Oberwolfach Seminar:
Interfaces: Modeling, Analysis, Numerics

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Program:

Interfaces separating different states appear in various mathematical problems and they frequently appear in applications, see [1-4]. Often the evolution of an interface is governed by a law relating the normal velocity to curvature quantities and possibly other variables that are defined on the interface or in the domains away from the interface. Equations for these variables will in turn typically depend on the interface evolution giving rise to a complex mathematical system. Such a system may even involve the solution of a PDE on the interface, a subject that has developed into a very active research area in recent years. An important starting point for the analytical and numerical solution of these problems is a suitable description of the interface. To do so, parameterizations, level set approaches, volume of fluid methods, measure theoretic approaches and phase field methods are possible ways to represent interfaces.

Using a parametric approach for the interface evolution typically leads to fully nonlinear or at least quasi-linear partial differential equations whose exploration requires knowledge about the theory of abstract evolution equations and maximal regularity. Weak approaches for interfaces often involve measure theoretic methods such as varifold theory and approaches that use Caccioppoli sets. An advantage of these approaches is that they allow for topological changes which are not directly possible in sharp interface methods based on parameterizations. In phase field methods the interface is described as a diffuse interfacial layer. The governing equations typically allow for quite smooth solutions which is beneficial both for the analysis and for the numerics. Although phase field methods can be derived using classical thermodynamical principles, it is important to relate them to classical sharp interface descriptions, and many analytical questions in this context are still open.

Different representations of the interface go with different numerical approaches. Recently, classical parametric descriptions of the interface came back into the focus of the international research community because new ideas entered the field yielding in particular a better mesh quality. Phase field approaches and level set methods in contrast allow for an implicit way to treat topological changes. In recent years, both parametric and implicit approaches have also successfully been applied to the solution of PDEs on stationary or evolving interfaces. Moreover numerical analysis for systems coupling a geometric evolution of an interface to a PDE defined on the moving interface has received increasingly attention.

The proposed seminar aims at giving an introduction to mathematical methods for dealing with interfaces as described above. We will address all three aspects mentioned in the title: modeling, analysis and numerics to give a broad picture of this dynamic field of research.

After a short introduction with motivating examples from applications (grain growth, solidification, bio-membranes, two-phase flow) the seminar is structured as follows:

- Some notions from differential geometry
  (Differential operators on surfaces, curvature quantities, evolving surfaces, transport theorems for evolving surfaces)
- Modeling
  (Derivation of mean curvature flow, gradient flows, Stefan problem, Willmore flow, anisotropic flows, two-phase flows, sharp interface models and phase field models)
- Analysis for problems involving interfaces
  (Classical solutions for mean curvature flow, approaches that use level set methods, viscosity solutions, BV approaches, relating phase field and sharp interface approaches, PDEs on surfaces)
- Numerical methods
  (Introduction to numerical methods for phase field approaches, interface tracking methods, interface capturing methods, parametric and level set methods for mean curvature flow, numerical methods for Willmore flow, parametric approaches for two-phase flow, numerical methods for anisotropic problems, numerical methods for PDEs on surfaces)

Reading:
The following optional texts may be helpful as introductory material. It is of course not necessary to read them before the seminar. However, a look at the first sections of these articles is possibly helpful. They can be obtained from the organizers on request by email to harald.garcke@ur.de

Prerequisite:
Basic knowledge of partial differential equations.

References

[1] J. W. Barrett, H. Garcke, R. Nürnberg. *Parametric finite element approximations of curvature-driven interface evolutions.* Handbook of Numerical Analysis **21**, 275–423 (2020).

[2] E. Bänsch, A. Schmidt. *Free boundary problems in fluids and materials.* Handbook of Numerical Analysis **21**, 555–619 (2020).

[3] K. Deckelnick, G. Dziuk, C. M. Elliott. *Computation of geometric partial differential equations and mean curvature flow.* Acta Numer. **14**, 139–232 (2005).

[4] H. Garcke. *Curvature driven interface evolution.* Jahresbericht der Deutschen Mathematiker-Vereinigung **115** (2), 63–100 (2013).

[5] P. Pozzi, B. Stinner. *Curve shortening flow coupled to lateral diffusion.* Numer. Math. **135** (4), 1171-1205 (2017).

Application deadline: September 4$^{\text{th}}$, 2022