ARBEITSGEMEINSCHAFT MIT AKTUELLEM THEMA: ELLIPTIC COHOMOLOGY ACCORDING TO LURIE MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH 31 MARCH - 6 APRIL 2019

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1. Introduction

Twenty or more years ago, Mike Hopkins and his collaborators began to uncover a very strong connection between arithmetic algebraic geometry and stable homotopy theory. This revolutionized stable homotopy theory and gave impetus to the emergence of the field of derived algebraic geometry. We learned, in particular, that core objects in arithmetic geometry, such as the Lubin-Tate deformation space and the Deligne-Mumford moduli stack of elliptic curves, have canonical lifts to derived algebraic geometry. The original constructions used very difficult homotopy theory, but Lurie has recently found a way to much more directly access the classical geometry. The overall goal of the Arbeitsgemeinschaft is to give an exposition of this viewpoint and constructions.

If algebraic geometry studies geometric spaces with sheaves of commutative rings, then spectral algebraic geometry studies spaces with sheaves of rings in some category amenable to homotopy theory. If we would work in characteristic zero, then we could work with commutative differential graded algebras, but arithmetic geometry and homotopy theory both very much seek integral data, so the natural setting here is that of $\mathbb{E}_\infty$-ring spectra, or $\mathbb{E}_\infty$-rings for short. Stable homotopy theorists have long studied $\mathbb{E}_\infty$-rings as they serve as a rigid refinement of multiplicative cohomology theories – classical cohomology theories such as complex K-theory and ordinary cohomology admit canonical refinements as $\mathbb{E}_\infty$-rings.

The key insight is this. There are important objects in algebraic geometry which arise as solutions to moduli- or deformation theoretic problems. It turns out that appropriate variations of the same problems have solutions in derived algebraic geometry which lift the original object in algebraic geometry. Elliptic cohomology is one key example. We will focus on explaining this philosophy and its technical incarnation. We aim to set up the pace and the prerequisites of the talks so that it should be possible, with some effort, to follow the story without a research background in homotopy theory or algebraic geometry. Some knowledge about spectra in general and highly structured ring spectra in particular would be helpful. We highly recommend that participants try to understand the material covered on the first day of the Arbeitsgemeinschaft (Lectures 1-4) in advance and familiarize themselves with the relevant notions. Reading (at least) the introduction of the paper [Lur18b], which is our main reference, might also help with following the story during the week.

2. Some words on the approach of Lurie

At the center of the connection between geometry and homotopy theory are formal groups. Every cohomology theory with a natural theory of Chern classes has an associated formal group and, starting in about 1970, Quillen and subsequent authors realized this connection was very rigid. In particular, there is a very checkable criterion, due to Landweber, which allows us to assign a cohomology theory to a formal group law.

There are two important examples of formal group laws for which this theory applies:

(1) The Lubin-Tate formal group is defined as a universal deformation of a formal group law of a given height n over a perfect field of characteristic p. Through Landweber's theorem one can assign a cohomology theory to the Lubin-Tate formal group that is called Morava E-theory (or Lubin-Tate theory) and denoted by E_n. By construction, E_n comes as a multiplicative cohomology theory, but not as an $\mathbb{E}_\infty$-ring. The classical story uses complicated obstruction theoretic argument due to Goerss, Hopkins and Miller to verify that E_n can be uniquely refined to an $\mathbb{E}_\infty$-ring.

(2) The formal completion of an elliptic curve at the marked point is a formal group. Under certain conditions on the elliptic curve (namely being étale over the moduli stack of elliptic curves) one can use Landweber's criterion to associate a cohomology theory with this elliptic curve which we refer to as elliptic cohomology. Another obstruction theoretic argument then shows that the elliptic cohomology theory obtained in this way admits a preferred refinement to an $\mathbb{E}_\infty$-ring which is even functorial in elliptic curves.

The approach sketched above is very indirect, at times ad hoc, and it relies on difficult and mysterious obstruction theory. The idea of Lurie is to describe E_n as a universal deformation of the (ordinary) height n formal group law in the world of $\mathbb{E}_\infty$-rings. This technique is quite flexible and can be extended to more global examples where the local deformation theory is governed by formal groups or, more generally, p-divisible groups with one-parameter formal subgroup. This includes elliptic cohomology.

3. Talks

We will now give an outline of the talks. In general the speakers should feel free to contact the organizers if they find the outline too rough or need help in preparing their talk. Also some of the talks are strongly intertwined or built on each other (e.g. Talk 15 and 16), so that we recommend that the speakers coordinate.

Day 1

Lecture 1: Define cohomology theories and even periodic cohomology theories. Use complex K-theory and periodic ordinary cohomology as examples. Review Quillen's construction of a formal group from an even periodic cohomology theory, explain how it works for complex K-theory and periodic ordinary cohomology. Define the notion of formal group law (and formal group) over a commutative ring R.

Describe Landweber's theorem (without stating it precisely): given a commutative ring R and a formal group G satisfying certain conditions, there is an essentially unique even periodic cohomology theory with $E_0(\mathrm{pt}) \cong R$ and associated formal group G. Mention the multiplicative group (over $\mathbf{Z}$) as an example, obtaining a "purely algebraic" reconstruction of complex K-theory. Basic references here include [Ada74, Part II], [Rav86, Section 1.3], or [Lur07, Lectures 4-6,10,11]. For the Landweber Exact Functor Theorem, see [Lan76], [Mil17], or [Lur07, Lecture 15]. See also Chapter 4 of [Lur18b] for a modern take of Quillen's results.

Lecture 2: Review the classical theory of 1-dimensional formal groups in more detail. Define the Hasse invariants v_n and the notion of height for a formal group. Define the formal group associated to an elliptic curve, and relate v_2 to the usual Hasse invariant (vanishing on supersingular elliptic curves). Use the Hasse invariants to give a precise formulation of Landweber's criterion. Show that it is satisfied for the multiplicative group over $\mathbf{Z}$ and for versal families of elliptic curves. State the classification of formal groups over an algebraically closed field. If time permits, define the Morava stabilizer groups and describe what they look like. Beyond the references for Lecture 1, see also [Lur07, Lectures 12-14] and [Goe08, Section 5]. For the Hasse invariant of an elliptic curve and the explicit connection to formal groups, see [Sil09].

Lecture 3: Define the Lubin-Tate universal deformation of a 1-dimensional formal group over a perfect field. Describe the Lubin-Tate ring, sketching some part of the proof (maybe that the Lubin-Tate ring is generated by the v_n). Apply Landweber's theorem to construct Lubin-Tate spectra. Explain that for the multiplicative group over $\mathbf{F}_p$, this recovers the p-adic completion of complex K-theory. Observe that the Morava stabilizer group acts on a Lubin-Tate spectrum in the category of cohomology theories, and the group $\langle \pm 1 \rangle$ acts on KU in the category of cohomology theories (mention relationship to complex conjugation). Explain why it might be useful to have more: for example, a strict action of $\langle \pm 1 \rangle$ on KU would allow us to reconstruct KO as the homotopy fixed points. The original Lubin-Tate paper is [LT66], but see also [Sch68]. A discussion for homotopy theorists can be found in [Rez98, Part 1]. Interpretations can also be found in [Lur07, Lecture 16] and [Goe08, Section 7].

Lecture 4: Review the notions of spectra and $\mathbb{E}_\infty$-ring. Remark that once you have an $\mathbb{E}_\infty$-ring you have a good theory of algebras and modules over it. State the theorem of Goerss-Hopkins-Miller about Lubin-Tate spectra having a unique $\mathbb{E}_\infty$-structure, with a strict action of the Morava stabilizer group. Explain that this allows you to construct the 2-adic completion of KO. Discuss the Goerss-Hopkins-Miller results for elliptic curves. The main references for the $\mathbb{E}_\infty$-structure on the Lubin-Tate spectrum are [GH04] and [GH03], but one of the main points of these lectures is that technical mastery of those daunting papers are not necessary. For an introduction to the point of view taken here, see the introduction to Chapter 5 of [Lur18b]. For the results about elliptic curves, see [Goe10] and [DFHH14], especially Chapter 12. We give references on spectra and $\mathbb{E}_\infty$-ring spectra below, after the list of lectures.

Day 2

Lecture 5: Review the classical theory of p-divisible groups. Explain how to extract a p-divisible group from an elliptic curve (or abelian variety). Explain that p-divisible groups over p-complete rings have associated formal groups, given by the identity component. Describe the classification over a field, discuss the notion of connected-étale sequence. Mention as motivation that p-divisible groups have a better behaved deformation theory than formal groups (because the height of a p-divisible group is locally constant, but the height of the formal group is not). State the Serre-Tate theorem. Some references: [Tat67], [Mes72], [Kat81].

Lecture 6: Give an overview of the "main result" of the week ([Lur18b, Theorem 0.0.8]). First ask about Lubin-Tate theory: given a p-divisible group G_0 over a commutative $\mathbb{F}_p$-algebra R_0, when does G_0 have a universal deformation $G \to \operatorname{Spec}(R)$? Define "nonstationary" and "F-finite", state that these conditions are sufficient (and explain why they are natural assumptions). Assert that in this case, $R = \pi_0(A)$ for some (weakly) even periodic $\mathbb{E}_\infty$-ring A. Give a rough idea of the strategy: A should classify (oriented) deformations of G_0 over $\mathbb{E}_\infty$-rings. Reference: [Lur18b] (mainly the introduction to the paper and the introduction to Section 3).

Lecture 7: Recall that a formal group over a commutative ring R can be identified with its "functor of points" (a functor from commutative R-algebras to abelian groups). Explain how to generalize this to the case where R is an $\mathbb{E}_\infty$-ring, explain why it reduces to the classical notion when R is discrete. Perhaps say something about the formalism of ∞-categories (which makes this sort of idea easy to implement).[1] Reference: [Lur18b] (Section 1, particularly subsection 1.6).

Lecture 8: Give some examples of formal groups. Explain how to define the formal multiplicative group over the sphere spectrum (hence over any $\mathbb{E}_\infty$-ring). State (maybe with explanation) that the additive group cannot be defined over the sphere spectrum. Construct the Quillen formal group associated to an even periodic $\mathbb{E}_\infty$-ring R. References: [Lur18b] (Subsections 1.6 and 4.1).

Lecture 9: Define the notion of p-divisible group over an $\mathbb{E}_\infty$-ring R (again using the functor of points perspective). Compare with classical theory of p-divisible groups. Explain connection with formal groups: if R is p-complete, every p-divisible group has an identity component which is a formal group over R. Mention connected-étale sequences and results that guarantee their existence in the complete local Noetherian case. References: [Lur18a] (Section 6), [Lur18b] (Section 2).

[1] The discussion of ∞-categories will be a balancing act. As they are essential to the narrative, they can't be left as mysterious, but neither are they the core subject of the week. The ideal would be to communicate enough of the theory to see why it's relevant for this application, and how it lets you give a definition of "spectral formal group". This should be precise and concrete – assuming that we've already given ourselves the ∞-categories of ring spectra and connective $\mathbb{Z}$-modules. The basics of ∞-categories can be found in [Lur09a, Sections 1 and 2]; however, there is too much material and it will require careful selection. Another discussion, without proofs, but in the Bourbaki style can be found in [Cis16]. One core confusion, mostly of language, is what the notions of limit and colimit are in this context and how we work with them.

Day 3

Lecture 10: Digression on the (topological) cotangent complex/(topological) André-Quillen homology. Define the cotangent complex of a functor X (referencing classical algebraic geometry for motivation). Give more concrete definitions in the case where $X = \mathrm{Spec}(R)$. State/sketch proof of finiteness properties of absolute cotangent complex in the case where R is Noetherian and F-finite. Note to potential speakers: the complexity and density of the lecturers has increased by this point; careful organization and time management are both required. References: [Lur18c] (Section 17.2), [Lur18b] (Subsection 3.3).

Lecture 11: Recall Lecture 6 to motivate the following question: given a functor X (from $\mathbb{E}_\infty$-rings to spaces), when is it representable by the formal spectrum of an $\mathbb{E}_\infty$-ring A (complete with respect to an ideal I in $\pi_0(A)$)? Explain the statement of Theorem 18.2.3.2 of [Lur18c], with a sketch of proof. References: it might be helpful to explain the relationship with Brown representability in algebraic topology or Schlessinger's criterion in algebraic geometry ([Bro62] and [Sch68]), though these are not relevant to the logic. Main reference: [Lur18c], sections 17.2, 17.3, and 18.2. (Beware that the proof in 18.2 is given in a lot more generality than is needed for these applications.)[2]

Day 4

Lecture 12: Given a p-divisible group G_0 over a commutative ring R_0, say precisely what it means to have a "universal deformation" of G_0 over a ring spectrum A. (Give two definitions: one mirroring the classical Lubin-Tate presentation, the other in terms of a deformation problem.) Explain why the hypotheses of Lecture 6 guarantee that such an object exists (over a connective $\mathbb{E}_\infty$-ring R^{un}). References: [Lur18b] (Section 3, particularly the introduction and subsection 3.1).

Lecture 13: The ring R^{un} is not the one we promised in Lecture 6. Recall that in the case of an even periodic ring spectrum, there is a canonical choice of formal group (the Quillen formal group). Formulate a universal property of the Quillen formal group in terms of the notion of (pre)orientation. Sketch construction of the universal orientation of a formal group G. Define the ring spectrum R^{or}. Note: it is not at all obvious that R^{or} is nonzero. Use this to motivate the notion of a *balanced* formal group. References: [Lur18b] (Subsections 4.2, 4.3, 4.4, introduction to section 6).

Lecture 14: Return to the Lubin-Tate case. Let $R_0 = k$ be a perfect field of characteristic p and let G_0 be a 1-dimensional formal group of finite height n. Give two definitions of what it means to "be" the associated Lubin-Tate spectrum: one in terms of homotopy groups, the other in terms of universal mapping properties

[2] This is a good deal of material, but it is at this point where the derived algebraic geometry approach aligns well with classical ideas from homotopy theory and algebraic geometry. Given a deformation problem, there's a natural procedure for producing a representing object, and a natural assumptions that will guarantee that the procedure applies. The historical precedents of this result are Brown's representability theorem in topology and Schlessinger's criterion in algebraic geometry.

(in the $K(n)$-local setting). Explain why $L_{K(n)}R^{or}$ satisfies the second characterization; state that it also satisfies the first. Explain the contrast with the original proof of the Goerss-Hopkins-Miller theorem.[3]

Lecture 15: Recall Quillen's theorem relating complex bordism to the Lazard ring (using periodic bordism throughout). Then frame Construction 5.4.6 as a black box: there exists an algebra $\widetilde{A}$ over the connective cover of MP having a certain universal mapping property (articulated in Definition 5.2.1). Then proceed through the rest of 5.4, going back to section 5.2.3 for the explanation of Lemma 5.4.7.

Day 5

Lecture 16: Explain why the algebra $\widetilde{A}$ exists. This can be broken into two parts: a general existence criterion for formal thickenings (which is the contents of 5.2; and can be framed in terms of material from Lecture 11), and verifying that the criterion applies in this case (which is the contents of Lemmas 5.4.4 and 5.4.5). In the paper, the first part takes up a lot more space. But pedagogically, it makes more sense to give them equal time (with a lot of emphasis on Lemma 5.4.4).[4]

Lecture 17: Sketch the proof that the formal group over R^{un} is always balanced (so that R^{or} satisfies the promises of Lecture 6). This uses the same proof technique as Lecture 13: reduce to the case of a point by completion, then a connected-étale sequence to reduce to the Lubin-Tate case, where we can use information from Lecture 16. It also needs the result of Lecture 13 (to handle the case where the "point" is not in the vanishing locus of p). References: [Lur18b] (Section 6, particularly subsections 6.1, 6.2, and 6.4).

Lecture 18: Introduce the notion of an elliptic curve over an $\mathbb{E}_\infty$-ring R. Define the moduli stack of elliptic curves (in the classical and spectral setting), explain why the results of Lecture 11 guarantee representability in the spectral setting. Recall the classical Serre-Tate theorem and state a spectral analogue, maybe with a sketch of proof. References: [Lur18a] (Sections 1, 2, and 7).

Lecture 19: Explain some applications. Show that the formal multiplicative group over the (non-completed) sphere is balanced; deduce Snaith's theorem on the structure of KU. Show that the formal group of the universal elliptic curve over the (spectral) moduli stack of elliptic curve is balanced. Define topological modular forms. References: [Lur18b] (Subsection 6.5, Section 7).

4. Notes on spectra and $\mathbb{E}_\infty$-ring spectra.

It would be nice if there existed, somewhere in the literature, a concise expository article on what spectra are, where they came from, and how they are used. This would include the notion of an $\mathbb{E}_\infty$-ring spectrum, which are commutative rings for a good notion of tensor product in spectra. As far as we know, none such exists. However, it is not necessary to know the details of the point-set constructions in

[3]This talk is the fulcrum of the week. The speaker should be prepared to work closely with the organizers and be prepared to closely follow the introduction to Section 5 of [Lur18b].

[4]It is not necessary to explain what $K(n)$-localization is in general; the proof uses the characterization of $K(n)$-locality in the complex orientable case as completeness with respect to the ideal $(v_0, ..., v_{n-1})$, plus invertibility of v_n.

order to work with $\mathbb{E}_\infty$-ring spectra, but only necessarily to have some familiarity with the basic properties needed. These can be axiomatized. For spectra themselves, see [HHR16, Section 2.2]; this is written for spectra with a group action, but the axioms there apply for the trivial group. For structured ring spectra see [GH03, Section 1.2]. See also [Lur09b, Section 2.1] for a heuristic development. For those interested in the historical development of the subject, see [Ada74, Part III], then [BF78] for the homotopical foundations, [EKMM97] and [HSS00] for the introduction of modern point set models. At this point in time (2018) many people tend to use the orthogonal spectra of [MM02] and [MMSS01] as the standard model but for this workshop the ∞-categorical approach in [Lur17, Section 1.4] is used.

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