ARBEITSGEMEINSCHAFT: GEOMETRIC REPRESENTATION THEORY

SIMON RICHE, WOLFGANG SOERGEL, AND GEORDIE WILLIAMSON

1. Introduction

A fundamental problem in representation theory is the description of all irreducible representations. A first step in such a description is a parametrization of irreducible representations. Once we have a paramatrization we move onto more refined questions: dimensions, character formulas, explicit realizations, descriptions of categories, etc.

This Arbeitsgemeinschaft will be concerned with these questions for algebraic representations of (split) reductive algebraic groups. We start with a split reductive algebraic group G (like GL_n or Sp_{2n}) and consider representations

$$\rho : G \to \mathrm{GL}(V)$$

in the category of algebraic groups. These are the representations that arise in algebraic geometry (on rings of functions, on cohomology of equivariant bundles, . . .). Thus these representations arise when pursuing "harmonic analysis in algebraic geometry".

If our base field is $\mathbb{C}$ (or more generally of characteristic zero), then the theory is very mature. One has a parametrization via highest weight, the characters of irreducible representations are given by Weyl's character formula, and one has a realization of all simple modules as global sections of line bundles on flag varieties (the Borel-Weil theorem). Moreover the category of representations is semi-simple. In fact, the whole theory runs parallel to the representation theory of compact Lie groups.

If our base field is of characteristic p then the situation is much more complicated. We still have a parametrization via highest weight (Chevalley's theorem), however our categories of representations are not semi-simple (unless G is a torus) and the dimensions and characters of our irreducible representations are unknown in general. One source of the complexity is the Frobenius morphism

$$\mathrm{Fr} : G \to G$$

in characteristic p. (For example, if $G = \mathrm{GL}_n$ this is the morphism which raises matrix entries to their p^{th} power.) Precomposing with the Frobenius morphism provides a new operation on representations called *Frobenius twist*. It is not difficult to check that the Frobenius twist of an irreducible module is again irreducible, but that its highest weight has been dilated by p. Nothing like this operation exists in characteristic 0, and we get a first hint of the complexities awaiting us. In general the representation theory of reductive algebraic groups is a fascinating mix of the characteristic zero theory (highest weights, Weyl character formula, . . .) with flavours coming from characteristic p (non-semi-simplicity, Frobenius twist, relation to finite groups of Lie type ...). This Arbeitsgemeinschaft will focus on the use of geometric techniques (perverse and parity sheaves, geometric Satake, Smith-Treumann theory, ...) to make progress on this difficult and fascinating subject.

1.1. **The IC paradigm in geometric representation theory.** Much of our deeper understanding of the representation theory of reductive algebraic groups draws inspiration from the Kazhdan-Lusztig conjecture for simple highest weight representations of complex semi-simple Lie algebras. For this reason, we take some time to recall this conjecture, before returning to reductive algebraic groups.

Let $\mathfrak{g}$ denote a complex semi-simple Lie algebra, and fix a choice of a Cartan and Borel subalgebra $\mathfrak{h} \subset \mathfrak{b}$. Any weight $\lambda \in \mathfrak{h}^*$ can be viewed as a one-dimensional representation $\mathbb{C}_\lambda$ of $\mathfrak{b}$ via the identification $\mathfrak{h} = \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$, inducing up to $\mathfrak{g}$ one obtains a Verma module

$$\Delta_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda.$$

This Verma module has a unique irreducible quotient $\Delta_\lambda \twoheadrightarrow L_\lambda$, and one obtains in this way all irreducible highest weight modules for $\mathfrak{g}$.

The classes of the modules $\{\Delta_\lambda \mid \lambda \in \mathfrak{h}^*\}$ and $\{L_\lambda \mid \lambda \in \mathfrak{h}^*\}$ both form bases for the Grothendieck group of highest weight $\mathfrak{g}$-modules. By the PBW theorem, the character of each Δ_λ is easily written down, and hence to know the character of L_λ it is enough to express its class in terms of Verma modules. This is the subject of the Kazhdan-Lusztig conjecture. In an important case (the "principal block") it reads:

$$[L_{x\cdot(-2\rho)}] = \sum_{y \in W_f} (-1)^{\ell(x)+\ell(y)} P_{y,x}(1) [\Delta_{y\cdot(-2\rho)}]$$

It is not important to understand all the details of this conjecture for the purposes of this introduction.[1] The most important point is the appearance of (evaluations at 1 of) Kazhdan-Lusztig polynomials $P_{x,y}$ on the right-hand side. These are certain polynomials which are computable by a combinatorial algorithm involving only the Weyl group and its simple reflections. These polynomials were introduced by Kazhdan and Lusztig in a seminal paper in 1979, and a year later they showed that these polynomials encode the local intersection cohomology groups of Schubert varieties. The Kazhdan-Lusztig conjecture was proved soon after by Beilinson-Bernstein, and Brylinski-Kashiwara using D-modules and the Riemann-Hilbert correspondence. Another proof was given a decade later by Wolfgang Soergel, using what came to be known as Soergel bimodules.

The Kazhdan-Lusztig conjecture gave rise to what one might call the "IC paradigm" in geometric representation theory. With disarming regularity topological invariants of singularities (in particular graded dimensions of stalks of IC sheaves) give important representation theoretic information.[2] This has been a central vein of research for over forty years.

[1]For the sake of completeness: W_f is the Weyl group, $\ell : W_f \to \mathbb{Z}_{\geqslant 0}$ is the length function; ρ denotes the half-sum of the positive roots and $\cdot$ denotes the dot action, i.e. $x \cdot \lambda = x(\lambda + \rho) - \rho$.

[2]For an impressive list of applications of the IC paradigm, the reader should consult Lusztig's superb lecture [Lus91] at the 1990 ICM.

1.2. **Lusztig's character formula.** How effective is the IC paradigm for studying modular (i.e. mod p) representations? This is the main problem which this Arbeitsgemeinschaft aims to address.

Let us return to the study of representations of a split reductive group G defined over a field k. As we mentioned above, the situation is well-understood if k is of characteristic 0, and so we might as well assume that k is of characteristic $p > 0$. To any dominant weight λ of our torus, we may associate a *Weyl module* Δ_λ. These are roughly analogous to Verma modules in category $\mathcal{O}$:

(1) their character is known (and given by Weyl's character formula);
(2) each Δ_λ admits a unique simple quotient L_λ;
(3) the L_λ constitute all simple G-modules.

As in the case of category $\mathcal{O}$, we know the character of L_λ if we can express its class in the Grothendieck group in terms of Weyl modules. Such an expression was conjectured by Lusztig, a year after the formulation of the Kazhdan-Lusztig conjecture, and is known as the *Lusztig character formula*:

$$[L_{x\cdot_p(-2\rho)}] = \sum_{y\in W} (-1)^{\ell(x)+\ell(y)} P_{y,x}(1)[\Delta_{y\cdot_p(-2\rho)}] \tag{1}$$

This formula is formally very similar to the Kazhdan-Lusztig conjecture. The main difference is that the finite Weyl group W_f is replaced by the *affine* Weyl group W, so the Kazhdan-Lusztig polynomials which appear on the right hand side are now those for the affine Weyl group. (Also, $\cdot_p$ denotes the "p-dilated dot action" and one should disregard any weights $y \cdot_p (-2\rho)$ appearing on the right hand side which are not dominant.)

Lusztig conjectured that (1) holds, as long as the characteristic is not too small.[3] An important consequence of Lusztig's conjecture is the rather remarkable prediction that the representation theory of G should be "uniform in p". Whilst the Kazhdan-Lusztig conjecture was solved remarkably quickly, Lusztig's conjecture was much more resistent to attack. In the mid 1990s, it was established that, once we fix a root system, (1) holds for p larger than some ineffective bound. (This result is due to Andersen-Jantzen-Soergel, and made crucial use of long and complicated works on representations of affine Lie algebras and quantum groups at roots of unity, by Kazhdan-Lusztig, Lusztig and Kashiwara-Tanisaki). However, several aspects of Lusztig's conjecture remained mysterious. For example, for a given G and given p, does Lusztig's conjecture hold? We still don't know in general!

1.3. **Modular geometric representation theory.** We do however have a good understanding of the geometry underlying Lusztig's character formula. For example, this new understanding means that we can answer the question at the end of the last paragraph in considerably more cases than we could a decade ago.

Recall that the "IC paradigm" can be summarized as the idea that stalks of intersection cohomology sheaves provide important information in representation theory. In order to attack questions in modular representation theory, it is natural to consider perverse sheaves with coefficients in fields of positive characteristic. (We work either over complex varieties with their classical metric topology, or with étale sheaves with torsion coefficients coprime to the characteristic of our

[3] His original condition on p is a little tricky to state. However results of Kato proved soon after Lusztig formulated his conjecture led to a widespread belief that (1) should hold as long as p is larger than the Coxter number h of G.

schemes.) A major drawback of modular perverse sheaves is that computations are extremely difficult! The main reason is the absence of a good theory of weights, and its associated package of miracles (e.g. hard Lefschetz and the Decomposition Theorem).

However, despite its difficulties, this is still a fruitful line of attack. The lectures of this Arbeitsgemeinschaft will explore the following ideas in some depth:

(1) The geometric Satake equivalence [MV07], which gives an equivalence between the category of representations of a split reductive algebraic group and certain perverse sheaves on the affine Grassmannian.
(2) The realization (that began with [Soe00]) that there is a close link between the IC paradigm and the validity of the Decomposition Theorem for certain Bott-Samelson resolutions with mod p coefficients. An important related idea is that the indecomposable summands ("parity sheaves") of resolutions are interesting, even when they fail to be IC sheaves.
(3) The realization (made explicit in [JMW14], but already implicit in [CG97] and [dCM02]) that certain intersection forms attached to the fibres of maps control the failure of the Decomposition Theorem. (This observation eventually led to the construction by Williamson of a systematic family of counter-examples to Lusztig's character formula [Wil17c, Wil17b].)
(4) The idea that the stalks of parity sheaves should replace IC data in modular representation theory. (This idea first appears in [Soe00] and was explored in depth in [RW18]).
(5) Points (2), (3) and (4) above are also intertwined with progress understanding the "Hecke category" (first in its geometric incarnation via Soergel bimodules, and later in its diagrammatic incarnation). This allows (slow!) algorithmic computation of the stalks of parity sheaves.

The Arbeitsgemeinschaft will culminate with a discussion of the recent paper [RW19] which uses Smith-Treumann theory (a sheaf theoretic operation unique to $\mathbb{F}_p$-sheaves and $\mathbb{Z}/p\mathbb{Z}$-actions) to give a new proof of Lusztig's character formula, passing through the geometric Satake equivalence. An advantage of this approach is that it gives character formulas in terms of the stalks of parity sheaves in all characteristics.

2. Highest weight categories, KL polynomials and reductive groups

2.1. **Category $\mathcal{O}$ and highest weight categories.** Familiarity with category $\mathcal{O}$ is invaluable in representation theory, and will serve as a running example in the talks that follow. Discuss category $\mathcal{O}$ in detail: Block decompositions, principal block, Verma modules, simples, example of $\mathfrak{sl}_2(\mathbb{C})$ including a discussion of all five indecomposables in the principal block. (Here [Hum08, Soe21] are excellent references.[4]) Introduce the notion of a highest weight category following [Ric, Appendix A]. (The origin of this version of highest-weight formalism is to be found in [BGS96, Section 3].) This beautiful classical example of category $\mathcal{O}$ is a motivation behind a lot of the more advanced material that we will discuss this week, so the speaker should be aware of where we are going, and highlight the general features evident in the example of category $\mathcal{O}$.

[4] The best human reference is probably W.S. so feel free to ask him!

2.2. **Coxeter groups and Kazhdan-Lusztig polynomials.** Discussion of the Hecke algebra of a Coxeter group. (This is an infamous exercise in [Bou68], and is treated in detail in [Hum90].) Kazhdan–Lusztig polynomials (including parabolic versions). The original reference for Kazhdan-Lusztig polynomials is [KL79]. The speaker should follow the treatment (and conventions) of [Soe97], and comment on the relationship to [KL79] in a remark. Discussion of positivity properties: positivity of coefficients of Kazhdan-Lusztig polynomials, positivity of structure constants of multiplication. State and discuss Kazhdan-Lusztig conjecture and its relation to BGG-reciprocity and translation functors.

Examples: Weyl groups of Kac–Moody groups, (extended) affine Weyl groups of reductive groups.

Optional extra: If time permits, the speaker might explain the computation of Kazhdan–Lusztig polynomials for dihedral groups. The speaker could also use Joel Gibson's LieVis software to give "live" computations of Kazhdan-Lusztig polynomials for affine Weyl groups: `https://www.jgibson.id.au/lievis/`

2.3. **Reductive groups I.** Structure of reductive groups over algebraically closed fields. Maximal tori, Borel subgroups, roots, root data etc. Classification of simple modules. Induced modules, Weyl modules. Weyl's character formula. Kempf's vanishing theorem. (The bible for representations of reductive algebraic groups is [Jan03], however the beginner might get lost. To help the reader navigate, the papers [Wil17a, CW21] are useful.) That representations of reductive algebraic groups forms a highest weight category should be emphasized.

Examples: As a basic example one should discuss the construction of induced modules for SL_2 via global sections of equivariant line bundles on $\mathbb{P}^1$. One might also like to discuss global sections of equivariant line bundles on $\mathbb{P}^{n-1}$ (to get symmetric powers, see [Jan03, §II.2.16]), as well as on Grassmannians (to obtain e.g. the fundamental representations of SL_n).

Optional extras: If time permits, the speaker could also discuss the Steinberg tensor product theorem.

2.4. **Reductive groups II.** State the Borel–Weil–Bott theorem, and linkage principle. Introduce translation functors, and establish their basic properties: (i) effect on simple/induced/Weyl modules[5]; (ii) that they provide equivalences when the stabilizer in the affine Weyl group is the same; (iii) conclude that for many questions (e.g. determination of characters) it suffices to analyse the principal block. References for this talk include [Jan03], and the references therein.

Optional extras: If time permits, the following could be discussed, and is left to the discretion of the speaker:

(1) Discuss the failure of the Borel–Weil–Bott theorem in positive characteristic (e.g. examples of Mumford and Griffiths).
(2) Discuss a proof of the linkage principle. Here there are two routes: one valid in large characteristic via consideration of the reduction modulo p of the centre of the enveloping algebra together with the central character ([Ric, §2.4]); the other in full generality due to Andersen, and discussed in detail in [Jan03]. (Note that one of the goals of this Arbeitsgemeinschaft is to give a geometric proof, via the geometry of the affine Grassmannian.)

[5] Effect on simple modules is hard in general! But some things are known, and this should be recalled.

3. Hecke categories

3.1. **Perverse sheaves on flag varieties.** Brief reminder on perverse sheaves, intersection cohomology complexes and the Decomposition Theorem. Brief discussion of the geometry of flag varieties (for Kac–Moody groups and affine versions): Bruhat decomposition, relation with Coxeter combinatorics. Categories of perverse sheaves (highest weight structure, classification of simples), including parabolic and Whittaker versions. (There are many references for this material but [Ric] provides a good starting point.) Discuss in detail the case of SL_2, including a discussion of all five indecomposable perverse sheaves. Discuss the computation of stalks of ICs in terms of Kazhdan-Lusztig polynomials following [Spr82].[6]

3.2. **The Hecke algebra and Hecke category.** Discuss the Hecke algebra of $B(\mathbb{F}_q)$-biinvariant functions on $G(\mathbb{F}_q)$ and establish isomorphism with the abstractly defined Hecke algebra from the second lecture (with v appropriately specialized). Briefly recall the equivariant derived category from [BL94]. Discuss the categorification of the Hecke algebra, following e.g. [Wil18, §2.1-2.2]. The speaker should discuss the categories of the previous talk as (right) modules over the Hecke category.

3.3. **Soergel bimodules.** Definition and first examples. State the classification of the indecomposable bimodules. (References for this material include [Soe07, EMTW20].) Discuss the relation to the Hecke category of the previous lecture (i.e. by taking hypercohomology). State that hypercohomology gives an equivalence of additive categories (see [Soe01, §3.4] and [BY13]).

Optional extra: The speaker may wish to spend the second part of their talk giving a survey of [SVW18], and in particular highlight the equivalence:

$$MTDer_{B\times B}(G) \cong K^b(SBim).$$

Emphasis should be placed on the idea that one can consider an additive category as a surrogate for an appropriate notion of weights (e.g. étale, Hodge, or motivic).

Optional extra: One could discuss the Soergel conjecture, and relation with KL conjecture (via Soergel's functor $\mathbb{V}$.)

3.4. **Parity sheaves.** Now we start to consider what happens when our sheaf coefficients are taken to be of characteristic p. Discuss the validity of the Decomposition Theorem with mod p coefficients, and the role of intersection forms (following [JMW14]). Examples should be given (e.g. from [JMW12] or [Wil18, §1.5]). Define parity sheaves (following [JMW14]) and state and prove their classification. As an application, use the Decomposition Theorem to show that parity sheaves coincide with IC sheaves when the coefficients are in $\mathbb{Q}$. Discuss what goes wrong when we try to compute the stalks of parity sheaves with mod p coefficients, as in Talk 3.1.

3.5. **Diagrammatic Hecke category.** Define the diagrammatic Hecke category, following e.g. [EMTW20]. State the classification of indecomposable objects, and the categorification theorem. Define the p-canonical basis. Give some sample computations. (It might be good to liase with the speaker of the previous section, so

[6]Pramod Achar's book is a another good reference for the computation performed via parity techniques. This books is currently being published, but speakers can request an electronic copy from the organizers.

that one example is discussed from two points of view. E.g. the only singular Schubert variety in Sp_4/B, see [Wil18, Example 2.25].)

Optional extra: Discussion of light leaves and algorithmic description of indecomposables. The reference here is [JW17]. For more examples, the reader may consult the last chapter of [EMTW20].

4. Character formulas and geometry

Now we return to reductive groups, and their character theory.

4.1. **Lusztig's conjecture.** Compute simple characters for SL_2 via hand + Steinberg's tensor product theorem. State Jantzen's sum formula and use it to deduce some simple characters in low rank, e.g. for SL_3 and Sp_4. (The more complicated case of SL_4 is discussed in §II.8.20 of [Jan03], where a reference to an old paper of Jantzen (in German) is given for SL_3 and Sp_4.) State Lusztig's conjecture, and discuss bounds (following e.g. the discussion in [Wil17a, §1.12]). Give a birds-eye view of the proof in large characteristic (e.g. following the discussion in [Wil17a, §1.15]).

Optional extras: If time permits, one could discuss Fiebig's bound in [Fie12] and roughly how he comes to it.

4.2. **Combinatorial origins of geometric Satake.** The speaker should explain how Lusztig's character formula leads to a prediction for the values of certain affine Kazhdan-Lusztig polynomials at 1. (This is discussed in Lusztig's notes to his papers available on the arxiv, and also in [Ric, Chapter 1, 3.5] and [Wil17a, §1.14]). State that this prediction was proved by Lusztig [Lus83] and Kato [Kat82].[7]

4.3. **Geometric Satake and Finkelberg-Mirković.** Discuss the geometry of affine Grassmannians. State the geometric Satake equivalence. Discuss one example (e.g. SL_2) in detail. Discuss the proof, following [MV07, BR18, Zhu17]. Discuss the Finkelberg-Mirković conjecture (references include [FM99], [AR18], and [Wil17a, 2.5]). Explain how Lusztig's conjecture for large p follows from the Finkelberg-Mirković conjecture.

4.4. **Torsion explosion.** Define modular category $\mathcal{O}$ following [Soe00]. Use the methods of [Soe00] to deduce that Lusztig's conjecture implies that that parity sheaves on finite flag varieties coincide with IC sheaves when the coefficients are of characteristic $> h$. (The crucial computation of the endomorphism of the "big projective" in modular category $\mathcal{O}$ can be left as a black box.) Explain the "torsion explosion" theorem, following [Wil17c, Wil17b]. (Note that the proof of [Wil17c] is diagrammatic and relies on a result of [HW]. The proof in [Wil17b] is geometric, and is probably closer to the spirit of this Arbeitsgemeinschaft.)

Optional extras: The appendix to [Wil17b] contains some interesting number theoretic problems related to torsion explosion. If time permits, these problems could be discussed briefly.

[7] Lusztig's result was also reproved by Knop in "On the Kazhdan-Lusztig basis of a spherical Hecke algebra".

5. Higher representation theory

5.1. **Tilting character formula.** Define and classify tilting objects in a highest weight category (see e.g. [Ric]). Explain classification of indecomposable tilting objects in the particular setting of representations of reductive algebraic groups. State Donkin's tilting tensor product theorem (see [Jan03, §II.E.9]) and use it to deduce the characters of all tilting modules for SL_2 (see [EH02] and [Ric, §3.2]). State Soergel's theorem computing the characters of tilting modules for quantum groups in terms of anti-spherical Kazhdan-Lusztig polynomials. (To this end the speaker should give a birds-eye picture of the representation theory of quantum groups, without going into too much detail.) Explain why knowledge of tilting characters implies knowledge of simples characters (under mild restrictions on p, see [RW21] and the references therein).

Optional extras: If time permits, one could also discuss the relationship between tilting modules for GL_n and decomposition numbers for symmetric groups.

5.2. **Categorical conjecture.** Explain the definition of the anti-spherical p-Kazhdan–Lusztig polynomials, and explain the relation with "usual" parabolic Kazhdan–Lusztig polynomials. State the "numerical" and "categorical" conjectures of Riche-Williamson. Define the categorified anti-spherical module. Explain why the categorical conjecture implies the numerical one. If the speaker wishes, they can discuss these conjectures in the context of higher representation theory. (For all of the above, see [RW18].)

Optional extra: Explain the simple character formula of Riche-Williamson [RW21], as another instance of higher representation theory.

6. The Iwahori-Whittaker model and Smith-Treumann theory

6.1. **The Iwahori-Whittaker model.** Discuss the Iwahori-Whittaker model of geometric Satake, following the original paper [BGM+19]. If time permits, it would be nice to discuss the relation with the "Casselmann-Shalika formula." Emphasis should be given to four aspects: (i) the Iwahori-Whittaker gives a model for $\mathrm{Rep}\, G$ which extends to derived categories; (ii) although one loses the monoidal structure, it is still a right module over the Satake category; (iii) parity sheaves correspond to tilting modules; (iv) the "shift by ρ" on parameters.

Optional extra: If the speaker has time, they might explain the connection between tilting modules and parity sheaves in the setting of geometric Satake [JMW16], and then how [BGM+19] deduce that the perverse cohomology groups of parity sheaves are always tilting.

6.2. **Smith-Treumann theory.** Explain the Smith quotient associated to a trivial action of μ_ℓ on a variety, and the Smith restriction functor $D^b_{\mu_\ell}(X) \to \mathrm{Sm}(X^{\mu_\ell})$, following Treumann, [Wil] and [RW19]. (The notation in this subject has not stabilized. For the benefits of the week it is probably best to stick as closely as possible to the notation of [RW19].) Explain that "Smith restriction commutes with functors" by proving it for a few functors. (If time permits, deduce classical theorems of Smith on fixed points of $\mathbb{Z}/p$-actions on homology p-spheres.) Explain the necessary modifications to consider Smith restriction in the étale setting, and in particular prove the crucial [RW19, Proposition 2.6].

6.3. **Smith–Treumann theory and the linkage principle.** This talk should cover the main results of [RW19]. Firstly, the loop rotation action on the affine Grassmannian should be discussed, and the fixed points under μ_ℓ should be described. Next, it should be proved that the Smith restriction functor preserves standard and costandard objects, and hence maps indecomposable tilting sheaves to indecomposable parity sheaves. Lastly, it should be explained that indecomposable parity sheaves on affine Grassmannians stay indecomposable in the Smith quotient.

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