ARBEITSGEMEINSCHAFT: QUANTITATIVE STOCHASTIC HOMOGENIZATION

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Homogenization means approximating the effective, i. e. macroscopic, behavior of a heterogeneous medium by a homogeneous one, which amounts to a substantial conceptual and practical reduction of complexity. Stochastic homogenization means that one is considering an ensemble of, i. e. a probability measure on, such heterogeneities (typically expressing a lack of knowledge of the details); and that the effective behavior is also deterministic next to being homogeneous. In this Arbeitsgemeinschaft, we focus on media that are described by a linear elliptic operator in divergence form, which may describe the conductive, elastic or viscous properties of the medium. Quantitative stochastic homogenization means that we focus on quantitative aspects of this theory, where there has been much recent progress. The subject, which is motivated by materials science, combines the theory of linear partial differential equations (PDE) with hands-on probability theory, and also has a life in computational mathematics and statistical physics.

We propose a program that starts with the general theory of oscillatory coefficients (with its rational mechanics flavor), and then addresses the qualitative theory of stochastic homogenization (with its functional analytic flavor). Depending on the participants, we may connect to the topic of random walks in random environments. Quantitative stochastic homogenization is intimately connected to elliptic regularity theory in Hölder and L^p-spaces: Large-scale regularity emerges and helps in the quantification. Stochastic estimates are typically derived from spectral gap or from finite range assumptions. The rate of the convergence depends on whether it is expressed in terms of strong or weak topologies. Depending on the participants, we then may branch out to computational aspects, to the design of electro-magnetic (meta-)materials, to the effective viscosity of dilute suspensions (Einstein's PhD thesis), to boundary layer effects, and to wave propagation.

The following detailed list of topics with references provides suggestions. We give a wide selection so that participants have options – we are aware that only a small part can be covered within a week. We also encourage participants to present other work, provided it is not from their own group.

1. General theory of homogenization

We propose three introductory presentations on the (periodic then general) qualitative theory.

1.1. **Two-scale expansion.** The best way to heuristically identify the macroscopic behavior of the solution of a PDE with microscopically periodic coefficients is to make an ansatz in terms of a microscopic and a macroscopic variable and to use formal asymptotic

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expansions – see [18, Chapter 1, Section 2]. The key object of a corrector already appears in this context.

1.2. **Two-scale convergence.** At least in the case of periodically oscillating coefficient fields, this two-scale expansion is the basis of an elegant rigorous theory, see the reference article [4] based on [72].

1.3. **H-convergence.** In the general case, homogenization amounts to identifying the macroscopic limit of *products* of microscopically oscillatory sequences, relating fluxes (of controlled divergence) and fields (of controlled rotation). By constructing oscillating test functions, this can be tackled via compensated compactness. It is natural to recast homogenization as convergence of solution operators. This leads to the notion of H-convergence on the level of the coefficient fields, as studied in the original article [69] and in the monograph [77]. We suggest [69] or [5, Chapter 1, Sections 1.2 & 1.3] as material. The set of all uniformly elliptic coefficient fields is compact under this topology.

2. Qualitative stochastic homogenization

In the second part, we specify to random coefficient fields. We propose two presentations for the main part, and – depending on the participants – two presentations on alternative approaches.

2.1. **Probability measures on the space of coefficients.** Considering a probability measure on the space of coefficient fields requires fixing a σ-algebra of measurable functions/observables suitable for homogenization. There are several approaches: the abstract ergodic approach [58, Section 7.1], a constructive approach [75], and an approach based on H-convergence, [53, Lemma 18], motivated by Γ-convergence [24]. We propose to consider the last approach. The minimal requirements for the qualitative theory are stationarity and ergodicity, which will also be introduced in this talk.

2.2. **Construction of correctors.** As we learned from the two-scale expansion, correctors are the key objects in homogenization. There are two approaches to constructing correctors with stationary gradients: one based on infra-red regularization [75], another on decomposition of a vector field in solenoidal and potential parts, see [58, Chapter 7] and [51, Lemma 1]; we propose to consider the last approach. In preparation of the next section, this talk also constructs the convenient flux correctors. Equipped with the correctors, one can show that stochastic homogenization amounts to a special instance of H-convergence, cf. [58, Chapter 7].

2.3. **Subadditive ergodic theorem and homogenization.** Instead of using the PDE (via correctors and compensated compactness), one might consider the variational problem when coefficients are symmetric (in which case one has compactness in terms of Γ-convergence of integral functionals rather than compactness of operators in terms of H-convergence). Correctors are by-passed via the use of the subadditive ergodic theorem in [24] (short lecture notes will be available for the reduction to the linear setting).

2.4. **Stochastic two-scale convergence.** There are extensions of two-scale convergence to the random setting. The closest such notion is the quenched stochastic two-scale convergence of [78] (which is streamlined in [55, Section 1]).

3. Random walks in random environments

Instead of considering the solution of an elliptic PDE with random coefficients, one can consider a random walk in a random environment. There are two sources of randomness: the environment (like coefficients in a PDE) and the random itself walk in that environment. When the environment is constant, the law of the (suitably rescaled) trajectory of the random walker converges to the law of a Brownian motion. What about the random walker in a random environment ?

3.1. **The Kipnis-Varadhan argument (annealed invariance principle).** We start with a first type of convergence: the law of the random walker converges to the law of a Brownian motion (with covariance given by the homogenized coefficients) after averaging in probability over the environments. This is an annealed invariance principle, cf. [61, 26]

3.2. **The quenched invariance principle.** The annealed convergence result can be upgraded into an almost sure result (that is, convergence in law to Brownian motion of almost every realization of the environment), cf. [67].

3.3. **Statistical mechanics.** Homogenization has been used to study models of mathematical physics. A very inspiring work (which is somehow the starting point of quantitative homogenization) is the study of the large-scale behavior of statistical mechanical models defined by convex Hamiltonians in [70]. The $\nabla\phi$-model is a statistical mechanical model for the evolution of an interface, it is studied using homogenization in [42], see also the recent work [13] for quantitative results.

4. Quantitative stochastic homogenization

Sections 4 and 5 are closely related. In order not to overload the program, we first make strong assumptions that allow us to separate the two subjects, and rely solely on deterministic regularity results to prove quantitative homogenization results. In the general case, we have to appeal to large-scale regularity instead, which we postpone to Section 5. In the present section,

- We quantify oscillations by estimating how close the gradient of the two-scale expansion is to the gradient of the solution itself in a strong norm;
- We quantify fluctuations: In stochastic homogenization, the solution does not only oscillate (like in periodic homogenization) but it does also randomly fluctuate. The aim is to characterize the asymptotic behavior of these fluctuations (which amounts to estimates in weak norms).

In this section we shall assume that the coefficient field is a perturbation of the identity to the effect that we may assume (weighted) Meyers' estimates (that is, L^p-regularity with weight) for p as large as needed and weights in the (standard) Muckenhoupt class.

4.1. **Functional calculus in probability.** The corrector is a random object that depends nonlinearly and non locally of the coefficient field. A convenient way to investigate correlations / stochastic properties of the corrector is to use functional calculus in probability, such as variance estimates, logarithmic-Sobolev inequalities, Stein's inequality, the Helffer-Sjöstrand representation formula, or Malliavin calculus. Such tools are described in [64, 23, 33]. We propose to focus here on Gaussian fields with integrable correlations and on the associated Malliavin calculus in form of [34, Section 4 and Appendix].

4.2. **Variational approach for small convergence rate.** Functional calculus in probability does not necessarily hold. For coefficients with a finite range of dependence (but not iid), one has to proceed differently, more in the renormalization group perspective (relying on local averages and a decomposition of scales). The first quantitative result in this context was obtained in [14] and [11, Section 2] using a variational approach and Meyers' estimate. It proves the strict sublinearity of the correctors in this context.

4.3. **Bounds on correctors.** In order to quantify the two-scale expansion error, one needs to characterize the growth of correctors (which is necessarily strictly sublinear by qualitative ergodicity). The shortest proof of optimal bounds goes via the CLT scaling and a buckling argument when the coefficients fields that have integrable correlations, see [59, Section 4].

4.4. **Pathwise structure of fluctuations.** To characterize fluctuations of the observables of the gradient of the solution (that is, average with a given test function), one may appeal to two-scale expansions as for the characterization of oscillations. For fluctuations however, one should not consider the gradient of the solution itself, but rather the homogenization commutator introduced in [33] (another quantity built on the coefficients, the solution, and the homogenized coefficients). Using the two-scale expansion of the commutator allows to characterize the leading order of the fluctuations of the gradient of the solution by (suitable) fluctuations of the standard homogenization commutator (that of the corrector). Here we propose to follow a streamlined version of the approach of [59, Section 6] for weakly correlated field by assuming as strong deterministic L^p regularity as needed.

4.5. **Scaling law of commutator.** Once we have described the leading-order fluctuations of the homogenization commutator of the solutions by the fluctuations of the standard homogenization commutator, it remains to characterize the latter as done in [33, 34]. As above, we propose to streamline the approach of [29, Sections 3 and 5] based on Stein-Malliavin calculus for weakly correlated field by assuming as strong deterministic L^p regularity as needed, and on the Helffer-Sjöstrand representation formula for covariances.

5. Large-scale regularity

This section focuses on large-scale regularity results which are proxies for the deterministic regularity used in Section 4 (which does not hold in more general cases). The general idea is to lift the nice regularity properties of the homogenized (i.e. constant coefficient) operator at the level of the original operator at sufficiently large scales.

5.1. **Large-scale $C^{1,\alpha}$-regularity.** For equations with measurable coefficients, Schauder theory does not apply, and one does not have $C^{1,\alpha}$-regularity for solutions. Whereas classical regularity locally compares functions to affine functions, in the context of homogenization, one should rather locally compare functions to affine functions plus their correctors. In these terms, one can prove regularity at large scales, cf. the original work [15] in the periodic setting, and [14], [11, Chapter 3], and [51, 39] in the random setting. This regularity cannot hold at the scale of the oscillations (since then, coefficients are only measurable), but it holds from a minimal scale onwards – which we call minimal radius, a stationary random field. We propose to follow the approach of [51, Sections 3.2–3.6].

5.2. **Quenched large-scale Calderón-Zygmund estimates.** For equations with constant coefficients, next to $C^{k,\alpha}$-regularity, we also have L^p-regularity theory (Calderón-Zygmund estimates). In line with large-scale $C^{1,\alpha}$-theory, one can introduce large-scale Calderón-Zygmund estimates, see [16] in the periodic setting, and [11, Chapter 3], [51] and [34, Proposition 6.4] in the random setting – where we have to locally square-average functions at the scale of the minimal radius before taking the L^p-norm. We propose to follow the approach of [51, Sections 3.7 & 3.8].

5.3. **Systematic argument for bounds on the minimal radius.** When coefficients have integrable correlations and one has functional calculus in probability at hand, the same argument giving bounds on correctors allows to control the minimal radius based on CLT scaling, deterministic regularity, and a buckling argument, see [59]. The stochastic integrability one gets is however suboptimal. For the optimal result, one has to use large-scale regularity itself (rather than deterministic regularity) to control the minimal radius, and we refer to the argument of [51, Sections 4 & 5].

5.4. **Annealed Calderón-Zygmund estimates.** In the quenched Calderón-Zygmund estimates (which hold almost surely), we first need to locally square-average at the minimal scale. The smaller the minimal radius, the stronger the estimate. The minimal radius varies from point to point, so that the size of the local average in the quenched Calderón-Zygmund estimates depends on the point in space. A way to make this dependence uniform on the point is to take stochastic moments, which leads to annealed Calderón-Zygmund estimates, as obtained in [9, 34, 59, 11]. We propose to follow the approach of [34, Section 6].

5.5. **Bourgain's result.** Bourgain's only result in homogenization concerns the expectation of the solution of a discrete elliptic equation with random coefficients: it shows that it can be expanded up to order $2d$, whereas one can only go up to $d/2$ for the solution itself (that is, without expectation). The original articles are [22, 60]. They are rephrased in more standard homogenization terms in [37], see also [28] for nonperturbative results.

6. Numerical approaches to homogenization

6.1. **Approximation of homogenized coefficients.** To use the homogenized equation one needs to approximate the homogenization coefficients, and therefore to approximate the solution of the corrector equation. There are two contributions to the error when approximating homogenization: a systematic error and a random error. Let us focus on the systematic error. Methods based on periodization or on the use of a massive term and filtering are analyzed in [50, Section 5] and [48, 49]. Methods based on a multiscale decomposition are analyzed in [54].

6.2. **Variance reduction techniques.** The random error is due to fluctuations, it is fully analyzed in [52]. A possible way to reduce this error is to use variance reduction methods, see [19, 63], and the analysis of [38].

6.3. **Approximation of solutions.** The numerical approximation of small-scale oscillations in the homogenization regime goes beyond homogenized coefficients. An iterative method is proposed and analyzed in [10]. In [65], an artificial boundary condition is introduced to localize the effect of heterogeneities outside a domain of interest.

6.4. **Orthogonal decomposition of scales.** When there is no scale separation or the law of the coefficients is not precisely known, methods based on homogenization cannot be used. One can then use compactness properties of the elliptic operator to devise suitable low-dimensional spaces where the solution can be well-approximated. This is done in [66, 73].

7. Homogenization in material sciences

7.1. **Bounds on homogenized coefficients.** When homogenization is viewed as a convergence of operators, one may ask oneself what limits one can reach starting from the mixture of two given materials in given proportion. There are two ways of doing this: by finding bounds on the possible limits or by identifying the set of possible limits (the so-called H-closure, related to relaxation via the calculation of convex / quasi-convex hulls), see [41, 40] and the monograph [58, Chapter 6] and [68].

7.2. **Meta-materials.** Meta-materials are composite materials that display original physical properties hat are not encountered in nature, for instance invisibility cloaks for submarines for certain frequencies. This can be rephrased as homogenization of the Maxwell equations, and results heavily depend on the microstructure, e.g. flat rings in [62, 21], and other micro-structures [20].

8. Homogenization in fluids

8.1. **From (Navier-)Stokes to Brinkman.** The homogenization of PDEs on domains with small holes (with nontrivial boundary conditions) yields the occurrence of capacitary terms – such as the Brinkman force for the homogenization of the (Navier-)Stokes equation with fixed obstacles:

- Stokes with periodic inclusions: [76, Chapter 7 and Appendix by Tartar].
- Navier-Stokes with periodic inclusions: [3, 2]
- scalar equation with random inclusions: [74, 46]
- Stokes with random inclusions: [47]

8.2. **Homogenization of random suspensions and the Einstein formula.** In one of his most cited papers of 1905, Einstein derived a first order formula for the effective viscosity of a Stokes fluid with small rigid particles in the regime of small volume fraction. The very notion of effective density involves the homogenization of the steady Stokes equation with a random suspension of rigid particles interacting with the fluid: see [30] (with uniform separation of particles), [35] (with moment bounds on the distance between particles), [31] (without minimal assumption on the minimal distance between particles, but with percolation estimates). For the justification of the Einstein formula, see [44, 36].

8.3. **Sedimentation of random suspensions.** When the particles in the random suspension are heavier than the fluid, they fall by gravity. The quantity of interest is the speed of sedimentation, which is analyzed in [32], and requires stochastic cancellations from the very beginning.

9. Homogenization and boundaries

9.1. **Convergence rate and boundaries.** Boundary conditions are not necessarily compatible with the oscillations the solution would develop in the whole space, and this generates boundary layers, cf. [6, 45, 12].

9.2. **Effective interface conditions.** When heterogeneous media are coupled via an interface, homogenization does not only yield homogenized media but also an effective transmission conditions. Examples are given in [56, 57, 71].

9.3. **Homogenization of boundary conditions in fluids.** Consider a fluid in a domain with a rough and oscillating boundary with some standard boundary condition. Homogenization then aims at replacing the oscillating boundary by a flat boundary and effective boundary conditions, as done in [1, 43, 25] for instance.

10. Homogenization of the wave equation

This section considers the classical wave equation with heterogeneous coefficients and the main question concerns the long-time description of the flow.

10.1. **Spectral theory and Bloch waves in the periodic setting.** For constant coefficients, the Fourier transform allows to explicitly solve the wave equation on the whole space. For periodic coefficients, one may rely on the Floquet-Bloch theory: [8, 7].

10.2. **Approximate spectral theory in the random setting.** The Floquet-Bloch theory is limited to periodic coefficients. By relating it to correctors, it can be partly extended to the random setting: [17, 27].

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