Arbeitsgemeinschaft: Higher rank Teichmüller theory

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Riemann surfaces are of fundamental importance in many areas of mathematics and theoretical physics. The study of the moduli space of Riemann surfaces of a fixed topological type is intimately related to the study of the Teichmüller space of that surface, together with the action of the mapping class group. Classical Teichmüller theory has many facets and involves the interplay of various methods from geometry, analysis, dynamics, and algebraic geometry.

Due to the uniformization theorem, the Teichmüller space of a surface can be realized as the space of marked hyperbolic structures. In this way the Teichmüller space can be identified with a connected component of the space of representations of the fundamental group of the surface into the Lie group $\mathsf{PSL}(2,\mathbb{R})$, which consists entirely of faithful representations with discrete image.

In fact, this is a more general phenomenon: there are also higher rank semisimple Lie groups admitting connected components of the character variety consisting only of faithful representations with discrete image, the so-called higher rank Teichmüller spaces. The study of this phenomenon is often referred to as higher rank Teichmüller theory. As in classical Teichmüller theory, higher rank Teichmüller theory builds on a combination of methods from various areas of mathematics. And the very different techniques from bounded cohomology, Higgs bundles, positivity, cluster algebras, harmonic maps, incidence structures, geodesic currents, real algebraic geometry, and dynamics applied in higher Teichmüller theory add to the richness of the topic.

In this Arbeitsgemeinschaft we will study some aspects of higher Teichmüller theory, starting from a short review of key properties of the classical Teichmüller space. The program will focus on geometric aspects, but explore also relations to algebraic structures (e.g. cluster algebras), Higgs bundles, and dynamical properties. The Arbeitsgemeinschaft will include some exercise sessions.

Below is a preliminary version of the program; it might be slightly adapted or made more precise later. Please do not hesitate to contact the organizers if you have questions.

Some general information: "Arbeitsgemeinschaft" means "study group". All lectures at the Arbeitsgemeinschaft are given by participants and the goal is communal learning by active participation. All applicants are required to volunteer to give lectures, although there will be more participants than lectures.

Applications for participation should be addressed to the director by email to ag@mfo.de prior to the deadline, which is May 31, 2022. Applications should include your full name and postal address, a CV, a publication list, and a paragraph justifying why you would be interested in participating.

You should also indicate which talks you are willing to give:

- First choice: talk no. ...
- Second choice: talk no. ...
- Third choice: talk no. ...

You will be informed as soon as possible after the deadline if your participation is possible and whether you have been chosen to give one of the lectures.

1 Monday

1.1 Teichmüller space I

Goal of the talk: Review classical Teichmüller theory from the point of view of hyperbolic geometry.
Suggested plan: Introduce the Teichmüller space of a closed oriented surface S as the moduli space of marked hyperbolic structures on S. Discuss holonomy representations, coordinates (shear and Fenchel–Nielsen coordinates; gluing construction for hyperbolic structures, interpretations as amalgamation of representations). Discuss the action of the mapping class group. Time permitting, mention Thurston's compactification. References include [Hub06, Ch. 3], [Pap07], and [FM12].

1.2 Teichmüller space II

Goal of the talk: Review classical Teichmüller theory from the point of view of complex analysis.
Suggested plan: Define Riemann surfaces and state the classical Poincaré–Koebe uniformization theorem. Define holomorphic quadratic differentials and their associated half-translation surface, see [Hub06]. Build explicit deformations of the conformal structure using this half-translation surface and state Teichmüller's theorem [Hub06, Th. 5.3.8]. Identify the cotangent space of Teichmüller space with the space of quadratic differentials [Hub06, Prop. 6.6.2]. Define the Weil–Petersson Kähler metric [Hub06, §7.7]. State the works of Bonahon [Bon88], Goldman [Gol84] and Wolpert [Wol86], relating the real part of this metric to the Hessian of Bonahon's intersection number, and the imaginary part to Goldman's symplectic form on the character variety. State Wolpert's theorem [Hub06, Th. 7.8.1] showing that Fenchel–Nielsen coordinates are symplectic with respect to the imaginary part of the Weil–Petersson metric. Time permitting, discuss harmonic maps and particularly [Wol89].

1.3 Hitchin component

Goal of the talk: Introduce the Hitchin component of the character variety of a closed surface group into $\mathsf{PSL}(n,\mathbb{R})$ (which was first considered and studied in [Hit92]). Discuss the basic properties of the corresponding surface group representations, known as Hitchin representations. Describe analogies between the Hitchin component and the classical Teichmüller space.
Suggested plan: Define Hitchin representations of a closed surface group into $\mathsf{PSL}(n,\mathbb{R})$ (or more generally into a real split simple Lie group).

For $n = 3$, explain their geometric interpretation as holonomies of convex projective structures on the surface, which follows from the work of Koszul [Kos68] (openness), Goldman [Gol90] (connectedness), and Choi–Goldman [CG93] (closedness); this implies that Hitchin representations into $\mathsf{PSL}(3,\mathbb{R})$ are faithful with discrete image.

For general n, explain how the existence of continuous equivariant boundary maps (established in [FG06] and [Lab06]) implies that Hitchin representations are faithful with discrete image. Mention that Hitchin representations are Anosov [Lab06] and positive [FG06], as will be explained later in talks 2.1 and 4.1. Describe the characterization of Hitchin representations in terms of Frenet maps from [Lab06, Gui05].

Give analogies between the Hitchin component and the classical Teichmüller space: topology [Hit92], mapping class group action [Lab08], collar lemma [LZ17], simple root entropy [PS17], coordinates (see talk 2.4), etc. Mention also differences: e.g. sequences with entropy going to zero [Zha15].

The survey [Can21] might be useful.

1.4 Maximal representations

Goal of the talk: Introduce maximal representations of surface groups into simple Lie groups of Hermitian type. Discuss some of the geometric properties of these representations and of their parameter space.
Suggested plan: Introduce the Euler number as a characteristic number distinguishing Teichmüller space in the $\mathsf{PSL}(2,\mathbb{R})$ character variety, and the Toledo number as its generalization leading to maximal representations [BIW14, § 3.4 and § 5.1]. Describe some examples of Fuchsian loci in maximal character varieties [BILW05, § 3.1], and the characterization of maximal representations in terms of boundary maps [BIW10, Th. 8]. Deduce that maximal representations are faithful with discrete image [BIW10, § 4.3-4], and thus form higher Teichmüller spaces. You should focus on representations of fundamental groups of closed surfaces, but mention that the theory also works for surfaces with boundary, and possibly mention the additivity of the Toledo invariant under surface decompositions [BIW10, Th. 1]. Discuss the topology of connected components of maximal representations: in most cases these components have been counted with the aid of Higgs bundles [BGPG06], but it is possible to obtain a lower bound on their number also with topological methods [GW10]. Mention the existence of components consisting entirely of Zariski-dense representations

into $\mathsf{PSp}(4,\mathbb{R})$ (first discovered by Gothen [Got01a]; a good reference is [AC19], where the homeomorphism type of the components is also described). Time permitting, discuss analogies and differences with Teichmüller space as in talk 1.3: e.g. mapping class group action [Wie06], collar lemma [BP17, § 3], simple root entropy [PSW19, § 9.2], coordinates [Str15, AGRW19].

2 Tuesday

2.1 Anosov representations

Goal of the talk: Introduce Anosov representations — a large class of representations of closed surface groups or more generally of Gromov hyperbolic groups, including all the representations appearing in higher Teichmüller spaces. Give examples and discuss basic properties of Anosov representations.

Suggested plan: Define Anosov representations of a closed surface group into $\mathsf{SL}(n,\mathbb{R})$ or $\mathsf{PGL}(n,\mathbb{R})$, first introduced by Labourie [Lab06]. Mention that the definition extends to general Gromov hyperbolic groups, see [GW12].

Give various characterizations of Anosov representations, in particular in terms of singular value gaps: see [KLP16, GGKW17, KLP18b, BPS19] and [Can20, Kas19].

Discuss some important basic properties of Anosov representations: they have finite kernel and discrete image (same proof as in talk 1.3); they are in fact quasi-isometric embeddings; they form an open subset of the representation variety [Lab06], and in fact of the character variety (see [GGKW17, Prop. 1.8]).

Give examples of Anosov representations, including convex cocompact representations into rank-one simple Lie groups, Hitchin representations of closed surface groups into real split simple Lie groups, and maximal representations of closed surface groups into simple Lie groups of Hermitian type. Note that there exist Anosov representations of more general Gromov hyperbolic groups than just surface groups or free groups: e.g. any Gromov hyperbolic Coxeter group admits an Anosov representation, and such groups may not be isomorphic to uniform lattices in rank-one simple Lie groups [DGK18, LM19].

2.2 Domains of discontinuity and geometric structures

Goal of the talk: Explain how one can associate, to Anosov representations, cocompact domains of discontinuity in certain flag varieties, yielding compact quotient manifolds with a geometric structure. Describe these geometric structures more precisely in some examples for representations corresponding to higher Teichmüller spaces.

Suggested plan: Recall (from talk 1.3) that any Hitchin representation into $\mathsf{PSL}(3,\mathbb{R})$ acts properly discontinuously and cocompactly on some properly convex open subset Ω of the projective plane $\mathbb{P}(\mathbb{R}^3)$; the boundary of Ω is the limit set of the Hitchin representation, and the boundary of the dual of Ω is its dual limit set. Explain how, for $p > q \geq 1$, any P_1-Anosov (resp. P_q-Anosov) representation into $\mathsf{SO}(p,q)$ admits a cocompact domain of discontinuity in the space of isotropic q-subspaces (resp. lines) of $\mathbb{R}^{p,q}$, described explicitly in terms of the limit set in the space of isotropic lines (resp. q-subspaces) of $\mathbb{R}^{p,q}$, following [Fra05, GW12]. Briefly describe the general theory of Kapovich–Leeb–Porti [KLP18a] giving a construction of cocompact domains of discontinuity for any choice of a balanced ideal in a double quotient of the Weyl group.

Describe the associated geometric structures in a couple of cases where we have a better description: Hitchin representations into $\mathsf{SL}(4,\mathbb{R})$ (following [GW08]) and maximal representations into $\mathsf{SO}(n,2)$ (following [CTT19]).

Time permitting, discuss oriented flag manifolds [ST], and give an example where this gives something truly new.

2.3 Pressure metrics

Goal of the talk: Define the family of pressure semi-norms on the character variety of θ-Anosov representations ([BCLS15]). Discuss their geometric meaning in various contexts ($\mathsf{PSL}(2,\mathbb{R})$: Bonahon [Bon88], Bridgeman-Taylor [BT08], McMullen [McM08], Wolpert [Wol86]; higher rank: Bridgeman-Pozzetti-Sambarino-Wienhard [BPSW20], Dai [Dai19], Labourie-Wentworth [LT20]).
Suggested plan: Define entropy, intersection and normalized intersection for positive Hölder potentials over a metric-Anosov flow, see [BCLS18, § 2] or [BCS18, § 2.2].

Define the geodesic flow of a projective Anosov representation ([BCLS15, § 4]). It is a metric-Anosov flow ([BCLS15, § 5]) and given two such representations the corresponding flows are Hölder-reparametrizations one of the other. Follow then [PS17, § 4] to define, locally, a pressure form for each $\varphi \in (\mathfrak{a}_\theta)^*$ that is positive on Benoist's limit cone associated to the representation in question. In particular every $\varphi \in \mathfrak{a}^*$ which is non-negative on the chamber defines a pressure semi-norm on the space of Δ-Anosov representations.

State the non-degeneracy result [BCLS15, Th. 1.4] for the pressure metric associated to the spectral radius, on the space of irreducible, G-generic, projective Anosov representations.

Recall from talk 1.3 that in Teichmüller space the obtained form is the real part of the Weil–Petersson Kähler metric (Bonahon [Bon88] and Wolpert [Wol86]).

State the analogous result on the Fuchsian locus of a Hitchin representation from Labourie–Wentworth [LW18, Th. 1.0.1]: the decomposition of the tangent space in sum of holomorphic differentials is orthogonal for the spectral radius pressure metric and on each degree the metric is a multiple of the L^2-metric.

2.4 Fock–Goncharov coordinates and cluster varieties

Goal of the talk: Discuss Fock–Goncharov coordinates and their cluster structure. Shortly explain Bonahon–Dreyer coordinates for closed surfaces.
Suggested plan: Introduce Fock–Goncharov coordinates for framed and decorated local systems focusing on the case of $\mathsf{PSL}(n,\mathbb{R})$, in particular $\mathsf{PSL}(3,\mathbb{R})$

[FG06, FG07b]. (See [FG07a] for a review of the $\mathsf{PSL}(2,\mathbb{R})$ case. Describe the connection to total positivity and to cluster algebras. Discuss mutations and flips of a triangulation as a sequence of mutations. Time permitting, provide a short outlook beyond $\mathsf{PSL}(n,\mathbb{R})$ [FG06, Le19a, Le19b, GL19], relation to representations in the Hitchin component, and mention Bonahon–Dreyer coordinates for closed surfaces [BD14, BD17], which build on Fock–Goncharov's work.

3 Wednesday

3.1 Higgs bundles

Goal of the talk: Discuss the nonabelian Hodge correspondence for surfaces [Don87, Hit87, Cor88, Sim92]. Explain Morse theoretic tools to understand the topology of the moduli space of Higgs bundles.
Suggested plan: Explain how the nonabelian Hodge correspondence works, how given a reductive representation of the fundamental group of a surface into a reductive Lie group G one constructs an equivariant harmonic map from the universal cover of the surface into the symmetric space of G. Then explain how the choice of a complex structure on the surface allows to construct a Higgs bundle, see [Li19]. Explain how a Higgs bundle leads to a local system on the surface. Discuss some examples, see e.g. [Col19]. The moduli space of Higgs bundles often allows for explicit parametrizations, see [Hit92] for the Hitchin component, and [BGPG06] for maximal representations. Discuss how the L^2-norm of the Higgs field can be used as a Morse function on the moduli space of Higgs bundles, and allows to give information on the number of connected components of the space of maximal representations [Got01b, BGPG08].

3.2 Projections to Teichmüller space

Goal of the talk: Describe rank-2 higher Teichmüller spaces as mapping-class-group-equivariant fiber bundles over the classical Teichmüller space of the surface [AC19, CTT19, Lab17, Lof01].
Suggested plan: Start with Loftin's [Lof01] and Labourie's [Lab07b] construction for $\mathsf{PSL}(3,\mathbb{R})$. One may follow [Cal15, § 6].

The fibrations of [AC19, CTT19, Lab17] are found via the uniqueness of a minimal surface, associated to the representation, on the symmetric space of the ambient group. Cyclic Higgs bundles introduced by Baraglia ([Bar, § 3.5] and [Bar15]) play a central role in proving uniqueness.

Vector bundles of holomorphic differentials over Teichmüller space carry natural Kähler structures ([KZ17] and more specifically [Lab17, Prop. 9.0.1]).

Compare with Labourie–Wentworth [LW18] stating that the obvious choices of $\varphi \in \mathfrak{a}^*$ do not provide compatible pressure metrics.

4 Thursday

4.1 Θ-positivity I

Goal of the talk: Define the notion of Θ-positivity for arbitrary noncompact real semisimple real groups, explicit earlier known examples, state the classification result by Guichard–Wienhard [GW18, Th. 4.4]. Emphasize that the space of positive triples with fixed extremes $\{x, y\}$ is a union of connected components of the space of flags that are transverse to both x and y, and each connected component has the structure of a semi-group [GW18, Th. 4.7].
Suggested plan: Define total positivity for $\mathsf{SL}(d, \mathbb{R})$, recall that a minor of a matrix corresponds to a coefficient of an exterior power of it, deduce the existence of a fixed flag.

State the main results from Lusztig [Lus94], one could follow [GW18, § 2.1–2.2] or [Sam20, § 5.1–5.3] for brief summaries. Recall the characterization of Hitchin representations in terms of positive maps from Fock–Goncharov [FG06].

Quickly recall the Maslov index [GW18, § 3.2] and the characterization of maximal representations in terms of positive maps from [BIW10].

Follow [GW18, § 4].

4.2 Θ-positivity II

Goal of the talk: Introduce Θ-positive representations, and the Higgs bundles perspective on their moduli spaces.
Suggested plan: Introduce Θ-positive representations [GW18], and discuss the cases in which it is known that Θ-positive representations form higher rank Teichmüller spaces [GLW21, BP21]: Θ-positive representations form an open subset of the character variety, which is closed within the set of non-parabolic representations [GLW21], and is always closed if the target group is $\mathsf{PO}(p, q)$ [BP21]. Discuss the parallel work on magical $\mathfrak{sl}(2)$ triples from the Higgs bundles perspective [AABC$^+$19, BCG$^+$21]; since the special components consist entirely of non-parabolic representations, they are higher rank Teichmüller spaces [GLW21]. Explain what is known about the number of connected components of Θ-positive representations, and the existence of components consisting entirely of Zariski-dense Θ-positive representations in the $\mathsf{SO}(n, n+1)$-character variety.

4.3 Compactifications via currents

Goal of the talk: Discuss different geodesic currents associated to representations in higher rank Teichmüller theories, and discuss how this leads to various compactifications of the spaces.
Suggested plan: Quickly discuss Bonahon's work on geodesic currents, mention weighted multicurves and the Liouville current for a hyperbolic metric as examples of geodesic currents, introduce the intersection function on the space of geodesic currents, and Bonahon's compactification of Teichmüller space [Bon88]. Introduce the notion of positively ratioed representations by Martone–Zhang

[MZ19]: explain how for these representations some length functions, more precisely the length functions associated to the fundamental weights, can be computed as intersection with a geodesic current [MZ19, Th. 1.1]. Mention that not only Hitchin and maximal representations [MZ19, § 3], but also Θ-positive representations into $\mathsf{SO}(p,q)$ [BP21, § 4], are positively ratioed with respect to all the Anosov roots. Discuss how this leads to a compactification of the higher rank Teichmüller spaces [BIPP19]. Explain how, in the case of rank-2 groups, it is possible to associate to representations in various higher rank Teichmüller spaces a different geodesic current, induced by the negatively curved metric on a suitable minimal surface: this links with talk 3.2, and was used in the work of Ouyang and Tamburelli on compactifications [Ouy19, OT21, OT20]. Discuss mixed structures, and their appearance in the Ouyang–Tamburelli compactification.

4.4 Compactifications, buildings and geometric structures

Goal of the talk: Discuss the link between compactifications of higher rank Teichmüller spaces and actions on $\mathbb{R}$-buildings, as well as the geometric structures associated to some points in the compactifications in low rank.
Suggested plan: Discuss ultralimits of representations as representations with values in Robinson fields [Par12], the natural appearance of $\mathbb{R}$-buildings as asymptotic cones of symmetric spaces, and the real spectrum perspective to organize such representations in a compact space [BIPP21a]. Illustrate the associated geometric structures, mentioning Parreau's examples of $\mathfrak{a}^+$-structures for some points in the Hitchin component of open surfaces [Par15], and singular flat structures for maximal representations into $\mathsf{SL}(2,\mathbb{R}) \times \mathsf{SL}(2,\mathbb{R})$ (for this compare the work of Martone–Ouyang–Tamburelli [MOT21] and Burger–Iozzi–Parreau–Pozzetti [BIPP21b], also in relation to talk 4.3).

5 Friday

5.1 Symplectic geometry of higher rank Teichmüller spaces

Goal of the talk: Introduce the Goldman symplectic structure on character varieties. Discuss applications for higher rank Teichmüller spaces.
Suggested plan: Introduce the Goldman symplectic structure on the character variety of a surface group into a reductive Lie group [Gol84]. Discuss shortly that the pressure metric is not compatible with the symplectic form, see [LW18]. Introduce twist flows along separating curves and show that twist flows are Hamiltonian functions of appropriate length functions [Gol86]. A consequence is that Fenchel–Nielsen coordinates on Teichmüller space are Darboux coordinates, and the symplectic form has a nice formula in terms of length and twist coordinates (Wolpert's formula). Hyperbolic structures on a pair of pants are entirely determined by the length of the boundary curves, this is not true anymore in higher rank Teichmüller spaces. There are interesting new flows arising,

which deform the structure on a pair of pants. Introduce eruption flows for the Hitchin component of $\mathsf{PSL}(3,\mathbb{R})$ [WZ18], and if time permits give an outlook on the general case [SWZ20, SZ17].

5.2 Higher complex structures

Goal of the talk: Describe the partly conjectural work of Fock and Thomas on higher complex structures.
Suggested plan: Classical Teichmüller theory describes the space of complex structures on a surface. This complex analytic side is still largely missing in higher rank Teichmüller theory. In recent years, Fock and Thomas introduced the concept of higher complex structures [FT21]. They conjectured that the space of higher complex structures is isomorphic to the Hitchin components and established several steps towards this conjecture. Introduce the concept of higher complex structures, define its moduli space, state the conjecture and explain what is proven [FT21, Tho20a, Tho20b].

5.3 Universal Teichmüller space

Goal of the talk: Describe various proposals that have been made to construct universal higher rank Teichmüller spaces.
Suggested plan: Describe various attempts to define generalizations of universal Teichmüller spaces. One or more of the following aspects can be discussed: the study of a universal Hitchin component, understood as the limit as n diverges by Hitchin [Hit16, Biq19, Hit20]; a non-equivariant Hitchin/maximal representation variety, understood as space of rank-n cross ratios by Labourie, resp. as space of positive curves by Labourie–Toulisse [Lab07a, LT20] Sullivan's perspective on universal Teichmüller space, and the link with Hitchin representations discovered by Tholozan [Tho19].

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