G-COMPLETE REDUCIBILITY, GEOMETRIC INVARIANT THEORY AND SPHERICAL BUILDINGS

MICHAEL BATE, BENJAMIN MARTIN, AND GERHARD RÖHRLE

Programme for Oberwolfach Seminar 2223b

Abstract. This seminar covers recent developments and applications of Serre's notion of G-complete reducibility for algebraic groups. The focus is on giving an overview of G-complete reducibility and related tools in geometric invariant theory and the theory of spherical buildings, aiming towards a discussion of cutting-edge research in the area and open problems. The seminar is specifically aimed at PhDs and postdocs within the research field of algebraic groups.

1. Concept and Introduction

The Oberwolfach Seminar is in a core area of algebraic group theory and at the interdisciplinary crossroads of algebra and representation theory on the one hand, geometry and geometric invariant theory on the other. The notion of *G-complete reducibility* for subgroups of a reductive algebraic group G was introduced by J-P. Serre in the 1990s as a natural generalization of the notion of complete reducibility in representation theory (which corresponds to the case where $G = \mathrm{GL}(V)$ is the general linear group). Since its introduction, and especially over the last 15 years or so, this notion has been widely studied, both as an important notion in its own right, with applications to the classification and structure of linear algebraic groups, and also as a useful tool with applications in representation theory, geometric invariant theory, the theory of buildings, and number theory. The aim of this Oberwolfach seminar is to introduce participants to G-complete reducibility and explain some of its many applications across pure mathematics — participants will learn some rich and deep modern algebra, and leave equipped with an understanding of how this mathematics continues to be applied to solve a diverse range of problems, particularly in the theory of algebraic groups.

2. Course Details

Three complementary courses will run in parallel during the week. Each course will draw on materials from the other two, but will also have its own focus and direction, culminating in an understanding of recent published work which relies on the material presented.

Prerequisites. We will assume that participants have a working knowledge of the theory of linear algebraic groups over algebraically closed fields, as laid out for example in the books of Borel [2] and Springer [7]. This should include knowledge of reductive groups, Borel subgroups and parabolic subgroups, maximal tori and the root system, the classification of reductive groups by root data.

Participants wishing to do further preparation for the courses can also consult the survey papers [5], [6] and the notes [3]. Similarly, further background in Geometric Invariant Theory can be found in [4, §3.1–§3.3], and for the theory of buildings, see [1, §1–§4, §6]. Note,

2010 *Mathematics Subject Classification.* 20G15 (20G40, 20E45, 14L24).

however, that anything needed in the courses from these sources will be introduced during the week.

Preparatory Reading

[1] P. Abramenko, K. Brown, *Buildings. Theory and applications*, Graduate Texts in Mathematics, **248**, Springer, New York, 2008. xxii+747 pp.

[2] A. Borel, *Linear Algebraic Groups*, Graduate Texts in Mathematics, **126**, Springer-Verlag 1991.

[3] B. Martin, *Algebraic groups and G-complete reducibility: a geometric approach*, notes from the International Workshop on Algorithmic problems in group theory and related areas, Novosibirsk, 2016, available at https://www.ruhr-uni-bochum.de/imperia/md/content/mathematik/lehrstuhlvi/comp_red.pdf.

[4] P.E. Newstead, *Introduction to moduli problems and orbit spaces*. Tata Institute of Fundamental Research Lectures on Mathematics and Physics, 51. Tata Institute of Fundamental Research, Bombay 1978.

[5] J-P. Serre, *The notion of complete reducibility in group theory*, Moursund Lectures, Part II, University of Oregon, 1998.

[6] J-P. Serre, *Complète Réductibilité*, Séminaire Bourbaki, 56ème année, 2003-2004, n° 932.

[7] T.A. Springer, *Linear Algebraic Groups*, Second edition. Progress in Mathematics, 9. Birkhäuser Boston, Inc., Boston, MA, 1998.

Course 1: Basic theory of complete reducibility and subgroup structure of algebraic groups. This course will introduce the basic theory of G-complete reducibility for reductive groups over algebraically closed fields, and explain how it has become a core component of the ongoing effort to classify subgroups of reductive algebraic groups. This course will also explain the key link between complete reducibility and GIT which allows one to apply geometric techniques to problems in algebraic group theory. The course will also service the other two courses, by introducing much of the key notation and terminology for the week.

Topics: definition of G-complete reducibility and first properties; examples and non-examples; complete reducibility versus reductivity; G-complete reducibility and simultaneous conjugation; classification techniques and results.

Fundamental papers: [5, 6, 8–10]; Recent works: [11].

References for Course 1

[8] M. Bate, B. Martin, G. Röhrle, *A geometric approach to complete reducibility*, Invent. Math., **161** no. 1, (2005), 177–218.

[9] M. Bate, B. Martin, G. Röhrle, *Complete reducibility and commuting subgroups*, J. Reine Angew. Math., **621**, (2008), 213–235.

[10] M.W. Liebeck, G.M. Seitz, *Reductive subgroups of exceptional algebraic groups*. Mem. Amer. Math. Soc. no. **580** (1996).

[11] A. J. Litterick, A. R. Thomas, *Complete reducibility in good characteristic*, Transactions of the American Mathematical Society **370** (2018), no. 8, 5279–5340.

Course 2: The geometric approach. In the course we will explore further key ideas of Richardson, including his tangent space argument and the notion of a reductive pair, and the optimality formalism of Kempf-Rousseau-Hesselink.

Topics: Optimality results and consequences; the tangent space argument and separability.

Fundamental papers: [8, 12, 15, 17, 18, 20, 21]; Recent works: [13, 16, 19].

References for Course 2

[12] M. Bate, S. Herpel, B. Martin, G. Röhrle, *G-complete reducibility and semisimple modules*, Bull. Lond. Math. Soc. **43** (6), (2011), 1069–1078.

[13] M. Bate, B. Martin, G. Röhrle, *Overgroups of regular unipotent elements in reductive groups*, Preprint 2021. https://arxiv.org/abs/2107.01925

[14] M. Bate, B. Martin, G. Röhrle, R. Tange, *Complete reducibility and separability*, Trans. Amer. Math. Soc., **362** (2010), no. 8, 4283–4311.

[15] M. Bate, B. Martin, G. Röhrle, R. Tange, *Closed orbits and uniform S-instability in geometric invariant theory*, Trans. Amer. Math. Soc. **365** (2013), no. 7, 3643–3673.

[16] S. Herpel, D.I. Stewart, *On the smoothness of normalisers, the subalgebra structure of modular Lie algebras, and the cohomology of small representations.* Doc. Math. **21** (2016), 1–37.

[17] W.H. Hesselink, *Uniform instability in reductive groups*, J. Reine Angew. Math. **303/304**,(1978), 74–96.

[18] G.R. Kempf, *Instability in Invariant Theory*, Ann. Math. **108** (1978), 299–316.

[19] B. Martin, *Generic stabilisers for actions of reductive groups*, Pacific J. Math. **279**, (2015), 397–422.

[20] R. W. Richardson, *Conjugacy classes of n-tuples in Lie algebras and algebraic groups*, Duke Math. J. **57**, (1988), no. 1, 1–35.

[21] G. Rousseau, *Immeubles sphériques et théorie des invariants*, C.R.A.S. **286** (1987), 247–250.

Course 3: Complete reducibility over an arbitrary field. In this course, we will explore the study of complete reducibility over a field which is not necessarily algebraically closed. In order to work in this setting, one needs to understand the link with spherical buildings, and also the notion of cocharacter-closed orbits, which has wider applications across GIT. We will explore how complete reducibility behaves in the presence of field extensions, making links with representation theory (in the case $G = \mathrm{GL}_n$).

Topics: working over non-algebraically closed fields; G-complete reducibility over k; key examples; link to buildings and Tits' Centre Conjecture; separable field extensions; semisimplification.

Fundamental papers: [21–23, 26]; Recent works: [24, 27].

References for Course 3

[22] M. Bate, S. Herpel, B. Martin, G. Röhrle, *Cocharacter-closure and the rational Hilbert-Mumford Theorem*, Math. Z. **287** (2017), 39–72.

[23] M. Bate, B. Martin, G. Röhrle, *Complete reducibility and separable field extensions*, C. R. Math. Acad. Sci. Paris **348** (2010), 495–497.

[24] M. Bate, B. Martin, G. Röhrle, *Semisimplification for subgroups of reductive algebraic groups*, Forum Math. Sigma **8** (2020), Paper No. e43, 10pp.

[25] B. Mühlherr, J. Tits, *The center conjecture for non-exceptional buildings*, J. Algebra **300** (2006), no. 2, 687–706.

[26] C. Ramos-Cuevas, *The center conjecture for thick spherical buildings*, Geom. Dedicata **166** (2013), 349–407.

[27] T. Uchiyama, *Complete reducibility of subgroups of reductive algebraic groups over nonperfect fields I*, J. Algebra **463** (2016), 168–187.

Department of Mathematics, University of York, York YO10 5DD, United Kingdom
Email address: michael.bate@york.ac.uk

Department of Mathematics, University of Aberdeen, King's College, Fraser Noble Building, Aberdeen AB24 3UE, United Kingdom
Email address: b.martin@abdn.ac.uk

Fakultät für Mathematik, Ruhr-Universität Bochum, D-44780 Bochum, Germany
Email address: gerhard.roehrle@rub.de