DERIVED GALOIS DEFORMATION RINGS AND COHOMOLOGY OF ARITHMETIC GROUPS

ORGANIZERS: FRANK CALEGARI, SOREN GALATIUS, AKSHAY VENKATESH

In the 1980s, Mazur introduced [16, 17] the deformation ring of Galois representations in order to shed light on how Galois representations deform in p-adic families. Mazur was inspired, in part, by the construction of Hida [12] of surjective Galois representations $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\mathbf{Z}_p[\![x]\!])$ associated to ordinary families of modular forms. Mazur's formalism also allowed for the study of deformation rings R which capture (conjecturally) all Galois representations which come from automorphic forms of a fixed level and weight.

These works were primarily concerned with representations of the Galois group of $\mathbf{Q}$ into GL_2. More recent work ([7, 10]) has suggested that (in the general case) one should study instead a derived version of the ring R, and that this derived R acts on the homology of arithmetic groups (shifting homological degree).

The purpose of the workshop is to explain this story. The action of R remains conjectural, but there is very strong evidence for it based on the Taylor–Wiles method and known properties of automorphic forms.

We will split into four series of lectures, discussed below.

Overall background: A reasonable background would be to know the theory of holomorphic forms and something about the Galois representations associated to them e.g. at the level of [9]. For each of the lecture courses, we give below some additional background that might help you get the most out of the lectures.

- (§A): Cohomology of arithmetic groups, and how to compute them using "Hodge theory":

  The whole picture depends in an essential way on a numerical coincidence between a dimension computation that occurs in $(\mathfrak{g}, K)$ cohomology and a dimension computation in Galois cohomology. The goal of §A is to outline the dimension computation in $(\mathfrak{g}, K)$ cohomology and how it manifests itself in the cohomology of arithmetic groups.

  Useful background might be: learn the definition of a $(\mathfrak{g}, K)$-module and some examples.
- (§B): The Taylor–Wiles method (with the case of GL(1) serving as a motivating example).

  The Taylor–Wiles method is a way of analyzing the structure of the deformation space of a Galois representation by slowly enlarging the deformation space — allowing extra ramification — until it becomes formally smooth. In this process, one needs a way to bound the size of the deformation space from below, and this typically comes because one knows there are plenty of Galois representations coming from modular forms. We will introduce this method in a simple (abelian) case in §B.

Useful background: duality and Euler characteristic formulas in Galois cohomology of number fields.

- (§C): The definition of derived deformation rings.

  The material of §B will already require the definition of the "usual" (i.e. underived) deformation space of a Galois representation. However, this can be enriched in a natural way to a derived space — that is to say, the ring of functions becomes a simplicial ring rather than a usual ring. To describe the usual deformation space we must describe its A-points for A a test ring; to describe the derived deformation space we must describe A-points for A a *simplicial* test ring.

  Many constructions with derived rings are not canonical up to unique isomorphism in the strict sense, but rather "up to contractible choice". We shall roughly follow the approach of [10] which tries to make such choices explicit.

  Useful background: some prior exposure to simplicial sets, and the definition of simplicial objects in a category.

- (§D): Taylor-Wiles method in general.

  We will apply the Taylor–Wiles method, as explained in §B, to the cohomology of an arithmetic group in the higher rank case (focussing on GL_n). This will eventually relate the derived deformation ring to the cohomology in a direct way (and in fact this is the only way of accessing the derived deformation ring that we currently know about).

  This brings together the three previous parts.

1. Cohomology of arithmetic groups (4 lectures)

The standard reference for this material is [4]. Unfortunately it is not easy to read; studying it involves a lot of background in the representation theory of reductive groups, and it is easy to get lost and never be found again. So please always do the following:

- **Restrict to the case of the trivial coefficient system**.
- **Illustrate all results with the examples of SL_2 or GL_n**

1.1. **Lecture A.1.** Introduction considering products of hyperbolic spaces ([22, §3.2, 3.3]): Explain how the Hodge $*$ operator acting on individual factors leads to a decomposition of cohomology. Our goal is to understand this decomposition in general, which comes from $(\mathfrak{g}, K)$ cohomology.

Start of general case: define $(\mathfrak{g}, K)$-cohomology and its relationship to cohomology of spaces G/K ([4, I.1]). Matsushima's formula in the cocompact case with trivial coefficients [4, VII, Theorem 3.2].

For context, define $(\mathfrak{g}, K)$-modules and mention that these "capture" irreducible unitary representation (e.g. irreducible unitary is determined by the underlying $(\mathfrak{g}, K)$ module as in [14, Corollary 9.2]).

Finally, hint where we are going — our eventual goal is [4, III, Theorem 5.1] which classifies the cohomology of tempered representations. For now explain the definition of the invariant ℓ_0, note that the answer is an exterior algebra $\wedge^* \mathbf{C}^{\ell_0}$, and explain what ℓ_0 was in the example with hyperbolic spaces.

1.2. **Lecture A.2.** Prove that the $(\mathfrak{g}, K)$ cohomology vanishes unless infinitesimal character is trivial ([4, I, Cor. 4.2]) and state without proof Kuga's lemma ([4, II, Proposition 3.1] and surrounding discussion) which asserts that all the differentials in the complex computing $(\mathfrak{g}, K)$ cohomology vanish, in the cases of interest.

Worked example:

(1) Describe the $(\mathfrak{g}, K)$-modules underlying irreducible representations for $\mathrm{SL}_2(\mathbf{R})$. A picture with raising and lowering operators would be good, as in Proposition 8.7 of `https://www.math.ubc.ca/~cass/research/pdf/Irr.pdf` and above.
(2) Work out the $(\mathfrak{g}, K)$-cohomology for representations of SL_2. There are only three representations with nontrivial $(\mathfrak{g}, K)$ cohomology.
(3) Finally, use this with Matsushima's formula to recover the relationship between cohomology of $\Gamma\backslash\mathbf{H}$ (for $\Gamma \leqslant \mathrm{SL}_2(\mathbf{R})$ a cocompact lattice) and weight 2 modular forms.

1.3. **Lecture A.3.** This lecture covers representation theoretic background. (A good reference is Knapp's book [14]; however we are only covering basic things.)

Describe parabolic induction of representations, and explain how the $(\mathfrak{g}, K)$-modules from §A.2 occur in parabolic induction. (For the example of $\mathrm{SL}_2(\mathbf{R})$ see [14, Chapter 2,7]) – it is a good exercise to work out the $(\mathfrak{g}, K)$-module structure of these examples.)

Explain the meaning of "tempered" and "discrete series" and explain, with reference to the *Worked example* from §A.2, what are the tempered and discrete series representation of $\mathrm{SL}_2(\mathbf{R})$. Note that 'most" representations occurring in $\Gamma\backslash G$ are tempered (see e.g. [19] for a precise statement), which partly justifies their special significance.

If time allows, state the Langlands classification for a general group G [14, Theorem 8.54].

1.4. **Lecture A.4.** Cohomology of tempered representations and the integer ℓ_0:

State [4, Theorem 5.1] and explain carefully all the terms appearing in the special case of F_λ trivial and $G = \mathrm{SL}_n(\mathbf{R})$. Outline the proof, with particular emphasis on the appearance of the exterior algebra.

To conclude, if there is time, *state without proof* the Vogan–Zuckerman classification of cohomological representations ([23, Theorem 2.5]).

2. Deformation rings and the Taylor–Wiles method for GL_1

2.1. **Lecture B.1.** Mazur's universal deformation ring R and its basic properties, in particular relation of tangent space and obstruction space to Galois cohomology. Mazur's article [17] surveys various proofs (see §10); the proof via Schlessinger criterion is Mazur's original approach in [16] and is worthwhile because the same proof will be repeated in the derived setting. (Other notes on Galois representations include [3]).

Explain also the local analogue of the deformation ring and the global deformation ring subject to prescribed local conditions.

If there is time, start on the Review at start of B.2.

2.2. **Lecture B.2.** Review of Galois cohomology of local fields and number fields: state local and global duality theorems and Euler characteristic formula.

Introduce the adelic quotient associated to GL_n and define the Hecke ring $\mathbf{T}$ associated to it. We will use this setup and notation often so do it carefully. Briefly state Scholze's result about Galois representations ([20, V.2.7, V.4.4], which also defines the adelic quotient and Hecke ring), mentioning also [1, §4,§5]. In particular explain that one expects to have a map $R \to \mathbf{T}$ after localizing at suitable maximal ideals.

Now is a good time to discuss the crucial numerical equality:

> *(*) The expected dimension of the Galois deformation ring that controls automorphic forms for a group G is equal to $-\ell_0$.*

When $\ell_0 > 0$ this corresponds to derived behavior. You can verify this at least in the case of GL_1 over an arbitrary number field F.

Here is more background discussion: If R^{univ} denotes the universal global deformation ring, and R^{loc} denotes the universal local deformation ring, and R^{cris} the closed subscheme of R^{loc} cut out by local conditions (conjecturally) associated to automorphic forms of a fixed weight and level, then $R = R^{\mathrm{univ}} \otimes_{\mathbf{R}^{\mathrm{loc}}} R^{\mathrm{cris}}$, and so (suitably interpreted in terms of rigid analytic spaces)

$$\mathrm{exp.dim}(R[1/\varpi]) \stackrel{?}{=} \mathrm{exp.dim}(R^{\mathrm{cris}}[1/\varpi]) + \mathrm{exp.dim}(R^{\mathrm{univ}}[1/\varpi]) - \mathrm{exp.dim}(R^{\mathrm{loc}}[1/\varpi]) = -l_0,$$

and so only in the $l_0 = 0$ situation should we expect that this intersection is transverse. In the case $\ell_0 > 0$ the lack of transversality should be interpreted as a derived structure for R. As noted above you can check this explicitly for $\mathrm{GL}(1)/F$.[1]

[1]If $p = \prod \mathfrak{p}^{e_i}$ with $\sum_{i=1}^r e_i f_i = d = [F:\mathbf{Q}] = r_1 + 2r_2$, then $\mathrm{exp.dim}(R^{\mathrm{univ}}[1/\varpi]) = r_2 + 1$, $\mathrm{exp.dim}(R^{\mathrm{loc}}[1/\varpi]) = \sum_{i=1}^r (1 + e_i f_i)$, $\mathrm{exp.dim}(R^{\mathrm{cris}}[1/\varpi]) = \sum_{i=1}^r 1$, and so $\mathrm{exp.dim}(R[1/\varpi]) = 1 + r_2 - d = -(r_1 + r_2 - 1) = -l_0$.

(For some vague hints on how to compute $\exp.\dim(R^{\mathrm{cris}}[1/\varpi])$ in general in the ordinary case, see [6, §2.8].)

2.3. **Lecture B.3.** Taylor–Wiles for GL_1 (minimal case, $\ell_0 = 0$), following [6] and [7, §8.2]: Specialize to the case of GL_1; in particular explain how the definition of the Hecke ring T simplifies. Explain the definition of a set of Taylor–Wiles primes and how this allows one to increase the ramification without increasing the size of the tangent space of R. In particular, for Q a set of Taylor-Wiles primes, one gets surjections $R_Q \to T_{\mathbf{Q}}$ where R_Q has the same tangent space as $R_\emptyset$ and T_Q acts on a module M_Q which is free over $\mathbf{Z}_p[\Delta_Q]$ with co-invariants $M_\emptyset$ (on which $T_\emptyset$ acts faithfully). Explain the Taylor–Wiles limit process involving a sequence of such Q and deduce that $R \to T$ is an isomorphism, in particular that the largest unramified abelian p-extension of F (for $F = \mathbf{Q}$ or F an imaginary quadratic field, and $p \nmid w_F$) comes from the class group, although the reflection formulas have class field theory built into them so this is somewhat circular.

2.4. **Lecture B.4.** Taylor–Wiles method for GL_1 in the general case ($\ell_0 > 0$) following [6] and [7, §8.2]. Explain the modifications required for general F, namely: to address the following two problems:

(1) The existence of units implies that M_Q is not in general free over $\mathbf{Z}_p[\Delta_Q]$, and the patched module is not free over S_∞. Find a way of capturing the "size" by working instead with a suitable complex (which for GL_1 can be transparently constructed).
(2) The Selmer group and dual Selmer group now differ in dimension — the difference in dimension l_0 mirrors the lengths of the complexes considered above.

Show how these two issues are resolved via [6, Desiderata 2.7], and how this leads to both a minimal $R = \mathbf{T}$ theorem and an identification of the cohomology groups with $\mathrm{Tor}_{S_\infty}(M_\infty, \mathbf{Z}_p)$.

3. Derived deformation rings

This section will define the derived deformation ring of a Galois representation, by using the derived Schlessinger criterion of Lurie [15]. We will follow [10] in giving an exposition of this criterion. See also [2] for a different perspective.

3.1. **Lecture C.1: simplicial sets and rings.** Start by recalling the definition of the category $s\mathcal{C}$ of simplicial objects in a category $\mathcal{C}$. Of particular importance to us are the categories of sets, abelian groups, and commutative rings. Recall geometric realization of simplicial sets and define homotopy groups of a pointed simplicial sets as homotopy groups of its realization. State the Dold–Kan correspondence and explain how homology of a chain complex corresponds becomes homotopy groups of the underlying pointed simplicial set.

Introduce the category SCR of simplicial commutative rings, and explain how to make the homotopy groups of an object into a graded-commutative ring.

Finally, introduce what it means for a functor SCR $\to$ sSets to be (derived) *representable*. You can take the definition in [10, Definition 2.14] but for simplicity replace Art_k by SCR in this talk. This requires first discussing *homotopy invariance* and *simplicial enrichment* (e.g. as in [10, §2.3] but again replacing Art_k by SCR for simplicity) and some discussion of (fibrant and) cofibrant replacement. There will not be time for a lengthy discussion of simplicial model categories, but at least explain the "problem": the simplicial enrichment is not homotopy invariant unless the domains are cofibrant; and the "solution": cofibrant replacement. The analogy to internal Hom of chain complexes may be helpful: homotopy invariance holds when the domain is projective. Ideally mention a simple example (e.g. the forgetful functor, sending a simplicial commutative ring to its underlying simplicial set, is representable in this sense).

3.2. **Lecture C.2: Simplicial Artin rings and pro-representability.** Define the category Art_k and what it means for a (simplicially enriched) functor $\mathrm{Art}_k \to s\mathrm{Sets}$ to be pro-representable, [10, Definition 2.19]. Perhaps include a simple example, e.g. from [10, §3.3].

Explain how any object $R \in \mathrm{Art}_k$ is (equivalent to one) built from the objects $k \oplus k[n]$ by iterated homotopy pullbacks, [10, §2.2]. The cardinality of $\pi_*(R)$ determines the minimal number of steps needed.

The rest of this lecture should explain Lurie's "derived Schlessinger criterion" ([15, Proposition 6.2.3]), as outlined in [10, §4.6][2].

Any pro-representable functor $\mathcal{F} : \mathrm{Art}_k \to s\mathrm{Sets}$ must necessarily preserve all homotopy pullbacks and its value on a homotopy discrete object must be homotopy discrete. The derived Schlessinger criterion is a converse to this statement: in fact it suffices to preserve only *certain* homotopy pullbacks[3] and that $\mathcal{F}(k \oplus k[0])$ be homotopy discrete. (After discussing the tangent complex $\mathfrak{t}\mathcal{F}$ the latter condition may be stated as $\pi_i \mathfrak{t}\mathcal{F} = 0$ for $i > 0$, as in [10, §4.6].)

If there is time left, sketch the inductive step in the (transfinite) induction proof of [10, Theorem 4.33], omitting details as necessary.

3.3. **Lecture C.3: Cotangent complexes of rings and tangent complexes of functors.** Briefly sketch Quillen's definition of the *cotangent*

---

[2] [15] avoids mentioning pro-objects in its statement, at the cost of a stronger hypothesis and more steps in its proof. Namely, the subcategory pro-Art_k on objects with finite dimensional tangent complex may be identified with the subcategory of $\mathrm{SCR}_{/k}$ consisting of "Complete Noetherian local" objects, after translating to the corresponding ∞-categories [15, §6.2]. See also [18, §2.30].

[3] in [10, Definition 3.8(i)] this weaker assumption is misleadingly called "preserve homotopy pullback"

complex $L_{B/A}$ for a morphism $A \to B$ in SCR. It is a (simplicial) module over B, well defined up to contractible choices. If k is a field and $\epsilon : \pi_0(B) \to k$ is a ring homomorphism, we may regard $\mathrm{Hom}_B(L_{B/A}, k)$ as a chain complex of k-vector spaces (concentrated in negative degrees if we use homological grading conventions). It is the *tangent complex*, and its homology is the *André–Quillen cohomology* $D^*_A(B; k)$. Discuss the behavior of $B \mapsto \mathrm{Hom}_B(L_{B/A}, k)$[4].

The case of interest to us has k a finite field and $A = \mathbf{Z}$ (or $A = W(k)$). Then homology of the tangent complex depends only on the functor $\mathrm{Hom}_{\mathrm{SCR}_{/k}}(B, -) : \mathrm{Art}_k \to s\mathrm{Sets}$ (with B made cofibrant): degree $-n$ may be identified with $\pi_0(\mathcal{F}(k \oplus k[n]))$. Motivated by this example, define the tangent complex of a functor $\mathcal{F} : \mathrm{Art}_k \to s\mathrm{Sets}$ which is "formally cohesive" in the sense[5] of [15, Definition 6.2.1], [10, §3.4]. It is easy to explain why the tangent complex is a *spectrum*, but there may not be time for explaining in detail how to make this spectrum into a k-linear chain complex (but see e.g. [10, §4.4], [15, Proposition 2.5.5], or [5, Definition 4.2.2].)

An important point is that the tangent complex detects weak equivalences: if $\mathcal{F} \to \mathcal{F}'$ is a natural transformation of formally cohesive functors $\mathrm{Art}_k \to s\mathrm{Sets}$ which induces a weak equivalence of tangent complexes, then $\mathcal{F}(A) \to \mathcal{F}'(A)$ is a weak equivalence for all $A \in \mathrm{Art}_k$.[6]

3.4. **Lecture C.4: representation functors.** This lecture should define the representation functors as in [10, §5.2,§7.3] and sketch the calculation of its tangent complex, [10, §5.3].

The output is then a representing $\mathcal{R} \in \text{pro-}\mathrm{Art}_k$; the connection to lecture B.2 (namely an isomorphism $\pi_0\mathcal{R} \cong R$, as well as how the deformation functors relate) should be explained.

[4]For cofibrant $R \in \text{pro-}\mathrm{Art}_k$, the tangent complex measures how to build the represented functor $\mathcal{F}_R : \mathrm{Art}_k \to s\mathrm{Sets}$ by iterated homotopy pullbacks or, equivalently how to build R by iterated homotopy *pushouts* (i.e., derived tensor product). The homotopy groups $\pi_*(R)$ measure how to build R by iterated homotopy *pullback*; in contrast with the the tangent complex, the homotopy groups of R are not easily expressed by evaluating the represented functor.

[5]The notion of formally cohesive functor is a special case of the later notion of a *formal moduli problem.*

[6]An interesting question that unfortunately will be outside the scope of this workshop is what kind of *structure* the chain complex $\mathfrak{t}\mathcal{F}$ has. In the analogous case where k has characteristic zero, it is an old insight that it should have a Lie bracket. More precise statements, due to Pridham and Lurie, assert that the functor $\mathcal{F}$ may be reconstructed from $\mathfrak{t}\mathcal{F}$ with its structure of differential graded Lie algebras. Recent work of Brantner and Mathew [5] establishes an analogous statement in characteristic p, except now $\mathfrak{t}\mathcal{F}$ is a *partition Lie algebra* (a new notion defined in op. cit.).

4. The general Taylor–Wiles method and the derived Galois deformation ring

The goal here is to explain how the Taylor–Wiles method, when applied now to a general group G, gives (assuming certain conjectures about the existence of Galois representations that are not currently known in general) an action of $\pi_*\mathcal{R}$ on the homology of an arithmetic group associated to G. Here $\mathcal{R}$ is the derived deformation ring for Galois representations with target in the dual group of G.

This action was constructed implicitly in [7], without explicitly identifying what was acting; the identification of this acting object with a derived deformation ring was made in [10, Theorem 14.1].

Again, to avoid technicalities to do with general groups, it may be best to restrict to the case of GL_n**.**

References: A very clear and concise outline of the Calegari–Geraghty setup is given in Jack Thorne's notes [21]. This will be a useful reference for this part; the only downside is it restricts to PGL_2, although you can also use the corresponding discussions in [10] (which are formulated for more general groups).

Other references: the original paper of Calegari-Geraghty [7], Toby Gee's Arizona winter school notes [11], Clozel–Harris–Taylor [8], and finally the "10 author paper" [1]. The latter paper, in particular, contains the state of the art on what is actually *proven* towards the various conjectures about Galois representations. **For our purposes, however, we should simply take these conjectures as an assumption.**

The argument requires various pieces and one makes various careful choices to make all the pieces fit together. For maximal comprehensibility, we will try to focus each lecture on one of the major pieces, and not spend too much time on the choices.

Lecture D.1: Underived R and the homology of locally symmetric spaces. This lecture (which overlaps with B.2) seeks to explain (i) why underived R acts on homology in the case of the modular curve and (ii) (in more detail than B.2) what one needs to be true in general to run the method.

Begin by explaining the basic example of modular curves: For congruence subgroups of $\mathrm{SL}_2(\mathbf{Z})$, state the properties of the ℓ-adic Galois representation attached to each Hecke eigenform f. Define the Hecke algebra T and explain why, for a maximal ideal $\mathfrak{m}$ of T, we get a Galois representation valued in $T_\mathfrak{m}$. (References: The theorems about modular forms and Galois representations can be found in e.g. [9, §12]. To construct a Galois representation valued in $T_\mathfrak{m}$ it is easiest to proceed by imposing a mild technical condition and using Carayol's lemma to glue together the representations from individual f, as explained on [11, p 34]).

Next outline what is expected to be true in general and recall what is currently known. A short account is given by §2 of [21], up to and including Conjecture A; you should modify the statements to apply to GL_n.

4.1. **Lecture D.2: more on local conditions.** Discuss what crystalline means, explain when the Galois representation associated to a classical modular form is crystalline, and outline the (underived) "Fontaine–Laffaille" deformation ring (e.g. §2.4 of CHT).

Redo the dimension equality from Lecture B.2, now in the context of GL_n over a base field F. Finally explain how to lift the underived condition to the derived setting (§9 of [10]).

4.2. **Lecture D.3: Construction of complexes that can be patched.** The goal is to define complexes M_Q as in *Desiderata 2.7* of [6] to which one can apply the patching method.

First of all, define Taylor–Wiles sets of primes Q relative to a Galois representation ρ. Explain the map $\mathcal{O}[\Delta_Q] \to R_Q$ and the fact that $R_Q/\mathfrak{a}_Q = R_\emptyset$ in the notation of Desiderata 2.7. (This is all discussed in §1 of [21], and just transpose to GL_n; alternately [10, §13.6] and references therein).

The desired complexes M_Q are constructed in [21, §5], where they are called C_Q (also see [10, §13.6, Lemma 13.2] and [13]). Explain this construction, again replacing GL_2 from [21] by GL_n as appropriate. Explain Conjecture C of [21].

4.3. **Lecture D.4.** This is the concluding lecture (although we will try to leave some extra time in case it takes longer). State Theorem 14.1 of [10] and recall the conjectures that are required for this Theorem; mention the independence result in §15.

Present the high-level outline of the proof as it relates to the Taylor–Wiles method (discussion around (1.11) in the introduction to [10]) and state Theorem 11.1 (say a word or two about proof if time permits). Explain the proof of the derived patching statement in Theorem 12.1.

References

[1] Patrick Allen, Frank Calegari, Ana Caraiani, Toby Gee, David Helm, Bao Le Hung, James Newton, Peter Scholze, Richard Taylor, and Jack Thorne. Potential automorphy over CM fields. 2018. `http://www.math.uchicago.edu/~fcale/papers/Ramanujan.pdf`.

[2] Jorge Antonio. Moduli of p-adic representations of a finite group. `https://arxiv.org/pdf/1709.04275.pdf`.

[3] Gebhard Böckle. Deformations of Galois representations. In *Elliptic curves, Hilbert modular forms and Galois deformations*, Adv. Courses Math. CRM Barcelona, pages 21–115. Birkhäuser/Springer, Basel, 2013.

[4] A. Borel and N. Wallach. *Continuous cohomology, discrete subgroups, and representations of reductive groups*, volume 67 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2000.

[5] Lukas Brantner and Akhil Mathew. Deformation theory and partition lie algebras. *arXiv preprint arXiv:1904.07352*, 2019.

[6] Frank Calegari. Motives and L-functions. http://www.math.uchicago.edu/~fcale/papers/CDM.pdf. notes from the Current Developments in Mathematics conference, 2018.
[7] Frank Calegari and David Geraghty. Modularity lifting beyond the Taylor–Wiles method. *Inventiones mathematicae*, 211(1):297–433, Jan 2018.
[8] Laurent Clozel, Michael Harris, and Richard Taylor. Automorphy for some l-adic lifts of automorphic mod l Galois representations. *Pub. Math. IHES*, 108:1–181, 2008.
[9] Fred Diamond and John Im. Modular forms and modular curves. In *Seminar on Fermat's Last Theorem (Toronto, ON, 1993–1994)*, volume 17 of *CMS Conf. Proc.*, pages 39–133. Amer. Math. Soc., Providence, RI, 1995.
[10] S. Galatius and A. Venkatesh. Derived Galois deformation rings. *Adv. Math.*, 327:470–623, 2018.
[11] Toby Gee. Arizona Winter School lectures notes on modularity lifting. http://wwwf.imperial.ac.uk/~tsg/Index_files/ArizonaWinterSchool2013.pdf.
[12] Haruzo Hida. Galois representations into $\mathrm{GL}_2(\mathbf{Z}_p[[X]])$ attached to ordinary cusp forms. *Invent. Math.*, 85(3):545–613, 1986.
[13] Chandrashekhar B. Khare and Jack A. Thorne. Potential automorphy and the Leopoldt conjecture. *Amer. J. Math.*, 139(5):1205–1273, 2017.
[14] Anthony W. Knapp. *Representation theory of semisimple groups*, volume 36 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1986. An overview based on examples.
[15] Jacob Lurie. *Derived algebraic geometry*. ProQuest LLC, Ann Arbor, MI, 2004. Thesis (Ph.D.)–Massachusetts Institute of Technology.
[16] B. Mazur. Deforming Galois representations. In *Galois groups over* $\mathbf{Q}$ *(Berkeley, CA, 1987)*, volume 16 of *Math. Sci. Res. Inst. Publ.*, pages 385–437. Springer, New York, 1989.
[17] Barry Mazur. An introduction to the deformation theory of Galois representations. In *Modular forms and Fermat's last theorem (Boston, MA, 1995)*, pages 243–311. Springer, New York, 1997.
[18] J. P. Pridham. Unifying derived deformation theories. *Adv. Math.*, 224(3):772–826, 2010.
[19] François Sauvageot. Principe de densité pour les groupes réductifs. *Compositio Math.*, 108(2):151–184, 1997.
[20] Peter Scholze. On torsion in the cohomology of locally symmetric varieties. *Ann. of Math. (2)*, 182(3):945–1066, 2015.
[21] Jack Thorne. Beyond the Taylor–Wiles method. https://www.dpmms.cam.ac.uk/~jat58/beyondtw.pdf.
[22] Akshay Venkatesh. Cohomology of arithmetic groups and periods of automorphic forms. *Jpn. J. Math.*, 12(1):1–32, 2017.
[23] David A. Vogan, Jr. and Gregg J. Zuckerman. Unitary representations with nonzero cohomology. *Compositio Math.*, 53(1):51–90, 1984.