $a > b > 0$ , where  $\psi = \sum_{i=1}^n \psi_i$  is the total  $\psi$  class. It remains open to study positivity of other divisor classes on the compactified strata.

We remark that the above results and questions can be generalized to the strata of  $k$ -differentials for all  $k$  ([2, 3, 4, 5]).

## REFERENCES

- [1] M. Bainbridge, D. Chen, Q. Gendron, S. Grushevsky, and M. Möller, *Compactification of strata of Abelian differentials*, Duke Math. J. **167** (2018), no. 12, 2347–2416.
- [2] M. Bainbridge, D. Chen, Q. Gendron, S. Grushevsky, and M. Möller, *Strata of  $k$ -differentials*, Algebr. Geom., to appear.
- [3] D. Chen, *Affine geometry of strata of differentials*, J. Inst. Math. Jussieu, to appear.
- [4] D. Chen, *Tautological ring of strata of differentials*, Manuscripta Math., to appear.

- [5] D. Chen, *Positivity of divisor classes on the strata of differentials*, arXiv:1803.11268.
- [6] M. Cornalba and J. Harris, *Divisor classes associated to families of stable varieties, with applications to the moduli space of curves*, Ann. Sci. École Norm. Sup. (4) **21** (1988), no. 3, 455–475.
- [7] S. Diaz, *A bound on the dimensions of complete subvarieties of  $\mathcal{M}_g$* , Duke Math. J. **51** (1984), no. 2, 405–408.

## The Masur-Veech volume of principal strata of quadratic differentials

VINCENT DELECROIX

(joint work with Élise Goujard, Anton Zorich, Peter Zograf)

The cotangent space to the moduli space  $\mathcal{M}_{g,n}$  of complex curve of genus  $g$  and  $n$  marked points can be identified to  $\mathcal{Q}_{g,n}$  the moduli space of pairs  $(C, q)$ , where  $C$  is a hyperbolic surface of genus  $g$  with  $n$  cusps and  $q$  is a holomorphic quadratic differential on  $C$  with finite volume  $\int_C |q| < \infty$  (it can be checked that this condition is equivalent to the fact that  $q$  has at most simple poles at the punctures).

This cotangent space  $\mathcal{Q}_{g,n}$  comes with a natural symplectic form and associated volume form called the Masur–Veech volume form. In this article we provide a formula for the volume of the level hypersurface of quadratic differentials of area 1. We also provide a formula of similar nature for the so called Siegel–Veech constant of  $\mathcal{Q}_{g,n}$ .

As a concrete application, we get a large table of exact numerical values of the volumes and Siegel–Veech constants for all small  $g$  and  $n$  extending previously known data of Goujard [4, 5] based on completely different approach designed by Eskin and Okounkov [3].

Both the volume and the Siegel–Veech constant are expressed as polynomials in the intersection numbers of  $\psi$ -classes supported on the boundary components of the Deligne-Mumford compactification  $\overline{\mathcal{M}}_{g,n}$ . The formula we obtain are derived from lattice point counting involving the Kontsevich [6] volume polynomials  $N_{g,n}(b_1^2, \dots, b_n^2)$  that also appear in Mirzakhani [7] topological recursion for the Weil–Petersson volumes of the moduli space  $\mathcal{M}_{g,n}$ . See also [8].

One can recover the  $g = 0$  formulas from [1] by using the explicit values of the intersection of  $\psi$ -classes in genus 0.

## REFERENCES

- [1] J. Athreya, A. Eskin, and A. Zorich, *Counting generalized Jenkins–Strebel differentials*, Geometriae Dedicata, **170:1** (2014), 195–217.
- [2] A. Eskin, H. Masur, A. Zorich, *Moduli spaces of Abelian differentials: the principal boundary, counting problems and the Siegel–Veech constants*, Publications Mathématiques de l’IHÉS, **97** (1) (2003), 61–179.
- [3] A. Eskin, A. Okounkov, *Pillowcases and quasimodular forms*, Algebraic Geometry and Number Theory, Progress in Mathematics 253 (2006), 1–25.
- [4] E. Goujard, *Siegel–Veech constants for strata of moduli spaces of quadratic differentials*, GAFA, **25:5** (2015), 1440–1492.
- [5] E. Goujard, *Volumes of strata of moduli spaces of quadratic differentials: getting explicit values*, Ann. Inst. Fourier, **66** no. 6 (2016), 2203–2251.

- [6] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*, Comm. Math. Phys. **147** (1992), 1–23.
- [7] M. Mirzakhani, *Weil–Petersson volumes and intersection theory on the moduli space of curves*, Journal of Amer. Math. Soc. **20** (2007), no. 1, 1–23.
- [8] P. Norbury, *Counting lattice points in the moduli space of curves*, Math. Res. Lett. **17** (2010), no. 3, 467–481.

## Structural stability in the moduli space of algebraic foliations of the complex projective plane

BERTRAND DEROIN

(joint work with Aurélien Alvarez)

I showed the existence of a non trivial stability component in the space of degree two algebraic foliations of the complex projective plane, consisting of foliations having a non trivial Fatou component: a fibration by discs over a Riemann surface of genus three. This result can be opposed to the famous result of Y.Iliashenko (addressed at the ICM 78) establishing minimality, ergodicity and rigidity for generic foliations of the plane that preserve a line, and its generalizations by F.Loray/J.Rebelo or T.Golenishcheva-Kutusova/V.Kleptsyn.

## Lattice point enumeration in moduli spaces of higher differentials

PHILIP ENGEL

Let  $N = 1, 2, 3, 4$ , or  $6$ , and let  $\mu = (\mu_i)$  be a list of non-zero integers such that  $\mu_i > -N$ . Define a *stratum* of  $N$ -ic differentials as a moduli space of pairs

$$\mathcal{H}_N(\mu) := \{(\Sigma, \omega) \mid \Sigma \text{ a Riemann surface, } \omega \in H^0(\Sigma, K_\Sigma^{\otimes N}) \text{ such that } \operatorname{div}(\omega) = \sum \mu_i p_i\}.$$

There are local flat coordinates  $\zeta$  on  $\Sigma$  away from  $p_i$  such that  $d\zeta = \omega^{1/N}$ . This endows the pair  $(\Sigma, \omega)$  with a flat metric with cone singularities at the points  $p_i$  whose monodromy lies in the order  $N$  rotation group  $\langle \zeta_N \rangle$ . When  $N = 3, 4, 6$  there is a natural subset of  $\mathcal{H}_N(\mu)$  consisting of flat surfaces that admit a tiling into regular hexagons, squares, or triangles respectively. This subset is defined by  $\int_{p_i}^{p_j} \omega^{1/N} \in \mathbb{Z}[\zeta_N]$ , and therefore forms a lattice in the period coordinates. The curvatures of the tiling are determined by the numbers  $\mu_i$ . The main focus of this talk is the following generalization of work of Eskin and Okounkov [EO06]:

**Theorem 1.** [Eng17] *Consider a tile with cyclic order  $N$  symmetry whose translates tessellate  $\mathbb{R}^2$ . Let  $c_d(\mu)$  be the weighted number of tilings of a compact oriented surface with  $d$  tiles and non-zero curvatures  $\mu_i$ . Under the substitution  $q = e^{2\pi i\tau}$ , the generating function*

$$h_N(\mu, q) = \sum_{d \geq 1} c_d(\mu) q^d$$

*is in the ring of quasimodular forms for  $\Gamma_1(N)$ .*

Since quasimodular forms of given level and bounded weight form a finite-dimensional space, the generating function for a tiling problem can be determined by finitely many coefficients. Most of the moduli spaces  $\mathcal{H}_N(\mu)$  are not arithmetic, so the arithmetic techniques of my joint work with P. Smillie [ES18] are no longer available. Instead, the proof relies on representation-theoretic techniques pioneered by Bloch, Eskin, and Okounkov. They proved Theorem 1 in the cases  $N = 1$  [EO01, BO00] and  $N = 2$  [EO06], generalizing work of Dijkgraaf [Dij95].

Every surface tiled by (resp.) equilateral triangles, squares, or vertex-bicolored hexagons admits a locally isometric map to an *elliptic orbifold*, the quotient of an elliptic curve by a cyclic group of order (resp.) 6, 4, or 3. Conversely, given a covering of one of the elliptic orbifolds satisfying certain ramification conditions, the cover admits a natural tiling. See Figure 1 for an example of such a branched cover. Thus, the Hurwitz numbers of these elliptic orbifolds are counts of tiled surfaces, with the ramification profile encoding the list of curvatures.

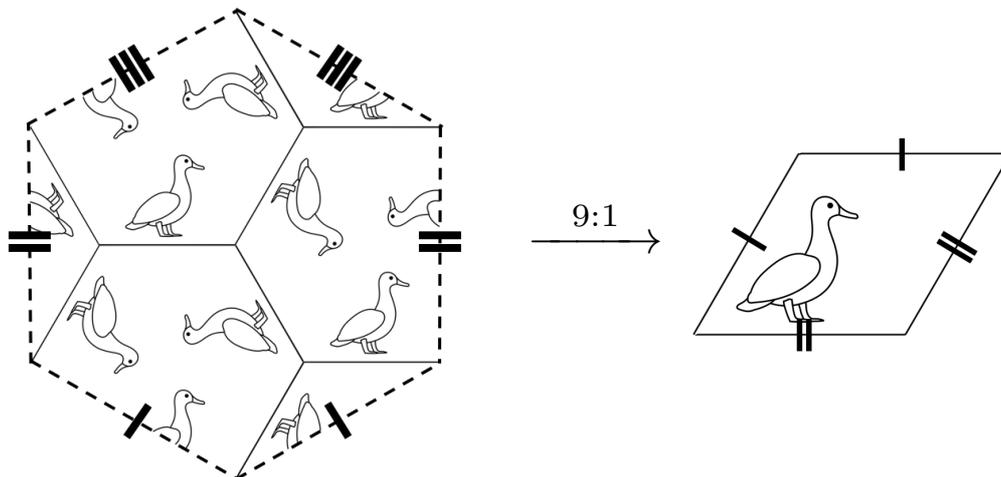


FIGURE 1. A degree 9 branched cover of  $\mathbb{P}_{3,3,3}$  which produces a surface tiled by hexagons that have order 3 rotational symmetry.

There is an approach to computing Hurwitz numbers which connects to physics. Let  $V := \text{span}\{\underline{i} : i \in \frac{1}{2} + \mathbb{Z}\}$  be a vector space with basis indexed by half-integers. Define the (charge zero subspace of) the *half-infinite wedge* or *Fock space* to be the span of formal symbols

$$\Lambda_0^{\infty/2} V := \text{span}\{\underline{i}_1 \wedge \underline{i}_2 \wedge \cdots \mid i_1 > i_2 > \cdots \text{ and } i_N = -N + \frac{1}{2} \text{ for } N \gg 0\},$$

which are declared to be orthonormal for an inner product  $\langle \cdot | \cdot \rangle$ . There are two canonical bases of this space,  $\{w_\mu\}$  and  $\{v_\lambda\}$ , both indexed by the set of all partitions. The representation theory of all symmetric groups is encoded by the formula  $\langle w_\mu | v_\lambda \rangle = \chi^\lambda(\mu)$  for the character of the irreducible representation indexed by  $\lambda$ , evaluated on the cycle type  $\mu$ . The *energy* is the operator  $Hv_\lambda = |\lambda|v_\lambda$ . One defines *vertex operators*  $\mathcal{E}(x)$  acting on Fock space, that depend on an analytic parameter  $x$ . The Hurwitz numbers of the elliptic orbifold of order  $N$  can be

extracted from the Taylor series expansion about  $(x_1, \dots, x_n) = (0, \dots, 0)$  of the so-called  $n$ -point function

$$F_N(x_1, \dots, x_n) := \frac{\operatorname{tr} q^H \mathcal{E}(x_1) \dots \mathcal{E}(x_n) \mathfrak{W}_N}{\operatorname{tr} q^H \mathfrak{W}_N}.$$

Here  $\mathfrak{W}_N$  is an operator generalized from the  $N = 2$  case [EO06]. Proving that this trace encodes the Hurwitz numbers relies on some combinatorics, in particular, the notions of an  $N$ -quotient and  $N$ -core—partition-theoretic analogues of the quotient and remainder for integer division.

All these operators are tensor operators with respect to a tensor product decomposition of Fock space. Thus, the above trace is an infinite product over the tensor factors, and each factor is an explicitly computable rational function in  $x_i$  and  $q$ . These factors combine via the Jacobi triple product formula to give theta functions, whose Taylor series expansions consist of quasi-modular forms, proving Theorem 1. Furthermore, this gives an explicit formula

$$F_N(\ln x_1, \dots, \ln x_n) = \frac{1}{(2\pi i)^n} \oint_{|y_i|=c_i} \dots \oint \prod_i \frac{dy_i}{y_i} \cdot \frac{\vartheta(x_i y_i)}{\vartheta(x_i) \vartheta(y_i)} \\ \prod_{i < j} \frac{\vartheta(y_i/y_j) \vartheta(x_i y_i/x_j y_j)}{\vartheta(x_i y_i/y_j) \vartheta(y_i/x_j y_j)} \prod_{i,r} \frac{\vartheta(\zeta_N^r y_i)^{1/N}}{\vartheta(\zeta_N^r x_i y_i)^{1/N}}$$

for the  $n$ -point function, where  $\vartheta(x) = \vartheta(x, q)$  is the Jacobi theta function, with the argument  $q$  suppressed in the notation. The proof of Theorem 1 fails for triangulations, because of some representation-theoretic complications, but my recent work gives the following generalization:

**Theorem 2.** [Eng18] *All natural generating functions of Hurwitz numbers of the quotient of an elliptic curve by  $\langle \zeta_N \rangle$  are quasimodular forms of level  $\Gamma(N)$ . In particular, the generating function  $\Delta(\mu, q)$  whose  $q^n$  coefficient is the number of triangulations with  $2n$  triangles and non-zero curvatures  $\mu_i$  is a mixed weight quasimodular form for  $\Gamma_1(6)$ .*

The tiled surfaces equidistribute in the moduli space with respect to the Masur-Veech volume. We can conclude the following formula:

$$\operatorname{Vol}(\mathcal{H}_N^1(\mu)) = \lim_{q \rightarrow 1} \frac{h_N(\mu, q)(1-q)^{\dim}}{(\text{Area of Tile})^{\dim} \dim!}.$$

Using the (quasi)modular transformation rule for  $\tau \mapsto -1/\tau$  allows us to extract the asymptotic behavior of  $h_N(\mu, q)$  as  $q \rightarrow 1$ . The rationality of  $h_N(\mu, q)$  implies:

**Theorem 3.** *Let  $\theta_N := \zeta_N - \zeta_N^{-1}$ . When  $N = 3, 4, 6$ ,*

$$(i\theta_N)^{-\dim \mathcal{H}_N} \operatorname{Vol}(\mathcal{H}_N^1(\vec{\mu})) \in \mathbb{Q}[2\pi i\theta_N] \cup \{\infty\}.$$

## REFERENCES

- [BO00] Spencer Bloch and Andrei Okounkov. The character of the infinite wedge representation, *Advances in Mathematics* 149.1:1-60, 2000.
- [Dij95] R. Dijkgraaf. Mirror symmetry and elliptic curves, *Progress in Math*, Vol. 129, The Moduli Space of Curves, 1995.
- [Eng17] Philip Engel. Hurwitz theory of elliptic orbifolds, I. arXiv:1706.06738, 2017.
- [Eng18] Philip Engel. Hurwitz theory of elliptic orbifolds, II. arXiv:1809.07434, 2018.
- [ES18] Philip Engel and Peter Smillie. The number of convex tilings of the sphere by triangles, squares, or hexagons. *Geometry and Topology*, Vol. 22: 2839-2864, 2018.
- [EO01] Alex Eskin and Andrei Okounkov. Asymptotics of numbers of branched coverings of a torus and volumes of moduli spaces of holomorphic differentials, *Inventiones Mathematicae* 145.1:59-103, 2001.
- [EO06] Alex Eskin and Andrei Okounkov. Pillowcases and quasimodular forms, *Algebraic geometry and number theory*, Birkhäuser Boston: 1-25, 2006.

**On stationary measure rigidity and orbit closures for actions of non-abelian groups.**

ALEX ESKIN

(joint work with Aaron Brown, Federico Rodriguez-Hertz, Simion Filip)

I described joint work in progress with Aaron Brown, Federico Rodriguez-Hertz and Simion Filip. Our aim is to find some analogue, in the context of smooth dynamics, of Ratner's theorems on unipotent flows. This would be a (partial) generalization of the results of Benoist-Quint and my work with Elon Lindenstrauss in the homogeneous setting, the results of Brown and Rodriguez-Hertz in dimension 2, and my results with Maryam Mirzakhani in the setting of Teichmüller dynamics.

**An introduction to the geometry of the moduli space of curves**

GAVRIL FARKAS

Following a principle due to Mumford, most moduli spaces that appear in algebraic geometry (classifying curves, abelian varieties,  $K3$  surfaces) are of general type, with a finite number of exceptions, which are unirational, or at least uniruled. Understanding the transition from negative Kodaira dimension to being of general type is usually quite difficult. With one exception (the moduli space of spin curves), for all these moduli spaces there are notorious open cases, when the Kodaira dimension is not known. The aim of the talk was to shed some light on this change of the birational nature of the moduli space of curves.

In a series of landmark papers [7], [6], [2] published in the 1980s, Harris, Mumford and Eisenbud proved that  $\overline{\mathcal{M}}_g$  is a variety of general type for  $g > 23$ . This contrasts with the classical result of Severi [9] that  $\overline{\mathcal{M}}_g$  is unirational for  $g \leq 10$  (see [1] for a beautiful modern treatment) and with the more recent results of Chang-Ran, Sernesi, Verra [10] and Schreyer, which summarized, amount to the statement that  $\overline{\mathcal{M}}_g$  is uniruled for  $g \leq 16$ . The Slope Conjecture of Harris and

Morrison predicted that the Brill-Noether divisors are the effective divisors on  $\overline{\mathcal{M}}_g$  having minimal slope  $6 + \frac{12}{g+1}$ . This led people to expect that the moduli space changes from uniruledness to being of general type precisely at genus  $g = 23$ . However the Slope Conjecture turned out to be false and there are instances of effective divisors on  $\overline{\mathcal{M}}_g$  for infinitely many genera  $g \geq 10$  having slope less than  $6 + \frac{12}{g+1}$ , see [4], [5]. In view of these examples it is to be expected that there should be an effective divisor of slope less than  $\frac{13}{2} = 6 + \frac{12}{24}$  on  $\overline{\mathcal{M}}_{23}$  as well, which would imply that  $\overline{\mathcal{M}}_{23}$  is of general type. The best known result on  $\overline{\mathcal{M}}_{23}$  is the statement  $\kappa(\overline{\mathcal{M}}_{23}) \geq 2$ , proven in [3] via a study of the relative position of the three Brill-Noether divisors.

The main aim of the talk was to discuss this transition case and reduce the calculation of the Kodaira dimension of  $\overline{\mathcal{M}}_{23}$  to a transversality statement for generic curves of genus 23. In two very recent breakthrough papers, this transversality statement, also known as the *Strong Maximal Rank Conjecture* has been established independently by Liu, Osserman, Teixidor, Zhang and Jensen, Payne respectively.

We begin by describing our construction of an effective divisor on  $\mathcal{M}_{23}$ . By Brill-Noether theory, a general curve  $C$  of genus 23 carries a two-dimensional family of linear series  $L \in W_{26}^6(C)$ , all satisfying  $h^1(C, L) = 3$ . Each of these linear series is complete and very ample. Consider the multiplication map

$$\phi_L : \text{Sym}^2 H^0(C, L) \rightarrow H^0(C, L^{\otimes 2}).$$

By Riemann-Roch  $h^0(C, L^{\otimes 2}) = 30$ , whereas  $\dim \text{Sym}^2 H^0(C, L) = 28$ . Imposing the condition that  $\phi_L$  be non-injective, one expects a codimension 3 locus inside the parameter space of pairs  $[C, L]$ . Since this parameter space has 2-dimensional fibres over  $\mathcal{M}_{23}$ , by projection, one expects a divisor inside the moduli space  $\mathcal{M}_{23}$ .

**Theorem 1.** *The following locus consisting of curves of genus 23*

$$\mathfrak{D} := \left\{ [C] \in \mathcal{M}_{23} : \exists L \in W_{26}^6(C) \text{ with} \right. \\ \left. \text{Sym}^2 H^0(C, L) \xrightarrow{\phi_L} H^0(C, L^{\otimes 2}) \text{ not injective} \right\}$$

*is a virtual divisor on  $\mathcal{M}_{23}$ . The virtual class of its compactification inside  $\overline{\mathcal{M}}_{23}$  equals*

$$[\tilde{\mathfrak{D}}]^{\text{virt}} = \frac{4}{9} \binom{19}{8} \left( 470749\lambda - 72725 \delta_0 - 401951 \delta_1 - \sum_{j=2}^{11} b_j \delta_j \right) \in CH^1(\overline{\mathcal{M}}_{23}),$$

*where  $b_j \geq b_1$  for  $j \geq 2$ . In particular,  $s([\tilde{\mathfrak{D}}]^{\text{virt}}) = \frac{470749}{72725} = 6.473\dots < \frac{13}{2}$ .*

**Corollary 2.** *Assume the virtual class of  $\mathfrak{D}$  equal its actual cohomology class. Then  $\overline{\mathcal{M}}_{23}$  is a variety of general type.*

The question whether the virtual divisor  $\mathfrak{D}$  is an actual divisor is much related to the Maximal Rank Conjecture, originally due to Eisenbud and Harris

and predicting that for a pair  $[C, L]$ , where  $C$  is a general curve of genus  $g$  and  $L \in W_d^r(C)$  is a general linear system, the multiplication of global sections  $\phi_L : \text{Sym}^2 H^0(C, L) \rightarrow H^0(C, L^{\otimes 2})$  is of maximal rank. The conjecture has been the focus of much attention, both a couple of decades ago using embedded degenerations in projective space, as well as recently using tropical geometry, or limit linear series.

A refined version of the Maximal Rank Conjecture, taking into account *every* linear series  $L \in W_d^r(C)$  on a general curve (rather than the general one), has been put forward by myself. The *Strong Maximal Rank Conjecture*, motivated by applications to the birational geometry of the moduli space of curves, predicts that for a general curve  $C$  of genus  $g$  and for positive integers  $r, d$  such that  $0 \leq \rho(g, r, d) \leq r - 2$ , the determinantal variety

$$\left\{ L \in W_d^r(C) : \phi_L : \text{Sym}^2 H^0(C, L) \rightarrow H^0(C, L^{\otimes 2}) \text{ is not of maximal rank} \right\}$$

has the expected dimension. The Strong Maximal Rank Conjecture in the case  $g = 23$ ,  $d = 26$  and  $r = 6$  amounts to the statement that the virtual divisor  $\mathfrak{D}$  on  $\mathcal{M}_{23}$  is a genuine divisor. This case of the Strong Maximal Conjecture has been recently proved in two breakthrough papers by Liu, Osserman, Teixidor, Zhang and by Payne and Jensen.

#### REFERENCES

- [1] E. Arbarello and M. Cornalba, *Footnotes to a paper of Beniamino Segre*, *Mathematische Annalen* **256** (1981), 341–362.
- [2] D. Eisenbud and J. Harris, *The Kodaira dimension of the moduli space of curves of genus  $\geq 23$*  *Inventiones Math.* **90** (1987), 359–387.
- [3] G. Farkas, *The geometry of the moduli space of curves of genus 23*, *Math. Annalen* **318** (2000), 43–65.
- [4] G. Farkas, *Syzygies of curves and the effective cone of  $\overline{\mathcal{M}}_g$* , *Duke Math. Journal* **135** (2006), 53–98.
- [5] G. Farkas, *Koszul divisors on moduli spaces of curves*, *American Journal of Math.* **131** (2009), 819–869.
- [6] J. Harris, *On the Kodaira dimension of the moduli space of curves II: The even genus case*, *Inventiones Math.* **75** (1984), 437–466.
- [7] J. Harris and D. Mumford, *On the Kodaira dimension of  $\overline{\mathcal{M}}_g$* , *Inventiones Math.* **67** (1982), 23–88.
- [8] F. Liu, B. Osserman, M. Teixidor and N. Zhang, *The strong maximal rank conjecture and moduli of curves*, preprint.
- [9] F. Severi, *Sulla classificazione delle curve algebriche e sul teorema d'esistenza di Riemann*, *Rendiconti della R. Accad. Naz. Lincei* **24** (1915), 877–888.
- [10] A. Verra, *The unirationality of the moduli space of curves of genus  $\leq 14$* , *Compositio Mathematica* **141** (2005), 1425–1444.

## Geometric Invariant Theory for syzygies, with applications to $\mathcal{M}_g$

MAKSYM FEDORCHUK

Geometric Invariant Theory (GIT) was developed by Mumford [6] to construct orbit spaces of group actions in algebraic geometry. It quickly found applications in moduli theory, where its most spectacular achievements hinged on Hilbert stability, that is, GIT stability of Hilbert points of embedded varieties. For example, Hilbert stability has played a key role in the construction of the moduli space  $\overline{\mathcal{M}}_g$  of stable curves (by Mumford [7], and Gieseker [3]).

While Hilbert stability is a powerful tool, the moduli spaces that it produces tend to have complicated global geometry. This is evident already in the case of  $\overline{\mathcal{M}}_g$ , whose birational geometry quickly becomes very complex as  $g$  grows. However, GIT can also be used to produce much simpler birational models of  $\overline{\mathcal{M}}_g$ , with the GIT moduli space of plane quartics being one such example in genus 3.

In order to generalize the example of plane quartics to higher genus, we need to extend the notion of Hilbert stability using ideas coming from the Koszul cohomology theory of Green [4]. If a Hilbert point of an embedded variety  $X$  is simply a vector space of a fixed degree equations cutting out  $X$ , then syzygy points of  $X$  should encode relations among these equations. A machinery for defining syzygy points is developed in [2]. We briefly describe it in what follows.

Consider a subscheme  $X$  of a fixed projective space  $\mathbb{P}V^\vee$ . Consider now the  $p^{\text{th}}$ -order linear syzygies among the quadrics cutting out  $X$ . These are given by the Koszul cohomology group

$$K_{p+1,1}(X) \simeq \ker(\wedge^p V \otimes (I_X)_2 \rightarrow \wedge^{p-1} V \otimes (I_X)_3),$$

which is naturally a subspace of  $\frac{\wedge^{p+1} V \otimes V}{\wedge^{p+2} V}$ . The resulting point of the Grassmannian

$$\text{Grass}\left(\dim K_{p+1,1}(X), \frac{\wedge^{p+1} V \otimes V}{\wedge^{p+2} V}\right)$$

is called the  $(p, 2)$ -syzygy point of  $X$ , and will be denoted  $\text{Syz}_{(p,2)}(X)$ .

Since  $\frac{\wedge^{p+1} V \otimes V}{\wedge^{p+2} V}$  is a representation of  $\text{SL}(V)$ , we have an  $\text{SL}(V)$ -action on the corresponding Grassmannian, linearized by the Plücker line bundle, and it is natural to ask the following question: When is the  $(p, 2)$ -syzygy point of  $X$  semistable with respect to the  $\text{SL}(V)$ -action in the sense of GIT? This question is wide open even when  $X$  is a canonical curve of genus  $g$  in  $\mathbb{P}^{g-1}$ .

The first instance where the consideration of syzygy points leads to a genuinely new moduli space is the case of genus 6 curves, which is the smallest genus for which the  $(1, 2)$ -syzygy point of a canonical curve is well-defined and non-trivial. What aids the GIT stability analysis here is the beautiful geometry of canonical genus 6 curves, given by a well-known story, which we now recall. A smooth genus 6 curve can be exactly one of the following: hyperelliptic, trigonal, bielliptic, a plane quintic, or a quadric section of an anti-canonically embedded degree 5 (possibly singular) del Pezzo in  $\mathbb{P}^5$ . A generic curve appears only in the last case, and only

on a smooth del Pezzo. Quadric sections of singular del Pezzos form a divisor in the moduli space called the Gieseker-Petri divisor  $D_{6,4}$  (we follow the taxonomy of [1] for the Gieseker-Petri divisors; in particular  $D_{6,4}$  is the divisor in  $\mathcal{M}_6$  of curves with a base-point-free  $g_4^1$  for which the Petri map is not injective). Since a smooth del Pezzo  $\Sigma$  of degree 5 is unique up to an isomorphism, and has a group of automorphisms isomorphic to  $S_5$ , there is a distinguished birational model of  $\overline{\mathcal{M}}_6$  given by

$$(1) \quad X_6 := \mathbb{P}H^0(\Sigma, -2K_\Sigma)/S_5.$$

This was the model used by Shepherd-Barron to prove rationality of  $\overline{\mathcal{M}}_6$  [8]. It has also reappeared recently in the context of the Hassett-Keel program of  $\overline{\mathcal{M}}_6$ , as the ultimate non-trivial log canonical model of  $\overline{\mathcal{M}}_6$  [5].

It turns out that we can reinterpret  $X_6$  using GIT of  $(0, 2)$  and  $(1, 2)$ -syzygy points of canonical genus 6 curves. This allows us to also construct the penultimate log canonical model of  $\overline{\mathcal{M}}_6$ , which was not known previously, and to realize the contraction of the Gieseker-Petri divisor  $D_{6,4}$  as a VGIT two-ray game.

To state our main result, we introduce some notation. Let  $V = \mathbb{C}^6$ . Let  $\mathbb{H} = \text{Grass}(6, \text{Sym}^2 V) \times \text{Grass}(5, V \otimes \wedge^2 V / \wedge^3 V)$  be the parameter space for  $(0, 2)$  and  $(1, 2)$ -syzygy points of canonical curves of genus 6. Then  $\mathbb{H}$  is equipped with a two-dimensional  $\text{SL}(V)$ -ample cone, whose elements we denote by  $\mathcal{O}(1) \boxtimes \mathcal{O}(\beta)$ .

For every smooth canonically embedded curve  $C \subset \mathbb{P}V^\vee$  of genus six and Clifford index 2, we have a well-defined point  $h(C) = (\text{Syz}_{(0,2)}(C), \text{Syz}_{(1,2)}(C)) \in \mathbb{H}$ . Denote by  $\mathcal{Q} \subset \mathbb{H}$  the Zariski closure of the locus of all such  $h(C)$  in  $\mathbb{H}$ .

We have a natural  $\text{SL}(V)$ -action on  $\mathcal{Q}$ , and we denote by  $\mathcal{Q}^{\text{ss}}(\beta)$  the semistable locus in  $\mathcal{Q}$  with respect to the linearization  $\mathcal{O}(1) \boxtimes \mathcal{O}(\beta)$ . We also let  $\mathcal{G}(\beta) := [\mathcal{Q}^{\text{ss}}(\beta)/\text{SL}(V)]$  be the corresponding GIT quotient stack and denote by  $\overline{\mathcal{G}}(\beta)$  its moduli space. Namely, we have that

$$\overline{\mathcal{G}}(\beta) = \mathcal{Q}^{\text{ss}}(\beta) // \text{SL}(V).$$

In [2], we obtain the following result:

**Theorem 1** (Contraction of the Gieseker-Petri divisor in  $\overline{\mathcal{M}}_6$  via VGIT).

(1) For  $\beta > 4$ ,  $\mathcal{Q}^{\text{ss}}(\beta) = \{h(C) \mid C \in \mathbb{P}H^0(\Sigma, -2K_\Sigma)\}$ . Moreover,

$$\mathcal{G}(\beta) \simeq [\mathbb{P}H^0(\Sigma, -2K_\Sigma)/S_5]$$

is a Deligne-Mumford stack and  $\overline{\mathcal{G}}(\beta) \simeq X_6 \simeq \overline{\mathcal{M}}_6(\alpha)$ , where  $\alpha \in (\frac{16}{47}, \frac{35}{102})$ .

(2) For  $\beta = 4$ ,  $\mathcal{Q}^{\text{ss}}(4) = \mathcal{Q}^{\text{ss}}(4 + \epsilon) \cup \{h(C) \mid C \in \mathcal{U}\}$ . Moreover,

$$\overline{\mathcal{G}}(4) \simeq \overline{\mathcal{M}}_6(35/102).$$

(3) For  $\beta \in (4 - \epsilon, 4)$ ,

$$\mathcal{Q}^{\text{ss}}(\beta) = \{h(C) \mid C \in \mathcal{U}, C \neq C_0\} \cup \{h(C) \mid C \in \mathbb{P}H^0(\Sigma, -2K_\Sigma), C \neq C'_0\}.$$

The stack  $\mathcal{G}(\beta)$  is Deligne-Mumford, and we have

$$\overline{\mathcal{G}}(\beta) \simeq \overline{\mathcal{M}}_6 \left( \frac{35}{102} + \epsilon \right).$$

(4) We have a commutative diagram

$$(2) \quad \begin{array}{ccccc} \mathcal{G}(4 - \epsilon) & \xrightarrow{\subset} & \mathcal{G}(4) & \xleftarrow{\supset} & \mathcal{G}(4 + \epsilon) \\ \downarrow & & \downarrow & & \downarrow \\ \overline{M}_6\left(\frac{35}{102} + \epsilon\right) & & \overline{M}_6\left(\frac{35}{102}\right) & \xleftarrow{\sim} & \overline{M}_6\left(\frac{35}{102} - \epsilon\right) \end{array}$$

Moreover,  $\overline{M}_6\left(\frac{35}{102} + \epsilon\right)$  is isomorphic to  $\overline{M}_6$  at the generic point of the Gieseker-Petri divisor  $D_{6,4}$  and  $\overline{M}_6\left(\frac{35}{102} + \epsilon\right) \rightarrow \overline{M}_6\left(\frac{35}{102}\right)$  is the contraction of this divisor to  $[C_0] \in \overline{M}_6\left(\frac{35}{102}\right)$ .

## REFERENCES

- [1] Gavril Farkas and Richard Rimányi. Quadric rank loci on moduli of curves and K3 surfaces, 2017. arXiv:1707.00756.
- [2] Maksym Fedorchuk. Geometric invariant theory of syzygies, with applications to moduli spaces. In *Geometry of Moduli*, volume 14 of *Abel Symposia*, pages x+230. Springer International Publishing, 2018. arXiv:1712.02776, in print.
- [3] David Gieseker. Geometric invariant theory and applications to moduli problems. In *Invariant theory. Proceedings of the 1st 1982 Session of the Centro Internazionale Matematico Estivo (CIME), Montecatini, June 10–18, 1982*, volume 996 of *Lecture Notes in Mathematics*, pages v+159. Springer-Verlag, Berlin, 1983.
- [4] Mark L. Green. Koszul cohomology and the geometry of projective varieties. *J. Differential Geom.*, 19(1):125–171, 1984.
- [5] Fabian Müller. The final log canonical model of  $\overline{M}_6$ . *Algebra Number Theory*, 8(5):1113–1126, 2014.
- [6] David Mumford, John Fogarty, and Frances Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]*. Springer-Verlag, Berlin, third edition, 1994.
- [7] David Mumford. Stability of projective varieties. *Enseignement Math. (2)*, 23(1-2):39–110, 1977.
- [8] Nicholas I. Shepherd-Barron. Invariant theory for  $S_5$  and the rationality of  $M_6$ . *Compositio Math.*, 70(1):13–25, 1989.

## Around Lyapunov exponents

SIMION FILIP

Consider the linear ODE  $\dot{v} = Av$  where  $A$  is a constant matrix and  $v$  is a vector. The exponential growth rate at infinity of the solutions, i.e.  $\lim_{t \rightarrow \infty} \frac{1}{t} \log \|v(t)\|$  is determined by the eigenvalues of  $A$ , denoted  $\lambda_1 > \dots > \lambda_n$ : for a typical  $v$  the exponential growth rate is  $\lambda_1$ , in a proper subspace typical solutions grow at rate  $\lambda_2$ , and so on. In other words, one gets a *filtration* of the space of solutions according to the exponential growth rate.

It is rare that one gets constant matrix coefficients, however a general result of Oseledets guarantees that in many situations one does have such asymptotic

growth rates. Concretely, the Multiplicative Ergodic Theorem starts with a probability measure-preserving ergodic flow  $g_t: (X, \mu) \rightarrow (X, \mu)$  and a cocycle, i.e. a vector bundle  $V \rightarrow X$  such that the flow acts on the fibers by linear maps  $g_t(x): V(x) \rightarrow V(g_t x)$ . Under a natural boundedness assumption on the linear maps, there exist *Lyapunov exponents*  $\lambda_1 > \dots > \lambda_k$  such that for  $\mu$ -a.e.  $x$ , there exists a decomposition

$$V(x) = \bigoplus_{\lambda_i} V(x)^{\lambda_i}$$

such that if  $v \in V(x)^{\lambda_i} \setminus \{0\}$  we have

$$\lim_{t \rightarrow \pm\infty} \frac{1}{t} \|g_t(x)v\| = \lambda_i.$$

Note that flowing backwards one gets exponents with opposite signs; the two filtrations given by flowing into the future/past refine to give a decomposition.

One has a priori symmetries on the Lyapunov spectrum, e.g. if the cocycle preserves a symplectic pairing on the vector bundle, then

$$\lambda_1 \geq \dots \geq \lambda_g \geq -\lambda_g \geq \dots \geq -\lambda_1.$$

In general it is difficult to compute the Lyapunov exponents and one of the central questions is to show that they *don't vanish*, and furthermore are distinct, subject to the a priori symmetries.

**Teichmüller dynamics.** A key observation of Kontsevich [Kon97] was that when the vector bundle  $V$  admits a weight 1 variation of Hodge structures, and the base manifold  $X$  admits a compatible action of  $\mathrm{SL}_2(\mathbf{R})$ , then there is a formula

$$\lambda_1 + \dots + \lambda_g = \frac{\deg V^{1,0}}{\chi}$$

where  $\chi$  denotes an appropriate ‘‘Euler characteristic’’ type term, and the degree of the Hodge bundle is interpreted appropriately (when  $X$  is the unit tangent bundle of a Riemann surface, the terms are the usual ones).

This observation has been clarified and generalized by many, including Forni [For02] and Eskin–Kontsevich–Zorich [EKZ14]. It has been used in many instances to compute Lyapunov exponents of the Teichmüller geodesic flow and has found a diverse range of applications to the study of billiards when the tables and obstacles have rational angles.

**Higher weight.** For a weight 1 variation of Hodge structure, the complexified vector bundle decomposes as  $V_{\mathbf{C}} = V^{1,0} \oplus V^{0,1}$  and the cocycle preserves a symplectic form. In weight 2, the decomposition is  $V_{\mathbf{C}} = V^{2,0} \oplus V^{1,1} \oplus V^{0,2}$  (with dimensions  $d, k, d$ ) and the cocycle preserves an orthogonal pairing of signature  $(2d, k)$ . The Lyapunov exponents are then (assuming  $k \geq 2d$ ):

$$\lambda_1 \geq \dots \lambda_{2d} \geq \underbrace{0 \geq \dots \geq 0}_{k-2d \text{ zeros}} \geq -\lambda_{2d} \geq \dots \geq -\lambda_1$$

and the analogous formula (proved by the speaker for  $d = 1$ , and then in general by the speaker [Fil18] and in M. Constantini’s thesis)

$$\lambda_1 + \cdots + \lambda_d = \frac{\deg V^{2,0}}{\chi}$$

with the degree and Euler characteristic interpreted appropriately. The techniques in both weight 1 and weight 2 are essentially the same.

Interesting behavior begins in weight 3. Kontsevich performed numerical experiments with 14 examples of representations to  $\mathrm{Sp}_4(\mathbf{Z})$ , coming from monodromies of families of Calabi–Yau 3-folds. The expected formula is

$$\lambda_1 + \lambda_2 = \frac{\deg V^{3,0} + \deg V^{2,1}}{\chi}$$

and by a general result of Eskin–Kontsevich–Möller–Zorich [EKMZ18] one has the inequality  $\geq$  above. Numerically, the 14 examples were of two sorts:

- **7 good examples:** The formula holds, and the monodromy is infinite index in  $\mathrm{Sp}_4(\mathbf{Z})$ .
- **7 bad examples:** The inequality is strict, and the monodromy is finite index in  $\mathrm{Sp}_4(\mathbf{Z})$ .

It was proved by Daniel–Deroin [DD18], and independently by the speaker, that the inequality is strict in the bad examples. The equality case in the good examples is the subject of work in progress by the speaker.

## REFERENCES

- [Kon97] KONTSEVICH M. — “Lyapunov exponents and Hodge theory”. in *The mathematical beauty of physics (Saclay, 1996)* — vol. 24 of *Adv. Ser. Math. Phys.*, pp. 318–332. World Sci. Publ., River Edge, NJ — 1997.
- [For02] FORNI G. — “Deviation of ergodic averages for area-preserving flows on surfaces of higher genus”. *Ann. of Math. (2)* **155** no. 1, (2002) 1–103.
- [EKZ14] ESKIN A., KONTSEVICH M., & ZORICH A. — “Sum of Lyapunov exponents of the Hodge bundle with respect to the Teichmüller geodesic flow”. *Publ. Math. Inst. Hautes Études Sci.* **120** (2014) 207–333.
- [EKMZ18] ESKIN A., KONTSEVICH M., MÖLLER M. & ZORICH A. — “Lower bounds for Lyapunov exponents of flat bundles on curves”. *Geometry & Topology* 2018.
- [DD18] DANIEL J. & DEROIN B. — “Lyapunov exponents of Brownian motion on a Kähler manifold”. *IMRN* (2018).
- [Fil18] FILIP, S. — “Families of K3 surfaces and Lyapunov exponents”. *Israel Journal of Mathematics* (Apr, 2018) .

## Slicing the pillowcase

ELISE GOUJARD

(joint work with Martin Möller)

Counting torus and sphere covers is very useful to evaluate volumes of moduli spaces, in particular strata of moduli spaces of flat surfaces (abelian or quadratic differentials). Evaluation of area Siegel-Veech constants for strata of abelian or quadratic differentials can also be performed by counting these covers, with a so-called Siegel-Veech weight.

We prove the quasimodularity of generating functions for counting such covers, with and without Siegel-Veech weight, analyzing decompositions of flat surfaces into horizontal cylinders ("slicing" the torus or the pillowcase), see [6] and [7]. We show how these quasimodular forms arise as contour integral of generalized Jacobi forms. This work provides an alternative proof of the quasimodularity results of Bloch-Okounkov [2], Eskin-Okounkov ([4] and [5]) and Chen-Möller-Zagier [3], with the following refinement: we show that quasimodularity holds for every type of decomposition into cylinders (encoded in a Feynman graph). It generalizes the results of Böhm-Bringmann-Buchholz-Markwig [1] for simple ramification covers, and it provides a practical method to compute area Siegel-Veech constants.

A main new technical tool is a quasi-polynomiality result for 2-orbifold Hurwitz numbers with completed cycles.

## REFERENCES

- [1] J. Böhm, K. Bringmann, A. Buchholz and H. Markwig, *Tropical mirror symmetry for elliptic curves*, Journal für die Reine und Angewandte Mathematik. [Crelle's Journal] **732** (2017), 211–246.
- [2] S. Bloch and A. Okounkov, *The character of the infinite wedge representation*, Advances in Mathematics **149** (2000), 1–60.
- [3] D. Chen and M. Möller and D. Zagier, *Quasimodularity and large genus limits of Siegel-Veech constants*, Journal of the American Mathematical Society **31** (2018), 1059–1163.
- [4] A. Eskin and A. Okounkov, *Asymptotics of numbers of branched coverings of a torus and volumes of moduli spaces of holomorphic differentials*, Inventiones Mathematicae **145** (2001), 59–103.
- [5] A. Eskin and A. Okounkov, *Pillowcases and quasimodular forms*. In *Algebraic geometry and number theory*, volume **253** of Progr. Math., 1–25. Birkhuser Boston, Boston, MA, (2006).
- [6] E. Goujard and M. Möller, *Counting Feynman-like graphs: Quasimodularity and Siegel-Veech weight*, Journal of the EMS, to appear (2017).
- [7] E. Goujard and M. Möller, *Pillowcase covers: Counting Feynman-like graphs associated with quadratic differentials*, arXiv:1809.05016 (2018).

## Quotients of the orbifold fundamental group of strata of abelian differentials

URSULA HAMENSTÄDT

The moduli space of *abelian differentials* on a closed oriented surface  $S$  of genus  $g \geq 2$  decomposes into *strata*. Each such stratum is determined by a partition  $2g-2 = \sum_{i=1}^m k_i$  for some numbers  $k_i \geq 1$ , and it consists of all abelian differentials with the same number  $m \geq 1$  of zeros of the same multiplicities  $k_i$ . Such a stratum, denoted by  $\mathcal{H}(k_1, \dots, k_m)$ , is a complex orbifold of complex dimension  $2g-1+m$ . Strata are not necessarily connected, but their connected components were classified by Kontsevich and Zorich [2].

They showed that each stratum consists of at most three connected components. *Hyperelliptic components* are components of differentials on hyperelliptic surfaces, and they are preimages of strata of quadratic differentials on  $\mathbb{C}P^1$  with simple poles at the images of all or all but one Weierstrass point. They are fairly well understood.

For genus  $g = 3$ , Looijenga and Mondello [3] used methods from algebraic geometry to investigate the topology of some components of strata in genus 3. They found that the components  $\mathcal{H}(4)$  and  $\mathcal{H}(1, 3)$  are classifying spaces for the quotient of the Artin group  $E_6$  and  $E_7$ , respectively, by its center.

For a component  $\mathcal{Q}$  of a stratum consisting of differentials with  $m \geq 2$  zeros call an abelian differential  $q \in \mathcal{Q}$  *completely periodic admissible* for  $\mathcal{Q}$  if the following conditions are satisfied.

- (1) The differential  $q$  is horizontally and vertically periodic. Equivalently, every regular leaf of the horizontal or the vertical foliation is closed.
- (2) The core curves of the horizontal and vertical cylinders decompose  $S$  into  $m$  disks.
- (3) Any two curves from the collection  $\mathcal{C}$  of core curves of horizontal or vertical cylinders intersect in at most one point. Furthermore, the graph whose vertices are the components of  $\mathcal{C}$  and where two such vertices  $c, d$  are connected by an edge if they intersect is a tree.

It is not hard to construct explicitly such differentials for every component of a stratum.

The main result discussed in the talk is

**Theorem 1.** *Let  $\mathcal{Q}$  be a component of a stratum of abelian differentials and let  $q \in \mathcal{Q}$  be completely periodic admissible for  $\mathcal{Q}$ . Then the image of the orbifold fundamental group of  $\mathcal{Q}$  in the mapping group of the surface  $S$  with  $m$  marked points is the group which is generated by the Dehn twists about the core curves of the horizontal and vertical cylinders of  $q$ .*

## REFERENCES

- [1] U. Hamenstädt, *Quotients of the orbifold fundamental group of strata of abelian differentials*, preprint, available at [www.math.uni-bonn.de/ursula](http://www.math.uni-bonn.de/ursula)
- [2] M. Kontsevich, A. Zorich, *Connected components of the moduli space of abelian differentials with prescribed singularities*, *Invent. Math.* 153 (2003), 631–678.
- [3] E. Looijenga, G. Mondello, *The fine structure of the moduli space of abelian differentials in genus 3*, *Geom. Dedicata* 169 (2014), 109–128.

**How can a line bundle on a curve degenerate?**

JESSE LEO KASS

(joint work with Nicola Pagani and David Holmes)

The moduli stack  $M_{g,n}$  of smooth  $n$ -marked curves of genus  $g$  is the base of the universal Jacobian  $J_{g,n}^d$ , or moduli space of degree  $d$  line bundles, and a basic problem in algebraic geometry is to extend  $J_{g,n}^d$  to a family over the moduli space  $\overline{M}_{g,n}$  of stable marked curves by adding degenerate fibers called compactified Jacobians. This problem is challenging because the obvious extension, the moduli space of line bundles on stable marked curves, does not provide a well-behaved moduli space. Since at least 1956, mathematicians such as Altman, Caporaso, D’Souza, Esteves, Igusa, Kleiman, Mayer, Melo, Mumford, Oda, Pandharipande, Seshadri, Simpson, and Viviani have worked to modify this idea to create well-behaved extensions of  $J_{g,n}$ . In [KP17], we extend formalism developed Oda–Seshadri to define and study extensions.

The essential step in defining a well-behaved extension of  $J_{g,n}^d$  is specifying which line bundles to include, and the most important invariant for doing this is the multidegree of a line bundle, which is the integral vector whose components are the partial degrees on the irreducible components of the curve. In [OS79], Oda and Seshadri defined a linear algebra object that describes ways of selecting multidegrees in a manner that specifies a compactified Jacobian  $\overline{J}_C$  of a nodal curve. Given an integer  $d$ , let  $\mathbb{R}^{\#\text{Vert}(\Gamma_C)}$  denote the vector space of real valued functions on the vertex set  $\text{Vert}(\Gamma_C)$  of the dual graph and let  $V^d(C) \subset \mathbb{R}^{\#\text{Vert}(\Gamma_C)}$  denote the affine subspace of vectors  $\phi$  satisfying  $\sum \phi(v) = d$ . Define a line bundle  $L$  to be  $\phi$ -(semi)stable if it satisfies

$$(1) \quad \left| \deg_{C_0}(L) - \sum_{v \in \Gamma_{C_0}} \phi(v) \right| < \frac{\#(C_0 \cap C_0^c)}{2} \quad (\text{resp. } \leq \frac{\#(C_0 \cap C_0^c)}{2}).$$

Here  $C_0 \subset C$  runs over the subcurves and  $C_0^c$  denotes the closure of the complement of  $C_0$ .

The notation of  $\phi$ -(semi)stability extends in a natural way to rank 1, torsion-free sheaves (certain degenerate line bundle), and one of Oda and Seshadri’s theorems is that there is a projective scheme  $\overline{J}(\phi)$  that is a suitably defined coarse space for the moduli functor of  $\phi$ -semistable sheaves. For our purposes, the most important part of this result is that  $\overline{J}(\phi)$  in fact represents the moduli functor when  $\phi$ -stability coincides with  $\phi$ -semistability, a condition we call nondegenerate.

A major goal in [KP17] is to package the different  $\overline{\mathcal{J}}(\phi)$ s to produce a family over  $\overline{M}_{g,n}$ . We do so by defining the **stability space** to be the subspace

$$V_{g,n}^d \subset \prod_{\Gamma} V^d(\Gamma)$$

consisting of vectors such that the component indexed by  $\Gamma$  is invariant under  $\text{Aut}(\Gamma)$  and the components are compatible with graph contractions. With this definition, we can use the components of  $\phi \in V_{g,n}^d$  to define  $\phi$ -stability for rank 1, torsion-free sheaves on genus  $g$  stable,  $n$ -marked curves. A basic result is then that, under the assumption that  $\phi$  is nondegenerate, the moduli space  $\overline{\mathcal{J}}_{g,n}(\phi)$  of  $\phi$ -stable sheaves exists as a proper Deligne–Mumford stack.

We also describe how  $\overline{\mathcal{J}}_{g,n}(\phi)$  depends on  $\phi$  by decomposing of  $V_{g,n}^d$  into bounded polytopes such that  $\overline{\mathcal{J}}_{g,n}(\phi)$  changes precisely when  $\phi$  passes from one polytope to another. This result is [KP17, Theorem 2], and it takes a particularly nice form when specialized to the case where  $d = g - 1$ :

**Theorem 1.** *For  $g \geq 2, n \geq 1$ , there is an identification  $V_{g,n}^{g-1} = \mathbb{R}^N \times \mathbb{R}^n$  such that  $\overline{\mathcal{J}}_{g,n}(\phi_1) = \overline{\mathcal{J}}_{g,n}(\phi_2)$  if and only if  $\phi_1, \phi_2$  lie in a common polytope for the polytope decomposition that is the product of the decomposition of  $\mathbb{R}^{N-1}$  by integer translates of the hyperplanes*

$$(2) \quad \{\vec{x} \in \mathbb{R}^N : x_\ell = 0\} \text{ for } \ell = 0, \dots, N$$

and the decomposition of  $\mathbb{R}^n$  by integer translates of

$$(3) \quad \left\{ \vec{x} \in \mathbb{R}^n : \sum_{i \in S} x_i - \frac{\ell}{2g-2} \sum_{i=1}^n x_i = 0 \right\} \text{ for } \ell = 0, \dots, 2g-3, S \subseteq \{1, \dots, n\}.$$

The second collection of hyperplanes is a refinement of a hyperplane arrangement known as the resonance hyperplane arrangement.

A long-term goal is to apply the description in (2) to describe how natural Chow classes on  $\overline{\mathcal{J}}_{g,n}(\phi)$  change as we change  $\phi$ . Pagani, Holmes, and I [HKP17] together with Pagani and Ricolfi [PR18] and Chiodo [Chio02] have proved results describing how the double ramification cycle and related classes change. Recall that the double ramification cycle  $\text{DR}(\underline{a})$  associated to weights  $(a_1, \dots, a_n)$ ,  $a_1 + \dots + a_n = 0$ , is an extension of the fundamental class of  $\{(C, p_1, \dots, p_n) : \mathcal{O}(a_1 p_1 + \dots + a_n p_n) = \mathcal{O}\} \subset M_{g,n}$  to a class on  $\overline{M}_{g,n}$ . The double ramification cycle  $\text{DR}(\underline{a})$  exhibits beautiful structural properties including a formula expressing  $\text{DR}(\underline{a})$  as a sum over stable graphs with terms given by products over markings and edges. The formula is known as Pixton’s conjecture, but it is now a theorem that is proven in [JPPZ17].

In loc. cit., the double ramification cycle is defined using the formalism of virtual fundamental classes and the moduli space of stable maps. The class can alternatively be interpreted in terms of  $\overline{\mathcal{J}}_{g,n}(\phi)$ . Observe that  $\{(C, p_1, \dots, p_n) : \mathcal{O}(a_1 p_1 + \dots + a_n p_n) = \mathcal{O}\}$  is the pullback of the locus of trivial line bundles under the map  $\sigma_{\underline{a}} : M_{g,n} \rightarrow J_{g,n}^0$  defined by  $(C, p_1, \dots, p_n) \mapsto \mathcal{O}(a_1 p_1 + \dots + a_n p_n)$ . Thus we can

extend the fundamental class by picking a  $\phi$  and considering  $\sigma_{\underline{a}}$  as a morphism  $\overline{M}_{g,n} \dashrightarrow \overline{J}_{g,n}(\phi)$ . For many choices of  $\phi$ , this class will not equal  $DR(\underline{a})$ , but we prove [HKP17, Theorem 5]:

**Theorem 2.** *If  $\phi \in V_{g,n}^0$  is nondegenerate and such that every multidegree 0 line bundle is  $\phi$ -stable, we have  $[DR(\underline{a})]$  is the pullback of the locus of trivial line bundles under  $\sigma_{\underline{a}}: \overline{M}_{g,n} \dashrightarrow \overline{J}_{g,n}(\phi)$ .*

Strictly speaking, rather than working with the locus of trivial line bundles, in loc. cit., we work with a related Brill–Noether cycle  $w(\phi)$  that has better formal properties.

An important issue that we have not emphasized is that  $\sigma_{\underline{a}}$  is typically a rational map. In proving [HKP17, Theorem 5], an essential input is work of Holmes [Hol17] that constructs an explicit resolution of indeterminacy.

In view of Theorem 2, an important project is to understand how the pullback of the locus of the trivial line bundle, or better  $w(\phi)$ , changes as we change  $\phi$ . An important step in this direction was taken by Pagani and Ricolfi in [PR18] and Chiodo [Chio02]. While the map  $\sigma_{\underline{a}}$  is, in general, only a rational map and pullbacks by rational maps can be difficult to compute, for some important special choices of  $\underline{a}$  the map is regular. For these choices, Chiodo, Pagani, and Ricolfi compute the pullback of the zero section as well as more general Brill–Noether cycles. The most general formula is given as [PR18, Theorem 1], and we omit it as it is rather complicated. The stability parameter  $\phi$  considered in loc. cit. is typically different from the one appearing Theorem in 2, and the formula appearing in loc. cit. is different from Pixton’s formula. Both formulas do, however, display rich combinatorial structure, and an ongoing challenge is to understand how to interpolate between the two structures when varying  $\phi$ .

## REFERENCES

- [Chio02] A. Chiodo, *Towards an enumerative geometry of the moduli space of twisted curves and  $r$ -th roots*, Compos. Math. 144 (2008), no. 6, 1461–1496.
- [Hol17] D. Holmes, *Extending the double ramification cycle by resolving the Abel–Jacobi map*, arXiv:1707.02261, 2017.
- [HKP17] D. Holmes, J. L. Kass, and N. Pagani, *Extending the Double Ramification Cycle using Jacobians*. *European Journal of Mathematics*, 2018, 1–13.
- [JPPZ17] F. Janda, R. Pandharipande, A. Pixton, and D. Zvonkine, *Double ramification cycles on the moduli spaces of curves*, Publ. Math. Inst. Hautes Études Sci. **125** (2017), 221–266.
- [KP17] J. L. Kass, N. Pagani, *The stability space of compactified universal Jacobians*. E-print: arXiv:1707.02284 (July 7, 2017). 36 Pages.
- [OS79] T. Oda and C. S. Seshadri, *Compactifications of the generalized Jacobian variety*. *Trans. Amer. Math. Soc.*, 253:1–90, 1979.
- [PR18] N. Pagani, A. T. Ricolfi, *Pullbacks of universal Brill–Noether classes via Abel–Jacobi morphisms*. arXiv:1809.10668 (Sep 27, 2018). 14 Pages.

## Complete cycles in the moduli space of curves

IGOR KRICHEVER

The goal of the talk is to present new upper bound on the dimension of a complete (complex) cycle in the moduli space  $\mathcal{M}_g^{ct}$  of genus  $g$  stable curves of compact type.

The upper bound for the case of moduli  $\mathcal{M}_g$  space of *smooth* curves is celebrated Diaz' theorem (1986):

- *There does not exist a complete (complex) cycle in  $\mathcal{M}_g$  of dimension greater than  $g - 2$ .*

Then easy induction arguments prove (Diaz):

- *There is no complete cycle in the moduli space  $\mathcal{M}_g^{ct}$  of stable curves of compact type of dimension greater than  $2g - 3$ .*

Keel and Sadun improved this bound:

- *For  $g \geq 3$  there do not exist complete complex subvarieties of  $\mathcal{M}_g^{ct}$  of dimension greater than  $2g - 4$ .*

This seemingly humble improvement of the previous bound is quite significant in relation to Faber's conjectures or, more precisely, in relation to their analog for  $\mathcal{M}_g^{ct}$  (see extensive discussion in [1]). In this connection, our main result looks even more striking:

**Theorem 1.** *There do not exist complete complex subvarieties of  $\mathcal{M}_g^{ct}$  having non empty intersection with  $\mathcal{M}_g$  of dimension greater than  $g - 1$ . For  $g \geq 2$  the maximum dimension of complete complex subvarieties in  $\mathcal{M}_g^{ct}$  is  $\frac{3}{2}g - 2$ .*

The proof is based on a study of the period map defined by *real normalized differentials*. By definition a real normalized meromorphic differential is a differential whose period over any cycle on the curve is real. As easily follows from the positive-definiteness of the imaginary part of the period matrix, for any fixed singular parts at marked points with imaginary residues summing up to zero there exists a unique meromorphic differential with prescribed singular parts, and with all periods real.

The general notion of real normalized differential was introduced in the framework of the Whitham perturbation theory of soliton equations [2]. Their systematic study and applications to the study of geometry of moduli spaces of curves was initiated in our joint paper [3] with S. Grushevsky.

Let  $\mathcal{M}_{g,1}^{\leq n, \tau} := \{C, \Psi, \Psi_1\}$  be the moduli space of smooth genus  $g$  algebraic curves  $C$  with a pair of real normalized differentials having pole of order at most  $n + 1$  at one marked point  $p_0$ , whose singular parts  $\sigma$  and  $\sigma_1$  satisfy the equation  $\sigma = \tau \sigma_1$  with  $\tau \in \mathbb{C}$ ,  $\text{Im} \tau > 0$ . This space was central in the construction of elliptic families of solutions of the Kadomtsev-Petvishavili equation [5].

The (local) period map is defined as

$$\Pi : \mathcal{M}_{g,1}^{\leq n, \tau} \rightarrow \left( \oint_{\gamma_i} \Psi, \oint_{\gamma_i} \Psi_1 \right) \in \mathbb{R}^{2g} \oplus \mathbb{R}^{2g}$$

For any  $Y \subset \mathcal{M}_g$  the preimage of  $Y$  under the forgetful map  $\mathcal{M}_{g,1}^{\leq n, \tau} \rightarrow \mathcal{M}_g$  will be denoted by  $Y^{\leq n, \tau}$ . It is of dimension  $\dim Y + n + 1$

The following unexpected result is crucial in the proof of our main theorem:

**Lemma 2.** *Let  $Z$  be a complete cycle in  $\mathcal{M}_g^{ct}$  of dimension  $g - 1$ , then for any  $\tau$  the period map*

$$\Pi : (Z \cap \mathcal{M}_g)^{\leq g, \tau} \rightarrow \mathbb{R}^{2g} \oplus \mathbb{R}^{2g}$$

*fails to be injective at every point.*

#### REFERENCES

- [1] Keel, S., Sadun, L.: *Oort's conjecture for  $\mathcal{A}_g$* , J. Amer. Math. Soc. **16** (2003), 887–900.
- [2] I. Krichever, *The spectral theory of "finite-gap" nonstationary Schrödinger operators. The nonstationary Peierls model*, Funct. Anal. Appl. **20** (1986), no. 3, 42–54.
- [3] S. Grushevsky, I. Krichever, *The universal Whitham hierarchy and geometry of the moduli space of pointed Riemann surfaces*, in: Surveys in Differential Geometry **14** (2010).
- [4] S. Grushevsky, I. Krichever, *Foliations on the moduli space of curves, vanishing in cohomology, and Calogero-Moser curves*, arXiv:1108.4211
- [5] A. Akhmetshin, I. Krichever, Yu. Volvovski, *Elliptic families of solutions of the Kadomtsev-Petviashvili equation, and the field analogue of the elliptic Calogero-Moser system*, Funct. Anal. Appl. **36:4** (2002), 117

### Cohomology classes of singularity loci in Hurwitz spaces

SERGEY LANDO

(joint work with Maxim Kazarian, Dmitry Zvonkine)

The subject of the talk are the universal formulas for cohomology classes in spaces of meromorphic functions on complex curves represented by subspaces of functions with singularities of prescribed types. The study is based on the principles by R. Thom and M. Kazarian that guarantee the existence of such universal expressions in the case when all the local singularities are isolated. We extend these principles to the case of nonisolated singularities, allowing contracting of a rational curve. We deduce recurrence relations for such cohomology classes in Hurwitz spaces of rational functions and use the results to obtain new formulas for rational double Hurwitz numbers.

Double Hurwitz numbers enumerate possible ways to represent the identical permutation as the product of two permutations of given cyclic types and a given number of transpositions. In topological terms, they describe topologically distinct meromorphic functions on Riemann surfaces of given genus with prescribed orders of poles and zeroes and prescribed nonzero simple critical values. In the case when the underlying surface has genus zero, and one of the two distinguished permutations is identical, a closed formula for these numbers was proposed by Hurwitz more than a century ago. In the general case, explicit formulas not for the numbers themselves but for the generating function of these numbers are known. In spite of existence of such formulas and a variety of modern proofs, many natural questions concerning these numbers remain open.

Hurwitz's argument was algebraic, based on the study of combinatorics of the permutation group. On the other hand, Hurwitz numbers are related to the geometry of spaces of rational functions. We propose a new way to compute Hurwitz numbers that has a topological origin: it is derived from cohomological information contained in the stratification of the Hurwitz space according to multisingularities of the functions. We expect that variations of this approach could be adopted to other families of Hurwitz numbers for which effective formulas are not known at the moment, including those for curves of higher genera.

According to our approach, the classes of (multi)singularities of maps are expressed by universal formulas involving characteristic classes of manifolds participating in the mapping. The terms of these formulas are actually independent of the particular mapping, they are totally determined by the degeneracy types of the mapping. The formulas can be applied provided the mapping is generic, that is, certain transversality condition must be satisfied. We formulated these transversality conditions explicitly. The cyclic types of the two permutations used in the definition of the double Hurwitz numbers enter our formulas in a non-symmetric way: the cyclic type of one of them denotes the multisingularity type of a rational function which is simplified step by step in the recursive procedure, while the other one is just used in the definition of the suitable ambient Hurwitz space and could be arbitrary; it does not affect the structure of the recursive relation.

#### REFERENCES

- [1] M. Kazarian, S. Lando, *On intersection theory on Hurwitz spaces*, Izv. Ross. Akad. Nauk Ser. Mat., **68**, no. 5, 91–122 (2004); translation in Izv. Math. **68**, no. 5, 935–964 (2004)
- [2] M. Kazarian, S. Lando, *Thom polynomials for mappings of curves with isolated singularities*, in Tr. Mat. Inst. Steklova (2007) Anal. i Osob. Ch. 1; translated in: Proc. Steklov Inst. Math. (2007), no. 1, 93–106
- [3] M. Kazarian, S. Lando, D. Zvonkine, *Universal cohomological expressions for singularities in families of genus 0 stable maps*, arXiv:1512.03285
- [4] S. K. Lando, D. Zvonkine, *Counting Ramified Coverings and Intersection Theory on Spaces of Rational Functions I (Cohomology of Hurwitz Spaces)*, Moscow Math. J., **7** (1), 85–107 (2007)

### The shape of large genus translation surfaces

HOWARD MASUR

(joint work with Kasra Rafi, Anja Randecker)

For this talk we consider the stratum  $\mathcal{H}(2g - 2)$  of genus  $g$  surfaces with a single zero of order  $2g - 2$ . This is the minimal stratum in that genus. We are interested in the expected or average diameter of a surface  $(X, \omega) \in \mathcal{H}(2g - 2)$  as  $g \rightarrow \infty$ . We let  $\nu$  be normalized Lebesgue measure on the stratum.

Our result is

$$\frac{\int_{\mathcal{H}} \text{diam}(X, \omega) d\nu(X, \omega)}{\nu(\mathcal{H})} = O\left(\sqrt{\frac{\log g}{g}}\right)$$

as  $g \rightarrow \infty$ .

The proof consists of using several ideas. The first is the Siegel-Veech formula which says that if one wants to count cylinders, for example, on a translation surface,  $(X, \omega)$ , each cylinder has a holonomy vector  $v$  so that one gets a discrete set  $\Lambda_\omega$ . There is a constant  $c$  called the Siegel-Veech constant so that for  $f$  a compactly supported function on  $\mathbb{R}^2$  if we let  $\hat{f}(X, \omega) = \sum_{v \in \Lambda_\omega} f(v)$  then

$$\int_{\mathcal{H}} \hat{f} = c \int_{\mathbb{R}^2} f.$$

The constant  $c$  depends on volumes of strata [EMZ] and their asymptotics have recently been found [A],[S]. This information put together allows one to compute volumes of sets in terms of their diameter and this is what allows us to prove the theorem.

#### REFERENCES

- [A] A. Aggarwal *Large Genus Asymptotics for Volumes of Strata of Abelian Differentials*, With an appendix by Anton Zorich, arXiv:1804.05431, 2018.
- [EMZ] A. Eskin, H.Masur, A.Zorich *Moduli spaces of abelian differentials, the principal boundary, counting problems, and the Siegel-Veech constants* Publication IHES (2004) 61-179
- [S] A. Sauvaget *Volumes and Siegel-Veech constants of  $\mathcal{H}(2g-2)$  and Hodge integrals* arXiv:1801.01744, 2018.

### Arithmetic Kontsevich–Zorich monodromies

CARLOS MATHEUS

(joint work with Pascal Hubert)

Let  $G$  be an algebraic Lie group and consider a  $\mathbb{Z}$ -form of  $G_{\mathbb{Z}}$ . Given  $\Gamma \subset G_{\mathbb{Z}}$  Zariski-dense in  $G$ , we say that  $\Gamma$  is *thin*, resp. *arithmetic* if the index of  $\Gamma$  in  $G_{\mathbb{Z}}$  is infinite, resp. finite. For example, when  $G_{\mathbb{Z}} = SL(2, \mathbb{Z})$ , one has that  $\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \rangle$  is arithmetic and  $\langle \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \rangle$  is thin.

The notion of thin groups was introduced by Sarnak [10] partly because some counting or sieving problems in Number Theory become easier / harder when dealing with arithmetic / thin groups: in a nutshell, arithmeticity is relevant thanks to the so-called property (T), while thinness creates some difficulties especially in the regime of small Hausdorff dimension of certain limit sets.

Moreover, it is often interesting to establish the arithmeticity of certain Zariski-dense subgroups because this might provide new extra informations: for instance, the proof of arithmeticity of Rauzy–Veech groups by Gutiérrez-Romo [5] led him to discover new tools allowing to obtain in [6] the first cases of the analog of Avila–Viana theorem [1] for moduli spaces of half-translation surfaces (almost 10 years after the first attempts by Avila, Lanneau and Resende).

1. HOW OFTEN IS  $\Gamma$  THIN?

It is known that the presence or absence of thin groups depends on the precise setting. For example, Fuchs and Rivin [4] showed that the group  $\langle A, B \rangle$  generated by two matrices  $A, B \in SL(n, \mathbb{Z})$  is “typically” thin, and Singh–Venkataramana [11] and Brav–Thomas [2] proved that we have 7 arithmetic cases and 7 thin cases among the 14 examples of monodromy representations coming from Dwork families of Calabi–Yau 3-folds.

In this direction, Sarnak asked Eskin how frequently are thin Kontsevich–Zorich monodromies. For the sake of simplicity, we shall restrict our discussion of this question to the context of square-tiled surfaces  $X$  of genus  $g \geq 2$  without automorphisms in what follows. In this setting, we recall that the *Kontsevich–Zorich monodromy*  $\Gamma_X$  is simply the image  $\rho_X(SL(X))$  of the Veech group  $SL(X)$  of  $X$  under the natural monodromy representation (Kontsevich–Zorich cocycle)  $\rho_X : SL(X) \rightarrow Sp(H_1^{(0)}(X)) \simeq Sp(2g - 2, \mathbb{Z})$ , where  $H_1^{(0)}(X)$  is the subspace of zero holonomy elements of  $H_1(X, \mathbb{R})$ .

As it was pointed out by Möller, the Kontsevich–Zorich monodromy  $\Gamma_X$  of a square-tiled surface  $X$  of genus 2 is always arithmetic: roughly speaking, this happens because the period map gives a non-constant holomorphic map from  $\mathbb{H}/SL(X)$  to  $\mathbb{H}/\Gamma_X$ .

Therefore, the next question is: what happens in genus 3? Here, it is worth to point out that the question makes sense because  $\Gamma_X$  is “usually” Zariski-dense when  $X$  has genus 3: for instance,  $\Gamma_X$  is Zariski-dense in  $Sp(4, \mathbb{R})$  for “most” square-tiled surfaces  $X$  in the minimal stratum  $\mathcal{H}(4)$  thanks to our joint work [8] with Möller and Yoccoz and the work [9] of Prasad–Rapinchuk.

## 2. THIN KONTSEVICH–ZORICH MONODROMIES IN GENUS 3?

If  $X$  is a square-tiled surface of genus 3 such that  $SL(X)$  is virtually free,  $\rho_X : SL(X) \rightarrow Sp(4, \mathbb{Z})$  is *faithful* (i.e., injective), and  $\Gamma_X = \rho_X(SL(X))$  is Zariski-dense in  $Sp(4, \mathbb{R})$ , then  $\Gamma_X$  is thin (because  $Sp(4, \mathbb{Z})$  has property (T)).

On the other hand,  $\rho_X$  is *not* faithful when  $X$  decomposes into a single cylinder in some rational direction (because the multi-twist stabilising such a cylinder belongs to the kernel of  $\rho_X$ ).

Hence, it is natural to look for some examples of thin  $\Gamma_X$  among square-tiled surfaces of genus 3 without one-cylinder decompositions. After a quick computer search (with Sage), Delecroix discovered that the square-tiled surface  $O$  associated to the pair of permutations  $h = (1)(2, 3, 4, 5)(6, 7, 8, 9)$  and  $v = (1, 8, 9, 2)(3, 5, 4, 6)(7)$  has these properties.

Thus, it is tempting to investigate the features of  $\Gamma_O$ . For this sake, we note that  $\Gamma_O$  is generated by two matrices: indeed, one can check that the Veech group  $SL(O)$  of  $O$  is

$$SL(O) = \langle a \rangle * \langle b \rangle := \left\langle \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \right\rangle * \left\langle \begin{pmatrix} 1 & -3 \\ 1 & -2 \end{pmatrix} \right\rangle \simeq \mathbb{Z}/3 * \mathbb{Z}/3,$$

so that  $\Gamma_O$  is generated by  $A := \rho_O(a)$  and  $B := \rho_O(b)$ . In particular, one can use [8] and [9] to show that  $\Gamma_O = \langle A, B \rangle$  is Zariski-dense in  $Sp(4, \mathbb{R})$ .

Nevertheless, Kohl found out that  $\rho$  is *not* faithful because

$$\rho \begin{pmatrix} -24587 & 42408 \\ 15048 & -25955 \end{pmatrix} = \rho((aba^{-1}ba^{-1}bab^{-1})^3) = \text{Id}_{4 \times 4}.$$

As it turns out, there is a good reason for the non-faithfulness of  $\rho$ : in our joint work [7] with Hubert, we showed that  $\Gamma_O$  is arithmetic.

The proof of this result relies on the following particular case of a conjecture of Margulis recently established by Benoist–Miquel [3] (after important partial results by Raghunathan, Venkataramana and Oh): if  $\Gamma \subset Sp(4, \mathbb{Z})$  is a Zariski-dense subgroup intersecting the unipotent radical of the parabolic subgroup stabilising the flag  $\{0\} \subset \mathbb{R}e_1 \subset (\mathbb{R}e_1)^\dagger \subset \mathbb{R}^4$  (where  $V^\dagger$  stands for the symplectic orthogonal of  $V$ ) in a lattice, then  $\Gamma$  is arithmetic.

In summary, despite testing some natural candidates (such as  $O$  above), Hubert and the author were not able to locate thin Kontsevich–Zorich monodromies among square-tiled surfaces of genus 3.

## REFERENCES

- [1] A. Avila and M. Viana, *Simplicity of Lyapunov spectra: proof of the Zorich-Kontsevich conjecture*, Acta Math. 198 (2007), 1–56.
- [2] C. Brav and H. Thomas, *Thin monodromy in  $Sp(4)$* , Compos. Math. 150 (2014), 333–343.
- [3] Y. Benoist and S. Miquel, *Arithmeticity of discrete subgroups containing horospherical lattices*, preprint (2018) available at arXiv:1805.00045.
- [4] E. Fuchs and I. Rivin, *Generic thinness in finitely generated subgroups of  $SL_n(\mathbb{Z})$* , Int. Math. Res. Not. 2017, 5385–5414.
- [5] R. Gutiérrez-Romo, *Classification of Rauzy–Veech groups: proof of the Zorich conjecture*, preprint (2017) available at arXiv:1706.04923.
- [6] R. Gutiérrez-Romo, *Simplicity of the Lyapunov spectra of certain quadratic differentials*, preprint (2017) available at arXiv:1711.02006.
- [7] P. Hubert and C. Matheus, *An origami of genus 3 with arithmetic Kontsevich–Zorich monodromy*, preprint (2018) available at arXiv:1806.09855.
- [8] C. Matheus, M. Möller and J.-C. Yoccoz, *A criterion for the simplicity of the Lyapunov spectrum of square-tiled surfaces*, Invent. Math. 202 (2015), 333–425.
- [9] G. Prasad and A. Rapinchuk, *Generic elements in Zariski-dense subgroups and isospectral locally symmetric spaces*, Thin groups and superstrong approximation, 211–252, Math. Sci. Res. Inst. Publ., 61, Cambridge Univ. Press, Cambridge, 2014.
- [10] P. Sarnak, *Notes on thin matrix groups*, Thin groups and superstrong approximation, 343–362, Math. Sci. Res. Inst. Publ., 61, Cambridge Univ. Press, Cambridge, 2014.
- [11] S. Singh and T. Venkataramana, *Arithmeticity of certain symplectic hypergeometric groups*, Duke Math. J. 163 (2014), 591–617.

## Exploring the moduli space of spherical surfaces with conical points

GABRIELE MONDELLO

(joint work with Dmitri Panov)

We consider compact, connected, oriented surfaces  $S$  of genus  $g$  with  $n$  distinct marked points  $\mathbf{x} = (x_1, \dots, x_n)$  on it and an  $n$ -uple  $\boldsymbol{\vartheta} = (\vartheta_1, \dots, \vartheta_n)$  of positive number. A *spherical* metric on  $(S, \mathbf{x})$  with conical singularities of angles  $2\pi\boldsymbol{\vartheta}$  is a Riemannian metric  $h$  on  $\dot{S} := S \setminus \mathbf{x}$  with curvature 1 such that it has a conical point at  $x_i$  of angle  $2\pi\vartheta_i$  for all  $i = 1, \dots, n$ .

**Monodromy and  $\mathbb{CP}^1$ -structures.** Here we recall that, since metrics of  $K = 1$  are locally isometric to  $\mathbb{S}^2$ , a spherical surface  $(S, \mathbf{x}, h)$  come endowed with a  $\mathrm{SO}_3(\mathbb{R})$ -valued monodromy representation of  $\pi_1(\dot{S})$  and a locally isometric developing map to  $\mathbb{S}^2$ . In particular, a spherical metric induces a  $\mathbb{CP}^1$ -structure on  $S$  with singularities at  $\mathbf{x}$ . The case in which the monodromy of  $h$  is *co-axial* (i.e. contained in a one-parameter subgroup of  $\mathrm{SO}_3(\mathbb{R})$ ) is somehow a degenerate one, whereas the study of non-coaxial spherical metrics are essentially equivalent to the study of their associated  $\mathbb{CP}^1$ -structures.

**Simple cases.** We remark that the case  $n = 0$  with no singularities is not very interesting, since all such surfaces would be isometric to the standard sphere  $\mathbb{S}^2$ . In fact, for every Riemann surface  $S$  of genus 0 there exist a family of  $\mathbb{H}^3 = \mathrm{PSL}_2(\mathbb{C})/\mathrm{SO}_3(\mathbb{R})$  conformal spherical metrics. Troyanov [10] showed that for  $(g, n) = (0, 1)$  there are no spherical metrics (unless smooth), and that for  $(g, n) = (0, 2)$  the metrics look like a double bigon (with  $\vartheta_1 = \vartheta_2 > 0$  and  $d(x_1, x_2) = \pi$ ) or a cyclic cover of degree  $r \geq 2$  of the standard  $\mathbb{S}^2$  branched at two points (so that  $\vartheta_1 = \vartheta_2 = r$  and  $d(x_1, x_2) \in (0, \pi)$ ).

**Spherical triangles.** In the case  $(g, n) = (0, 3)$  all such surfaces are double triangles. Eremenko [5] showed that there exist at most one such triangle for each  $\boldsymbol{\vartheta}$  and he determine exactly for which  $\boldsymbol{\vartheta}$  such triangle exists.

**Moduli space and forgetful map.** We denote by  $\mathcal{MSph}_{g,n}(\boldsymbol{\vartheta})$  the moduli space of spherical surfaces of genus  $g$  with  $n$  conical points of angles  $2\pi\boldsymbol{\vartheta}$ , and by  $\mathcal{M}_{g,n}$  the moduli space of Riemann surfaces of genus  $g$  with  $n$  distinct marked points. Let  $F_{g,n,\boldsymbol{\vartheta}} : \mathcal{MSph}_{g,n}(\boldsymbol{\vartheta}) \rightarrow \mathcal{M}_{g,n}$  be the map that sends forgets the metric and only remembers the underlying conformal structure. It can be shown that such forgetful map is a real-analytic map between real-analytic orbifolds (both of which have real dimension  $6g - 6 + 2n$ , if the metrics are not co-axial). Most problems can be formulated in terms of properties of  $\mathcal{MSph}_{g,n}(\boldsymbol{\vartheta})$  or of  $F_{g,n,\boldsymbol{\vartheta}}$ . As a warning, we stress that the moduli space of spherical surfaces displays quite different features than the analogous moduli spaces of flat or hyperbolic surfaces. Indeed, the existence of a metric in each conformal class is not guaranteed due to possible bubbling phenomena; moreover, the uniqueness of a metric in each conformal class is not guaranteed either, since the positive sign of the curvature makes the maximum principle not available.

**Non-emptiness problem: is  $\mathcal{MSph}_{g,n}(\boldsymbol{\vartheta}) \neq \emptyset$ ?**

Certainly one needs the obvious Gauss-Bonnet condition to be satisfied, namely

$\chi(S, \boldsymbol{\vartheta}) := 2 - 2g + n + \|\boldsymbol{\vartheta}\|_1 > 0$  with  $\|\boldsymbol{\vartheta}\|_1 := \vartheta_1 + \dots + \vartheta_n$ . For  $g > 0$ , the answer is yes [9] and the construction of a particular spherical metric is rather explicit. For  $g = 0$  non-emptiness depends on  $\boldsymbol{\vartheta}$ . In particular, in [8] it is proven that a non-coaxial metric in such space exists if and only if  $d_1(\boldsymbol{\vartheta} - \mathbf{1}, \mathbb{Z}_o^n) > 1$ , where  $\boldsymbol{\vartheta} - \mathbf{1} = (\vartheta_1 - 1, \dots, \vartheta_n - 1)$ ,  $\mathbb{Z}_o^n = \{\mathbf{m} \in \mathbb{Z}^n \mid m_1 + \dots + m_n \text{ odd}\}$  and  $d_1(\mathbf{p}, \mathbf{m}) = \sum_i |p_i - m_i|$  is the standard  $L^1$  distance in  $\mathbb{R}^n$ . There it is also proven that  $d_1(\boldsymbol{\vartheta} - \mathbf{1}, \mathbb{Z}_o^n) < 1$  implies that  $\mathcal{MSph}_{g,n}(\boldsymbol{\vartheta})$  is empty. The obstruction to the existence here relies in the monodromy, and in particular in the possibility of realizing a canonical  $SU(2)$ -lift of the monodromy representation associated to the spherical surface of genus 0. The analysis of the case  $d_1(\boldsymbol{\vartheta} - \mathbf{1}, \mathbb{Z}_o^n) = 1$  (in which metrics are necessarily co-axial by [8]) was carried out by Eremenko [4].

**Surjectivity problem: is  $F_{g,n,\boldsymbol{\vartheta}}$  surjective?** In other words, does there exist a spherical metrics in each conformal class? Analytically, the problem reduces to finding a suitable conformal factor (with respect to a smooth background metric on  $S$ ), and equivalent to solving a suitable singular Liouville equation on  $S$ . Using a variation approach, Troyanov [11] proved that the answer is yes in the subcritical case  $0 < \chi(S, \boldsymbol{\vartheta}) < \tau(S, \boldsymbol{\vartheta}) := \min\{2, 2\vartheta_1, \dots, 2\vartheta_n\}$ , since the functional associated to the above Liouville equation is coercive. Bartolucci-De Marchis-Malchiodi [1] showed that the answer is again yes if  $g > 0$  and all  $\vartheta_i \geq 1$ , again by considering the variational problem and showing that its sublevels have different topology. In this way they actually gave a lower bound (which increases with  $g$  and with  $\|\boldsymbol{\vartheta}\|_1$  on the number of metrics in the general conformal class). On the other hand, if one conical angle is small enough, it is possible to show with some minor exception that  $F_{g,n,\boldsymbol{\vartheta}}$  cannot be surjective [9].

**Holomorphicity problem: is  $F_{g,n,\boldsymbol{\vartheta}}$  holomorphic for some complex structure on  $\mathcal{MSph}_{g,n}(\boldsymbol{\vartheta})$ ?**

In some case the answer is yes, for instance if  $g = 0$  and  $\vartheta_4 = \vartheta_5 = \dots = \vartheta_n \in \mathbb{N}$ , since the monodromy representation is the same for every metric [6]. The answer is yes in the case  $g > 0$  and  $\vartheta_1 = \dots = \vartheta_n = 2$ . In general, the answer is no and an explicit example with  $(g, n) = (0, 4)$  and  $\boldsymbol{\vartheta} = (\vartheta_1, \frac{1}{2}, \frac{1}{2}, \frac{3}{2})$  with  $\vartheta_1$  small can be found in [9]. In such example the map  $F_{0,4,\boldsymbol{\vartheta}}$  displays some topological behaviour that is incompatible with holomorphicity.

**Connectedness problem: is  $\mathcal{MSph}_{g,n}(\boldsymbol{\vartheta})$  connected?**

This happens to be the case, for instance, if  $g = 0$  and all  $\vartheta_i \in \mathbb{N}$  by Liu-Osserman [7]. In general, both  $\mathcal{MSph}_{g,n}(\boldsymbol{\vartheta})$  and its image via  $F_{g,n,\boldsymbol{\vartheta}}$  can have many connected components. An explicit series of examples with  $(g, n) = (0, 3 + k)$  and at least  $3^k$  connected components is given in [9]. The angles are  $\boldsymbol{\vartheta} = (m_1 + \frac{1}{2}, m_2 + \frac{1}{2}, m_3 + \frac{1}{2}, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_k)$  for integers  $m_1, m_2, m_3 \in \mathbb{N}$  and small  $\varepsilon_1, \dots, \varepsilon_k > 0$ . In these examples, the conical points  $x_4, \dots, x_{3+k}$  must stay close to exactly one of the points  $x_1, x_2, x_3$ : the combinatorics of the configuration permits to separate (at least)  $3^k$  connected components.

**Properness problem: is  $F_{g,n,\boldsymbol{\vartheta}}$  proper?**

In other words, is the space of such spherical metrics in a fixed conformal class compact?

This is not always the case, due to possible bubbling phenomena. Thus, we introduce a *non-bubbling parameter*  $\text{NB}_{g,n}(\boldsymbol{\vartheta}) := d_{\mathbb{R}}(\chi(\dot{S}), \text{Crit}_{\boldsymbol{\vartheta}})$ , where  $d_{\mathbb{R}}$  is the usual distance in  $\mathbb{R}$  and  $\text{Crit}_{\boldsymbol{\vartheta}} := \left\{ \sum_{i \in I} \vartheta_i - \sum_{j \notin I} \vartheta_j + 2b \mid b \in \mathbb{N}, I \subseteq \{1, 2, \dots, n\} \right\}$ . A Gauss-Bonnet-type argument shows that a family of spherical metrics in a fixed conformal class may degenerate only if  $\text{NB}_{g,n}(\boldsymbol{\vartheta}) = 0$ : in this case properness of  $F_{g,n,\boldsymbol{\vartheta}}$  follows. A more precise quantitative statement [9] (“systole inequality”) is that: if  $\text{NB}_{g,n}(\boldsymbol{\vartheta}) \geq \varepsilon$  and  $\text{Extsys} \geq \frac{2\pi\|\boldsymbol{\vartheta}\|_1}{\log(1/\varepsilon)}$ , then  $\text{sys} \geq (\varepsilon/4\pi\|\boldsymbol{\vartheta}\|_1)^{-3\chi(\dot{S})}$ , where the *extremal systole*  $\text{Extsys}$  is the minimum of the extremal lengths of simple closed curves on  $\dot{S}$  and the *spherical systole*  $\text{sys}$  is the minimum of the lengths (with respect to the given spherical metric) of smooth geodesic arcs with endpoints in  $\boldsymbol{x}$ . Since both  $\text{Extsys} : \mathcal{M}_{g,n} \rightarrow \mathbb{R}_+$  and  $\text{sys} : \mathcal{MSph}_{g,n}(\boldsymbol{\vartheta}) \rightarrow \mathbb{R}_+$  decay to zero at infinity, the systole inequality easily implies properness of the forgetful map. Another consequence of the systole inequality, which reminds of Carlotto’s non-existence result [2] for solutions of singular Liouville equations is the following [9]: given  $\hat{\boldsymbol{\vartheta}} = (0, \vartheta_2, \vartheta_3, \dots, \vartheta_n)$  such that  $\text{NB}_{g,n}(\hat{\boldsymbol{\vartheta}}) > 0$  and fixed a conformal structure on  $(S, \boldsymbol{x})$ , there exists a small  $\vartheta_1^* > 0$  such that no spherical metric in the given conformal class and angles  $2\pi(\vartheta_1, \dots, \vartheta_n)$  exists if  $\vartheta_1 < \vartheta_1^*$ .

#### REFERENCES

- [1] Daniele Bartolucci, Francesca De Marchis, and Andrea Malchiodi, *Supercritical conformal metrics on surfaces with conical singularities*, Int. Math. Res. Not. IMRN (2011), no. 24, 5625–5643.
- [2] Alessandro Carlotto, *On the solvability of singular Liouville equations on compact surfaces of arbitrary genus*, Trans. Amer. Math. Soc. **366** (2014), no. 3, 1237–1256.
- [3] Chiun-Chuan Chen and Chang-Shou Lin, *Mean field equation of Liouville type with singular data: topological degree*, Comm. Pure Appl. Math. **68** (2015), no. 6, 887–947.
- [4] Alexandre Eremenko, *Co-axial monodromy*, preprint arXiv:1706.04608.
- [5] ———, *Metrics of positive curvature with conic singularities on the sphere*, Proc. Amer. Math. Soc. **132** (2004), no. 11, 3349–3355 (electronic).
- [6] Alexandre Eremenko and Vitaly Tarasov, *Fuchsian equations with three non-apparent singularities*, SIGMA Symmetry Integrability Geom. Methods Appl. **14** (2018), 058, 12 pages.
- [7] Fu Liu and Brian Osserman, *The irreducibility of certain pure-cycle Hurwitz spaces*, Amer. J. Math. **130** (2008), no. 6, 1687–1708.
- [8] Gabriele Mondello and Dmitri Panov, *Spherical metrics with conical singularities on a 2-sphere: angle constraints*, Int. Math. Res. Not. IMRN (2016), no. 16, 4937–4995.
- [9] Gabriele Mondello and Dmitri Panov, *Spherical surfaces with conical points: systole inequality and moduli spaces with many connected components*, preprint arXiv:1807.04373.
- [10] ———, *Metrics of constant curvature on a sphere with two conical singularities*, Differential geometry (Peñíscola, 1988), Lecture Notes in Math., vol. 1410, Springer, Berlin, 1989, pp. 296–306.
- [11] ———, *Prescribing curvature on compact surfaces with conical singularities*, Trans. Amer. Math. Soc. **324** (1991), no. 2, 793–821.

## Cones of effective cycles in moduli spaces of curves

SCOTT MULLANE

We report on recent progress on the structure of the cones of effective algebraic cycles in  $\overline{\mathcal{M}}_{g,n}$  through the use of flat geometry and the strata of abelian differentials.

Investigating the codimension-one subvarieties, known as divisors, of a moduli space is an important element in understanding the birational geometry of such spaces and corresponds to considering the rational morphisms of the moduli space. Harris and Mumford [HMu], Eisenbud and Harris [EH], Farkas [F1][F2], Farkas and Verra [FV], Chen and Coskun [CC1] and many others have used geometrically defined divisors to study many aspects of moduli spaces of curves including the Kodaira dimension and the cone of effective divisors. Though comparatively little is known, there has recently also been growing interest in understanding finer aspects of the birational geometry encoded in the cones of higher codimension cycles [CC2][FL1][FL2].

The stratum of canonical divisors of type  $\kappa = (k_1, \dots, k_n)$  is defined as

$$\mathbb{P}\Omega\mathcal{M}_g(\kappa) = \left\{ [C, p_1, \dots, p_n] \in \mathcal{M}_{g,n} \mid \sum_{i=1}^n k_i p_i \sim K_C \right\}.$$

This subvariety has codimension  $g - 1$  for a holomorphic signature (all  $k_i \geq 0$ ) and codimension  $g$  for a meromorphic signature. Taking the closure of this loci and pushing forward under morphisms forgetting marked points provides cycles of lower codimension in moduli spaces of curves with fewer marked points. In [M1] the author presents a closed formula for the divisor class  $D_\kappa$  in  $\overline{\mathcal{M}}_g$  of the closure of all possible codimension one loci of this type. Computations for bounded genus give bounds on the slope of these divisors

$$8 + \frac{1}{2^{g-3}} = s(D_{(4,2^{g-3})}^{\text{even}}) \leq s(D_\kappa) \leq s(D_{(g+1,1^{g-3})}) = 9 + \frac{2}{g(g+1)},$$

suggesting that asymptotically the slopes are bounded between 8 and 9.

Recently, the full compactification of the strata initiated by Farkas and Pandharipande [FP] was completed by Bainbridge, Chen, Gendron, Grushevsky and Möller [BCGGM] providing a useful tool in furthering our understanding of the birational geometry of  $\overline{\mathcal{M}}_{g,n}$ .

An old trick to show that a divisor is rigid and extremal in the effective and pseudo-effective cones employs a covering curve. If irreducible curves with class equal to  $[B]$  cover a Zariski dense subset of irreducible divisor  $D$  and  $[B] \cdot [D] < 0$  then  $D$  is rigid and extremal.

Considering meromorphic signatures  $\kappa = (d_1, d_2, d_3, 1^{2g-3})$  and

$$\varphi : \overline{\mathcal{M}}_{g,2g} \longrightarrow \overline{\mathcal{M}}_{g,g+1}$$

the forgetful morphism that forgets the last  $g - 1$  points, we obtain the divisors

$$D_\kappa^{g+1} = \varphi_*(\mathbb{P}\Omega\overline{\mathcal{M}}_g(\kappa))$$

in  $\overline{\mathcal{M}}_{g,g+1}$ . By constructing covering curves in [M2] we obtain these divisors are rigid and extremal for  $g \geq 2$ . Hence  $\overline{\text{Eff}}(\overline{\mathcal{M}}_{g,n})$  is not rational polyhedral for  $g \geq 2$ ,  $n \geq g + 1$ . Hence in these cases the Cox ring is not finitely generated, giving the new result that  $\overline{\mathcal{M}}_{2,n}$  is not a Mori dream space.

In [M3] using the extremal divisors from [M2] and gluing morphisms we are able to construct infinitely many codimension two cycles supported in the boundary of  $\overline{\mathcal{M}}_{g,n}$  for  $g \geq 3$  and  $n \geq g - 1$  via the strategy of [CC2].

To obtain rigid and extremal cycles supported in the interior of  $\overline{\mathcal{M}}_{g,n}$  we use an inductive argument on the rigidity of the cycles using the above base case of divisors. Let

$$\varphi_j : \overline{\mathcal{M}}_{g,2g} \longrightarrow \overline{\mathcal{M}}_{g,2g-j}$$

be the forgetful morphism forgetting the last  $j$  points. For  $g \geq 2$  the cycle

$$[\varphi_{j*} \mathbb{P}\Omega \overline{\mathcal{M}}_g(d_1, d_3, d_3, 1^{2g-3})]$$

is rigid and extremal for  $j = 0, \dots, g - 1$  with some  $d_i = 1$  if  $g = 2$ .

Finally we examine the genus  $g = 1$  case. In this case the meromorphic strata of canonical divisors have codimension one and to produce rigid and extremal higher codimension cycles we intersect the pullbacks of strata under forgetful morphisms to obtain infinitely many extremal codimension  $k$  cycles in  $\overline{\mathcal{M}}_{1,n}$  for  $k \leq n - 2$ .

#### REFERENCES

- [BCGGM] M. Bainbridge, D. Chen, Q. Gendron, S. Grushevsky, and M. Möller, Compactification of strata of abelian differentials, *Duke. Math. J.* **167** (2018), no. 12, 2347–2416.
- [CC1] D. Chen and I. Coskun, Extremal effective divisors on  $\overline{\mathcal{M}}_{1,n}$ , *Mathematische Annalen* **359** (2014), 891–908.
- [CC2] D. Chen and I. Coskun. Extremal higher codimension cycles on moduli spaces of curves. *Proc. London Math. Soc.* **111**(1)(2015), 181–204.
- [EH] D. Eisenbud, and J. Harris, The Kodaira dimension of the moduli space of curves of genus  $\geq 23$ , *Invent. Math.* **90** (1987), no. 2, 359–387.
- [F1] G. Farkas, The geometry of the moduli space of curves of genus 23. *Math. Ann.* **318** (2000), no. 1, 43–65.
- [F2] G. Farkas, Koszul divisors on moduli spaces of curves. *Amer. J. Math.* **131** (2009), no. 3, 819–867.
- [FP] G. Farkas and R. Pandharipande, The moduli space of twisted canonical divisors, with an appendix by F. Janda, R. Pandharipande, A. Pixton, and D. Zvonkine, *J. Institute Math. Jussieu* **17** (2018), 615–672.
- [FV] G. Farkas and A. Verra, The geometry of the moduli space of odd spin curves, *Ann. of Math. (2)* **180** (2014), no. 3, 927–970.
- [FL1] M. Fulger, and B. Lehmann, Morphisms and faces of pseudo-effective cones *Proc. Lon. Math. Soc.* **112** (2016), no. 4, 651–676.
- [FL2] M. Fulger, and B. Lehmann, Positive cones of dual cycle classes, *Alg. Geom.* **4** (2017), no. 1, 1–28
- [HMu] J. Harris and D. Mumford, On the Kodaira dimension of the moduli space of curves, *Invent. Math.* **67** (1982), no.1, 23–88.
- [M1] S. Mullane, Divisorial strata of abelian differentials, *Int. Math. Res. Notices* **6** (2017), 1717–1748.
- [M2] S. Mullane, On the effective cone of  $\overline{\mathcal{M}}_{g,n}$ , *Adv. in Math.* **320** (2017), 500–519.
- [M3] S. Mullane, On the effective cone of higher codimension cycles in  $\overline{\mathcal{M}}_{g,n}$ , arXiv:1710.09044

## Veech dichotomy, embedded triangles, and tessellations of the hyperbolic plane

DUC-MANH NGUYEN

Let  $M = (X, \omega)$  be a translation surface, and  $\Sigma$  a finite subset of  $M$  that contains all the singularities. A *saddle connection* for the pair  $(M, \Sigma)$  is a geodesic segment of the flat metric that joins two points in  $\Sigma$  and does not contain any point of  $\Sigma$  in its interior. Note that the endpoints of a saddle connection are not necessarily distinct.

Given any direction  $\theta \in \mathbb{RP}^1 = \mathbb{R} \cup \{\infty\}$ , one has a foliation  $\mathcal{F}_\theta$  on  $M$  by straight lines in direction  $\theta$ . If all the leaves of this foliation are either saddle connections, or closed geodesics,  $\mathcal{F}_\theta$  is said to be *periodic*. On the other hand, if each of its leaf is equidistributed in  $M$ , then  $\mathcal{F}_\theta$  is said to be *uniquely ergodic*.

Define  $\text{Aff}^+(M, \Sigma)$  to be the group of orientation preserving homeomorphisms of  $M$  that map  $\Sigma$  to itself and are given by affine maps  $v \mapsto A(v) + c$ , with  $A \in \text{SL}(2, \mathbb{R})$  and  $c \in \mathbb{R}$  constant, in local charts associated with the flat metric. The *Veech group*  $\Gamma(M, \Sigma)$  of the pair  $(M, \Sigma)$  is defined to be the image in  $\text{SL}(2, \mathbb{R})$  of the group  $\text{Aff}^+(M, \Sigma)$  under the derivative mapping.

In [4], Veech showed that if  $\Gamma(M, \Sigma)$  is a lattice in  $\text{SL}(2, \mathbb{R})$ , then  $(M, \Sigma)$  satisfies the following property: for any  $\theta \in \mathbb{RP}^1$ ,  $\mathcal{F}_\theta$  is either periodic, or uniquely ergodic. This is called the *Veech dichotomy*.

We will say that  $(M, \Sigma)$  satisfies the *topological Veech dichotomy* if for any  $\theta \in \mathbb{RP}^1$ , either  $\mathcal{F}_\theta$  is periodic, or every leaf of  $\mathcal{F}_\theta$  is dense in  $M$ . The topological Veech dichotomy is equivalent to the following property: for any  $\theta \in \mathbb{RP}^1$ , if there is a saddle connection in direction  $\theta$ , then  $\mathcal{F}_\theta$  is periodic. By definition, if  $(M, \Sigma)$  satisfies the Veech dichotomy, then it satisfies the topological Veech dichotomy as well. It is worth noticing that there are examples of surfaces satisfying the Veech dichotomy, but the Veech group is not a lattice of  $\text{SL}(2, \mathbb{R})$  (see [1, 3]).

Let  $T$  be an Euclidean triangle whose vertices are denoted by  $v_1, v_2, v_3$ . Let  $T^* = T \setminus \{v_1, v_2, v_3\}$ . We will say that  $T$  is an *embedded triangle* for the pair  $(M, \Sigma)$ , if there exists a map  $\varphi : T \rightarrow M$  satisfying the following

- $\varphi(v_i) \in \Sigma$ ,
- $\varphi(T^*) \subset M \setminus \Sigma$ ,
- $\varphi^*\omega = dz$ ,
- the restriction of  $\varphi$  to  $T^*$  is an embedding.

Let  $\mathbb{E}(M, \Sigma)$  denote the set of Euclidean triangles that be embedded in  $(M, \Sigma)$  (in the above sense).

To each embedded triangle  $T$ , we have a corresponding ideal triangle  $\Delta_T$  in the hyperbolic plane  $\mathbb{H}$  defined as follows: let  $k_1, k_2, k_3$  be the slopes of the sides of  $T$  (here, the slope of a vector  $v = (x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$  is defined to be  $\frac{x}{y} \in \mathbb{R} \cup \{\infty\}$ ), then  $\Delta_T$  is the ideal triangle whose vertices are  $k_1, k_2, k_3$ .

Let  $\mathcal{I}$  denote the set of ideal triangles in  $\mathbb{H}$  that are associated with the triangles in  $\mathbb{E}$ . Let  $\mathcal{L}$  denote the set of hyperbolic geodesics that are sides of the ideal triangles in  $\mathcal{I}$ , and  $\mathcal{C} \subset \partial\mathbb{H}$  the set of vertices of those ideal triangles.

**Example:** If  $M$  is the standard torus  $\mathbb{C}/\mathbb{Z}^2$ , and  $\Sigma = \{0\}$ , then it is not difficult to see that  $\mathcal{L}$  is the Farey tessellation of the hyperbolic plane.

**Theorem [2]** If  $(M, \Sigma)$  satisfies the topological Veech dichotomy, then  $\mathcal{L}$  defines a tessellation  $\Pi$  of  $\mathbb{H}$ . More precisely, every connected component of the complement of  $\mathcal{L}$  is a hyperbolic polygon with finitely many sides and area bounded by  $\pi$ .

Moreover, if  $\Gamma$  is a lattice then we have

$$\text{vol}(\mathbb{H}/\Gamma) \leq \pi \#(\mathcal{I}/\Gamma).$$

### Remarks

- a) Since the ideal triangles in  $\mathcal{I}$  may overlap, in most of the cases  $\Pi$  is not a triangulation of  $\mathbb{H}$ .
- b) Since  $(M, \Sigma)$  satisfies the topological Veech dichotomy,  $\mathcal{C}$  is actually the set of the periodic directions on  $M$ .
- c)  $\Gamma$  is a subgroup of the group of automorphisms of  $\Pi$ , denoted by  $\text{Aut}(\Pi)$ . If  $\Gamma$  is a lattice then its index in  $\text{Aut}(\Pi)$  is finite.
- d) One can exhibit an algorithm to compute  $\#(\mathcal{I}/\Gamma)$  and get a bound on the volume of  $\mathbb{H}/\Gamma$ .
- e) Let  $(M', \Sigma')$  be another translation surface with marked points, where  $M' = (X', \omega')$ . Suppose that there is a (ramified) covering map  $f : X' \rightarrow X$  such that  $f^*\omega = \omega'$ , and  $f^{-1}(\Sigma) = \Sigma'$ , then  $\mathbb{E}(M', \Sigma') = \mathbb{E}(M, \Sigma)$ . Hence  $\Pi(M', \Sigma') = \Pi(M, \Sigma)$ .

### REFERENCES

- [1] P. Hubert and T. Schmidt: Infinitely generated Veech groups, *Duke Math. J.* **123** (2004), no. 1, pp. 49–69.
- [2] D.-M. Nguyen: Veech dichotomy and tessellation of the hyperbolic plane, *preprint*, arXiv:1808.09329.
- [3] J. Smillie and B. Weiss: Veech dichotomy and the lattice property, *Ergo. Th. Dyn. Syst.* **28** (2008), no. 6, pp. 1959–1972.
- [4] W.A. Veech: Teichmüller curves in moduli space, Eisenstein series, and an application to triangular billiards, *Invent. Math.* **97** (1989), pp. 553–583.

## A new cohomology class on the moduli space of stable curves and maps to elliptic curves.

PAUL NORBURY

In this talk we define a collection of cohomology classes on the moduli space of curves. A generating function for the intersection numbers involving these cohomology classes is a tau function of the KdV hierarchy, analogous to the Kontsevich-Witten theorem. These classes naturally pair with cycles defined by branched covers of elliptic curves.

1. COHOMOLOGY CLASSES ON  $\overline{\mathcal{M}}_{g,n}$ .

Let  $\overline{\mathcal{M}}_{g,n}$  be the moduli space of genus  $g$  stable curves—curves with only nodal singularities and finite automorphism group—with  $n$  labeled points disjoint from nodes. Define  $\psi_i = c_1(L_i) \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$  the first Chern class of the line bundle  $L_i \rightarrow \overline{\mathcal{M}}_{g,n}$  with fibre above  $[(C, p_1, \dots, p_n)]$  given by  $T_{p_i}^*C$ . Consider the natural maps given by the gluing maps

$$\overline{\mathcal{M}}_{g-1,n+2} \xrightarrow{\phi_{\text{irr}}} \overline{\mathcal{M}}_{g,n}, \quad \overline{\mathcal{M}}_{h,|I|+1} \times \overline{\mathcal{M}}_{g-h,|J|+1} \xrightarrow{\phi_{h,I}} \overline{\mathcal{M}}_{g,n}, \quad I \sqcup J = \{1, \dots, n\}$$

and the forgetful map

$$\overline{\mathcal{M}}_{g,n+1} \xrightarrow{\pi} \overline{\mathcal{M}}_{g,n}.$$

In this talk we describe cohomology classes  $\Theta_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n})$  for  $g \geq 0$ ,  $n \geq 0$  and  $2g - 2 + n > 0$  satisfying the following four properties:

- (i)  $\Theta_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n})$  is of pure degree,
- (ii)  $\phi_{\text{irr}}^* \Theta_{g,n} = \Theta_{g-1,n+2}$ ,  $\phi_{h,I}^* \Theta_{g,n} = \pi_1^* \Theta_{h,|I|+1} \cdot \pi_2^* \Theta_{g-h,|J|+1}$ ,
- (iii)  $\Theta_{g,n+1} = \psi_{n+1} \cdot \pi^* \Theta_{g,n}$ ,
- (iv)  $\Theta_{1,1} = 3\psi_1$ .

A consequence of (i) and (ii) is that either  $\deg \Theta_{g,n} = 0$  or  $\deg \Theta_{g,n} = 4g - 4 + 2n$ . The first case gives rise to the trivial cohomology class  $1 \in H^0(\overline{\mathcal{M}}_{g,n})$ . The class  $1 \in H^0(\overline{\mathcal{M}}_{g,n})$  satisfies conditions (i), (ii), (iii')  $\Theta_{g,n+1} = \pi^* \Theta_{g,n}$  (which replaces (iii)) and (iv')  $\Theta_{1,1} = 1$  (which replaces (iv)). In the second case we produce the classes  $\{\Theta_{g,n}\}$  of this talk.

**Theorem 1** ([2]). *For  $2g - 2 + n > 0$ , there exist classes  $\Theta_{g,n}$  satisfying (i) - (iv) and furthermore any such classes satisfy the following properties.*

- (1)  $\Theta_{g,n} \in H^{4g-4+2n}(\overline{\mathcal{M}}_{g,n})$ .
- (2)  $\Theta_{0,n} = 0$  for all  $n$ .
- (3)  $\Theta_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n})^{S_n}$ , i.e. it is symmetric under the  $S_n$  action.
- (4) The following intersection numbers are uniquely determined:

$$(5) \quad \int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n} \prod_{i=1}^n \psi_i^{m_i} \prod_{j=1}^N \kappa_{\ell_j}.$$

$$Z^\Theta(\hbar, t_0, t_1, \dots) = \exp \sum_{g,n,\vec{k}} \frac{\hbar^{g-1}}{n!} \int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n} \cdot \prod_{j=1}^n \psi_j^{k_j} \prod t_{k_j}$$

is a tau function of the KdV hierarchy.

*Remarks:* The non-constructive uniqueness result (4)—which relies on the existence of non-explicit tautological relations—follows from the more general property that the intersection numbers  $\int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n} \prod_{i=1}^n \psi_i^{m_i} \prod_{j=1}^N \kappa_{\ell_j}$  are uniquely determined by *any* initial value  $\int_{\overline{\mathcal{M}}_{1,1}} \Theta_{1,1} \in \mathbb{C}$ . The existence proof of  $\Theta_{g,n}$ , which is constructed via the push-forward of a class over the moduli space of spin curves, requires the initial value property (iv) given by  $\Theta_{1,1} = 3\psi_1$ . The existence of  $\Theta_{g,n}$  with (iv) replaced by  $\Theta_{1,1} = \lambda\psi_1$  for general  $\lambda \in \mathbb{C}$  is unknown. For the initial value

$\int_{\overline{\mathcal{M}}_{1,1}} \Theta_{1,1} = \frac{1}{8}$ , (4) is strengthened by (5) which allows one to recursively calculate all intersection numbers  $\int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n} \prod_{i=1}^n \psi_i^{m_i}$  via recursive relations coming out of the KdV hierarchy. The proof of (5) does not directly use the KdV hierarchy. Instead it identifies the proposed KdV tau function  $Z^\Theta$  with a known KdV tau function—the Brezin-Gross-Witten KdV tau function  $Z^{\text{BGW}}$  defined in [1]. This identification of  $Z^\Theta(\hbar, t_0, t_1, \dots)$  with  $Z^{\text{BGW}}(\hbar, t_0, t_1, \dots)$  uses a set of tautological relations, known as Pixton’s relations, obtained from the moduli space of 3-spin curves and proven in [5].

## 2. GROMOV-WITTEN INVARIANTS

Define the moduli space of stable maps to a variety  $X$  by

$$\mathcal{M}_{g,n}(X, \beta) = \{(C, p_1, \dots, p_n) \xrightarrow{f} X \mid f_*[C] = \beta\} / \sim$$

where  $\beta \in H_2(X, \mathbb{Z})$ ,  $C$  is a genus  $g$  nodal curve containing  $n$  labeled point  $\{p_i\}$  which avoid the nodes. Furthermore, we require  $f$  to be *stable* meaning that it has only finitely many automorphisms. The moduli space of stable maps has irreducible components of different dimensions but it has a virtual class of dimension

$$\dim[\mathcal{M}_{g,n}(X, \beta)]^{\text{virt}} = (\dim X - 3)(1 - g) + \langle c_1(X), \beta \rangle + n.$$

Define the Gromov-Witten invariants of  $X$  coupled to  $\Theta_{g,n}$  by

$$\left\langle \Theta \cdot \prod_{i=1}^n \tau_{b_i}(\alpha_i) \right\rangle_d^g := \int_{[\mathcal{M}_{g,n}(X, \beta)]^{\text{virt}}} p^* \Theta_{g,n} \cdot \prod_{i=1}^n \psi_i^{b_i} ev_i^*(\alpha_i)$$

for  $\alpha_i \in H^*(X)$  and  $ev_i : \mathcal{M}_{g,n}(X, \beta) \rightarrow X$ , the evaluation map for  $i = 1, \dots, n$ . Line bundles  $L_i$  are defined analogously to those above by  $L_i \rightarrow \mathcal{M}_{g,n}(X, \beta)$  with fibre above  $[(f, C, p_1, \dots, p_n)]$  given by  $T_{p_i}^* C$  and  $\psi_i = c_1(L_i) \in H^2(\mathcal{M}_{g,n}(X, \beta))$ . The usual Gromov-Witten invariants of  $X$  replace  $p^* \Theta_{g,n}$  with the trivial class 1.

It is useful to view the Gromov-Witten invariants of  $X$  coupled to  $\Theta_{g,n}$  via the dimension of the virtual class capped with the pull-back of  $\Theta_{g,n}$ :

$$(1) \quad \dim \{[\mathcal{M}_{g,n}(X, \beta)]^{\text{virt}} \cap p^{-1}(\Theta_{g,n}^{\text{PD}})\} = (\dim X - 1)(1 - g) + \langle c_1(X), \beta \rangle$$

which shows that  $\dim X = 1$  is rather special.

The Gromov-Witten invariants coupled to  $\Theta_{g,n}$  have been calculated for target  $X = \{\text{pt}\}$  in Theorem 1, where its partition function is shown to be related to the KdV equation, and for  $X = \mathbb{P}^1$  in [3] where its partition function is shown to be related to the Toda equation. Here we make a few comments regarding the case when  $X = E$  is an elliptic curve.

When  $X = E$  we have  $\dim[\mathcal{M}_{g,n}(E, d)]^{\text{virt}} = 2g - 2 + n = \deg p^* \Theta_{g,n}$  so the two pair to give a number. This is equivalent to vanishing of the dimension given in (1) for  $X = E$  since  $\dim E = 1$  and  $c_1(E) = 0$ . It ends up that these invariants vanish. Instead, in place of  $\Theta_{g,n} = \psi_1 \dots \psi_n \pi^* \Theta_g$  remove the product  $\psi_1 \dots \psi_n$  and

simply consider the pull-back of  $\Theta_g \in H^*(\overline{\mathcal{M}}_g)$ :

$$\int_{[\overline{\mathcal{M}}_{g,n}(E,d)]^{\text{vir}}} p^* \Theta_g \cdot \prod_{i=1}^n ev_i^*(\omega), \quad \omega \in H^2(E)$$

which we can show does not always vanish. Oberdieck and Pixton [4] proved that the series  $\sum_{d \geq 0} q^d p_* \left( [\mathcal{M}_{g,n}(E,d)]^{\text{virt}} \cdot ev_i^*(\omega) \right) \in H^*(\overline{\mathcal{M}}_{g,n}) \otimes \text{QMod}$  is a cycle valued quasi-modular form. This implies that

$$\sum_{d \geq 0} q^d \int_{[\overline{\mathcal{M}}_n^g(E,d)]^{\text{vir}}} p^* \Theta_g \cdot \prod_{i=1}^n ev_i^*(\omega), \quad \omega \in H^2(E)$$

is quasimodular which should make it calculable.

#### REFERENCES

- [1] D. Gross and E. Witten, *Possible third-order phase transition in the large- $N$  lattice gauge theory*. Phys. Rev. D21 (1980) 446.
- [2] P. Norbury, *A new cohomology class on the moduli space of curves*. arXiv:1712.03662
- [3] P. Norbury, *Gromov-Witten invariants of  $\mathbb{P}^1$  coupled to the Brezin-Gross-Witten KdV tau function*. Preprint.
- [4] Oberdieck, G. and Pixton, A. *Holomorphic anomaly equations and the Igusa cusp form conjecture*. Inventiones mathematicae **213** (2018), 507-587.
- [5] R. Pandharipande, A. Pixton and D. Zvonkine, *Relations on  $\overline{\mathcal{M}}_{g,n}$  via 3-spin structures*. J. Amer. Math. Soc. **28** (2015), 279-309.

### Around enumerative aspects of moduli of curves and the double ramification cycle

AARON PIXTON

(joint work with Georg Oberdieck)

Enumerative geometry, broadly speaking, is the study of numerical invariants of moduli spaces. Instead of trying to count the number of geometric objects with certain properties directly, we look at the moduli space of such objects and compute an Euler characteristic or intersect classes in cohomology (or in the Chow ring) to get a number. In some situations this number will match with an enumerative count, but this isn't necessarily the case.

Gromov-Witten theory is a good example of this method. Let  $X$  be a projective variety. We want to count maps  $f : C \rightarrow X$  from a curve to  $X$  (maybe satisfying certain constraints), so we work in the (co)homology of a moduli space of such objects. More concretely,  $\overline{\mathcal{M}}_{g,n}(X, \beta)$  is the “moduli space of stable maps” parametrizing maps  $f : C \rightarrow X$  where  $C$  has arithmetic genus  $g$ , there are  $n$  marked points  $x_1, \dots, x_n \in C$ , the map represents curve class  $\beta \in H_2(X, \mathbb{Z})$  (i.e.  $f_*[C] = \beta$ ), and the map satisfies certain stability conditions.

This moduli space comes with a pure-dimensional virtual class  $[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\text{vir}} \in H_*(\overline{\mathcal{M}}_{g,n}(X, \beta))$  and morphisms

$$p : \overline{\mathcal{M}}_{g,n}(X, \beta) \rightarrow \overline{\mathcal{M}}_{g,n}, \text{ev}_1, \dots, \text{ev}_n : \overline{\mathcal{M}}_{g,n}(X, \beta) \rightarrow X.$$

Gromov-Witten invariants are defined by integrating cohomology classes pulled back along  $p, \text{ev}_1, \dots, \text{ev}_n$  against the virtual class.

I will discuss certain cycle-valued Gromov-Witten invariants, which are a variant of the above construction in which we consider the classes

$$p_* \left( [\overline{M}_{g,n}(X, \beta)]^{\text{vir}} \cap \prod_i \text{ev}_i^*(\gamma_i) \right) \in H^*(\overline{M}_{g,n})$$

in the cohomology of the moduli space of curves, where  $\gamma_1, \dots, \gamma_n \in H^*(X)$ . The numerical Gromov-Witten invariants can be reconstructed from these by pairing with cohomology classes of complementary degree. In this way it is natural to focus our attention on the cohomology of the moduli space of curves; the forgetful morphisms  $p : \overline{M}_{g,n}(X, \beta) \rightarrow \overline{M}_{g,n}$  mean that all of the Gromov-Witten moduli spaces sit above the moduli space of curves.

In particular, I will explain recent work [2] showing that these cycle-valued Gromov-Witten invariants are quasimodular forms in the case where  $X = E$  is an elliptic curve (and the invariants are summed over the degree of the map to form formal power series). The key ingredient in the proof is information about the double ramification cycle, a family of cohomology classes on  $\overline{M}_{g,n}$  that is very interesting in its own right.

Let  $A = (a_1, \dots, a_n)$  be a sequence of integers with sum zero. By taking the positive  $a_i$  or the absolute values of the negative  $a_i$ , we obtain two partitions  $\mu, \nu$  of the same number  $d$ . Intuitively the double ramification cycle parametrizes curves  $[C, x_1, \dots, x_n]$  of genus  $g$  that admit a degree  $d$  map to  $\mathbb{P}^1$  with ramification profiles  $\mu, \nu$  over  $0, \infty$  respectively and ramification points equal to the corresponding  $x_i$ . The precise definition of this over  $\overline{M}_{g,n}$  is usually done using the virtual class in relative Gromov-Witten theory. The result is a cycle

$$\text{DR}_{g,A} \in H^{2g}(\overline{M}_{g,n}).$$

A explicit formula for the double ramification cycle  $\text{DR}_{g,A}$  was given in [1].

#### REFERENCES

- [1] F. Janda, R. Pandharipande, A. Pixton, and D. Zvonkine, *Double ramification cycles on the moduli spaces of curves*, Publ. Math. Inst. Hautes Études Sci. **125** (2017), 221–266.
- [2] G. Oberdieck and A. Pixton, *Holomorphic anomaly equations and the Igusa cusp form conjecture*, Invent. Math. **213** (2018), no. 2, 507–587.

### The double ramification hierarchy

PAOLO ROSSI

The double ramification cycle can be used, together with a given cohomological field theory, to construct an integrable hierarchy of Hamiltonian PDEs and its quantization. This integrable system is called the double ramification hierarchy associated to the CohFT and was introduced at the classical level by A. Buryak and quantized and studied in a series of works with myself. I will explain how this construction works.

## Mazur-Veech volumes and intersection theory on the Hodge bundle

ADRIEN SAUVAGET

(joint work with Dawei Chen, Martin Möller)

**Mazur-Veech volumes.** Let  $g$  and  $n$  be non-negative integers such that  $2g - 2 + n > 0$ . We denote by  $\mathcal{M}_{g,n}$  the moduli space of smooth complex curves of genus  $g$  with  $n$  distinct marked points. This moduli space admits a smooth compactification  $\overline{\mathcal{M}}_{g,n}$  (of stable nodal curves) that is endowed with a universal curve  $\pi : \overline{\mathcal{C}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}$ . The *Hodge bundle*  $\overline{\mathcal{H}}_{g,n}$  is the total space of  $R^0\pi_*\omega_{\overline{\mathcal{C}}/\overline{\mathcal{M}}}$ : a point in  $\overline{\mathcal{H}}_{g,n}$  is a tuple  $(X, \alpha, x_1, \dots, x_n)$  of a curve endowed with an abelian differential and  $n$  distinct markings. We denote by  $\mathcal{H}_{g,n}$  its restriction to the smooth locus.

Let  $\mu = (m_1, \dots, m_n)$  be non-negative integers such that  $\sum_{i=1}^n m_i = 2g - 2$ . We denote by  $\mathcal{H}(\mu) \subset \mathcal{H}_{g,n}$  the locus of objects  $(X, \alpha, x_1, \dots, x_n)$  such that  $x_i$  is a zero of  $\alpha$  of order  $m_i$ .

Let  $(X_0, \alpha_0, x_1, \dots, x_n) \in \mathcal{H}(\mu)$  and let  $(\gamma_1, \dots, \gamma_{2g-1+n})$  be a basis the relative homology group  $H_1(X_0, \{x_1, \dots, x_n\}, \mathbb{Z})$ . Locally in  $\mathcal{H}(\mu)$ , one can identify the curves to  $X_0$  and define the periods  $v_i = \int_{\gamma_i} \alpha$ . The periods  $(v_1, \dots, v_{2g-1+n})$  are the local coordinates around  $(X_0, \alpha_0, x_i)$  (see [3]).

The period coordinates endow  $\mathcal{H}(\mu)$  with an affine integral structure and thus with a volume form  $\nu$ . We will consider the subspace  $\mathcal{H}(\mu)_{\leq 1} \subset \mathcal{H}(\mu)$  of objects  $(X, \alpha, x_1, \dots, x_n)$  satisfying

$$\text{area}(X, \alpha) := \frac{i}{2} \int_X \alpha \wedge \bar{\alpha} \leq 1.$$

We denote by  $\text{Vol}(\mu)$  the total volume of  $\mathcal{H}(\mu)_{\leq 1}$  for  $\nu$ . It is finite (see [4] and [7]) and we call it the Mazur-Veech volume of  $\mathcal{H}(\mu)$ .

**Expression of  $\text{Vol}(\mu)$  as an intersection number.** The space  $\mathcal{H}(\mu)$  is invariant under the  $\mathbb{C}^*$ -action on the Hodge bundle. Thus we denote by  $\mathbb{P}\mathcal{H}(\mu) \subset \mathbb{P}\mathcal{H}_{g,n}$  the projectivization of  $\mathcal{H}(\mu)$  and by  $\mathbb{P}\overline{\mathcal{H}}(\mu)$  its closure in  $\mathbb{P}\overline{\mathcal{H}}_{g,n}$ . The space  $\mathbb{P}\overline{\mathcal{H}}(\mu)$  is proper, of dimension  $2g - 2 + n$  and singular in general. We will consider two types of classes in  $A^*(\mathbb{P}\overline{\mathcal{H}}_{g,n}, \mathbb{Q})$ :

- The class  $\xi \in A^1(\mathbb{P}\overline{\mathcal{H}}_{g,n}, \mathbb{Q})$  is the Chern class of the canonical line bundle  $\mathcal{O}(1) \rightarrow \mathbb{P}\overline{\mathcal{H}}_{g,n}$ .
- For  $1 \leq i \leq n$ , we denote by  $\sigma_i : \overline{\mathcal{M}}_{g,n} \rightarrow \overline{\mathcal{C}}_{g,n}$  the section corresponding to the  $i$ -th marking. Then we define  $\psi_i \in A^1(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$  as the Chern class of the line bundle  $\sigma_i^*\omega_{\overline{\mathcal{C}}/\overline{\mathcal{M}}}$  (the cotangent line at the marking). Besides we denote by the same letter the pull-back of  $\psi_i$  under the forgetful map  $\mathbb{P}\overline{\mathcal{H}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}$ .

**Conjecture A.** For all  $\mu$  the following equality holds

$$\text{Vol}(\mu) = \frac{2(2i\pi)^{2g}}{(2g - 3 + n)!} \int_{\mathbb{P}\overline{\mathcal{H}}(\mu)} \xi^{2g-2} \prod_{i=1}^n \psi_i.$$

Such a formula was expected to exist since the 90's (see [2] for example) but it is only with the methods developed in [5] that one could evaluate the intersection numbers on right and side to make a precise statement. The case  $n = 1$  of this theorem is proved in [6] under a regularity assumption of a natural metric on the line bundle  $\mathcal{O}(1)$ . The extension to  $n \geq 1$  is the object of the current work in progress with D. Chen and M. Möller.

**Counting geodesics.** We assume that  $\mu$  is of length  $\geq 2$ . Let  $(X, \alpha, x_1, \dots, x_n)$  be a point in  $\mathcal{H}(\mu)$ . The differential  $\alpha$  endows the curve with Riemannian metric outside the zeros. A *saddle-connection* is a geodesics in  $X \setminus \{x_1, \dots, x_n\}$  that links two markings. We say that two saddle-connections  $\gamma_1$  and  $\gamma_2$  are homologous if  $X \setminus \{\gamma_1, \gamma_2\}$  is disconnected.

For all  $L \in \mathbb{R}_{\geq 0}$ , we denote by  $\mathcal{N}_{1 \rightarrow 2}(X, \alpha, L)$  the number of homology classes of saddle-connections between  $x_1$  and  $x_2$  of length  $\leq L$

**Conjecture B.** *For almost all  $(X, \alpha, x_1, \dots, x_n)$  in  $\mathcal{H}(\mu)$ , the following equivalence holds*

$$\mathcal{N}_{1 \rightarrow 2}(X, \alpha, L) \underset{L \rightarrow +\infty}{\sim} \frac{(m_1 + 1)(m_2 + 1)}{\text{area}(X, \alpha)} \pi L^2.$$

Surprisingly, we showed that Conjecture B implies Conjecture A. The argument is based on the relation between the growth of the number saddle connections and Mazur-Veech volumes established in [1].

## REFERENCES

- [1] Alex Eskin, Howard Masur, and Anton Zorich. *Moduli spaces of abelian differentials: the principal boundary, counting problems, and the Siegel-Veech constants*. Publ. Math. Inst. Hautes Études Sci., (97):61–179, 2003.
- [2] M. Kontsevich. *Lyapunov exponents and Hodge theory*. In The mathematical beauty of physics (Saclay, 1996), volume 24 of *Adv. Ser. Math. Phys.*, pages 318–332. World Sci. Publ., River Edge, NJ, 1997.
- [3] Maxim Kontsevich and Anton Zorich. *Connected components of the moduli spaces of Abelian differentials with prescribed singularities*. Invent. Math., 153(3):631–678, 2003.
- [4] Howard Masur. *Interval exchange transformations and measured foliations*. Ann. of Math. (2), 115(1):169–200, 1982.
- [5] Adrien Sauvaget. *Cohomology classes of strata of abelian differentials*. Preprint, 2017, arXiv:1701.07867.
- [6] Adrien Sauvaget. *Volumes and Siegel-Veech constants of  $\mathcal{H}(2g - 2)$  and Hodge integrals*. Preprint, 2018, arXiv:1801.01744.
- [7] William A. Veech. *Gauss measures for transformations on the space of interval exchange maps*. Ann. of Math. (2), 115(1):201–242, 1982.

## Gothic Teichmüller curves on Hilbert modular surfaces

DAVID TORRES-TEIGELL

(joint work with Martin Möller)

Teichmüller curves are complex geodesics in the moduli space of curves  $\mathcal{M}_g$ . They arise as the projection of  $\mathrm{SL}_2(\mathbb{R})$ -orbits of flat surfaces with optimal dynamics, called Veech surfaces. If the Veech surface is not obtained by a covering construction from a lower genus surface, the resulting Teichmüller curve is called primitive. There are very few infinite families of primitive Teichmüller curves: the Weierstrass curves in genus two (found independently by Calta and McMullen), the Prym curves in genus three and four (found by McMullen), Teichmüller curves uniformised by a triangle group (studied and classified by Bouw and Möller) and two new series of Teichmüller curves in genus four (found by McMullen-Mukamel-Wright and Eskin-McMullen-Mukamel-Wright respectively).

Each of the families lying in a fixed genus stems from an invariant submanifold ‘like the minimal stratum  $\Omega\mathcal{M}_2(2)$ ’ in genus two. While the geometry of  $\Omega\mathcal{M}_2(2)$  and of the Prym loci is well-understood now, the geometry of the two invariant submanifolds recently discovered by Eskin-McMullen-Mukamel-Wright ([EMMW18]) is basically unexplored. We will focus on the gothic locus  $\Omega G \subset \Omega\mathcal{M}_4(2^3)$ , introduced already in [MMW17].

**The gothic locus.** The gothic locus  $\Omega G$  is defined as the set of flat surfaces  $(X, \omega)$  in the stratum  $\Omega\mathcal{M}_4(2^3)$  that admit (1) an involution  $J$  sending  $\omega$  to  $-\omega$  and fixing the three zeros of  $\omega$  and three other marked points and (2) a degree three ‘odd’ map  $\pi_B : X \rightarrow B$  to an elliptic curve  $B$  mapping all the zeros to a single point. Under these conditions, the involution  $J$  induces a degree two map  $\pi_A : X \rightarrow A$  to another elliptic curve  $A$ , and the quotient  $P(X)$  of the Jacobian of  $X$  by the subvariety generated by  $\pi_A^*A$  and  $\pi_B^*B$  is called the Prym variety of  $X$ , and inherits a natural polarization of type  $(1, 6)$ .

It is proved in [MMW17] that the gothic locus is a 4-dimensional affine invariant manifold. Moreover, for each discriminant  $D > 0$  the subset  $\Omega G_D$  consisting of those flat surfaces  $(X, \omega)$  for which the Prym variety  $P(X)$  admits real multiplication by the quadratic order  $\mathcal{O}_D$  with  $\omega$  as an eigenform, is a closed (possibly empty)  $\mathrm{SL}_2(\mathbb{R})$ -orbit, and its projection  $G_D$  to  $\mathcal{M}_4$  is a finite union of Teichmüller curves. These Teichmüller curves are primitive if and only if  $D$  is not a square, which we assume throughout the talk.

**Hilbert modular surfaces and Hilbert modular forms.** The space of  $(1, 6)$ -polarised abelian varieties with a choice of real multiplication by  $\mathcal{O}_D$  is parameterised by the Hilbert modular surfaces  $X_D(\mathfrak{b})$ , where  $\mathfrak{b}$  is an  $\mathcal{O}_D$ -ideal of norm 6. These are quotients of the bi-disc  $\mathbb{H}^2$  by the action of a discrete group of matrices  $\mathrm{SL}(\mathfrak{b} \oplus \mathcal{O}_D^\vee) \subset \mathrm{SL}_2(\mathbb{Q}(\sqrt{D}))$  acting on the two copies of  $\mathbb{H}$  by Galois-conjugate Möbius transformations. In particular, via the Prym-Torelli map  $X \mapsto P(X)$ , one can see the Teichmüller curves  $G_D$  as curves inside some Hilbert modular surface

$X_D(\mathfrak{b})$ . We will denote by  $G_D(\mathfrak{b})$  the components of  $G_D$  that lie inside the Hilbert modular surface  $X_D(\mathfrak{b})$ . Our main objective is to cut  $G_D(\mathfrak{b})$  out as the vanishing locus of certain Hilbert modular form  $\mathcal{G}_D$  on  $X_D(\mathfrak{b})$ .

In order to do so, we first study the image of  $X$  inside its Prym variety  $P(X)$  under the Abel-Prym map  $\varphi : X \rightarrow P(X)$  and determine the existence of an odd theta function  $\theta_X \in H^0(\mathcal{L})_-$  such that  $\varphi(X) = \text{div}(\theta_X)$ , where  $\mathcal{L}$  denotes the line bundle  $\mathcal{O}_{P(X)}(\varphi(X))$  determined by the image of  $X$ . This theta function fulfills five conditions, expressed in terms of the vanishing of certain derivatives at special points.

Now, the line bundle  $\mathcal{O}_{P(X)}(\varphi(X))$  on  $P(X)$  can be consistently defined everywhere on  $X_D(\mathfrak{b})$ , and the fulfillment of three of these five conditions determines a unique theta function  $\theta_X$  for each point  $\tau$  in an open subset  $X_D(\mathfrak{b}) \setminus \mathcal{R}(\mathfrak{b})$ . The last two conditions are fulfilled by being in the zero locus of a Hilbert modular form. More precisely, for each  $\tau \in \mathbb{H}^2$ , there are four functions  $\theta_j(\tau, \mathbf{u}) : \mathbb{C}^2 \rightarrow \mathbb{C}$  generating  $H^0(\mathcal{L})_-$ , where  $\mathbf{u} = (u_1, u_2)$ . One can then define the gothic modular form as

$$\mathcal{G}_D(\tau) = \frac{\partial}{\partial u_2} \theta_0(\tau, 0) \cdot \frac{\partial}{\partial u_2} \theta_1(\tau, 0) - \frac{\partial}{\partial u_2} \theta_2(\tau, 0) \cdot \frac{\partial}{\partial u_2} \theta_3(\tau, 0).$$

The form  $\mathcal{G}_D$  is a Hilbert modular form of weight  $(1, 3)$  with the property that, if  $\theta_X$  satisfies the extra two conditions at a point  $\tau \in X_D(\mathfrak{b}) \setminus \mathcal{R}(\mathfrak{b})$ , then  $\mathcal{G}_D(\tau) = 0$ . In particular,  $G_D(\mathfrak{b}) \subset \text{div}(\mathcal{G}_D)$ .

**The vanishing locus of the modular form.** Contrary to the expectation from the situation in genus two and in the Prym loci of genus three and four (see [Möll14]), the vanishing locus of the gothic modular form  $\mathcal{G}_D$  does not consist uniquely of the Teichmüller curve  $G_D(\mathfrak{b})$ , but contains other ‘spurious curves’. More precisely,

$$\text{div}(\mathcal{G}_D) = G_D(\mathfrak{b}) + 2\mathcal{R}(\mathfrak{b}).$$

The curve  $\mathcal{R}(\mathfrak{b})$  is formed by the points corresponding to  $(1, 6)$ -polarised abelian varieties isomorphic to a product  $E_1 \times E_2$  of elliptic curves with the natural  $(2, 3)$ -polarization. In particular, it is a collection of modular curves whose parameters can all be computed.

**Main results.** We use this description to calculate the Euler characteristic of  $G_D$ .

**Theorem 1** ([MTT]). *Let  $D$  be a non-square discriminant. The gothic Teichmüller curve  $G_D$  is non-empty if and only if  $D \equiv 0, 1, 4, 9, 12, 16 \pmod{24}$ . In this case,  $G_D$  consists of  $\iota_D$  different sub-curves  $G_D(\mathfrak{b})$  corresponding to the  $\iota_D$  different  $\mathcal{O}_D$ -ideals  $\mathfrak{b}$  of norm 6. The Euler characteristics of all these sub-curves agree and are equal to*

$$-\chi(G_D(\mathfrak{b})) = \frac{3}{2} \chi(X_D(\mathfrak{b})) + 2 \chi(\mathcal{R}(\mathfrak{b})).$$

The number  $\iota_D$  of sub-curves is one if  $D \equiv 0, 12 \pmod{24}$ , two if  $D \equiv 4, 9, 16 \pmod{24}$  and four if  $D \equiv 1 \pmod{24}$ . We strongly suspect the sub-curves  $G_D(\mathbf{b})$  to be irreducible. The formula in the theorem can be written explicitly as

$$-\chi(G_D(\mathbf{b})) = \frac{1}{20} \kappa_D \sum_{[a,b,c] \in \mathcal{P}_1(D)} a - \frac{1}{3\iota_D} \sum_{[a,b,c] \in \mathcal{P}_6(D)} a,$$

where, for a quadratic discriminant  $D = f^2 D_0$  with conductor  $f$  and  $g = \gcd(6, f)$ , we define  $\kappa_D = \sigma(g)/g$  and

$$\mathcal{P}_k(D) = \{[a, b, c] \in \mathbb{Z}^3 : a > 0 > c, D = b^2 - 4 \cdot k \cdot ac$$

$$\text{and } \gcd(f, b, c/c_0) = 1, \text{ where } c_0 \text{ is the square-free part of } c \}.$$

We are also able to compute the Lyapunov exponent corresponding to the Prym part of  $G_D$ . Recall that Lyapunov exponents of a flat surface  $(X, \omega)$  measure the growth rate of cohomology classes in  $H^1(X, \mathbb{R})$  under parallel transport along the geodesic flow. In the case of flat surfaces  $(X, \omega)$  in the Gothic locus, the existence of the maps  $\pi_A$  and  $\pi_B$  splits the four non-negative Lyapunov exponents into three subsets:  $\{\lambda_A\}$  and  $\{\lambda_B\}$  corresponding to  $A$  and  $B$ , and  $\{\lambda_1 = 1, \lambda_P\}$ , corresponding to the ‘Prym’ complement. The existence of spurious curves in the vanishing locus of  $\mathcal{G}_D$  causes a varying phenomenon of the exponent  $\lambda_P$ .

**Theorem 2** ([MTT]). *The Prym Lyapunov exponent of a gothic Veech surface on  $G_D$  is equal to  $\lambda_P(G_D(\mathbf{b})) = 1 + \chi(X_D(\mathbf{b}))/\chi(G_D(\mathbf{b}))$ .*

*In particular, the Prym Lyapunov exponent  $\lambda_P$  of a generic surface in the gothic locus is equal to  $3/13$ .*

The second part of the theorem is a direct consequence of some asymptotics formulas for  $\lambda_P(G_D(\mathbf{b}))$  and the convergence of individual Lyapunov exponents ([BEW17]), since the curves  $G_D$  equidistribute towards (the Lebesgue measure on) the Gothic locus by [EMM15].

## REFERENCES

- [BEW17] C. Bonatti, A. Eskin, and A. Wilkinson. Projective cocycles over  $\mathrm{SL}_2(\mathbb{R})$ -actions: measure invariant under the upper triangular group. Preprint, arXiv:1709.02521, 2017.
- [EFW18] A. Eskin, S. Filip, and A. Wright. The algebraic hull of the Kontsevich–Zorich cocycle. *Ann. of Math. (2)*, 188(1):281–313, 2018.
- [EMMW18] A. Eskin, C. McMullen, R. Mukamel, and A. Wright. Billiards, quadrilaterals and moduli spaces. Preprint, 2018.
- [EMM15] A. Eskin, M. Mirzakhani, and A. Mohammadi. Isolation, equidistribution, and orbit closures for the  $\mathrm{SL}(2, \mathbb{R})$  action on moduli space. *Ann. of Math. (2)*, 182(2):673–721, 2015.
- [MMW17] C. McMullen, R. Mukamel, and A. Wright. Cubic curves and totally geodesic subvarieties of moduli space. *Ann. of Math. (2)*, 185(3):957–990, 2017.
- [Möl14] M. Möller. Prym covers, theta functions and kobayashi geodesics in hilbert modular surfaces. *Amer. Journal. of Math.*, 135:995–1022, 2014.
- [MTT] M. Möller, D. Torres-Teigell. Euler characteristics of Gothic Teichmüller curves. Preprint, arXiv:1807.10260, 2018.

## Around tropical curves

MARTIN ULIRSCH

### 1. TROPICAL CURVES

At its core, the geometry of tropical curves is based on an almost serendipitous analogy between compact Riemann surfaces and finite graphs. For example, there is a rich theory of divisors on finite graphs that gives rise to analogues of the classical Riemann-Roch-Theorem and the Abel-Jacobi-Theorem [4].

Let  $X$  be a smooth projective curve over a non-Archimedean field  $K$  and let  $\mathcal{X}$  be a flat and proper semistable model of  $X$ , e.g. arising as the base change of a semistable model over the unit disc  $\Delta$  of a family of Riemann surfaces over the pointed unit disc  $\Delta^*$ . The *dual graph*  $G$  of the special fiber  $\mathcal{X}_0$  of  $\mathcal{X}$  is a finite graph whose vertices correspond to the components of  $\mathcal{X}_0$  and whose edges correspond to the nodes of  $\mathcal{X}_0$ .

We want to think of  $G$  as a piecewise linear limit of the degeneration  $\mathcal{X}$ . We further develop this analogy by endowing  $G$  with a vertex weight  $h: V \rightarrow \mathbb{Z}_{\geq 0}$  that associates to a vertex  $v$  the genus of (the normalization of) the corresponding component  $X_v$ . This way, if we define the *genus of*  $(G, h)$  to be

$$g(G, h) = b_1(G) + \sum_{v \in V(G)} h(v) ,$$

we find that the genus is constant throughout the whole family.

Moreover, we also endow  $(G, h)$  with an edge length  $|\cdot|: E \rightarrow \mathbb{R}_{>0}$  that associates to an edge  $e$  corresponding to a node with local equation  $xy = t_e$  the valuation  $\text{val}(t_e)$  of  $t_e$  as its length. The *geometric realization*  $|(G, |\cdot|)|$  of  $(G, |\cdot|)$  is the metric space that is given by glueing intervals of length  $|e|$  for every edge  $e$  according to the incidences given by  $G$ . The vertex weight  $h$  naturally extends to a function on  $|(G, |\cdot|)|$  with finite support.

**Definition 1.** An (*abstract*) *tropical curve*  $\Gamma$  is a connected singular metric space  $|\Gamma|$  together with function  $h: |\Gamma| \rightarrow \mathbb{Z}_{>0}$  with finite support that is of the form  $(|(G, |\cdot|)|, h)$  for a finite metric graph  $(G, |\cdot|)$  with a vertex weight  $h$ .

We refer to  $(G, |\cdot|, h)$  as a *model* of  $\Gamma$ . Given a smooth projective curve  $X$  as above, we write  $\Gamma_X$  for the dual tropical curve of  $X$ .

### 2. MODULI OF TROPICAL CURVES

Let  $g \geq 2$ . Not unlike Riemann surfaces, tropical curves naturally live in a moduli space, the *moduli space*  $M_g^{\text{trop}}$  of *stable tropical curves of genus*  $g$ . It parametrizes isomorphism classes of stable tropical curves of genus  $g$ . A tropical curve  $\Gamma$  is hereby said to be *stable* if there is a model  $(G, |\cdot|, h)$  of  $\Gamma$  such that

$$2h(v) - 2 + \text{val}(v) > 0$$

for all vertices  $v$  of  $G$ . See [1] for a discussion of the finer combinatorial structure of  $M_g^{\text{trop}}$  and [8] for the proper stack-theoretic treatment.

In order to formalize the somewhat informal association  $X \mapsto \Gamma_X$ , we make use of the non-Archimedean analytification  $\mathcal{M}_g^{an}$  of  $\mathcal{M}_g$  in the sense of Berkovich [6]. A point in  $\mathcal{M}_g^{an}$  is nothing but a  $K$ -valued point of  $\mathcal{M}_g$  for a suitable non-Archimedean extension of the trivially valued base field  $k$ , i.e. a smooth projective curve  $X$  over  $K$ . Since the Deligne-Mumford compactification is proper, there is a finite extension  $L$  of  $K$  and a unique semistable model  $\mathcal{X}$  of  $X_L$  as above. The association  $X \mapsto \Gamma_X$  defines a *tropicalization map*

$$\text{trop}_g: \mathcal{M}_g^{an} \longrightarrow M_g^{trop}$$

whose structural properties have been studied in [1].

**Theorem 2** ([1]). *The tropicalization map is a well-defined, continuous, proper and closed strong deformation retraction to the non-Archimedean skeleton of  $\mathcal{M}_g^{an}$ .*

Notice, in particular, that  $M_g^{trop}$  completely determines the topological homotopy type of  $\mathcal{M}_g^{an}$  which is the source of the recent application of tropical methods to study the top-weight cohomology of  $\mathcal{M}_g$  in [9].

### 3. DIVISORS ON TROPICAL CURVES

As usual in modern geometry, we describe a geometric object by studying the types of functions that naturally live on it: Let  $\Gamma$  be a tropical curve. A *rational function* on  $\Gamma$  is a continuous map  $f: \Gamma \rightarrow \mathbb{R}$  whose restriction to every edge –which we identify with the interval  $[0, |e|]$ – is a piecewise linear function with integer slopes. A *divisor* on a tropical curve  $\Gamma$  is a formal finite sum  $\sum_{i=1}^n a_i p_i$  over points in  $\Gamma$ ; its *degree* is given by  $\sum_{i=0}^n a_i$ . Given a rational function, we may define its associated *principal divisor*

$$\text{div}(f) = \sum_{p \in \Gamma} \text{ord}_p(f) \cdot p$$

where  $\text{ord}_p(f)$  is the sum of the outgoing slopes at the point  $p$ . Two divisors are *equivalent* if their difference is principal and the *(complete) linear system*  $|D|$  of a divisor  $D$  is the set of effective divisors that are equivalent to  $D$ .

Unlike in the classical world of Riemann surfaces, tropical linear systems are far more complicated than mere projective spaces. They are not equidimensional which requires us to use a more implicit definition of the *rank* of linear system.

**Definition 3.** The *rank*  $r(D)$  of the linear system  $|D|$  is the largest number  $r \geq 0$  such that  $|D - E| \neq \emptyset$  for all effective divisors  $E$  of degree  $r$ . If  $|D| = \emptyset$  we formally set  $r(D) = -1$ .

Let  $\Gamma$  be a tropical curve of genus  $g$ . Define the *canonical divisor* on  $\Gamma$  to be

$$K_\Gamma = \sum_{p \in \Gamma} (2h(v) - 2 + \text{val}(p)) \cdot p$$

where  $\text{val}(p)$  denotes the valence of  $p$ . The Riemann-Roch-Theorem now states:

**Theorem 4** ([4, 12, 10]). *For a divisor  $D$  of degree  $d$  on  $\Gamma$  we have*

$$r(D) - r(K_\Gamma - D) = d - g + 1 .$$

We refer the reader to [5] for a recent survey of divisor theory on tropical curves and its applications to the theory of limit linear series.

#### 4. SPECIALIZATION AND THE REALIZABILITY PROBLEM

Let  $D$  be an effective Cartier divisor on a smooth projective curve  $X$  over a non-Archimedean field  $K$ . By a version of the stable reduction theorem (see [11] for details) there is a finite extension  $L$  of  $K$  as well as a unique semistable model  $\mathcal{X}$  of  $X_L$  such that the special fiber  $\mathcal{D}_0$  of the closure  $\mathcal{D}$  of  $D_L$  in  $\mathcal{X}$  does not meet the nodes of  $\mathcal{X}_0$  and  $K_{\mathcal{X}} + \mathcal{D}$  is relatively ample over the valuation ring of  $L$ .

We then define the *specialization*  $\text{sp}(D)$  of  $D$  to  $\Gamma_X$  as the multidegree of  $\mathcal{D}_0$  on the semistable model of  $\Gamma_X$  induced by  $\mathcal{X}$ . If  $D$  is an arbitrary Cartier divisor on  $X$ , we write it as  $D_1 - D_2$  for effective divisors  $D_1$  and  $D_2$  on  $X$  and extend the specialization linearly. Theorem 2 generalizes to this situation (see [11] for details), which, in particular, shows that the association  $D \mapsto \text{sp}(D)$  is well-defined.

The following Lemma is known as *Baker's Specialization Lemma* and is an extension of the principle of semicontinuity of ranks to the tropical limit.

**Lemma 5** ([2]). *For a divisor  $D$  on  $X$  we have the inequality:*

$$r_X(D) \leq r_\Gamma(\text{sp}(D)) .$$

We can formulate the following *realizability problem* which lies at the heart of many of the successful applications of tropical divisors to Brill-Noether theory.

**Problem 1.** Let  $D = \sum_{i=1}^n a_i p_i$  be a divisor of degree  $d$  and rank  $r$  on a tropical curve  $\Gamma$ . Does there exist a smooth projective curve  $X$  over a non-Archimedean field  $K$  as well as a divisor  $\tilde{D}$  on  $X$  of multiplicity profile  $(a_1, \dots, a_n)$  and rank  $r$  such that  $\Gamma$  is the dual tropical curve of  $X$  and  $D$  is the specialization of  $\tilde{D}$ .

The realizability problem is, in general, very difficult, and, due to a version of *Murphy's Law* (see [7] for details), we do not expect to ever find a solution in full generality. Nevertheless, in specific situations, complete solutions are possible, e.g. in the case of effective canonical divisors.

**Theorem 6** ([11]). *Let  $g \geq 2$ . An effective canonical divisor  $D = K_\Gamma + \text{div}(f)$  on a tropical curve  $\Gamma$  of genus  $g$  is realizable if and only if:*

- (1) *For every horizontal edge  $e$  in the graph  $\Gamma_f$  of  $f$  there is a simple cycle in  $\Gamma_f$  containing  $e$  that does not pass through any level of  $\Gamma_f$  below  $e$ .*
- (2) *For every inconvenient point  $p$  of  $\Gamma_f$  there is a simple cycle in  $\Gamma_f$  based at  $p$  that does not pass through any level of  $\Gamma_f$  below  $p$ .*

Loosely speaking, a point is *inconvenient* if (the absolute value of) one of the negative slopes of  $f$  emanating from  $p$  is bigger than the sum of all positive slopes emanating from  $p$ . The proof of Theorem 6 is deeply rooted in the recent complex analytic results of [3] on the closure of strata of abelian differentials in  $\overline{\mathcal{M}}_{g,n}$  and only applies in characteristic zero.

## REFERENCES

- [1] D. Abramovich, L. Caporaso, S. Payne, *The tropicalization of the moduli space of curves*, Ann. Sci. Éc. Norm. Supér. (4) **48** (2015), no. 4, 765–809.
- [2] M. Baker, *Specialization of linear systems from curves to graphs*, Algebra Number Theory **2** (2008), Number 6, 613–653.
- [3] M. Bainbridge, D. Chen, Q. Gendron, S. Grushevsky, M. Möller, *Compactification of strata of abelian differentials*, Preprint, arXiv:1604.08834 (2016).
- [4] M. Baker, S. Norine, *Riemann-Roch and Abel-Jacobi theory on a finite graph*, Adv. Math. **215** (2007), Issue 2, 766–788.
- [5] M. Baker, D. Jensen, *Degeneration of Linear Series From the Tropical Point of View and Applications*, Preprint, arXiv:1504.05544 (2015).
- [6] V. Berkovich, *Spectral Theory and Analytic Geometry over Non-Archimedean Fields*, Mathematical Surveys and Monographs **33** (1990) 169 pp.
- [7] D. Cartwright, *Lifting matroid divisors on tropical curves*, Mathematical Sciences (2015) 2:23. <https://doi.org/10.1186/s40687-015-0041-x>
- [8] R. Cavalieri, M. Chan, M. Ulirsch, J. Wise, *A moduli stack of tropical curves*, Preprint, arXiv:1704.03806 (2017).
- [9] M. Chan, S. Galatius, S. Payne, *Tropical curves, graph homology, and top weight cohomology of  $M_g$* , Preprint, arXiv:1805.10186 (2018).
- [10] A. Gathmann, M. Kerber, *A Riemann-Roch theorem in tropical geometry*, Math. Zeit. **259** (2008), Issue 1, 217–230.
- [11] M. Möller, M. Ulirsch, A. Werner, *Realizability of tropical canonical divisors*, Preprint, arXiv:1710.06401 (2017).
- [12] G. Mikhalkin, I Zharkov, *Tropical curves, their Jacobians and theta functions*, Curves and abelian varieties, Contemp. Math. **465** (2008), 203–230.

## Around higher Teichmüller theory

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The Teichmüller space is the universal covering of the moduli space of complex curves. It also plays a key role in understanding flat surfaces. Higher Teichmüller spaces have been discovered and investigated in the past twenty years from different points of view. They share many interesting properties with classical Teichmüller space. However, many interesting questions about higher Teichmüller spaces are wide open, and some might be of interest to people interested in algebraic curves and flat surfaces.

Let  $\Sigma_g$  be a closed oriented surface of genus  $g \geq 2$ . The Teichmüller space  $\mathcal{T}(\Sigma_g)$  is the space of marked conformal structures on  $\Sigma_g$ . By the uniformization theorem, there is a unique hyperbolic metric in each conformal class, and thus  $\mathcal{T}(\Sigma_g)$  can be identified with the space  $\mathcal{H}(\Sigma_g)$  of marked hyperbolic structures. The holonomy of a marked hyperbolic structure is a group homomorphism  $\pi_1(\Sigma_g) \rightarrow \mathrm{PSL}(2, \mathbb{R})$ , where  $\mathrm{PSL}(2, \mathbb{R})$  is the group of isometries of the hyperbolic plane. This gives a map  $\mathrm{hol} : \mathcal{H}(\Sigma_g) \rightarrow \mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{PSL}(2, \mathbb{R}))/\mathrm{PSL}(2, \mathbb{R})$ , which is an embedding onto a connected component. The image consists entirely of discrete and faithful representations, i.e. injective homomorphisms with discrete image. In fact there are two such components - one which parametrizes hyperbolic structures on  $\Sigma_g$  and the other on the surface with the opposite orientation.

Viewing Teichmüller space as a connected component of the representation variety  $\text{Hom}(\pi_1(\Sigma_g), \text{PSL}(2, \mathbb{R}))/\text{PSL}(2, \mathbb{R})$  is the starting point for higher Teichmüller theory, where we replace  $\text{PSL}(2, \mathbb{R})$  by a general simple Lie group  $G$ , and in particular by a Lie group of *higher*<sup>1</sup> rank.

**Definition 1.** A higher Teichmüller space is a union of connected components  $\mathcal{T}(\Sigma_g, G) \subset \text{Hom}(\pi_1(\Sigma_g), G)/G$  that consists entirely of discrete and faithful representations.

Examples of higher Teichmüller spaces have been discovered through the work of several mathematicians coming from rather different mathematical areas, and only a posteriori it was shown that they satisfy the property that we here use as definition. [Hit92, Gol90, Lab06, FG05, BIW10].

Not every Lie group  $G$  admits a higher Teichmüller space. When  $G = \text{SL}(2, \mathbb{C})$ , or more generally a simply connected complex group, the representation variety is irreducible and hence has only one connected component. We currently know two families of higher Teichmüller spaces: *Hitchin components*, defined when  $G$  is a split real form, e.g.  $\text{PSL}(n, \mathbb{R})$  or  $\text{PSp}(2n, \mathbb{R})$ , and *maximal representations*, defined when  $G$  is a Lie group of Hermitian type, e.g.  $\text{PSp}(2n, \mathbb{R})$  or  $\text{PU}(n, n)$ . Representations in the Hitchin component as well as maximal representations can be characterized in terms of positivity of triples in (partial) flag varieties. This provides a connection between higher Teichmüller spaces and commutative as well as non-commutative cluster algebras. For a more detailed description of these aspects and a conjecture regarding the existence of other higher Teichmüller spaces, we refer to [GW18, Wie12].

Many recent results in higher Teichmüller theory generalize well known results about the space of hyperbolic structures on  $\Sigma_g$ . At the same time there are interesting features of higher Teichmüller spaces that do not appear for classical Teichmüller space. In particular, the complex analytic theory of Teichmüller space has not yet been generalized to the higher rank situation.

We end by discussing the action of the mapping class group and the relation of higher Teichmüller spaces with classical Teichmüller space. As on Teichmüller space, the mapping class group acts properly on Hitchin components and spaces of maximal representations. The quotients are of infinite volume, contrary to the moduli space of complex curves - the quotient of Teichmüller space by the mapping class group - which is of finite volume. The quotient spaces are expected to be fiber bundles over the moduli space of curves. For the Hitchin component for  $\text{PSL}(n, \mathbb{R})$  Labourie proposed a precise conjecture:

**Conjecture 2** (Labourie). *There is mapping class group equivariant projection from the Hitchin component for  $\text{PSL}(n, \mathbb{R})$  to  $\mathcal{T}(\Sigma_g)$ , where the fiber over a Riemann surface  $X \in \mathcal{T}(\Sigma_g)$  is  $\bigoplus_{i=3}^n \mathbb{H}^0(X, K_X^i)$ , where  $\mathbb{H}^0(X, K_X^i)$  is the vector space of holomorphic differentials on  $X$  of degree  $i$ .*

This conjecture has been proven by Labourie [Lab17] for all split real Lie groups of rank 2. An extension of the conjecture for maximal representations has been

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<sup>1</sup>This accounts for the name “higher” Teichmüller theory.

proven by Alessandrini-Collier [AC18] and Collier-Tholozan-Toulisse [CTT17] for Hermitian Lie groups of rank 2. It is wide open for Lie groups of rank greater than 2.

The representation variety of a surface group carries a natural symplectic structure, and the action of the mapping class group is by symplectomorphisms. We raise the following question:

**Question 3.** Is there group of symplectomorphisms acting properly discontinuously on the Hitchin component with finite volume quotient?

One way to approach this question might be to look at suitable families of commuting Hamiltonian flows, which give the Hitchin component the structure of completely integrable system, [SWZ17, SZ17]

#### REFERENCES

- [AC18] Daniele Alessandrini and Brian Collier, *The geometry of maximal components of the  $PSp(4, R)$  character variety*, arXiv:1708.05361, to appear in *Geometry and Topology*.
- [BIW10] Marc Burger, Alessandra Iozzi, and Anna Wienhard, *Surface group representations with maximal Toledo invariant*, *Ann. of Math.* **172** (2010), 517–566.
- [CTT17] Brian Collier, Nicolas Tholozan, and Jérémy Toulisse, *The geometry of maximal representations of surface groups into  $SO(2, n)$* , Preprint, arXiv:1702.08799, 2017.
- [FG05] Vladimir Fock and Alexander Goncharov, *Moduli spaces of local systems and higher Teichmüller theory*, *Publ. Math. Inst. Hautes Études Sci.* **103** (2006), 1–211.
- [Gol90] William M. Goldman, *Convex real projective structures on compact surfaces*, *J. Differential Geom.* **31** (1990), no. 3, 791–845.
- [GW18] Olivier Guichard and Anna Wienhard, *Positivity and higher Teichmüller theory*, Proceedings of the 7th ECM, Berlin 2016, arXiv:1802.02833.
- [Hit92] Nigel Hitchin, *Lie groups and Teichmüller space*, *Topology* **31** (1992), 449–473.
- [Lab06] François Labourie, *Anosov flows, surface groups and curves in projective space*, *Invent. Math.* **165** (2006), 51–114.
- [Lab17] François Labourie, *Cyclic surfaces and Hitchin components in rank 2*, *Ann. Math.(2)* **185** (2017), no. 1, 1–58.
- [SWZ17] Zhe Sun, Anna Wienhard, and Tengren Zhang, *Flows on the  $PSL(V)$ -Hitchin component*, Preprint, arXiv:1709.03580, 2017.
- [SZ17] Zhe Sun and Tengren Zhang, *The Goldman symplectic form on the  $PSL(V)$ -Hitchin component*, Preprint, arXiv:1709.03589, 2017.
- [Wie12] Anna Wienhard, *An invitation to higher Teichmüller theory*, *Proc. Int. Cong. of Math.* 2018, Rio de Janeiro, Vol. 1, 1007–1034.