ARBEITSGEMEINSCHAFT: ZIMMER'S CONJECTURE

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1. Introduction

Zimmer's program can be described as the study and classification of homomorphisms $\rho : \Gamma \to \mathrm{Diff}(M)$ where Γ is a lattice in a higher-rank Lie group (e.g. $\Gamma = \mathrm{SL}(n, \mathbb{Z}), n \geq 3$), M is a compact manifold, and $\mathrm{Diff}(M)$ is the group of diffeomorphisms of a manifold. Recent work of Brown, Fisher and Hurtado combines ideas from many areas to make dramatic progress on a specific conjecture of Zimmer's concerning the triviality of ρ when the dimension of M is small relative to the "size" of Γ; see [BFH16, BFH17]. The main sources of techniques and ideas are:

(1) classical rigidity theorems;
(2) smooth dynamics, particularly Lyapunov exponents and non-uniformly hyperbolic dynamics;
(3) homogeneous dynamics, particularly the study and rigidity of invariant measures;
(4) operator algebras, particularly Lafforgue's strong property (T).

The purpose of the program is to introduce and explain key ingredients from each of the four above areas and to understand how they are combined to prove of Zimmer's conjecture in [BFH16]. We hope the program will be of interest for geometers, dynamicists, and mathematicians from other areas and will lead to further interactions between both ideas and participants from different fields.

"Arbeitsgemeinschaft" means "study group". All lectures at the Arbeitsgemeinschaft are given by participants and the goal is communal learning by active participation. All applicants are required to volunteer to give lectures, though there will be more participants than lectures. Applications for participation should be addressed to the director by email to ag@mfo.de prior to the deadline of 31 May, 2019. Applications should include your full name and postal address, a CV and a publication list.

You should also indicate which talk you are willing to give:

First choice: talk no. : : :
Second choice: talk no. : : :
Third choice: talk no. : : :

You will be informed as soon as possible after the deadline if your participation is possible and whether you have been chosen to give one of the lectures.

Most of the lectures will be one hour long. Below, we provide outlines including what each lecture should attempt to describe and recommended references to follow. However, if the speaker finds a better way to explain the concepts, she/he is more than welcome to do it in her/his own way.

2. Background

We expect all participants to have some understanding of basic concepts from ergodic theory, dynamical systems, and Lie groups.

3. Tentative titles and schedule of Lectures

Lie theory and actions.

(1) Symmetric spaces and Lattices.
(2) Structure theory of Lie groups.
(3) Examples of actions of Lie groups and Lattices.
(4) Suspension space.

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Lyapunov exponents and smooth ergodic theory.

(5) The top Lyapunov exponent and its properties; subadditive ergodic theorem.
(6) Oseledec's theorem, Pesin theory, and entropy
(7) Ledrappier–Young
(8) Smooth ergodic theory in higher-rank actions
(9) Invariance principles

Superrigidity.

(10) Margulis superrigidity
(11) Cocycle superrigidity
(12) Towards proofs: ergodicity and Lyapunov exponents
(13) Towards proofs: centralizers and finite dimensional invariant subspaces

Properties (T).

(14) Property (T)
(15) Property (T) groups acting on the circle
(16) Strong Property (T) in the proof of Zimmer's conjecture
(17) Strong Property (T), idea of proof.

Unipotent dynamics.

(18) Ratner's measure classification theorem and equidistribution
(19) Ratner's orbit closure theorem and generalized equidistribution.
(20) $ax + b$ invariant measures are $\mathrm{SL}(2, \mathbb{R})$ invariant.

Proof of [BFH16].

(21) Combine averaging arguments to show that if an action $\alpha\colon \Gamma \to \mathrm{Diff}(M)$ fails to have uniform subexponential growth of derivatives then the induced action as a measure projecting to Haar with non-zero fiber exponent.

4. Description of Lectures: Lie theory and actions

Lecture 1. Symmetric spaces and lattices (1 hour).

Discussion.

- State definition of symmetric space (Def. 1.1.5. in [WM15] and describe example 1.1.3)
- Explain construction of symmetric spaces (Proposition 1.2.2. in [WM15]) and give examples.
- State definition of lattice (Def. 1.3.5.) and give examples. State Theorem 1.3.9. in [WM15].
- (Time allowing) Provide examples of lattices using restrictions of scalars (Examples 5.5.2, 5.5.3, 5.5.4 in [WM15])
- (Time allowing) Mention basic facts about lattices, Like Godement Compactness Criterion.
- Give geometric definition of $\mathbb{R}$-rank. Theorem 2.1.3. in [WM15].

Additional comments:

- It might be worth mentioning that different embeddings of a Lie group G into $\mathrm{SL}_n(\mathbb{R})$, give rise to different lattices (Theorem 1.3.9 [WM15]).
- One can also mention Margulis' Arithmeticity Theorem. (Without formally stating it).

Lecture 2. Structure Theory of Lie groups (1 hour).

Discussion.

- Main reference is [BQ16], Chapter 6.7 and the references therein. See also [Kna02] for the definition of parabolic subgroups and [WM15] Chapter 8.
- Discuss the definitions of algebraic groups, maximal compact subgroup, cartan subspaces and restricted roots as in [BQ16] Sec 6.7.1, 6.7.2.

- Discuss KAK decomposition as in [BQ16] Sec 6.7.4 and give geometrical interpretation.
- Discuss minimal parabolic subgroups and Iwasawa Decomposition. See beggining of [BQ16] Sec 6.7.5 or Theorem 8.4.9 in [WM15].
- Example: $\mathrm{SL}_n(\mathbb{R})$. Sec. 6.7.7. of [BQ16].
- State Proposition 8.2.5 of [WM15] and explain briefly.
- (Time allowing) Example: $\mathrm{SO}(p,q)$ Sec.6.7.8. of [BQ16].
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Lecture 3. Examples of actions of Lie groups and Lattices (1 hour).

Discussion.

- The discussion can be based on Lecture 1, [WZ08]. See also sections 2.2 and 2.3 in [Fis11].
- Discussion of algebraic actions, [WZ08] Example 1.7. Also mention lattice actions on tori and if time permits nilmanifolds.
- (Time allowing) Explain construction of isometric actions via restriction of scalars, see [WM15], Chapter 5.
- Actions via Induction, [WZ08] Ch.1.B, explain principle.
- Actions via blow up, [WZ08] Ch.1.B.
- (Time allowing) Discuss more in detail the Katok-Lewis [KL96], Benveniste examples [BF05]. See also [Fis11, §2.3]

Lecture 4. Suspension space (1 hour).

Discussion.

- Define the suspension space corresponding to an action by a lattice (as defined for example in Section 2, [BRHW16]). See [Bro18, 10.1].
- Explain correspondence between Γ-invariant measures on M and G-invariant measures on suspension space.
- Explain how root spaces of A corresponds to Lyapunov exponents for the abelian action of A in G. See [Bro18, 9.1].
- Define subexponential growth of derivatives for the Γ action on M and the G action on the suspension space and explain correspondence in the case Γ is cocompact. See 3.3 in [BFH16].
- (Time allowing) State Lubotzky-Mozes-Ragunathan Theorem [LMR93, LMR00] and explain correspondance in the case where Γ is not cocompact.
  and [BFH17]. This might need to just be sketch.

5. Description of Lectures: Smooth ergodic theory

Lecture 5. The top Lyapunov exponent (1 hour).

Discussion.

(1) Define the top Lyapunov exponent for C^1 measure preserving diffeomorphisms and general cocycles. Formulate Kingman subadditive ergodic theorem. See [Via14, Chapter 3].
(2) State and prove relationship between top Lyapunov exponent and subexponential growth for $\mathbb{Z}$-actions. ([Bro18, Proposition 6.3.])
(3) Semi-continuity (in the weak-$*$ topology of invariant measure and C^0-topology on cocycles) of top Lyapunov exponent, [Via14, Lemma 9.1].
(4) Properties of invariant measures and top Lyapunov exponent under averaging by commuting diffeomorphism and actions of more general amenable groups commuting with the dynamics. [Bro18, Claim 13.1, Remark 13.2.]

Lecture 6. Oseledec's theorem, Pesin manifolds, metric entropy.

Discussion.

(1) Formulate Oseledec's theorem for a diffeomorphism preserving a probability measure. Possibly sketch proof.
(2) Define (un)stable subspaces and global (un)stable Pesin manifolds. Sketch construction in uniformly hyperbolic case via graph transform.
(3) Define filtration of stable manifolds by fast stable manifolds ([BP06, §9.2, Thm. 9.3]; note the inequalities should be non-strict); Lipschitz property of smoothness of fast stables inside stables. Follow either [BP06, §9.3] and [BPS99, appendix] (in $C^{1+\beta}$ setting) or [LY85b, Lemma 8.2.5] and [LY85a, Lemma 4.2.1] (in C^2 setting)
(4) Define metric entropy via partitions subordinate to unstable manifolds. See [LY85a, Lemmas 3.1.1, 3.1.2]. State [LY85a, Cor 5.3]: entropy of a diffeomorphism in terms of partitions subordinate to unstable manifolds
(5) Formulate Margulis-Ruelle inequality

For much of the above, you may also follow [Bro18, §6.1, §6.4, §7.3]

Lecture 7. Ledrappier–Young (1 hour).

Discussion. We follow [Bro18, §6.1, §6.4] or [LY85a, LY85b]

(1) State the three main results: [LY85a, Cor 5.3], [LY85a, Theorem A] (see also [Led84, Theorem 3.4]), [LY85b, Theorem C'].
(2) For algebraic systems, formulate [LY85a, Theorem A] as a rigidity statement. See [Bro18, Theorem 8.5]
(3) Prove [LY85a, Theorem A] (or [Led84, Theorem 3.4]) in the following simple setting:
    Let $A \in \mathrm{SL}(n, \mathbb{R})$ be a hyperbolic matrix with 1 eigenvalue χ^+ outside the unit circle and $(n-1)$ eigenvalues inside the unit circle. Let $f = f_A : \mathbb{T}^n \to \mathbb{T}^n$ be the induced Anosov diffeomorphism. Let μ be an ergodic, f_A-invariant probability measure on $\mathbb{T}^n$. Write $\lambda^u = \log|\chi^+|$. Then
$$h_\mu(f_A) \leq \lambda^u.$$
    Let ξ be a measurable partition subordinate to the partition of $\mathbb{T}^n$ into unstable manifolds (for both the measure μ and the ambient haar measure m). Let $\{\mu_x^\xi\}$ and $\{m_x^\xi\}$ denote, respectively, families of conditional measures relative to ξ for μ and the Haar measure m.
    Show (a) $\Rightarrow$ (d) in the following proposition. Follow [Bro18, pf of Theorem 8.5]

    Proposition 5.1. *Let μ be an ergodic, f_A-invariant probability measure on $\mathbb{T}^n$. The following are equivalent:*
    (a) $h_\mu(f) = \lambda$;
    (b) *for a.e.* $x \in \mathbb{T}^n$, $\mu_x^\xi \ll m_x^\xi$.
    (c) *for a.e.* $x \in \mathbb{T}^n$, μ_x^ξ *is equivalent to* m_x^ξ.
    (d) *for a.e.* $x \in \mathbb{T}^n$, $\mu_x^\xi = m_x^\xi$.
    (e) μ *is invariant under translation by vectors in* E^u.
    (f) $\mu = m$.

Lecture 8. Higher-rank dynamics (1 hour).

Discussion.

(1) Describe examples of higher-rank abelian actions $\alpha: \mathbb{Z}^k \to \mathrm{Diff}^{1+\alpha}(M)$ via via commuting matrices in $\mathrm{SL}(n, \mathbb{Z})$. (See Dirichlet's unit theorem to obtain large centralizers, c.f. [KKS02, Proposition 3.7]).
(2) State higher-rank Oseledec's Theorem
(3) Define coarse Lyapunov exponents and coarse Lyapunov manifolds
(4) State entropy product structure: [Bro18, Theorem 7.8]
(5) Give an idea of proof of entropy product structure using Ledrappier–Young. Sketch the proof in case there are 3 Lyapunov exponents $\lambda_1, \lambda_2, \lambda_3$, none are positively proportional, and $a \in \mathbb{Z}^2$ such that $\lambda_1(a) > 0$, $\lambda_2(a) > 0$, and $\lambda_3(a) < 0$.
(6) If time: compare to product structure of measures obtained in [EK03, Lemma 8.1].

Lecture 9. Invariance principles.

Discussion.

(1) Formulate Ledrappier/Avila-Viana invariance principle for Linear cocycles and diffeomorphism-valued cocycles over measurable base dynamics with decreasing σ-algebra. See [Led86, AV10].
(2) Given a diagonalizable $a \in G$, interpret the a action on the suspension space M^α in terms of the Avila-Viana framework. See for instance [BDZ18]
(3) Discussion proof of Ledrappier/Avila-Viana invariance principle using Ledrappier-Young in the setting of the G-action on M^α. See the proof of [Bro18, Theorem 11.1].
(4) Formulate the higher-rank invariance principle, [Bro18, Proposition 11.5].
(5) Discussion proof of higher-rank invariance principle using entropy product structure [Bro18, §11.3].

6. Description of Lectures: Superrigidity

Lecture 10. Superrigidity (1 hour).

Discussion:

(1) References for this lecture are [Zim84, Chapter 6] and [WM15, Chapter 16]
(2) Let $\Gamma < G$ be a lattice. Given definition of a homomorphism of a lattice Γ almost extending to homomorphism of G. State superrigidity in these terms. Include p-adic targets in statement.
(3) Show that superrigidity implies that $H^1(\Gamma, V) = 0$ for all finite dimensional Γ modules V.
(4) State result of Selberg-Weil that $H^1(\Gamma, V) = 0$ implies Γ conjugate into $G(k)$ for k a number field.
(5) Assuming $k = \mathbb{Q}$ show why this implies that Γ is arithmetic.
(6) State that any infinite image representation of Γ with compact image is Galois conjugate to one that extends to G. State that this implies dimension bounds on dimension on isometric actions. If time permits sketch the proof.

Lecture 11. Superrigidity for cocycles (1 hour).

Discussion:

(1) Define cocycles over group actions and when two are cohomologous [Zim84].
(2) State superrigidity for cocycles in general form from [FM03].
(3) Show that is G is acting on a vector bundle E over a manifold M preserving a measure μ, then Lyapunov exponents for elements of G are determined by the superrigidity representation. ([Fur81]or [Zim84]).
(4) Conclude that if the dimension of the vector space is small enough, Lyapunov exponents vanish.

Lecture 12. Towards proofs: ergodicity and Lyapunov exponents (1 hour).

Discussion.

(1) State the Howe-Moore theorem [Zim84].
(2) If time permits, sketch a proof that elementary subgroups and diagonal subgroups in $\mathrm{SL}(2, \mathbb{R})$ or $\mathrm{SL}(n, \mathbb{R}$ are ergodic via the Mautner phenomenon. ([Mar91] or many other sources.)
(3) State the following which is Thm. V.5.15 from [Mar91]:

Theorem 6.1. *Let G be a higher rank Lie group acting on a measure space (X, μ) ergodically and acting on a vector bundle V over X via a cocycle $\alpha : G \times X \to SL(V)$. Assume that the cocycle is not cohomologous to one taking compact values. Then there is an element $a \in A$ such that $\lambda_1(a, \mu) > 0$.*

(4) Deduce that this gives a A invariant section of the $End(V)$ bundle over X given by projection on the top Lyapunov subspace. (Note that α taking values in $\mathrm{SL}(V)$ implies that the top Lyapunov subspace is not all of V.)

Lecture 13. Towards proofs: centralizers and finite dimensional invariant subspaces (60-75 minutes).

Remark: This lecture requires some coordination with the previous lecture.
Discussion.

(1) From the last lecture, we have an A invariant section s of $X \times End(V)$. Let $W = End(V)$ and $F(X, W)$ be the space of measurable maps from X to W, modulo the relation of being equal on null set for μ. The goal of the lecture is to use s to produce a finite dimensional subspace of $F(X, W)$ that is G invariant and so a finite dimensional G representation.
(2) To do this, we prove a Lemma that says:

Lemma 6.2. *Suppose a group D acts on a W vector bundle over (X, μ). Let $H_1 < G$ be subgroup that acts ergodically on (X, μ) and assume there is a finite dimensional subspace B of $F(X, W)$ that is H_1 invariant and let $H_2 < Z_D(H_1)$. The set $H_2 \times B$ spans a finite dimensional H_1H_2 invariant subspace of $F(X, W)$.*

The lemma is proven in [Ben, Mar91, WM15]. The account in [Mar91] is most detailed, but a bit over written. It is probably best to do the sequence of exercises in [WM15], looking at [Mar91] for hints if you get stuck. The account in [Ben] is quite clean but in French.
(3) Use that $G = H_1H_2 \dots H_n$ with $H_i < Z_G(H_{i+1})$ to produce a finite dimensional subspace $\tilde{W}$ of $F(X, W)$ that is G invariant. This uses the Howe-Moore theorem from the last lecture to see all H_i are ergodic.
(4) If time permits, you can show that if there is no G invariant sub-bundle of $X \times W$ this implies cocycle superrigidity. Or that if the action comes from $(G \times W)/\Gamma$ and an irreducible Γ action on W, then this G representation on $\tilde{W} < F(X, W)$ extends the Γ representation on W. Both involve checking that the evaluation map from $\tilde{W}$ to W is an isomorphism.

7. Description of Lectures: Unipotent Dynamics

Lecture 14. Ratner's measure classifcation theorem and equidistribution (1 hour).

Discussion:

(1) State Ratner's theorem on invariant measures for actions of one parameter unipotent groups on G/Γ from [Rat94].
(2) State Theorem 1 of [DM92] on unipotent orbits and the singular set.
(3) State [DM92] Theorem 2 on equidistribution but fixing a one parameter subgroup $u_t = u_t^(i)$ and a point $x = x_i$ in the good set.
(4) Sketch the the proof of Theorem 2 from Theorem 1 assuming Γ is cocompact. This removes several steps from the proof.

Lecture 15. Ratner's orbit closure theorem and generalized equidistribution. (1 hour).

Discussion.

(1) State Ratner's theorem on orbit closures for the action of one parameter unipotent group U on G/Γ from [Rat94]. Indicate the first step of the proof: that any orbit not contained in a closed orbit of a closed subgroup containing U equidistributes and so has dense orbit. This uses the version of [DM92, Theorem 2] from the last lecture.
(2) State result on equidistribution inside orbit closures that is Theorem 5.1 (c) in [BFH16]
(3) Show how to deduce the case for one parameter subgroups using the theorem of Ratner on orbit closures and the equidistribution result from the last lecture. This is basically a proof by induction on dimension of the orbit closure.
(4) Define $X(H, W)$ as in [DM92].Explain:
    - Connection to closed orbits.
    - Connection to normalizer of W
    - State the fact that there are only countable many subgroups H that can occur in $X(H, W)$. This is the set $\mathcal{H}$ in [DM92].

Lecture 16. $ax+b$ invariant measures are $\mathrm{SL}(2,\mathbb{R})$ invariant (90 minutes).

Discussion.

(1) Sketch a proof of Theorem 1 in [Moz95] assuming: $L = \mathrm{SL}(2,\mathbb{R})$ and H the upper triangular group in $\mathrm{SL}(2,\mathbb{R})$.
(2) Note: the three hypotheses are all easy to check. Checking the first one is elementary given the classification of finite dimensional SL(2) representations. The other two are obvious. Do not spend time on the classification of SL(2) representations.
(3) Do not prove any of the results used on the structure of $X(H,W)$ (called a tube here). All are proven in [DM92]. The results of Bien and Borel that prove results about H invariant vectors in L representations are more or less trivial in this case as remarked above.

8. Description of Lectures: Properties (T)

Lecture 17. Property (T) (1 hour).

Discussion:

- State well known definition of Property (T), section 1.1 [BdlHV08]
- Include discussion of which Lie groups (and lattices) have Property (T), section 1.6 and 1.7 [BdlHV08]
- Mention that Property (T) implies finite generation and as a consequence many lattices are finitely generated. Section 1.3 [BdlHV08]
- Statement of Property FH, section 2.1 and 2.2 [BdlHV08]
- State that Property FH is equivalent to Property (T) for σ-compact locally compact groups (Delorme-Guichardet Theorem 2.12.4 [BdlHV08]).
- Groups acting on trees. State and proof of Theorem. Sec.2.3 [BdlHV08].

Lecture 18. Property (T) groups acting on the circle (1 hour).

Discussion: Follow [Nav11, Sec.5.2], [BdlHV08, Section 2.9]

- Statement of [Nav11, Theorem 5.2.14]
- Definition of Property T [Nav11, Def. 5.2.1]
- Sketch of Proof [Nav11, Sec. 5.2.3]

Additional comments:

- A discussion in the C^2 category might be sufficient.
- The fundamental cocycle is related to Hilbert's transform on $L_2(\mathbb{S}^1)$, and this might be worth pointing out, see [Rez01, Sec.2.1]
- Including a complete proof of Lemma 5.2.16 might not be illuminating.
- It is worth pointing out there is not known whether there exists and infinite group of $Homeo(\mathbb{R})$ with property (T).

Lecture 19. Strong Property (T) in the proof of Zimmer's conjecture (1 hour).

Discussion: Following [del16, sec. 1.1-1.3,],[BFH16, Section 6]

- State definition of Strong property (T) (Def. 0.1. in Lafforgue's article.)
- Explain the relation between the definition of Property T and Strong Property T.
- Compare definition with the one given in Sec.6, [BFH16]. Explain relation.
- Introduce the space of Sobolev metrics.
- Explain Lemma 6.7. from [BFH16] in the case $M = \mathbb{R}$ and explain briefly why is true in higher dimensions.
- State that action in the space of Sobolev metrics have subexponential growth.
- Show Strong Property (T) gives a Sobolev inner product (a priori degenerate), which by Soboloev embedding Theorem is a smooth metric.

- (time allowing) State Lafforgue's Theorem 1.4 and state that no hyperbolic group has Strong property T.

Additional comments:

- The case $M = \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ might explain clearly what is happening in general.

See also two videos from Mikael de la Salle video talks about Strong Property T in workshop on Zimmer's program: http://www.ipam.ucla.edu/programs/workshops/new-methods-for-zimmers-conjecture/?tab=schedule)

Lecture 20. Strong Property (T): Idea of proof (90 minutes).

Discussion:

- The talk should give an outline of the proof of property (T) for $\mathrm{SL}_3(\mathbb{R})$ using Lafforgue's proof of strong property (T), following DeLaSalle Habilitation (Sec. 1-3) [del16].
- State Theorem 1.1. in Page 12 [del16], and explain how implies property (T).
- State Lemma 2.2. [Laf08] or Proposition 1.2. [del16], explain its geometric interpretation and why is not true for $\mathbb{S}^1$ (fails for harmonics) and mention that the proof is done by reducing to the harmonic subspaces and explicit computation. See also [del16] Sec.1.3.
- Sketch the proof of Proposition 2.3. ([Laf08], pages 573-576) or Theorem 1.1 [del16].

Some comments:

- The proof fails for $\mathrm{SL}_2(\mathbb{R})$ because Lemma 2.2. is false for $\mathbb{S}^1$.
- The proof that lattice subgroups have strong property (T) follows from inducing an action of G in the usual way, but when the lattice is non-uniform, it is much more involved and is a recent theorem of De La Salle.
- It might be worth mentioning that the proof of Lafforgue's Theorem 1.4. is similar to the proof that groups with property (T) do not act on trees.
- The only known groups to have strong property (T) come from Lie groups and their lattices. Lots of other groups have property (T).

See also two videos from Mikael de la Salle video talks about Strong Property (T) in workshop on Zimmer's program: http://www.ipam.ucla.edu/programs/workshops/new-methods-for-zimmers-conjecture/?tab=schedule)

9. Description of Lectures: Proof of theorem

Lecture 21. Proof of Zimmer's conjecture for cocompact Γ in $\mathrm{SL}(n,\mathbb{R})$ (60-75 min).

Outline

We aim to show the following:

Theorem 9.1. *For $n \geq 3$, let $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ be a cocompact lattice. Let M be a compact manifold. If $\dim(M) < n-1$ then any homomorphism $\Gamma \to \mathrm{Diff}^2(M)$ has finite image.*

(1) Recall reductions from previous lectures. Let $G = \mathrm{SL}(n,\mathbb{R})$, let $\Gamma \subset G$ a cocompact lattice. Let M^α denote the suspension space with the induced G-action. We have the following

Claim 9.2. *The action $\alpha \colon \Gamma \to \mathrm{Diff}^2(M)$ has uniform subexponential growth of derivatives if and only if the G-action on M^α has uniform subexponential growth of fiberwise derivatives: for every $\epsilon > 0$ there is C such that for any $x \in M^\alpha$,*

$$\|D_x g|_F\| \leq C e^{\epsilon d(g,e)}.$$

(2) By discussions in Lectures 9 and 11, it remains to show the following:

Theorem 9.3. *If the G-action on M^α fails to have uniform subexponential growth of fiberwise derivatives then there exists an A-invariant measure μ on M^α projecting to the Haar measure on G/Γ with a nonzero fiberwise Lyapunov exponent.*

(3) State [Bro18, Proposition 12.1, 12.2].

(4) Give proof of [Bro18, Proposition 12.2] when $G = \mathrm{SL}(3, \mathbb{R})$ following [Bro18, §13.4].

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