Rigidity of stationary measures

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1 Introduction

Stationary probability measures ν are useful when one wants to understand the dynamics of the action of a non-abelian group or semigroup Γ on a compact space X. The reason is that in this situation, there might not exist any Γ-invariant probability measures on X.

To overcome this issue, one chooses a probability measure μ on Γ and one defines the stationary measures as the probability measures on X which are invariant by convolution by μ. These measures control the asymptotic distributions of the associated random walk on X. Since they exist on the closure of any Γ-orbit, they are useful to describe the closure of the Γ-orbits on X. They are also useful to describe equidistribution properties of a sequence of finite Γ-orbits.

The aim of this arbeitsgemeinschaft is to understand the classification of stationary measures for semisimple random walks on a finite volume homogeneous space and its applications as described in [4].

In order to clarify the ideas we will mainly focus on the case where $X = \mathbb{T}^d$ is the torus (when the action is proximal, this case is due to Bourgain, Furman, Lindenstrauss and Mozes in [11]) and also on the case where $X = \mathrm{SL}(d,\mathbb{R})/\mathrm{SL}(d,\mathbb{Z})$ is the space of unimodular lattices in $\mathbb{R}^d$.

The structure of this arbeitsgemeinschaft relies mainly on the introductory text [8] and follows the strategy of [4]. The technical details will rely also on [3] and on the book [9]. Most of the lectures are expected to be one hour long. Many references are available on Benoist or Quint website.

2 Lectures

The first lecture is an overview lecture.

Lecture 1. Stationary measures on tori and applications. The aim of this introductory lecture is to state precisely the main results of [4]. For instance following the presentation of [8] beginning by the torus $X = \mathbb{T}^d$

then by the space $X = \mathrm{SL}(d,\mathbb{R})/\mathrm{SL}(d,\mathbb{Z})$ of unimodular lattices of $\mathbb{R}^d$, and by a general finite volume homogeneous space X. No proof. No strategy of proof. Follow [8, Section 1].

2.1 Invariant and stationary measures

We begin by a few preliminary lectures.

Lecture 2. Example of invariant measures. This talk is a warm up which focuses on actions of the group $\mathbb{Z}$. It presents a few classical dynamical systems and their invariant measures. These examples are relevant here, since some of the ideas in their proof will be reused later. Follow [8, Sec. 2]

Lecture 3. Dynamics on Heisenberg nilmanifolds. This talk will present an example of classification of invariant measure in the case of unipotent flows on the 3-dimensional Heisenberg nilmanifold. Follow [17, Chap. 3].

Lecture 4. Ratner ergodic theorem. Explain Ratner measure rigidity theorem, focusing mainly on the polynomial drift argument used in the classification of probability measures on $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$ that are invariant by a unipotent group. Follow [1, Section 12.1]. See also [17, Chap. 4].

Lecture 5. Examples of stationary measures. The aim of this lecture is to give examples of stationary measures. Existence on compact set. Maximum principle for countable spaces. Choquet-Deny theorem for abelian groups. Probability measures on compact groups. Follow [8, Sections 3.1, 3.2, 3.4 and 3.5] and [3, Lemme 8.4].

Lecture 6. Empirical measures. The aim of this lecture is to prove Breiman law of large number which says that the limits of the empirical measures are stationary. One could also discuss Raugi theorem for equicontinuous Markov operator. Follow [8, Sec. 3.7], [9, Cor. 3.7 and 3.8] and [7, Sec. 2].

2.2 Linear random walks

The random walk on linear groups plays a crucial role for controling the drift in the exponential drift argument.

Lecture 7. Proximal action. The aim is to prove Furstenberg proximality theorem: a strongly irreducible and proximal action on a projective space has a unique stationary measure ν. The tools are the limit measures ν_b, that, in this case, are Dirac masses. Follow [9, Sec. 2.5 and 4.2] or [10, Lemme II.2.1, Prop. II.3.3 and Theorem II.4.1].

Lecture 8. Positivity of the first Lyapunov exponent. The aim of this lecture is to prove that the first Lyapunov exponent is positive. This implies a uniform exponential growth for the average size of the image of a vector. These facts are due to Furstenberg. Follow [9, Thm 4.28 and Corol. 4.32] or [10, Theorem II.4.1].

Lecture 9. Central Limit theorem. The aim of this lecture is to state the central limit theorem and the local limit theorem of Lepage and Guivarch and to give the main ideas of their proof. Follow [9, Theorem 12.1 (i)] or [10, Theorem V.5.1].

2.3 Recurrence in law

Proving recurrence in law for the random walk will be crucial, both in the proof of the rigidity of stationary measure but also in the applications of these rigidity results. Even when the space X is compact!

Lecture 10. Recurrence in law. The aim of this lecture is to prove Eskin–Margulis recurrence theorem for a Zariski dense random walk, first in the space $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$, then in $\mathrm{SL}(d,\mathbb{R})/\mathrm{SL}(d,\mathbb{Z})$ and in arithmetic quotients $G_{\mathbb{R}}/G_{\mathbb{Z}}$ of semisimple groups. One deduces a new proof (due to Margulis) that these quotients have finite volume. Follow [2, Section 4, 5 and 6] and [1, Chapter 1], or [6] or [13].

Lecture 11. Orbit closure on $\mathbb{T}^d$. The aim of this lecture is to explain three applications of the rigidity of stationary measures on the torus $\mathbb{T}^d$. First the equidistribution of random trajectories. Second the finiteness or denseness of Γ-orbits (a theorem originally due to Muchnik–Guivarch–Starkov). Third the equidistribution of sequence of distinct finite orbits. The key point is a recurrence in law away from 0. Follow [8, Sec. 4.1 and 4.2].

Lecture 12. Non degeneracy of the limit measures ν_b. The aim of this talk is the fact that when a stationary measure on $\mathbb{T}^d$ is non atomic, the limit measures ν_b are also non atomic. This non degeneracy is the starting point to apply the exponential drift. The key point is a recurrence in law away from the diagonal. Follow [8, Lemma 4.5] which relies on [3, Prop. 3.9] or [4, Prop. 6.17 and Corol. 6.26].

2.4 The exponential drift

The key point of the proof of the rigidity of stationary measures is an exponential drift argument based on the martingale theorem which will be explained in the following five lectures. This argument is reminiscent of the polynomial drift argument in the proof of Ratner's measure rigidity theorem which was based on Birkhof ergodic theorem.

Lecture 13. The conditional measure along a group action. The aim of this talk is to define the conditional measures of a measure along the leaves of a locally free action, to explain how they can be seen as a map to a Borel space, and to prove their basic properties. Follow [3, Section 4].

Lecture 14. The exponential drift argument on $\mathbb{T}^d$. This talk explains what the exponential drift is, and why it implies the rigidity of stationary measures. It should also explain how all the arguments we have seen so far are used in this proof and also why we need the following two technical facts, the "equidistribution of pieces of fibers" and the "law of the angle", that will be explained in the next two lectures. Follow [8, Section 4.3]

Lecture 15. Equidistribution of pieces of fibers. This talk is devoted to the proof of the "equidistribution of pieces of fibers" which is used in the exponential drift argument. It will introduce the fibered dynamical system, prove a formula for the conditional expectation, and relate it to the martingale theorem. Follow [4, Sections 3.1 to 3.4].

Lecture 16. The law of the angles. The aim is to present the "law of the angle" which is used in the exponential drift argument. This law tells that the Furstenberg stationary measure on the projective space describes also the asymptotic law of the random walk on the projective space when one condition it by the size of the "norm cocycle". Follow [4, Section 4.7] using the results on random walks proven in the book [9] and [4, Section 4.6].

Lecture 17. Stationary measures on the space of lattices. This talk should reexplain in detail the whole strategy of the classification of stationary measures, focusing on the space $X = \mathrm{SL}(d, \mathbb{R})/\mathrm{SL}(d, \mathbb{Z})$ instead of $X = \mathbb{T}^d$. It also points out the new difficulties that occur and explains how one overcome them thanks to Ratner's measure rigidity theorem. Follow [8, Section 4.4] which relies on [4, Section 8].

2.5 Beyond the exponential drift

We will end the week by a few lectures presenting without proof recent rigidity results for stationary measures and various applications based on the exponential drift argument.

Lecture 18. Diophantine approximation on Cantor sets. The aim of this talk is to present the paper by D. Simmons and B. Weiss which relates "random walks on homogeneous spaces and diophantine approximation on fractals". Follow arXiv:1611.05899.

Lecture 19. Space of rank two discrete subgroups in $\mathbb{R}^3$. The aim of this talk is to present the paper by O. Sargent and U. Shapira which extends the classification of stationary measures to an homogeneous space with non-discrete stabilizer and studies the "dynamics on the space of 2-lattices in 3-space". Follow arXiv:1708.04464.

Lecture 20. Patterns in large subsets of $\mathbb{Z}^n$. The aim of this talk is to present the paper by M. Bjorklund and K. Bulinski which gives some application of the classification of stationary measures on the torus to the values taken by an integral quadratic forms on a large subset of $\mathbb{Z}^n$ by studying "Twisted patterns in large subsets of $\mathbb{Z}^n$". Follow arXiv:1512.01719.

Lecture 21. Stationary measures on moduli spaces. The aim of this talk is to present the celebrated work of A. Eskin, M. Mirzakhani and A. Mohammadi which describes the "invariant and stationary measures for the $\mathrm{SL}(2,\mathbb{R})$ action on moduli space". Follow arXiv:1302.3320.

Lecture 22. Stationary measures for group action on surfaces. The aim of this talk is to present the paper of A. Brown and F. Rodriguez Hertz, which replaces the space X by a compact surface and focuses on "measure rigidity for random dynamics on surfaces and related skew products". Follow arXiv:1506.06826.

References

[1] Y. Benoist, Réseaux des groupes de Lie *Benoist's web page* (2008).

[2] Y. Benoist, Recurrence on the space of lattices *Proceedings ICM* (2014) 16 p.

[3] Y. Benoist, J.-F. Quint, Mesures stationnaires et fermés invariants des espaces homogènes (I), *Ann. of Math.* **174** (2011), 1111-1162.

[4] Y. Benoist, J.-F. Quint, Stationary measures and invariant subsets of homogeneous spaces (II), *Journal American Math. Soc.* **26** (2013) 659-734.

[5] Y. Benoist, J.-F. Quint, Stationary measures and invariant subsets of homogeneous spaces (III), *Ann. of Math.* **178** (2011), 1017-1059.

[6] Y. Benoist, J.-F. Quint, Random walks on finite volume homogeneous spaces, *Invent. Math.* **187** (2012), 37-59.

[7] Y. Benoist, J.-F. Quint, Random walks on projective spaces, *Compos. Math.* **150** (2014) 1579-1606.

[8] Y. Benoist, J.-F. Quint, Introduction to random walks on homogeneous spaces Tenth Takagi Lecture in Kyoto, *Japan. J. Math.* **7** (2012) 135-166.

[9] Y. Benoist, J.-F. Quint, Random walks on reductive groups Ergebnisse (2016)

[10] P. Bougerol et J. Lacroix, Products of Random Matrices with Applications to Schrödinger Operators, *PM Birkhaüser* (1985).

[11] J. Bourgain, A. Furman, E. Lindenstrauss, S. Mozes, Invariant measures and stiffness for non-abelian groups of toral automorphisms, *Journ. Am. Math. Soc.* **24** (2011), 231-280.

[12] L. Breiman, The strong law of large numbers for a class of Markov chains, *Ann. Math. Statist.* **31** (1960), 801-803.

[13] A. Eskin, G. Margulis, Recurrence properties of random walks on finite volume homogeneous manifolds, *in Random walks and geometry*, W. de Gruiter (2004), 431-444.

[14] F. Ledrappier, Mesures Stationnaires sur les espaces homogènes d'après Y.Benoist et JF. Quint, *Seminaire Bourbaki* **1058** (2012), 1-17.

[15] G. Margulis, *Discrete subgroups of semisimple Lie groups*, Springer (1991).

[16] G. Margulis, G. Tomanov, Invariant measures for actions of unipotent groups over local fields on homogeneous spaces, *Invent. Math.* **116** (1994), 347-392.

[17] J.F. Quint Examples of unique ergodicity of algebraic flows, *Quint's web page* (2007).

[18] M. Raghunathan, *Discrete subgroups of Lie groups*, Springer (1972).

[19] M. Ratner, On Raghunathan's measure conjecture, *Ann. of Math.* **134** (1991), 545-607.

[20] R. Zimmer, Ergodic theory and semisimple groups, Birkhäuser (1984).