Mathematisches Forschungsinstitut Oberwolfach

Report No. 46/2018

DOI: 10.4171/OWR/2018/46

MiniWorkshop: Algebraic, Geometric, and Combinatorial Methods in Frame Theory

Organised by
Emily J. King, Bremen
Christopher Manon, Lexington
Dustin G. Mixon, Columbus
Cynthia Vinzant, Raleigh

30 September – 6 October 2018

Abstract. Frames are collections of vectors in a Hilbert space which have reconstruction properties similar to orthonormal bases and applications in areas such as signal and image processing, quantum information theory, quantization, compressed sensing, and phase retrieval. Further desirable properties of frames for robustness in these applications coincide with structures that have appeared independently in other areas of mathematics, such as special matroids, Gel'Fand-Zetlin polytopes, and combinatorial designs. Within the past few years, the desire to understand these structures has led to many new fruitful interactions between frame theory and fields in pure mathematics, such as algebraic and symplectic geometry, discrete geometry, algebraic combinatorics, combinatorial design theory, and algebraic number theory. These connections have led to the solutions of several open problems and are ripe for further exploration. The central goal of our mini-workshop was to attack open problems that were amenable to an interdisciplinary approach combining certain subfields of frame theory, geometry, and combinatorics.

Mathematics Subject Classification (2010): 42Cxx, 14Pxx, 05Bxx, 53Dxx.

Introduction by the Organisers

The mini-workshop *Algebraic, Geometric, and Combinatorial Methods in Frame Theory*, organised by Emily King (Bremen), Christopher Manon (Lexington), Dustin G. Mixon (Columbus), and Cynthia Vinzant (Raleigh) was attended by 17 researchers from Europe, North America, Asia, and Oceania. The mathematical background of the participants was equally broad, with experts in frame theory, harmonic analysis, discrete math, combinatorics, algebraic geometry, and symplectic geometry.

The main focus of the mini-workshop was to bring together this diverse group of researchers to make progress on a collection of open problems in harmonic analysis, quantum information theory, and non-linear inverse problems using tools from frame theory, (real algebraic, symplectic, and discrete) geometry, combinatorics, and algebraic number theory. We also explored other connections between frames, geometry, and combinatorics, like frame theoretic configuration spaces, combinatorial designs from frames, and tensor decomposition techniques. The approach was at its very essence interdisciplinary and is driven by numerous recent discoveries and conjectures.

As a consequence the mini-workshop was organized in a somewhat non-traditional manner. In the first two days, 4 plenary and two mini-plenary talks were given. The goal of these talks was to introduce the group to problems and techniques in various areas related to frame theory. In addition, each participant, including plenary speakers, was asked to give a 5 minute talk on a topic of their choosing broadly connected to frame theory.

At the end of the second day the group met to brainstorm on open problems. The outcome of this session was that the participants divided into into 4 working groups each focusing on a different problem in the area. For the remainder of the week, working group members worked on their problems and the entire mini-workshop met daily to report and provide feedback.

The meeting began on Monday with a talk by Emily King (Bremen) who introduced the basics of finite frame theory and some of the connections to real algebraic geometry, algebraic combinatorics, discrete geometry, combinatorial design theory, tensor geometry, symplectic geometry, algebraic number theory. She was followed by Dustin Mixon (Columbus) who presented a tour of open problems in frame theory. Cynthia Vinzant (Raleigh) gave an introduction to real algebraic geometry and its connections to frame theory. Alex Fink (London) then gave an introduction to matroid theory. The first day concluded with a talk by Martin Ehlers (Wien) on t-designs and cubatures.

The second day began with a talk by Clayton Shonkwiler (Ft. Collins) who spoke on joint work with Tom Needham on the use of symplectic geometry to study the space of finite unit norm tight frames (FUNTFs). Recent developments in the field have also suggested that tools from symplectic geometry will prove to be quite powerful, and the inclusion of such methods will be one of the key innovations of the mini-workshop. The final talk was by Markus Grassl (Erlangen) who spoke on his joint work with Andrew Scott using numerical techniques to give examples of maximal equiangular tight frames (ETFs).

During the second afternoon the workshop met for a brainstorming session where open problems were suggested. At the conclusion of the afternoon session the participants agreed to break up into four subgroups each working on a particular problem. The subgroup consisting of Martin Ehler, Milena Hering, Christopher Manon, Tom Needham and Clayton Shonkwiler worked on symplectic approaches to the Paulsen problem. The subgroup of Bernhard G. Bodmann, Sabine Burgdorf, Dan Edidin, and Markus Grassl worked on the problem of constructing structured small frames for phase retrieval. The group of Alex Fink, Emily King, Cynthia Vinzant and Shayne Waldron studied the question of when an (equiangular) tight frame can be non-trivially decomposed into a disjoint union of (equiangular) tight frames for their spans. The subgroup of Gary Greaves, Joseph W. Iverson, John Jasper, and Dustin G. Mixon considered the problem of constructing frames over finite fields of order p^{2k} which "look" like ETFs over the complex numbers. The motivation for passing to finite fields is to facilitate the study of the tough problems surrounding complex ETFs, such as the Fickus conjecture and Zauners conjecture.

A write-up of each group's findings to date is included with this report.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, "US Junior Oberwolfach Fellows". Moreover, the MFO and the workshop organizers would like to thank the Simons Foundation for supporting Shayne Waldron in the "Simons Visiting Professors" program at the MFO.

MiniWorkshop: Algebraic, Geometric, and Combinatorial Methods in Frame Theory

Table of Contents

Abstracts

Algebraic, Geometric, and Combinatorial Methods in Frame Theory

Emily J. King

Frames are generalizations of orthonormal bases which yield similar decompositions of data. Such systems are the foundation of applied harmonic analysis and are also closely related to quantum measurements and linear codes. When one wants an optimally robust representation of data, one often looks for frames that have some sort of spread, be it geometric (as non-parallel as possible) or algebraic (no nontrivial linear dependencies). Over the last few years, it has been discovered that the relationship between these two types of spread is more complicated than had previously been believed. Furthermore, methods from algebra, geometry, and combinatorics have recently proven themselves to be very useful in the study of frames.

For example, algebraic graph theory and combinatorial design theory have led to new characterizations and novel constructions of optimal line (or, more generally, subspace) configurations which are also frames. Also, almost all desirable frame properties are the solution systems of real polynomials, thus such classes of frames form real algebraic varieties, and certain known results in frame theory have also been found to be equivalent to concepts in matroid theory and arrangements of hyperplanes.

In this talk, the basics of finite frame theory will be introduced and the some of the connections to real algebraic geometry, algebraic combinatorics, discrete geometry, combinatorial design theory, tensor geometry, symplectic geometry, algebraic number theory, and more will be presented.

A tour through open problems in frame theory

Dustin Mixon

In linear algebra, we learn that orthonormal bases provide a convenient representation of members of a finite-dimensional Hilbert space. In particular, coefficients in such a basis can be obtained by easy-to-compute inner products. However, in many applications, an orthonormal basis fails to deliver the redundancy necessary to accomplish the desired signal processing task. For example, to perform time-frequency analysis on a police siren (say) over $\mathbb{C}^d$, we want an ensemble of d^2 time- and frequency-shifted bump functions to compute the short-time Fourier transform. Such an ensemble is too large to form a basis, let alone an orthonormal basis. This motivates a generalization of basis known as a **frame**.

In finite dimensions, a frame is simply a spanning set, but there are several application-motivated specifications for frames that make frame theory useful. The first oft-sought property of a frame is that the frame elements have equal norm. This provides a notion of democracy between the measurements that allows for robustness to a single erasure [35]. Another important property is **tightness**: We say $\{\varphi_i\}_{i=1}^n$ in $\mathbb{C}^d$ is a tight frame if the **frame operator** $\sum_{i=1}^n \varphi_i\varphi_i^*$ is a multiple of the identity operator. Intuitively, this means that the vectors are in isotropic position (we identify the frame operator with a covariance matrix). Practically, this is useful since it enables painless reconstruction [16]:

$$\sum_{i=1}^n \langle x, \varphi_i\rangle\varphi_i = \sum_{i=1}^n \varphi_i\varphi_i^* x = \text{const}\cdot x \qquad \forall x \in \mathbb{C}^d.$$

We are also interested in frames of small **coherence**, defined as

$$\max_{\substack{i,j\in\{1,\ldots,n\}\\ i\neq j}} |\langle\varphi_i,\varphi_j\rangle|.$$

Intuitively, frames of small coherence capture measurements that are particularly pairwise dissimilar. Another measure of dissimilarity is **spark**, defined as

$$\min_{\substack{\Phi x=0\\ x\neq 0}} \text{card}(\text{supp}(x)).$$

In words, spark is the size of the smallest linearly dependent subcollection of frame elements. Finally, **symmetry** is a useful feature in various applications. Here, we want a large group of unitary matrices U for which there exists a permutation σ and scalars c_i such that $U\varphi_i = c_i\varphi_{\sigma(i)}$.

Much of the literature in frame theory follows one of two lines of inquiry. First, one might study the tension between various desired properties, such as equal norm, tightness, low coherence, large spark, and symmetry. Alternatively, one might study certain application-specific properties of frames that enable a particular instance of signal processing. In this talk, we discuss three prominent examples of these lines of inquiry. First, we consider the tension between equal norm and tightness. Next, we review what is known about minimizing coherence. Finally, we discuss various instances of nonlinear signal processing. Throughout this talk, we identify various open problems, emphasizing the problems that appear to be within reach.

The tension between equal norm and tightness can be relaxed to a tension between norms of frame elements and the spectrum of the frame operator. If we pass to partial frame operators $\sum_{i=1}^k \varphi_i\varphi_i^*$ for $k \in \{1,\ldots,n\}$, then the spectra of these operators necessarily interlace, and the traces of these operators increase by the squared norm of the added frame element. Such sequences of spectra are known as **eigensteps**. Eigensteps have been used to construct every possible frame with a given spectrum and sequence of lengths [7], they have led to a constructive proof of the Schur–Horn theorem [28], they were used to solve the frame completion problem [25], and they also played a crucial role in the proof that finite unit norm tight frames are connected [8]. Recently, [47] provided an alternative proof of connectivity in the complex case using symplectic geometry, suggesting that symplectic geometry might provide a rich framework to further study the tension between equal norm and tightness. What follows is the main open problem along these lines:

Problem 1 (The Paulsen Problem)**.** If Φ is nearly tight with frame elements that are nearly unit norm, then how close is the nearest unit norm tight frame?

To date, there have been numerous attempts to estimate this worst-case distance [4, 9, 42, 34], but the best known upper and lower bounds do not match (see [34]).

Minimizing coherence can be thought of as the problem of packing n points in projective space so that the minimum distance is maximized. For example, $\mathbb{R}\mathbf{P}^1$ is isometrically isomorphic to $\mathbb{S}^1$, and so the optimal packing amounts to equally spaced points by the pigeonhole principle. Similarly, $\mathbb{C}\mathbf{P}^1$ is isometrically isomorphic to $\mathbb{S}^2$, and so in this case, the problem reduces to the **Tammes problem**, which is famously difficult; the solution is currently only known for $n \leq 14$ and $n = 24$ [46]. In general, a solution amounts to a lower bound on coherence along with a construction that achieves equality in that bound. To date, lower bounds have been obtained by isometric embedding [13], the polynomial method [43, 33], estimating isotropic moments [6], a generalized Lasserre hierarchy [11, 17], and Tarski–Seidenberg projection [22]. Overall, the complex case seems to be harder than the real case. For example, the following is open, whereas the corresponding problem in the real case was solved in 1965 by Fejes Tóth [19]:

Conjecture 2 (see [38])**.** The optimal coherence for $n = 5$ unit vectors in $\mathbb{C}^3$ is $\mu = \frac{\sqrt{13}-1}{6}$, achieved (for example) by any frame Φ with Gram matrix

$$\Phi^*\Phi = I + \mu \cdot \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & \omega & \omega^2 \\ 1 & 1 & 0 & \omega^2 & \omega \\ 1 & \omega^2 & \omega & 0 & 0 \\ 1 & \omega & \omega^2 & 0 & 0 \end{bmatrix}, \qquad \omega = e^{2\pi i/3}.$$

There are two approaches to finding optimal packings. First, one can prove a general lower bound, characterize when a packing achieves equality in that lower bound, and then hunt for packings that satisfy the characterizing properties. Alternatively, one can compute a putatively optimal construction, and then compute a matching lower bound. The first approach leads to infinite families of constructions (i.e., constructions for various choices of d and n), but leads multiple gaps in (d, n), whereas the second approach fills these gaps for one choice of (d, n) at a time. Interestingly, most infinite families are either **equiangular tight frames** (ETFs), which achieve equality in the so-called **Welch bound** [53, 49, 26], or use ETFs as building blocks to achieve equality in other lower bounds [5, 6].

To be explicit, the Welch bound says that if $n \geq d$, then the coherence $\mu(\Phi)$ of any packing Φ of n unit vectors in $\mathbb{C}^d$ satisfies

$$\mu(\Phi) \geq \sqrt{\frac{n-d}{d(n-1)}}.$$

Equality in the Welch bound occurs precisely when $\Phi = [\varphi_1 \cdots \varphi_n]$ has the property that there exists $a, b \geq 0$ such that $\Phi\Phi^* = aI$ and $|\langle \varphi_i, \varphi_j \rangle| = b$ whenever $i \neq j$ (i.e., Φ is an ETF). Real ETFs can be thought of as combinatorial objects: they are in one-to-one correspondence with a certain subclass of strongly regular graphs [52], and (d, n) must satisfy certain (strong) integrality conditions in order for real ETFs to exist. By contrast, the complex case does not appear to reduce to the well-studied field of combinatorial design. Existing constructions have been obtained by leveraging symmetry [54, 55, 37], generalizing small examples [29, 23, 27], "complexifying" real examples [14, 24, 36], or "combinatorifying" algebraic examples [39, 20]. In pursuit of integrality conditions for the complex case, Matt Fickus posed the following:

Conjecture 3 (The Fickus Conjecture)**.** Consider d, $n-d$ and $n-1$. There exists an n-vector ETF in $\mathbb{C}^d$ only if one of these quantities divides the product of the other two.

Fickus will award US$200 for a proof of his conjecture, and US$100 for a disproof [45]. The conjecture holds for $(d, n) = (3, 8)$ by Gröbner basis calculation [50].

ETFs that arise from symmetry enjoy a nice characterization in terms of the Gram matrix. In particular, if G acts on $\mathbb{C}^d$ by some representation, and $\varphi \in \mathbb{C}^d$ has stabilizer H, then the Gram matrix of $G \cdot \varphi$ lies in the Hecke algebra of (G, H) [10]. As such, a symmetric ETF can be discovered by restricting one's search to the idempotents of this Hecke algebra. This substantially reduces the search space when the Hecke algebra is commutative, since then there are only finitely many idempotents to consider. Furthermore, if G acts doubly transitively on the frame, then the Gram matrix is necessarily a scalar multiple of a primitive idempotent, reducing the search even further [37]. Sadly, the most important open problem in this line of inquiry exhibits a non-commutative Hecke algebra:

Conjecture 4 (Zauner's Conjecture, weak form)**.** There exists an ETF of d^2 vectors in $\mathbb{C}^d$ for every $d > 1$.

Such ETFs are known as **symmetric, informationally complete positive operator–valued measures** (SICs). After Zauner stated his conjecture in 1999 [55], there has been a flurry of research to find SICs, due in part to their consequences in quantum information theory [32]. To date, there are only finitely many dimensions d for which a SIC is known to exist [30], but numerical evidence suggests that they exist for every $d \leq 151$ [31]. All known SICs exhibit Heisenberg–Weyl symmetry, i.e., the SIC takes the form $G \cdot \varphi$, where G is a Heisenberg–Weyl group. The first exact SIC constructions were obtained by Gröbner basis calculation, all of which have the property that the entries of $\varphi\varphi^*$ lie in an abelian extension of $\mathbb{Q}(\sqrt{(d-3)(d+1)})$ [2]. This feature has since been exploited to promote numerical solutions to exact solutions by "rounding" [1]. While the coordinates of the resulting SICs are expressible by radicals, this expression is inefficient (e.g., the expression for one of the SICs in dimension 48 occupies nearly one thousand pages [1]). This leads one to wonder about the "correct" expression for these coordinates, which lies in the purview of Hilbert's 12th problem. With this perspective, Kopp recently produced an infinite family of SICs, conditioned on a strengthened version of the Stark conjectures:

Conjecture 5 (Kopp [41])**.** For every odd prime $d \equiv 2 \bmod 3$, there exists an explicit $\varphi \in \mathbb{C}^d$ in terms of Stark units such that the Heisenberg–Weyl orbit of φ forms a SIC.

It is conceivable that any explicit solution to Zauner's conjecture will necessarily factor through the Stark conjectures. Alternatively, one might hunt for an implicit existence proof, perhaps by leveraging an appropriate fixed point theorem.

While minimizing coherence provides a geometric approach to spreading out frame vectors, maximizing spark provides a corresponding algebraic approach. Paradoxically, frames of minimal coherence frequently fail to exhibit maximal spark. This compelled Emily King to pose the following problem [40]:

Problem 6 (King's Problem)**.** Characterize the spark of frames with minimum coherence.

Today, we know several things along these lines: in many cases, Zauner's SIC is not full spark [15]; real ETFs are full spark only if $n = 2d$ [21]; and $\operatorname{spark}(\Phi) \geq 1/\mu(\Phi) + 1$, with equality if and only if Φ contains an embedded simplex [18, 21]. Empirically, optimal packings in the real case are usually not full spark [48], but there is currently no theory to explain this phenomenon.

For nonlinear signal processing, we consider three instances in which algebraic geometry provides tools to identify thresholds in sampling theory. For the first instance, we consider the problem of **phase retrieval**, in which measurements take the form $|\Phi^* x|^2$, where $|\cdot|^2$ is taken entrywise. Notice that x cannot be distinguished from any other member of the equivalence class $[x] := \{cx : |c| = 1\}$. How many measurements does it take to determine every signal up to a global phase factor? It was conjectured in [3] that (a) if $n < 4d - 4$, then $[x] \mapsto |\Phi^* x|^2$ is not injective, and (b) if $n \geq 4d - 4$, then $[x] \mapsto |\Phi^* x|^2$ is injective for generic Φ. Later, it was proved in [12] that (a) holds when $d - 1$ is a power of 2 and (b) holds for all d. Finally, Cynthia Vinzant established in [51] that (a) fails for $d = 4$, and provided a quantitative version of the conjectured $4d - 4$ sampling threshold:

Conjecture 7 (Vinzant's Conjecture)**.** Draw $\operatorname{range}(\Phi^*)$ uniformly from $\operatorname{Gr}(\mathbb{C}^{4d-5}, d)$, and let p_d denote the probability that $[x] \mapsto |\Phi^* x|^2$ is injective. Then

(a) $p_d < 1$ for every d, and
(b) $\lim_{d \to \infty} p_d = 0$.

Vinzant will award Coca-Cola for a proof of (a) and US\$100 for a proof of (b) [45]. Next, we consider the following general problem, which enjoys a multitude of instances such as phase retrieval and blind deconvolution:

Problem 8 (Action Retrieval)**.** Given a representation $\rho\colon G \to \operatorname{GL}(\mathbb{F}^d)$, characterize the subspaces $S \subseteq \mathbb{F}^d$ such that $\{\rho(g)z : g \in G\}$ uniquely determines (almost) every $z \in S$ modulo trivial scalar ambiguities.

For example, if $G = \mathbb{T}^n$ and ρ maps G to all diagonal matrices with phases on the diagonal, then the orbit $G \cdot z$ is informationally equivalent to $|z|^2$, and so action retrieval in this case reduces to the phase retrieval problem with $S = \mathrm{range}(\Phi^*)$. Finally, we consider the problem of self-calibration. Here, the idea is that in the real world, sensor parameters aren't perfectly known, and so these parameters must somehow be determined from sensor measurements. For example, suppose Φ is a known $d \times n$ matrix (each column corresponding to a sensor), X is an unknown $d \times m$ matrix (each column corresponding to a test signal), and D is an unknown $n \times n$ diagonal matrix (each diagonal entry corresponding to an uncalibrated signal multiplier). Thomas Strohmer posed the following problem: When does $D\Phi^* X$ uniquely determine D and/or X up to a global scalar factor? Along these lines, the following provides a conjectured sampling threshold:

Conjecture 9 (Li, Lee, Bresler [44])**.** If $n > d$ and $m > \frac{n-1}{n-d}$, then for almost every Φ, it holds that almost every (D, X) is uniquely determined from $D\Phi^* X$ modulo scaling.

References

[1] M. Appleby, T.-Y. Chien, S. Flammia, S. Waldron, Constructing exact symmetric informationally complete measurements from numerical solutions, J. Phys. A 51 (2018) 165302.

[2] D. M. Appleby, H. Yadsan-Appleby, G. Zauner, Galois automorphisms of a symmetric measurement, Quantum Inf. Comput. 13 (2013) 672–720.

[3] A. S. Bandeira, J. Cahill, D. G. Mixon, A. A. Nelson, Saving phase: Injectivity and stability for phase retrieval, Appl. Comput. Harmon. Anal. 37 (2014) 106–125.

[4] B. G. Bodmann, P. G. Casazza, The road to equal-norm Parseval frames, J. Funct. Anal. 258 (2010) 397–420.

[5] B. G. Bodmann, J. Haas, Achieving the orthoplex bound and constructing weighted complex projective 2-designs with Singer sets, Linear Algebra Appl. 511 (2016) 54–71.

[6] B. Bukh, C. Cox, Nearly orthogonal vectors and small antipodal spherical codes, arXiv:1803.02949

[7] J. Cahill, M. Fickus, D. G. Mixon, M. J. Poteet, N. Strawn, Constructing finite frames of a given spectrum and set of lengths, Appl. Comput. Harmon. Anal. 35 (2013) 52–73.

[8] J. Cahill, D. G. Mixon, N. Strawn, Connectivity and irreducibility of algebraic varieties of finite unit norm tight frames, SIAM J. Appl. Algebra Geometry 1 (2017) 38–72.

[9] P. G. Casazza, M. Fickus, D. G. Mixon, Auto-tuning unit norm frames, Appl. Comput. Harmon. Anal. 32 (2012) 1–15.

[10] T. Ceccherini-Silberstein, F. Scarabotti, F. Tolli, Harmonic analysis on finite groups: Representation theory, Gelfand pairs and Markov chains, Cambridge University Press, 2008.

[11] H. Cohn, J. Woo, Three-point bounds for energy minimization, J. Amer. Math. Soc. 25 (2012) 929–958.

[12] A. Conca, D. Edidin, M. Hering, C. Vinzant, An algebraic characterization of injectivity in phase retrieval, Appl. Comput. Harmon. Anal. 38 (2015) 346–356.

[13] J. H. Conway, R. H. Hardin, N. J. A. Sloane, Packing lines, planes, etc.: Packings in Grassmannian spaces, Exp. Math. 5 (1996) 139–159.

[14] G. Coutinho, C. Godsil, H. Shirazi, H. Zhan, Equiangular lines and covers of the complete graph, Linear Algebra Appl. 488 (2016) 264–283.

[15] H. B. Dang, K. Blanchfield, I. Bengtsson, D. M. Appleby, Linear dependencies in Weyl–Heisenberg orbits, Quantum Inf. Process. 12 (2013) 3449–3475.

[16] I. Daubechies, A. Grossmann, Y. Meyer, Painless nonorthogonal expansions, J. Math. Phys. 27 (1986) 1271–1283.

[17] D. de Laat, F. Vallentin, A semidefinite programming hierarchy for packing problems in discrete geometry, Math. Program. 151 (2015) 529–553.

[18] D. L. Donoho, M. Elad, Optimally sparse representation in general (nonorthogonal) dictionaries via ℓ^1 minimization, Proc. Natl. Acad. Sci. U.S.A. 100 (2003) 2197–2202.
[19] L. Fejes Tóth, Distribution of points in the elliptic plane, Acta Math. Hungar. 16 (1965) 437–440.
[20] M. Fickus, J. Jasper, Equiangular tight frames from group divisible designs, arXiv:1803.07468
[21] M. Fickus, J. Jasper, E. J. King, D. G. Mixon, Equiangular tight frames that contain regular simplices, Linear Algebra Appl. (2018).
[22] M. Fickus, J. Jasper, D. G. Mixon, Packings in real projective spaces, SIAM J. Appl. Algebra Geometry 2 (2018) 377–409.
[23] M. Fickus, J. Jasper, D. G. Mixon, J. Peterson, Tremain equiangular tight frames, J. Combin. Theory A 153 (2018) 54–66.
[24] M. Fickus, J. Jasper, D. G. Mixon, J. D. Peterson, C. E. Watson, Polyphase equiangular tight frames and abelian generalized quadrangles, Appl. Comput. Harmon. Anal. (2017).
[25] M. Fickus, J. D. Marks, M. J. Poteet, A generalized Schur–Horn theorem and optimal frame completions, Appl. Comput. Harmon. Anal. 40 (2016) 505–528.
[26] M. Fickus, D. G. Mixon, Tables of the existence of equiangular tight frames, arXiv:1504.00253
[27] M. Fickus, D. G. Mixon, J. Jasper, Equiangular tight frames from hyperovals, IEEE Trans. Inform. Theory 62 (2016) 5225–5236.
[28] M. Fickus, D. G. Mixon, M. J. Poteet, N. Strawn, Constructing all self-adjoint matrices with prescribed spectrum and diagonal, Adv. Comput. Math. 39 (2013) 585–609.
[29] M. Fickus, D. G. Mixon, J. C. Tremain, Steiner equiangular tight frames, Linear Algebra Appl. 436 (2012) 1014–1027.
[30] S. Flammia, Exact SIC fiducial vectors, `physics.usyd.edu.au/~sflammia/SIC/`
[31] C. A. Fuchs, M. C. Hoang, B. C. Stacey, The SIC question: History and state of play, Axioms 6 (2017) 21.
[32] C. A. Fuchs, R. Schack, A quantum-Bayesian route to quantum-state space, Found. Phys. 41 (2011) 345–356.
[33] J. I. Haas, N. Hammen, D. G. Mixon, The Levenstein bound for packings in projective spaces, Proc. SPIE 10394 (2017) 103940V.
[34] L. Hamilton, A. Moitra, The Paulsen Problem Made Simple, arXiv:1809.04726
[35] R. B. Holmes, V. I. Paulsen, Optimal frames for erasures, Linear Algebra Appl. 377 (2004) 31–51.
[36] J. W. Iverson, J. Jasper, D. G. Mixon, Optimal projective packings from association schemes, Proc. SPIE 10394 (2017) 103940P.
[37] J. W. Iverson, D. G. Mixon, Doubly transitive lines I: Higman pairs and roux, arXiv:1806.09037
[38] J. Jasper, E. J. King, D. G. Mixon, `tinyurl.com/y8lvfjol`
[39] J. Jasper, D. G. Mixon, M. Fickus, Kirkman equiangular tight frames and codes, IEEE Trans. Inform. Theory 60 (2014) 170–181.
[40] E. J. King, Algebraic and geometric spread in finite frames, Proc. SPIE 9597 (2015) 95970B.
[41] G. S. Kopp, SIC-POVMs and the Stark conjectures arXiv:1807.05877
[42] T. C. Kwok, L. C. Lau, Y. T. Lee, A. Ramachandran, The Paulsen problem, continuous operator scaling, and smoothed analysis, STOC (2018) 182–189.
[43] V. I. Levenshtein, Bounds on the maximal cardinality of a code with bounded modulus of the inner product, Soviet Math. Dokl. 25 (1982) 526–531.
[44] Y. Li, K. Lee, Y. Bresler, A unified framework for identifiability analysis in bilinear inverse problems with applications to subspace and sparsity models, arXiv:1501.06120
[45] D. G. Mixon, Conjectures from SampTA, `dustingmixon.wordpress.com/2015/07/08/conjectures-from-sampta/`
[46] O. R. Musin, A. S. Tarasov, The Tammes problem for $N = 14$, Exp. Math. 24 (2015) 460–468.

[47] T. Needham, C. Shonkwiler, Symplectic Geometry and Connectivity of Spaces of Frames, arXiv:1804.05899
[48] N. J. A. Sloane, Packings in Grassmannian Spaces, `neilsloane.com/grass/`
[49] T. Strohmer, R. W. Heath Jr., Grassmannian frames with applications to coding and communication, Appl. Comput. Harmon. Anal. 14 (2003) 257–275.
[50] F. Szöllősi, All complex equiangular tight frames in dimension 3, arXiv:1402.6429
[51] C. Vinzant, A small frame and a certificate of its injectivity, Proc. Sampl. Theory Appl. (2015) 197–200.
[52] S. Waldron, On the construction of equiangular frames from graphs, Linear Algebra Appl. 431 (2009) 2228–2242.
[53] L. Welch, Lower bounds on the maximum cross correlation of signals, IEEE Trans. Inform. Theory 20 (1974) 397–399.
[54] P. Xia, S. Zhou, G. B. Giannakis, Achieving the Welch bound with difference sets, IEEE Trans. Inform. Theory 51 (2005) 1900–1907.
[55] G. Zauner, Quantum designs—Foundations of a non-commutative theory of designs, Ph.D. thesis, U. Vienna, 1999.

Real algebraic geometry

Cynthia Vinzant

I will give an introduction to the basics of algebraic geometry, with a special focus on polynomials and algebraic sets defined over the real numbers, and discuss computational tools to test small cases.

An algebraic variety over a field k is the set of solutions in k^n to finitely polynomial equations $f_j = 0$ where $f_j \in k[x_1, \ldots, x_n]$. This defines a topology, called the *Zariski topology*, on k^n whose closed are sets are the algebraic varieties. We say that a *generic* point with some property if there is a non-empty Zariski open set of points with this property. An important example for frame theory is the variety $\mathcal{M}_{r,n}$ of $n \times n$ matrices of rank $\leq r$, which is defined by the $(r+1) \times (r+1)$ minors.

Example. Consider the set $S = \{vv^T : v \in \mathbb{R}^2\}$ of 2×2 positive semidefinite matrices of rank ≤ 1. The Zariski-closure of S in $\mathbb{R}^{2\times 2}_{\text{sym}}$ is the variety of all matrices of rank ≤ 1 defined by the the determinant. The complement is a non-empty Zariski-open set. We can then say that a generic 2×2 matrix has rank two.

A variety is *irreducible* if it cannot be written as a union of two proper subvarieties. The *dimension* of an irreducible variety $V \subset k^n$ can be defined by any of the following equivalent conditions:

- the dimension of the tangent space T_pV of V at a generic point $p \in V$,
- the largest d such that the projection of V onto some d coordinates in Zariski-dense in k^d, and
- the largest d such that a generic $(n-d)$-dimensional affine space intersects V

  (assuming that k is algebraically closed, e.g. $k = \mathbb{C}$).

When k is algebraically closed, the intersection of V with a generic $(n-d)$-dimensional affine space is finite. The number of intersection points is the *degree* of V. For example, the dimension of $\mathcal{M}_{1,n}$ equals $2n-1$. Indeed, the projection of rank-one matrices onto the $2n-1$ entries in the first row and column is Zariski-dense and almost an choice of entries can be completed back to a matrix in $\mathcal{M}_{1,n}$. Similarly, projecting onto the first two rows and columns shows that $\mathcal{M}_{2,n}$ has dimension $4n-4$.

Computation and certification form other important tools from algebraic geometry. Given a variety $V \subset k^n$ defined by polynomials $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$, any polynomial of the form $h_1 f_1 + \ldots + h_s f_s$ where h_i also vanishes on V. One can use this to certify that V is empty. In fact, *Hilbert's Nullstellensatz* states that, over $\mathbb{C}$, the variety defined by $f_1, \ldots, f_s$ if and only if there exist multipliers h_j so that $1 = h_1 f_1 + \ldots + h_s f_s$. Moreover, there are symbolic algorithms based on Gröbner bases to test this condition.

Over $\mathbb{R}$, it more useful to consider polynomials that are nonnegative on V. Sums of squares of are always globally nonnegative. Therefore any polynomial of the form $\sum_{i=1}^{r} g_i^2 + \sum_{j=1}^{s} h_j f_j$ will be nonnegative on V. The *Positivstellensatz* states that the real variety is empty if and only if -1 has this form.

Sums of squares representations can also be computed using semidefinite programming. For example, if $f \in \mathbb{R}[x_1, \ldots, x_n]$ is a polynomial of degree $\leq 2d$, then f is a sum of squares if and only if there exists a positive semidefinite matrix G such that $f = m^T G m$, where m is the vector of all monomials in $x_1, \ldots, x_n$ of degree $\leq d$. Indeed, if G is positive semidefinite, it can be written as a sum of rank-one matrices $\sum_i \vec{g}_i \vec{g}_i^T$. This gives the sum of squares representation $m^T G m = \sum_i (\vec{g}_i^T m)^2$ to f.

Many of the important classes of frames are real algebraic varieties. It would be very interesting to use both the theoretical and computational tools of real algebraic geometry to better understand them.

Matroids and combinatorics

Alex Fink

A *matroid* is a combinatorial object of the sort necessary to keep track of all the linear dependency data of a collection $\phi_1, \ldots, \phi_n$ of vectors. For the purposes of this extended abstract, the collection will be finite (though there is a theory of infinite matroids).

Matroid theory is noted for the large number of ways matroids can be defined, many of them not clearly equivalent at first glance: Gian-Carlo Rota gave this phenomenon the name "cryptomorphism". One definition follows.

Definition 1. A *matroid* M on a finite set E is a set system $\mathcal{I} \subseteq 2^E$, known as the *independent* sets of M, such that:

(i) $\emptyset \in \mathcal{I}$;
(ii) $\mathcal{I}$ is closed under taking subsets;
(iii) Given $A, B \in \mathcal{I}$ with $|A| < |B|$, there exists $b \in B \setminus A$ such that $A \cup \{b\} \in \mathcal{I}$.

The only nontrivial clause is (iii), which is an incarnation of the Steinitz exchange lemma, bearing in mind our guiding example:

Example 2. If $\phi_1, \ldots, \phi_n$ is a list of vectors in a vector space over a field $\mathbb{K}$, let $E = \{\phi_1, \ldots, \phi_n\}$, and take $\mathcal{I}$ to be the set of linearly independent subsets. A matroid M obtained this way is called *representable* over $\mathbb{K}$.

Choosing coordinates for the span of the vectors ϕ_i, whose dimension we call r, we may write them as columns of a rank r matrix $\Phi \in \mathbb{K}^{r\times n}$. Then M depends only on the rowspace of Φ. In this way, a matroid is associated to each point of the Grassmannian $\mathrm{Gr}(r, \mathbb{K}^n)$.

Not every matroid is representable: nonexamples include the non-Pappus and Vámos matroids. Why has matroid theory been built this way?

A pessimist might answer "because it's not possible to do better". The metatheorem that if $\mathbb{K}$ is an infinite field then there exists no finite axiom system that will describe all and only the matroids representable over $\mathbb{K}$ has been proven in various incarnations, e.g. [2]. Over $\mathbb{Q}$, detecting representability is as hard as solving Diophantine equations [4]. (However, over an algebraically closed field, matroid representability can be checked by a Gröbner basis computation.)

But I take the optimist's view that the definition of matroids captures a level of generality greater than that of linear algebra at which a lot of deep combinatorics still expresses itself. To give some examples from algebraic geometry, the K-class in [3] and the Chow ring in [1] both exist for all matroids. In the representable case, both arise from concrete algebraic varieties that are constructed from a list of vectors, but both go on existing even when no such variety is to be found.

We have given one of the cryptomorphic definitions of matroids above, in terms of independent sets. Some others are cast in terms of:

- *bases*, maximal independent sets.
- *circuits*, minimal dependent sets. What frame theorists call the spark of a frame is the size of its smallest circuit (when matroid theorists need to refer to the same quantity, they call it *girth*). So a frame of n vectors in r-space is of full spark if and only if its matroid is the *uniform matroid*, whose independent sets are all sets of size $\leq r$.
- the *rank* function, assigning to each set the size of its largest independent subset.
- *flats*, sets which are maximal for their rank. These form a *lattice*, a partially ordered set in which any two elements have a greatest lower bound and a least upper bound. The cone over the order complex of this lattice, in a particular realisation, is known as the *Bergman fan*, and is the principal way matroids appear in tropical geometry.
- a *polytope*, which can be described as a convex hull coming from the bases or as a solution set of a system of inequalities coming from the rank function.

  Here's a third appearance of this polytope for a representable matroid. Given a point of $\mathrm{Gr}(r, \mathbb{K}^n)$ represented by a matrix Φ, rescaling the columns of Φ doesn't change the matroid. This rescaling describes an action of the algebraic n-torus $(\mathbb{K}^*)^n$ on $\mathrm{Gr}(r, \mathbb{K}^n)$. The closure of the orbit of our point is a toric variety, the image of whose moment map is the above polytope.

There are many operations on a matroid which produce other matroids. One which is already familiar to this audience is matroid *duality*, special cases of which are *Gale duality* of vector configurations and *Naimark complementation* of frames.

References

[1] Karim Adiprasito, June Huh, and Eric Katz. Hodge theory for combinatorial geometries. *Annals of Mathematics*, 188(2):381–452, 2018.
[2] Dillon Mayhew, Mike Newman, and Geoff Whittle. Yes, the "missing axiom" of matroid theory is lost forever. *Transactions of the American Mathematical Society*, 370(8):5907–5929, 2018.
[3] David E Speyer. A matroid invariant via the k-theory of the grassmannian. *Advances in Mathematics*, 221(3):882–913, 2009.
[4] Bernd Sturmfels. On the decidability of diophantine problems in combinatorial geometry. *Bulletin of the American Mathematical Society*, 17(1):121–124, 1987.

(Discrepancy) t-designs and cubatures

Martin Ehler

Given a Borel probability measure μ on $\mathbb{R}^d$ with compact support, a fundamental sampling problem is to allocate a suitable n-point set $\{x_1, \dots, x_n\} \subset \mathrm{supp}(\mu)$ such that the normalized counting measure

$$\nu_n := \frac{1}{n} \sum_{j=1}^{n} \delta_{x_j}$$

approximates μ. The measure ν_n is then constructed by numerically minimizing a suitable optimization problem (the discrepancy between μ and ν_n) among all n-point sets $\{x_1, \dots, x_n\} \subset \mathrm{supp}(\mu)$ for fixed $n \in \mathbb{N}$.

Cubature points and t-designs are near minimizers of this minimization with special additional properties related to polynomial approximation. A point set $\{x_1, \dots, x_n\} \subset \mathbb{R}^d$ with weights $\{\omega_1, \dots, \omega_n\} \subset \mathbb{R}$ is called a weighted t-design with respect to μ (or a cubature of strength t with respect to μ) if

$$\int_{\mathbb{R}^d} f(x) \mathrm{d}\mu(x) = \sum_{j=1}^{n} \omega_j f(x_j), \quad f \in \mathbb{R}_t[x],$$

where $\mathbb{R}_t[x]$ denotes the collection of polynomials on $\mathbb{R}^d$ of total degree less or equals t. If the weights are constant, i.e.,

$$\int_{\mathbb{R}^d} f(x) \mathrm{d}\mu(x) = \frac{1}{n} \sum_{j=1}^{n} f(x_j), \quad f \in \mathbb{R}_t[x],$$

then $\{x_1, \ldots, x_n\}$ is simply called a t-design with respect to μ. We shall provide the framework of t-designs for measures μ supported on arbitrary compact subsets of $\mathbb{R}^d$ with special attention to the Grassmannian manifold.

Spherical t-designs refer to t-designs with respect to μ being the surface measure of the unit sphere

$$\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : \|x\| = 1\},$$

see [3], for instance. Existence of spherical t-designs with optimal asymptotics $n \sim t^{d-1}$ have been proved in [2]. The latter was recently extended to Riemannian measures supported on affine algebraic manifolds in [6]. The Grassmannian

$$\mathcal{G}_{k,d} := \{x \in \mathbb{R}^{d\times d} : x^\top = x,\ x^2 = x,\ \mathrm{rank}(x) = k\},$$

is an explicit example of an affine algebraic manifold and, in the present context, considered as a subset in $\mathbb{R}^{d^2}$. Grassmannian t-designs, i.e., t-designs with respect to the unique orthogonally invariant probability measure on $\mathcal{G}_{k,d}$ (the orthogonal group acts transitively on $\mathcal{G}_{k,d}$ by conjugation) are discussed in [1]. Function approximation using Grassmannian t-designs is investigated in [4]. The concept of t-designs with respect to orthogonally invariant measures supported on unions of Grassmannians are introduced in [5].

References

[1] C. Bachoc, E. Bannai, and R. Coulangeon, *Codes and designs in Grassmannian spaces*, Discrete Mathematics **277** (2004), 15–28.

[2] A. Bondarenko, D. Radchenko, and M. Viazovska, *Optimal asymptotic bounds for spherical designs*, Ann. Math. **178** (2013), no. 2, 443–452.

[3] J. S. Brauchart, J. Dick, E. B. Saff, I. H. Sloan, Y. G. Wang, and R. S. Womersley, *Covering of spheres by spherical caps and worst-case error for equal weight cubature in Sobolev spaces*, J. Math. Anal. Appl. **431** (2015), no. 2, 782–811.

[4] A. Breger, M. Ehler, and M. Gräf, *Quasi Monte Carlo integration and kernel-based function approximation on Grassmannians*, Frames and Other Bases in Abstract and Function Spaces: Novel Methods in Harmonic Analysis, vol. 1, Birkhauser/Springer, 2017.

[5] M. Ehler and M. Gräf, *Reproducing kernels for the irreducible components of polynomial spaces on unions of Grassmannians*, Constr. Approx. (2018).

[6] U. Etayo, J. Marzo, and J. Ortega-Cerdà, *Asymptotically optimal designs on compact algebraic manifolds*, J. Monatsh. Math. **186** (2018), no. 2, 235–248.

Sympletic Geometry and Frame Theory

Clayton Shonkwiler

(joint work with Tom Needham)

Speaking loosely, a frame in a Hilbert space $\mathcal{H}$ is an overcomplete basis for $\mathcal{H}$. The overcompleteness of a frame allows for greater flexibility and greater robustness to data loss, both of which are of substantial importance in a variety of applications [7, 8, 4].

More precisely, a *frame* in $\mathbb{C}^d$ is a collection $F = \{f_j\}_{j=1}^N$ of vectors $f_j \in \mathbb{C}^d$ satisfying

$$a\|v\|^2 \leq \sum_{j=1}^{N} |\langle v, f_j\rangle|^2 \leq b\|v\|^2 \quad \forall\, v \in \mathbb{C}^d$$

for some numbers $0 < a \leq b$ called *frame bounds.* When $a = b$ the frame is *tight*, and when all the frame vectors have unit norm the frame is a *unit norm frame.* *(Finite) unit norm tight frames* (FUNTFs) are particularly interesting, providing optimal reconstructions in the context of measurements of equal power with additive white Gaussian noise [6]. Thinking of F as a $d \times n$ matrix with ith column f_i, the tight frame condition is equivalent to the *frame operator* FF^* being a multiple of the $d \times d$ identity matrix I_d, and the unit norm condition is equivalent to F^*F having 1's on the diagonal. Since $\operatorname{Tr} FF^* = \operatorname{Tr} F^*F$, it follows that the FUNTFs are precisely those frames for which

$$FF^* = \frac{N}{d} I_d \qquad \text{and} \qquad \left(\|f_1\|^2, \dots, \|f_N\|^2\right) = (1, \dots, 1). \tag{1}$$

A natural and surprisingly challenging question to ask is whether the space of FUNTFs with fixed d and N is connected; that the answer is "yes" is called the frame homotopy conjecture, posed by Larson in a 2002 REU and first appearing in the literature in Dykema and Strawn's 2006 paper [5]. This was recently proved by Cahill, Mixon, and Strawn [2].

Both of the conditions in (1) turn out to be natural to describe in the language of symplectic geometry, leading to a simple alternative proof of the frame homotopy conjecture. Briefly, a *symplectic manifold* is a pair (M, ω) where M is a smooth, even-dimensional manifold and $\omega \in \Omega^2(M)$ is a closed, non-degenerate 2-form on M; see [3] for a nice introduction to symplectic geometry. For example, $(\mathbb{R}^{2n}, \sum \mathrm{d}x_i \wedge \mathrm{d}y_i)$ is the standard example of a symplectic manifold; since $\mathbb{C}^{d\times N} \simeq \mathbb{R}^{2dN}$, the space of $d \times N$ complex matrices is also symplectic.

The action of a Lie group G on a symplectic manifold M is called *Hamiltonian* if there exists a *moment map* $\mu : M \to \mathfrak{g}^*$ such that

$$\omega_p(X_V, X) = D_p\mu(X)(V) \tag{2}$$

for all $p \in M$, $V \in \mathfrak{g}$, and $X \in T_pM$, where $X_V = \frac{d}{dt}\big|_{t=0} \exp(tV) \cdot p$ is the vector field on M induced by the infinitesimal transformation $V \in \mathfrak{g}$. By work of Marsden–Weinstein [9] and Meyer [10], the *symplectic reduction*

$$M /\!/_{\mathcal{O}(\xi)} G := \mu^{-1}(\mathcal{O}(\xi))/G$$

is naturally a symplectic manifold, where $\xi \in \mathfrak{g}^*$ and $\mathcal{O}(\xi)$ is its coadjoint orbit.

There is a natural action of $U(d) \times U(1)^N$ on the space $\mathbb{C}^{d\times N}$ of $d \times N$ complex matrices, where $U(d)$ acts by multiplication on the left and $U(1)^N$ acts by multiplication on the right by a diagonal unitary matrix. In fact the scalar matrices in $U(d)$ and $U(1)^N$ have the same effect, so there is some redundancy in this action. Taking the quotient of $U(1)^N$ by the subgroup of scalar matrices produces an effective action of $U(d) \times U(1)^{N-1}$. Since $\mathfrak{u}(d)^* \simeq \mathcal{H}(d)$, the $d \times d$ Hermitian matrices, and $\left(\mathfrak{u}(1)^{N-1}\right)^* = \left(\mathfrak{u}(1)^*\right)^{N-1} \simeq \mathbb{R}^{N-1}$, the corresponding moment map is a map from $\mathbb{C}^{d\times N}$ to $\mathcal{H}(d) \times \mathbb{R}^{N-1}$ which turns out to be given by

$$\mu : F \mapsto \left(FF^*, \left(-\frac{1}{2}\|f_1\|^2, \dots, -\frac{1}{2}\|f_{N-1}\|^2\right)\right).$$

Therefore, the FUNTFs are simply the level set $\mu^{-1}\left(\frac{N}{d}I_d, \left(-\frac{1}{2}, \dots, -\frac{1}{2}\right)\right)$ and, while this space is not itself symplectic, its quotient

$$\begin{aligned}\mathcal{Q}_{d,N} &:= \mu^{-1}\left(\frac{N}{d}I_d, \left(-\frac{1}{2}, \dots, -\frac{1}{2}\right)\right) / (U(d) \times U(1)^{N-1}) \\ &= C^{d\times N} /\!\!/_{\left(\frac{N}{d}I_d, -\frac{\vec{1}}{2}\right)} U(d) \times U(1)^{N-1} \dots\end{aligned}$$

is symplectic. Performing the reduction in stages yields

$$\mathcal{Q}_{d,N} \simeq \left(\mathbb{C}^{d\times N} /\!\!/_{\frac{N}{d}I_d} U(d)\right) /\!\!/_{-\frac{\vec{1}}{2}} U(1)^{N-1} \simeq \mathrm{Gr}_d(\mathbb{C}^n) /\!\!/_{-\frac{\vec{1}}{2}} U(1)^{N-1},$$

where $\mathrm{Gr}_d(\mathbb{C}^n)$ is the Grassmannian of d-dimensional linear subspaces of $\mathbb{C}^n$. But $\mathrm{Gr}_d(\mathbb{C}^n)$ is connected, and a theorem of Atiyah [1] implies that symplectic reductions of connected manifolds by tori are connected, so $\mathcal{Q}_{d,N}$ is connected. Since $\mathcal{Q}_{d,N}$ is the quotient of the space of FUNTFs by a connected group, this gives a simple symplectic proof of the frame homotopy conjecture:

Theorem 1 ([2, 11])**.** *The space of length-N FUNTFs in $\mathbb{C}^d$ is path-connected for all $N \geq d \geq 1$.*

This approach generalizes to spaces of frames with arbitrary prescribed frame operator and arbitrary prescribed frame vector norms. Specifically, if S is a positive-definite Hermitian $d \times d$ matrix and $\vec{r} = (r_1, \dots, r_N)$ with $r_i > 0$ for all i, let

$$\mathcal{F}_S^{d,N}(\vec{r}) = \{F \in \mathbb{C}^{d\times N} | FF^* = S, \|f_i\|^2 = r_i\}$$

be the space of frames with frame operator S and frame vector norms determined by $\vec{r}$. Then a suitable generalization of the above argument yields the following generalized frame homotopy theorem:

Theorem 2 ([11])**.** *For any $N \geq d \geq 1$, any S, and any admissible $\vec{r}$, $\mathcal{F}_S^{d,N}(\vec{r})$ is path-connected.*

This only scratches the surface of a potentially fruitful connection between frame theory and symplectic geometry: the symplectic machinery should naturally generalize to fusion frames and seems well-adapted to other frame theory questions like the Paulsen problem, phase recovery, the existence of maximal equiangular tight frames, and the problem of uniformly sampling random FUNTFs.

References

[1] Michael Francis Atiyah. Convexity and commuting Hamiltonians. *Bull. Lond. Math. Soc.*, 14(1):1–15, 1982.

[2] Jameson Cahill, Dustin G. Mixon, and Nate Strawn. Connectivity and irreducibility of algebraic varieties of finite unit norm tight frames. *SIAM J. Appl. Algebra Geometry*, 1(1):38–72, 2017.
[3] Ana Cannas da Silva. *Lectures on Symplectic Geometry*, volume 1764 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2001.
[4] Peter G. Casazza and Gitta Kutyniok, editors. *Finite Frames*. Applied and Numerical Harmonic Analysis. Birkhäuser, Boston, 2013.
[5] Ken Dykema and Nate Strawn. Manifold structure of spaces of spherical tight frames. *Int. J. Pure Appl. Math.*, 28(2):217–256, 2006.
[6] Vivek K. Goyal, Jelena Kovačević, and Jonathan A. Kelner. Quantized frame expansions with erasures. *Appl. Comput. Harmon. Anal.*, 10(3):203–233, 2001.
[7] Jelena Kovačević and Amina Chebira. Life Beyond Bases: The Advent of Frames (Part I). *IEEE Signal Process. Mag.*, 24(4):86–104, 2007.
[8] Jelena Kovačević and Amina Chebira. Life Beyond Bases: The Advent of Frames (Part II). *IEEE Signal Process. Mag.*, 24(5):115–125, 2007.
[9] Jerrold Marsden and Alan Weinstein. Reduction of symplectic manifolds with symmetry. *Rep. Mathematical Phys.*, 5(1):121–130, 1974.
[10] Kenneth R. Meyer. Symmetries and integrals in mechanics. In *Dynamical systems (Proc. Sympos., Univ. Bahia, Salvador, 1971)*, pages 259–272. Academic Press, New York, 1973.
[11] Tom Needham and Clayton Shonkwiler. Symplectic geometry and connectivity of spaces of frames. Preprint, 2018. `arXiv:1804.05899 [math.FA]`.

Seeking and Finding Numerical and Exact Maximal ETFs

Markus Grassl
(joint work with Andrew J. Scott)

We reported on methods that allowed us to find both numerical and exact algebraic solutions to the problem whether there are d^2 unit vectors in $\mathbb{C}^d$ such that their mutual inner product has constant norm. In the terminology of frames, those are complex equiangular tight frames (ETF) of maximal size. The existence of such vectors for all dimensions has been conjectured by Zauner [4] almost 20 years ago, but we still have no proof of their existence for any infinite family of dimensions.

To date, we have numerical solutions for all $d \leq 175$, as well as for some sporadic dimensions, including $d = 844, 1155, 1299, 2208$. Exact algebraic solutions have in first place been found with the help of Gröbner bases imposing additional symmetries. Earlier results were obtained for dimensions up to $d = 48$, more recently dimensions 124 and 323 have been added [3]. Conjectures concerning the underlying number fields allowed to convert high-precision numerical solutions into exact ones [2], and additional structures for specific dimensions [1] helped the authors to find an exact solution for $d = 120$, and some of the authors of [1] reported on solutions for $d = 195$.

Numerical search is based on minimizing a non-convex function of degree eight in $2d - 1$ variables. We impose additional symmetries to confine the search to a subspace of lower dimension. With increasing dimension, however, the minimization will most of the time hit local minima, and hence we need several millions of randomly chosen initial points and hundreds of days of CPU time to find the global minimum which yields a solution. More recently, we have been using a modified function that appears to have a higher chance of convergence to the global minimum. Additionally, we have implemented parallel algorithms that allow to increase the numerical precision, which is needed for the aforementioned conversion to exact solutions. In addition to a high-precision gradient descent, we have a simple algorithm using alternating projections.

The analysis of numerical results lead to new conjectures concerning possible additional symmetries, which are in turn conjectured to be linked to the Galois group of the underlying number field. We hope that these connections will eventually lead to a proof of Zauner's conjecture.

References

[1] M. Appleby, I. Bengtsson, I. Dumitru, and S. Flammia, *Dimension towers of SICs. I. Aligned SICs and embedded tight frames*, Journal of Mathematical Physics **58** (2017) 112201. DOI: 10.1063/1.4999844

[2] M. Appleby, T.-Y. Chien, S. Flammia, and S. Waldron, *Constructing exact symmetric informationally complete measurements from numerical solutions*, Journal of Physics A: Mathematical and Theoretical **51** (2018) 165302. DOI: 10.1088/1751-8121/aab4cd

[3] M. Grassl and A. J. Scott, *Fibonacci-Lucas SIC-POVMs*, Journal of Mathematical Physics **58** (2017) 122201. DOI: 10.1063/1.4995444

[4] G. Zauner, *Grundzüge einer nichtkommutativen Designtheorie*, Dissertation, Universität Wien (1999).

The Paulsen Problem made Symplectic

Martin Ehler, Milena Hering, Christopher Manon, Tom Needham, and Clayton Shonkwiler

1. Introduction

Symplectic geometry [3] provides powerful tools for studying special spaces of complex matrices. We propose that it will be fruitful to use this perspective on Parseval frames, complex fixed unit normal tight frames (FUNTFs), and other spaces of frames. The notion of Hamiltonian reduction of a sympelectic manifold by a compact Lie group is of central importance. This technology has already been used to prove that connectivity properties hold for spaces of FUNTFs (see [12]), and we argue below that it should have applications to the Paulsen problem.

2. Elements of Hamiltonian Actions on Vector Spaces

Let $V = \mathbb{C}^n$ be a complex vector space equipped with a linear left action by a compact subgroup $K \subset U(n) \subset \mathrm{GL}_n(\mathbb{C})$. Here $U(n)$ is the group of unitary $n \times n$ matrices. The space V comes equipped with a symplectic form $\omega_V = \frac{i}{2}\sum_{i=1}^n dz_i \wedge d\bar{z}_i$. Note that this is a closed differential 2-form on V when it is considered as a real $2n$-dimensional vector space. The action of K on V is said to be symplectic because $k^*(\omega_V) = \omega_V$ for any $k \in K$, where $k^*(\omega_V)$ is the pullback form under the isometry of V defined by k.

For any $v \in V$ we can consider the $n \times n$ matrix $-\frac{i}{2}vv^*$, which can be taken to be an element in the space $\mathcal{H}_n$ of $n \times n$ skew-Hermitian matrices. The space $\mathcal{H}_n$ can be identified with the Lie algebra $\mathfrak{u}_n$ of $U(n)$; and by using the non-degenerate inner product $\langle A, B\rangle = -\frac{i}{2}tr(AB)$, we can view $\mathcal{H}_n$ as the dual space of $\mathfrak{u}_n$. The latter carries a linear group action by $U(n)$ called the coadjoint action, which can be computed on a Hermitian matrix by $Ad_u(A) = uAu^{-1}$ for $u \in U(n)$. There is an inclusion $t_K : \mathfrak{k} \to \mathfrak{u}_n$ and a corresponding surjection on dual spaces $t_K^* : \mathfrak{u}_n^* \to \mathfrak{k}^*$ given by considering the Lie algebras as the tangent spaces to K and $U(n)$ at the identity element. The *moment map* of the action of K on V is a continuous, K-invariant map $\mu_K : V \to \mathfrak{k}^*$ defined by $\mu_K(v) = t_K^*(-\frac{i}{2}vv^*)$.

Let $X \in \mathfrak{k}$ be an element of the Lie algebra, then we can use X to define a vector field on V. At a point $v \in V$ we let $X_v = [\frac{d}{dt}e^{itX}v]_{t=0} \in T_v(V)$, where $T_v(V)$ is the tangent space to V at the point v. It can be verified that the following always holds:

$$d\langle \mu_K(v), X\rangle \mid_v = \omega_V \mid_v (-, X_v). \tag{1}$$

In particular, the differential 1-form associated to the function $\langle \mu_K(-), X\rangle : V \to \mathbb{R}$, where $\langle -, -\rangle : \mathfrak{k} \times \mathfrak{k}^* \to \mathbb{R}$ is the dual pairing, agrees with the 1-form $\omega_V(X_v, -)$ on the tangent space $T_v(V)$. An action by symplectomorphisms with this property is said to be Hamiltonian.

Theorem 1. *For any compact subgroup $K \subset U(n) \subset \mathrm{GL}_n(\mathbb{C})$, the induced action of K on $V = \mathbb{C}^n$ is Hamiltonian.*

Hamiltonian group actions are the "correct" group actions to consider in the category of symplectic manifolds; in particular these are the actions for which a good notion of "quotient space" exists. The technical term for such a quotient is "Hamiltonian reduction," and generally speaking when this operation is carried out the result also has the structure of a symplectic space by a result of Marsden and Weinstein [10]. Along the way to constructing Hamiltonian reduction of V by K, one considers the so-called level sets of the momentum map μ_K. Let $c \in \mathfrak{k}^*$ be a central element, this means that c is fixed by the coadjoint action of K on $\mathfrak{k}^*$: $Ad_k^*(c) = kck^{-1} = c\ \forall k \in K$. For $c \in \mathfrak{k}^*$ a central element we consider the subspace $\mu_K^{-1}(c) \subset V$. Level sets of the moment map have a number of nice properties, for example they are always connected and K-stable subspaces (see [3], [13]). This property was used by Needham and Shonkwiler [12] to reprove and generalize a theorem of Cahill, Mixon, and Strawn [2], which states the space of complex fixed unit norm tight frames (FUNTFs) is connected.

The Hamiltonian reduction of V at level c is defined to be the quotient space $\mu^{-1}(c)/K$, and it can be shown to carry a natural symplectic structure. Part of the utility of this theory is that if c is integral, the reduction coincides with a corresponding "Geometric Invariant Theory" construction in algebraic geometry (see [11]), where the quotient by the corresponding complex group $K^{\mathbb{C}}$ is considered. This is a well-known result due originally to Kempf and Ness [7], with important technical extensions to general Kähler manifolds and projective algebraic varieties due to Kirwan [8] and Sjamaar [13]. Let $f : V \to \mathbb{R}$ be $f(v) = -||\mu_K(v) - c||^2$; the flow ϕ_t of the gradient ∇f provides an important ingredient in the proof that these two notions of quotient coincide. We state the relevant result below, let ϕ_∞ be the limit map of the flow ϕ_t (see [13, Proposition 2.4]).

Theorem 2. *There is a dense, open set $V^{ss} \subset V$ such that for any $v \in V$,*

$$\phi_\infty(v) \in \mu_K^{-1}(c). \tag{2}$$

3. Parseval Frames and the Paulsen problem

Now we point to several examples from the world of frames the defining conditions of which can be rephrased in the language of moment maps. Let $M_{m\times n}(\mathbb{C})$ $m < n$ be the space of $m \times n$ complex matrices equipped with its left and right actions by $U(m)$ and $U(n)$, respectively. The set of diagonal matrices with modulus 1 entries in $U(n)$ is isomorphic to an n-torus $\mathbb{T}^n = U(1) \times \cdots \times U(1)$. With these identifications, $M_{m\times n}(\mathbb{C})$ has a Hamiltonian action by $U(m) \times \mathbb{T}^n$.

The moment map $\mu_{\mathbb{T}^n}$ is computed by sending $\Phi \in M_{m\times n}(\mathbb{C})$ to $-\frac{1}{2}(\cdots, ||\Phi_i||^2, \cdots)$, where Φ_i is the i-th column of Φ. The dual of the Lie algebra of $\mathbb{T}^n$ can be identified with $\mathbb{R}^n$. Since $\mathbb{T}^n$ is Abelian, every element of $\mathbb{R}^n$ is central. Choosing $c = (c_1, \ldots, c_n)$, we can consider $\mu^{-1}(c) \subset M_{m\times n}(\mathbb{C})$; this is the space of matrices whose i-th column has length $\sqrt{-2c_i}$. In particular, we must have $c_i \in R_{\leq 0}$ for $\mu_{\mathbb{T}^n}(c)$ to be non-empty. The moment map $\mu_{U(m)}$ can be shown to be the *frame operator*: $\mu_{U(m)}(\Phi) = \Phi\Phi^*$. The moment map of a product group is simply the product of the moment maps, so we have:

$$\mu_{U(m)\times\mathbb{T}^n}(\Phi) = \mu_{U(m)}(\Phi) \times \mu_{\mathbb{T}^n}(\Phi) = (\Phi\Phi^*, -\frac{1}{2}||\Phi_1||^2, \cdots, -\frac{1}{2}||\Phi_n||^2). \tag{3}$$

The central elements of the $U(m)$ action on $\mathcal{H}_m$ are precisely the multiples of the identity matrix aI. It follows that we can consider the set of matrices with frame operator a fixed multiple of the identity aI and prescribed column lengths $\sqrt{2c_1}, \ldots, \sqrt{2c_n}$ as the moment level set $\mu^{-1}_{U(m)\times\mathbb{T}^n}(aI, -c_1, \ldots, -c_n)$. Theorem 2 then implies that the gradient flow of the *frame potential* $p_{a,\vec{c}}(\Phi) = ||(\Phi\Phi^*, -\frac{1}{2}||\Phi_1||^2, \cdots, -\frac{1}{2}||\Phi_n||^2) - (aI, c_1, \ldots, c_n)||^2$ takes any matrix $\Phi \in M_{m\times n}(\mathbb{C})^{ss}$ into the space $\mu^{-1}_{U(m)\times\mathbb{T}^n}(aI, -c_1, \ldots, -c_n)$.

Let $a = 1$ and $c_i = \frac{m}{2n}$, then matrices $\Phi \in \mu^{-1}_{U(m)\times\mathbb{T}^n}(I, -\frac{m}{2n}, \ldots, -\frac{m}{2n})$ are known as Parseval frames. A matrix $\Psi \in M_{m\times n}(\mathbb{C})$ is said to be an ϵ-nearly equal norm Parseval frame if $(1-\epsilon)I \preceq \Psi\Psi^* \preceq (1+\epsilon)I$, $(1-\epsilon)\frac{m}{n} \leq ||\Psi e_j||^2 \leq (1+\epsilon)\frac{m}{n}$, where e_j is the j-th canonical basis member of $\mathbb{C}^n$. Recently Hamilton and Moitra [6] have shown that for any ϵ-nearly equal norm Parseval frame there is an actual Parseval frame Φ such that $||\Phi - \Psi||^2 \leq 40\epsilon d^2$, where $|| - ||$ denotes the Frobenius norm. Finding bounds for $||\Phi - \Psi||^2$ is known as the Paulsen problem.

We conjecture an inequality of the form $||\Phi - \Psi||^2 \leq C\epsilon d^{2-\alpha}$ with $0 \leq \alpha < 1$ using the gradient flow of the frame potential. Following observations of Lerman [9], we intend to bound the distance between a matrix Ψ and the end point of its flow $\phi_\infty(\Phi)$ using a Łojasiewicz estimate. Similar "constrained" gradient flow methods have been used in [1] and [4], which translate roughly to utilizing the gradient flow for the moment maps $\mu_{\mathbb{T}^n}$ or $\mu_{U(m)}$ alone, rather than in concert.

An alternate approach to the Paulsen problem can be seen in [5] and [6], where one normalizes the columns of $\Gamma\Psi$, where $\Gamma \in \mathrm{GL}_m(\mathbb{C})$ is a matrix depending on Ψ. We observe that the resulting matrix is in the same $\mathrm{GL}_m(\mathbb{C}) \times (\mathbb{C}^*)^n$ orbit as Ψ. Another consequence of the fact that symplectic reduction and GIT quotient coincide implies that the matrix constructed in [5] must be in the same $U(m) \times \mathbb{T}^n$ orbit as the matrix we construct [13, Proposition 1.15]. The same holds for the approaches in [6], [1] and [4]. It follows that the procedure consisting of first using the gradient flow, and then optimizing over the $U(m) \times \mathbb{T}^n$ orbit must produce the state-of-the art answer to the Paulsen problem.

We expect there are other applications of symplectic geometry to other spaces of frames. In particular, spaces of finite complex fusion frames should be be amenable to symplectic techniques.

References

[1] Bernhard G. Bodmann and Peter G. Casazza. The road to equal-norm Parseval frames. *J. Funct. Anal.*, 258(2):397–420, 2010.

[2] Jameson Cahill, Dustin G. Mixon, and Nate Strawn. Connectivity and irreducibility of algebraic varieties of finite unit norm tight frames. *SIAM J. Appl. Algebra Geom.*, 1(1):38–72, 2017.

[3] Ana Cannas da Silva. *Lectures on symplectic geometry*, volume 1764 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2001.

[4] Peter G. Casazza, Matthew Fickus, and Dustin G. Mixon. Auto-tuning unit norm frames. *Appl. Comput. Harmon. Anal.*, 32(1):1–15, 2012.

[5] Martin Ehler. Preconditioning filter bank decomposition using structured normalized tight frames. *J. Appl. Math.*, pages Art. ID 861563, 12, 2015.

[6] L. Hamilton and A. Moitra. The paulsen problem made simple. arXiv:1809.04726 [math.FA].

[7] George Kempf and Linda Ness. The length of vectors in representation spaces. In *Algebraic geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978)*, volume 732 of *Lecture Notes in Math.*, pages 233–243. Springer, Berlin, 1979.

[8] Frances Clare Kirwan. *Cohomology of quotients in symplectic and algebraic geometry*, volume 31 of *Mathematical Notes*. Princeton University Press, Princeton, NJ, 1984.

[9] Eugene Lerman. Gradient flow of the norm squared of a moment map. *Enseign. Math. (2)*, 51(1-2):117–127, 2005.

[10] Jerrold Marsden and Alan Weinstein. Reduction of symplectic manifolds with symmetry. *Rep. Mathematical Phys.*, 5(1):121–130, 1974.

[11] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]*. Springer-Verlag, Berlin, third edition, 1994.

[12] T. Needham and C. Shonkwiler. Symplectic geometry and connectivity of spaces of frames. arXiv:1804.05899 [math.FA].

[13] Reyer Sjamaar. Holomorphic slices, symplectic reduction and multiplicities of representations. *Ann. of Math. (2)*, 141(1):87–129, 1995.

A structured small frame for phase retrieval

Bernhard G. Bodmann, Sabine Burgdorf, Dan Edidin, and Markus Grassl

1. Introduction

Phase retrieval in a finite-dimensional Hilbert space aims to recover an unknown vector, up to an overall unimodular constant factor, from the magnitudes of linear measurements. The linear measurements can be thought of as inner products with vectors that form a spanning sequence, so they constitute a frame. One goal in the construction of frames for phase retrieval is to find the smallest number of frame vectors that permit recovery. In our contribution, we focus on phase retrieval for vectors in a complex Hilbert space of dimension $d = 4$. Previous results on small frames for phase retrieval include structured [1] or generically chosen [3] frames for a complex Hilbert space of dimension d with $n = 4d - 4$ vectors. For $d = 4$, this means the frame has $n = 12$ vectors. Surprisingly, a computer-aided search produced a frame of $n = 11$ vectors that provides phase retrieval for $d = 4$ [4]. Hereafter, we construct a structured 11-element frame for a complex 4-dimensional Hilbert space and establish a certificate that it provides phase retrieval. We expect that similarly structured frames can be constructed in other dimensions.

2. Phase retrieval for at most cubic polynomials

We consider the Hilbert space of complex polynomials that are at most cubic, $\mathcal{P}_3 = \{p : p(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3, a_i \in \mathbb{C}\}$, equipped with the sesquilinear inner product $\langle p, q\rangle = (1/2\pi)\int_0^{2\pi} p(e^{it})\overline{q(e^{it})}dt$ between $p, q \in \mathcal{P}_3$. A frame $\Phi = \{\varphi\}_{j=1}^n$ is conveniently specified by its analysis operator $V : p \mapsto (\langle p, \varphi\rangle)_{j=1}^n$. The frame provides phase retrieval if the map $A : [p] \mapsto (|\langle p, \varphi\rangle|^2)_{j=1}^n$ is injective on the quotient space that identifies vectors in each equivalence class $[p] = \{\alpha p : |\alpha| = 1\}$. The analysis operator for the frame we construct is based on point evaluations for the polynomial p and point evaluations of its first and second derivatives p' and p'' at primitive roots of unity or the origin.

Theorem. Let $\xi = e^{2\pi i/7}$ and $\omega = e^{2\pi i/3}$. The frame Φ for $\mathcal{P}_3$ with analysis operator $V : p \mapsto (p(1), p(\xi), \dots, p(\xi^6), p'(1), p'(\omega), p'(\omega^2), p''(0))$ provides phase retrieval, i. e., the map given by

$$A : [p] \mapsto \left(|p(1)|^2, |p(\xi)|^2, \dots, |p(\xi^6)|^2, |p'(1)|^2, |p'(\omega)|^2, |p'(\omega^2)|^2, |p''(0)|^2\right)$$

is injective.

Proof. We follow the general strategy outlined in [3]. To obtain injectivity, we need to show that there is no rank-2 Hermitian in the kernel of the measurement when it is extended to all Hermitians on $\mathcal{P}_3$ by $A : Q \mapsto (\langle Q\varphi_j, \varphi_j\rangle)_{j=1}^n$. For this, we consider the basis of monomials $e_k(z) = z^{k-1}$ and represent each Hermitian operator Q on $\mathcal{P}_3$ as

$$Q = \sum_{j=1}^{4} x_{j,j}\, e_j \otimes e_j^* + \sum_{j=1}^{3} \sum_{k=j+1}^{4} [(x_{j,k} + iy_{j,k}) e_j \otimes e_k^* + (x_{j,k} - iy_{j,k}) e_k \otimes e_j^*]$$

where $x_{11}, \dots, y_{34}$ are real numbers. With the convention $x_{j,k} = x_{k,j}$, $y_{j,j} = 0$ and $y_{j,k} = -y_{k,j}$, the condition that Q has at most rank two can be expressed by requiring that all 3×3 minors of the matrix $\tilde{Q} = (x_{j,k} + iy_{j,k})_{j,k=1}^4$, denoted by $m_{j,k}$, vanish identically.

Each vector φ_k of the frame gives rise to a real linear form $\ell_k(\tilde{Q}) = \langle Q\varphi_k, \varphi_k \rangle$, expressed in the variables $x_{j,k}, y_{j,k}$. The measurements associated with the frame Φ are injective if and only if the system of equations

$$(1) \qquad m_{11} = m_{12} = \ldots = m_{44} = \ell_1 = \ldots = \ell_{11} = 0$$

has no non-zero real solution.

Using the computer algebra system Magma, it can be shown that the ideal I generated by the polynomials in (1)

$$(2) \qquad I = \langle m_{11}, m_{12}, \dots, m_{44}, \ell_1, \dots, \ell_{11} \rangle \lhd \mathbb{C}[x_{11}, x_{12}, \dots, x_{44}, y_{12}, \dots, y_{34}]$$

has a Gröbner basis with rational coefficients. Over the rationals, $I = I_1 \cap I_2 \cap I_3 \cap I_4 \cap I_5 \cap I_6$, i. e., I has six irreducible components in of degrees $6, 4, 4, 2, 2, 2$, respectively.

Setting additionally $y_{24} = 0$, the only solution of (1) is $Q = 0$. Without loss of generality we can then choose $y_{24} = 1$. The six components then read:

$$\begin{aligned} I_1 = \Big\langle & x_{11} - 8x_{44},\ x_{12} - 8x_{44}y_{34}^2 - \tfrac{3}{2}x_{44},\ x_{13} + 4x_{44}y_{34} - x_{44},\ x_{14},\ x_{22} + 9x_{44}, \\ & x_{23} + 12x_{44}y_{34}^2 - 3x_{44}y_{34} + 3x_{44},\ x_{24} - 4x_{44}y_{34} + x_{44},\ x_{33}, \\ & x_{34} - 4x_{44}y_{34}^2 + 3x_{44}y_{34} - \tfrac{3}{2}x_{44},\ x_{44}^2 - \tfrac{8}{9}y_{34}^2 + \tfrac{2}{9}y_{34} + \tfrac{1}{9},\ y_{12} - 2y_{34} + \tfrac{3}{2}, \\ & y_{13} + 1,\ y_{14},\ y_{23} + 3y_{34} - \tfrac{3}{2},\ y_{24} - 1,\ y_{34}^3 - \tfrac{3}{4}y_{34}^2 + \tfrac{9}{16}y_{34} - \tfrac{5}{64} \Big\rangle \end{aligned}$$

$$\begin{aligned} I_2 = \Big\langle & x_{11},\ x_{12} - 3x_{34}y_{34} + \tfrac{9}{4}x_{34},\ x_{13} + 2x_{34}y_{34} - \tfrac{3}{2}x_{34},\ x_{14},\ x_{22},\ x_{23} + 3x_{34}y_{34} + \tfrac{3}{4}x_{34}, \\ & x_{24} - 2x_{34}y_{34} + \tfrac{3}{2}x_{34},\ x_{33},\ x_{34}^2 - \tfrac{9}{4}y_{34} + \tfrac{3}{2},\ x_{44},\ y_{12} - 2y_{34} + \tfrac{3}{2},\ y_{13} + 1, \\ & y_{14},\ y_{23} + 3y_{34} - \tfrac{3}{2},\ y_{24} - 1,\ y_{34}^2 - \tfrac{3}{4}y_{34} + \tfrac{1}{2} \Big\rangle \end{aligned}$$

$$\begin{aligned} I_3 = \Big\langle & x_{11} - 8x_{44},\ x_{12} - 4x_{44}y_{34} + 6x_{44},\ x_{13} + 8x_{44}y_{34},\ x_{14},\ x_{22} + 9x_{44},\ x_{23} - 9x_{44}, \\ & x_{24} - 8x_{44}y_{34},\ x_{33},\ x_{34} + 4x_{44}y_{34} + 3x_{44},\ x_{44}^2 + \tfrac{1}{18}y_{34} + \tfrac{1}{72},\ y_{12} - 2y_{34} + \tfrac{3}{2}, \\ & y_{13} + 1,\ y_{14},\ y_{23} + 3y_{34} - \tfrac{3}{2},\ y_{24} - 1,\ y_{34}^2 + \tfrac{1}{2}y_{34} + \tfrac{5}{8} \Big\rangle \end{aligned}$$

$$I_4 = \Big\langle x_{11},\ x_{12}+x_{34},\ x_{13}-2x_{34},\ x_{14},\ x_{22},\ x_{23},\ x_{24}+2x_{34},\ x_{33},$$
$$x_{34}^2+\tfrac{9}{4},\ x_{44},\ y_{12}+\tfrac{1}{2},\ y_{13}+1,\ y_{14},\ y_{23},\ y_{24}-1,\ y_{34}-\tfrac{1}{2}\Big\rangle$$
$$I_5 = \Big\langle x_{11}-8x_{44},\ x_{12}-14x_{44},\ x_{13}+4x_{44},\ x_{14},\ x_{22}+9x_{44},\ x_{23}+18x_{44},\ x_{24}-4x_{44},$$
$$x_{33},\ x_{34}-4x_{44},\ x_{44}^2+18,\ y_{12}+\tfrac{1}{2},\ y_{13}+1,\ y_{14},\ y_{23},\ y_{24}-1,\ y_{34}-\tfrac{1}{2}\Big\rangle$$
$$I_6 = \Big\langle x_{11},\ x_{12},\ x_{13},\ x_{14},\ x_{22},\ x_{23},\ x_{24},\ x_{33},\ x_{34},\ x_{44},\ y_{12}-2y_{34}+\tfrac{3}{2},$$
$$y_{13}+1,\ y_{14},\ y_{23}+3y_{34}-\tfrac{3}{2},\ y_{24}-1,\ y_{34}^2-\tfrac{3}{4}y_{34}+2\Big\rangle,$$

where the generating sets are Gröbner bases with respect to lexicographical order. In each of the ideals I_j we find a univariate polynomial g_j that is strictly positive for real arguments:

$$g_1 = x_{44}^6 + x_{44}^4 + \frac{1}{3}x_{44}^2 + \frac{1}{36} = \left((x^2+1)x^2+\frac{1}{3}\right)x^2 + \frac{1}{36}$$
$$g_2 = y_{34}^2 - \frac{3}{4}y_{34} + \frac{1}{2} = \left(y_{34}-\frac{3}{8}\right)^2 + \frac{23}{64}$$
$$g_3 = y_{34}^2 + \frac{1}{2}y_{34} + \frac{5}{8} = \left(y_{34}+\frac{1}{4}\right)^2 + \frac{9}{16}$$
$$g_4 = x_{34}^2 + \frac{1}{4}$$
$$g_5 = x_{44}^2 + \frac{1}{48}$$
$$g_6 = y_{34}^2 - \frac{3}{4}y_{34} + \frac{1}{2} = \left(y_{34}-\frac{3}{8}\right)^2 + \frac{23}{64}$$

Hence, there is no real solution to the equations (1). □

Acknowledgments. B.G.B. was supported in part by NSF grant DMS-1715735. D.E. was supported by Simons Collaboration grant 315460.

References

[1] B. G. Bodmann and N. Hammen, Stable phase retrieval with low-redundancy frames, Adv. Comput. Math. 41 (2015), 317–331.
[2] Symbolic computation package Magma, http://magma.maths.usyd.edu.au/magma/
[3] A. Conca, D. Edidin, M. Hering, and C. Vinzant, An algebraic characterization of injectivity in phase retrieval, Appl. Comput. Harmon. Anal. 38 (2015), 346–356.
[4] C. Vinzant, A small frame and a certificate of its injectivity, International Conference on Sampling Theory and Applications (SampTA) 2015, pp. 197-200, IEEE, Washington, D.C., 2015.

Decomposing (equiangular) tight frames into (equiangular) tight frames for their spans

Alex Fink, Emily King, Cynthia Vinzant, and Shayne Waldron

Given a frame, it is of interest to ask if a subset of frame vectors also have certain frame properties. The answer to this question has implications for the robustness of the frame representation to erasures [15, 8] and appropriateness of the frame in compressed sensing applications [11, 4] and is sometimes characterized by combinatorial structures [12, 5, 17, 3]. In order to state our main research question, we need a definition.

Definition 1. (see, e.g., [13]) Let $\Phi = \begin{pmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \end{pmatrix} \in \mathbb{C}^{d\times n}$. We call Φ (or, more precisely, its columns) a *tight frame for its span* if there exists an $A > 0$ such that $(\Phi\Phi^*)^2 = A\Phi\Phi^*$; that is, $\Phi\Phi^*$ is a positive multiple of an orthogonal projection. If $\Phi\Phi^* = AI$, then Φ is a *tight frame.* If further each of the columns of Φ has unit norm and there exists an $\lambda \geq 0$ such that $|\langle \varphi_j, \varphi_k\rangle| = \lambda$ for all $j \neq k$, then we call Φ an *equiangular tight frame for its span* (respectively, *equiangular tight frame*).

Question 2. If Φ is an (equiangular) tight frame, can it non-trivially be decomposed into the disjoint union of (equiangular) tight frames for their spans?

This generalizes questions worked on in [3, 12, 7, 6, 1, 16, 9, 21, 10] and more. Since Φ itself is assumed to be an (equiangular) tight frame, it is also an (equiangular) tight frame for its span, yielding one trivial decomposition. On the other extreme, a non-zero (unit norm) vector is trivially an (equiangular) tight frame for its span, and Φ is the disjoint union of its vectors.

A selection of results proven during the mini-workshop includes the following.

Proposition 3. *Let Φ be a tight frame. Then if $\Phi = \Psi_1 \sqcup \Psi_2$, with Ψ_1, Ψ_2 tight frames for their spans, one of the following must hold:*

(1) $\operatorname{span}\Psi_1 \perp \operatorname{span}\Psi_2$ *or*
(2) $\operatorname{span}\Psi_1 = \operatorname{span}\Psi_2 = \operatorname{span}\Phi$.

If further Φ as an equiangular tight frame with Ψ_1, Ψ_2 equiangular tight frames for their spans, then

(1') *Φ is an orthonormal basis of at least 2 vectors.*

Proof. It follows from the hypotheses that there exist $A, \alpha, \beta > 0$ and orthogonal projections P_1, P_2 such that

$$AI = \Phi\Phi^* = \Psi_1\Psi_1^* + \Psi_2\Psi_2^*,\ \Psi_1\Psi_1^* = \alpha P_1 \text{ and } \Psi_2\Psi_2^* = \beta P_2 \ \Rightarrow\ I - \frac{\alpha}{A}P_1 = \frac{\beta}{A}P_2.$$

By comparing the spectra of $I - \frac{\alpha}{A}P_1$ and $\frac{\beta}{A}P_2$, one obtains that either $P_1 = P_2 = I$ or $1 - \frac{\alpha}{A} = 0$ and thus $P_2 = P_1^\perp$. The second claim follows by comparing the inner products of the vectors using the so-called Welch bound (see, e.g. [23, 22]). □

Theorem 4. *Let $\Phi \in \mathbb{C}^{d\times n}$ be a tight frame with frame bound A. Assume that $\Phi = \bigsqcup_{j=1}^{3} \Psi_j$, with Ψ_j, $j = 1,2,3$ tight frames for their spans. Then for each $j = 1,2,3$ there exists $r_j, n_j \in \mathbb{N}$, $\alpha_j > 0$ and a rank-r_j orthogonal projection P_j such that $\Psi_j \in \mathbb{C}^{d\times n_j}$ and $\Psi_j\Psi_j^* = \alpha_j P_j$. Further, if Φ does not decompose in the sense of (2) of Proposition 3, then exactly one of the following must hold:*

(1) $r_1 + r_2 + r_3 = d$ *and* $\alpha_1 = \alpha_2 = \alpha_3 = A$*;*
(2) $r_1 = r_2$, $r_3 = d - r_1$, $\alpha_1 + \alpha_2 = A$, *and* $\alpha_3 = A$*;*
(3) $(d - r_1) + (d - r_2) + (d - r_3) = d$ *and* $\alpha_1 = \alpha_2 = \alpha_3 = \frac{A}{2}$*; or*
(4) $r_1 = r_2 = r_3 = \frac{d}{2}$, $\alpha_1 + \alpha_2 + \alpha_3 = 2A$, *and* $0 \le \alpha_1, \alpha_2, \alpha_3 \le A$.

If further Φ is an equiangular tight frame with Ψ_j, $j = 1,2,3$ equiangular tight frames for their spans, then one of the following must hold:

(1') Φ *is an orthonormal basis of at least* 3 *vectors.*
(3') $n \equiv 9 \mod 12$, $d = \frac{n+3}{4}$ *and for* $j = 1,2,3$, $n_j = \frac{n}{3}$, *and* $r_j = \frac{n+3}{6}$*; or*
(4') $n \equiv 3 \mod 12$, $d = \frac{n+1}{2}$ *and for* $j = 1,2,3$, $n_j = \frac{n}{3}$, *and* $r_j = \frac{n+1}{4}$.

Proof. The proof uses Knutson-Tao honeycombs [18, 19, 20] and the Welch bound. □

There are examples of each of the configurations in Theorem 4.

(1') Trivial.
(3') Let ζ be a primitive 3rd root of unity. Set

$$\Phi = \left(\begin{array}{c|c|c} \Psi_1 & \Psi_2 & \Psi_3 \end{array} \right) = \frac{1}{\sqrt{2}} \left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & -1 & -\zeta & -\zeta^2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & -1 & -\zeta & -\zeta^2 \\ -1 & -\zeta & -\zeta^2 & 1 & 1 & 1 & 0 & 0 & 0 \end{array} \right).$$

Then Φ is an equiangular tight frame of 9 vectors in $\mathbb{C}^3$, and for each $j = 1,2,3$, $(\Psi_j\Psi_j^*)^2 = \frac{3}{2}\Psi_j\Psi_j^*$. This Φ is an example of a symmetric, informationally complete, positive operator-valued measure [16, 9] and a Gabor-Steiner equiangular tight frame [2, 14].

(4') Let ζ be a primitive 15th root of unity. Set

$$\Phi = \frac{1}{\sqrt{8}} \left(\zeta^{k\ell}\right)_{k\in D, \ell\in\{0,1,\dots,14\}}, \quad D = \{0,1,2,3,5,7,8,11\}$$

and for $j = 0,1,2$

$$\Psi_j = \frac{1}{\sqrt{8}} \left(\zeta^{k(j+3\ell)}\right)_{k\in D, \ell\in\{0,1,\dots,4\}}.$$

Then Φ is an equiangular tight frame of 15 vectors in $\mathbb{C}^8$, and for each $j = 0,1,2$, $(\Psi_j\Psi_j^*)^2 = \frac{5}{4}\Psi_j\Psi_j^*$. This Φ is an example of a Naimark complement (see, e.g. [23]) of an equiangular tight frame generated by $(3,2)$-Singer difference set (see, e.g. [24]) and the decomposition appears as Example 7.2 in [12].

In on-going work started at Oberwolfach, we are concerned with higher order decompositions, nested decompositions, the relationship of the decompositions with various dualities in frame theory, and constructions of infinite classes of (equiangular) tight frames which yield each possible type of configuration.

References

[1] Marcus Appleby, Ingemar Bengtsson, Irina Dumitru, and Steven Flammia. Dimension towers of SICS. I. Aligned SICs and embedded tight frames. Preprint, 2017.

[2] Bernhard Bodmann and Emily J. King. Optimal arrangements of classical and quantum states with limited purity. In preparation, 2018.

[3] Marcin Bownik, Kurt Luoto, and Edward Richmond. A combinatorial characterization of tight fusion frames. *Pacific J. Math.*, 275(2):257–294, 2015.

[4] Alfred M. Bruckstein, David L. Donoho, and Michael Elad. From sparse solutions of systems of equations to sparse modeling of signals and images. *SIAM Rev.*, 51(1):34–81, 2009.

[5] Jameson Cahill. Flags, frames, and Bergman spaces. Master's thesis, San Francisco State University, May 2009.

[6] Peter G. Casazza, Matthew Fickus, Andreas Heinecke, Yang Wang, and Zhengfang Zhou. Spectral tetris fusion frame constructions. *J. Fourier Anal. Appl.*, 18(4):828–851, 2012.

[7] Peter G. Casazza, Matthew Fickus, Dustin G. Mixon, Yang Wang, and Zhengfang Zhou. Constructing tight fusion frames. *Appl. Comput. Harmon. Anal.*, 30(2):175–187, 2011.

[8] Peter G. Casazza and Jelena Kovačević. Equal-norm tight frames with erasures. *Adv. Comput. Math.*, 18(2-4):387–430, 2003. Frames.

[9] Hoan Bui Dang, Kate Blanchfield, Ingemar Bengtsson, and D. M. Appleby. Linear dependencies in Weyl–Heisenberg orbits. *Quantum Information Processing*, 12(11):3449–3475, Nov 2013.

[10] Ken Dykema and Nate Strawn. Manifold structure of spaces of spherical tight frames. *Int. J. Pure Appl. Math.*, 28(2):217–256, 2006.

[11] Michael Elad. *Sparse and redundant representations.* Springer, New York, 2010. From theory to applications in signal and image processing, With a foreword by Alfred M. Bruckstein.

[12] Matthew Fickus, John Jasper, Emily J. King, and Dustin G. Mixon. Equiangular tight frames that contain regular simplices. *Linear Algebra Appl*, 555:98–138, October 2018.

[13] Matthew Fickus, Dustin G. Mixon, and John Jasper. Equiangular tight frames from hyperovals. *IEEE Trans. Inform. Theory*, 62(9):5225–5236, 2016.

[14] Matthew Fickus, Dustin G. Mixon, and Janet C. Tremain. Steiner equiangular tight frames. *Linear Algebra Appl.*, 436(5):1014–1027, 2012.

[15] Vivek K. Goyal, Jelena Kovačević, and Jonathan A. Kelner. Quantized frame expansions with erasures. *Appl. Comput. Harmon. Anal.*, 10(3):203–233, 2001.

[16] Lane Hughston. $d = 3$ SIC-POVMs and elliptic curves. Perimeter Institute, Seminar Talk, `http://pirsa.org/07100040/`, October 2007.

[17] Emily J. King. Algebraic and geometric spread in finite frames. In Manos Papadakis and Vivek K. Goyal, editors, *Wavelets and Sparsity XVI*, volume 9597 of *Society of Photo-Optical Instrumentation Engineers (SPIE) Conference Series*, 2015.

[18] Allen Knutson and Terence Tao. The honeycomb model of $\mathrm{GL}_n(\mathbf{C})$ tensor products. I. Proof of the saturation conjecture. *J. Amer. Math. Soc.*, 12(4):1055–1090, 1999.

[19] Allen Knutson and Terence Tao. Honeycombs and sums of Hermitian matrices. *Notices Amer. Math. Soc.*, 48(2):175–186, 2001.

[20] Allen Knutson, Terence Tao, and Christopher Woodward. The honeycomb model of $\mathrm{GL}_n(\mathbb{C})$ tensor products. II. Puzzles determine facets of the Littlewood-Richardson cone. *J. Amer. Math. Soc.*, 17(1):19–48, 2004.

[21] Jakob Lemvig, Christopher Miller, and Kasso A. Okoudjou. Prime tight frames. *Adv. Comput. Math.*, 40(2):315–334, 2014.

[22] Thomas Strohmer and Robert W. Heath, Jr. Grassmannian frames with applications to coding and communication. *Appl. Comput. Harmon. Anal.*, 14(3):257–275, 2003.
[23] Shayne F. D. Waldron. *An introduction to finite tight frames.* Applied and Numerical Harmonic Analysis. Birkhäuser/Springer, New York, 2018.
[24] Pengfei Xia, Shengli Zhou, and Georgios B. Giannakis. Achieving the Welch bound with difference sets. *IEEE Trans. Inform. Theory*, 51(5):1900–1907, 2005.

Equiangular tight frames over finite fields

Gary Greaves, Joseph W. Iverson, John Jasper, and Dustin G. Mixon

Pick a prime p and power $q = p^k$, and put $F = \mathbb{F}_{q^2}$. Define conjugation by $z \mapsto z^q$, and given a (column) vector $v \in F^d$, we write v^* for the conjugate transpose of v. Define the sesquilinear form $(\cdot,\cdot)\colon F^d \times F^d \to F$ by $(u,v) = u^*v$. Given $a,b,c \in \mathbb{F}_q$, we say $\{\varphi_i\}_{i=1}^n$ in F^d is an (a,b,c)**-equiangular tight frame (ETF)** if

(a) $(\varphi_i,\varphi_i) = a$ for every $i \in \{1,\dots,n\}$,
(b) $(\varphi_i,\varphi_j)^{q+1} = b$ for every $i,j \in \{1,\dots,n\}$ with $i \neq j$, and
(c) $\sum_{i=1}^n \varphi_i\varphi_i^* = cI$.

Finite field ETFs and complex ETFs enjoy many of the same algebraic properties, though with some technicalities:

Lemma 1. *Suppose there exists an (a,b,c)-ETF of n vectors in F^d. Then*
(a) *$dc = na$, and*
(b) *if $n \not\equiv 0 \bmod p$, then $a(c-a) = (n-1)b$.*

Lemma 2. *Take any $a,b,c \in \mathbb{F}_q$ such that $a,c \neq 0$ and $b \neq a^2$. Then there exists an (a,b,c)-ETF of n vectors in F^d only if $n \leq d^2$.*

In fact, the following machine converts complex ETFs into finite field ETFs:

Lemma 3. *Let $\{\psi_i\}_{i=1}^n$ be an ETF in $\mathbb{C}^d$, take any ring $R \subseteq \mathbb{C}$ containing the entries of $\{\psi_i\psi_i^*\}_{i=1}^n$ that is closed under complex conjugation, and suppose there exists a ring homomorphism $\sigma\colon R \to F$ such that*

$$\sigma(1) = 1, \qquad \sigma(\|\psi_1\|^2) \neq 0, \qquad \sigma(\overline{z}) = \sigma(z)^q \ \forall z \in R.$$

Then there exists an ETF $\{\varphi_i\}_{i=1}^n$ in F^d such that $\varphi_i\varphi_i^* = \sigma(\psi_i\psi_i^*)$ for every $i \in \{1,\dots,n\}$.

This machine converts the ETF of 4 vectors in $\mathbb{C}^2$ with Heisenberg–Weyl symmetry into an ETF in F^2 precisely when $q \equiv 11 \bmod 12$. However, one may also solve the defining polynomial equations to show that an ETF of 4 vectors in F^2 exists precisely when $q \equiv 5 \bmod 6$. Interestingly, these additional finite field ETFs can be obtained by expressing the original complex ETF in a different basis before applying a different choice of ring homomorphism.

The motivation for passing to finite fields is to facilitate the study of the tough problems surrounding complex ETFs, such as the Fickus conjecture and Zauner's conjecture. For example, can we solve these problems in the finite field setting? More interestingly, can we transfer solutions from the finite field setting back to the original complex setting? We are attempting to address these questions in an ongoing follow-up collaboration.

Reporter: Dan Edidin