OPERATOR-ADAPTED SPACES IN HARMONIC ANALYSIS AND PDES

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1. Topic

Many function spaces measuring integrability and smoothness can be seen as classes of distributions that have special properties under the action of the Laplace operator. Littlewood–Paley theory is one striking example. Nowadays, this provides a comprehensive theory for treating boundedness of operators in well-understood classes, such as Fourier multipliers and Calderón–Zygmund operators. For differential operators that have less regularity or solution operators to dispersive PDEs with (non-)smooth coefficients, these spaces may not be useful in all their scale since

(1) there is no meaningful theory of distributions for operators with merely measurable coefficients,
(2) solution operators to non-smooth PDEs are bounded only in a limited range of Lebesgue and Hardy spaces and hence must fall out of standard multiplier theory,
(3) solution operators to dispersive PDEs are not bounded on Lebesgue spaces but exhibit a loss of derivatives.

Oftentimes, such phenomena can be understood by first working on adapted spaces which incorporate the differential operator L or the differential equation under investigation itself. The result is a framework of abstract spaces on which functions built from L have the best possible properties. Smoothing properties of the PDE translate immediately into embeddings between abstract and concrete function spaces and the failure of good estimates in certain norms as in (2) becomes transparent because abstract and concrete spaces may diverge from each other.

2. Thematic focus of the planned lectures

In the lectures we aim to give an introduction into the abstract construction of spaces adapted to operators with a well-behaved L^2-theory. This includes a certain form of decay and a good functional calculus in order to overcome (1). We then demonstrate the versatility of this approach by discussing recent advances for boundary value problems for elliptic systems with measurable coefficients and wave equations with low regularity coefficients.

M. Egert will describe the general construction of Hardy spaces that are adapted to a given operator with spectrum in a bisector. This theory links with the tent spaces of Coifman–Meyer–Stein and the classical Hardy spaces of Fefferman–Stein when applied to the Laplacian. On the adapted spaces, the underlying operator will be seen to admit a bounded functional calculus on algebras of bounded holomorphic functions.

P. Auscher will take on the construction of adapted Hardy spaces that are designed for divergence-form operators and describe applications to boundary value problems for such operators. The main goal is to identify such spaces as concrete spaces of Hardy or Hardy-Sobolev type in a range of exponents in order to be able to construct solution formulæ. This also allows one to obtain abstract fundamental solutions and boundary layer operators. In the last part of the lectures it will be shown how well-posedness of boundary value problems can be obtained.

D. Frey will describe the hyperbolic counterpart of Hardy spaces that are adapted to Fourier integral operators. The theory builds upon dyadic-parabolic decompositions and wave packet transforms of Córdoba–Fefferman. It will be shown that the spaces are invariant under the half-wave group and admit sharp Sobolev embeddings. We will conclude with applications to the well-posedness of wave equations with low-regularity coefficients.

Recommended Reading for Auscher's and Egert's lectures

[1] P. Auscher and A. Amenta: *Elliptic boundary value problems with fractional regularity data*, vol. 37 of CRM Monograph Series, American Mathematical Society, Providence, 2018.

[2] P. Auscher and M. Egert: *Boundary value problems and Hardy spaces for elliptic systems with block structure*, book preprint available online at `https://hal.archives-ouvertes.fr/hal-03037758v1`

[3] R. R. Coifman, Y. Meyer and E. M. Stein: *Some new function spaces and their applications to harmonic analysis*, J. Funct. Anal. 62 (1985), no. 2, 304–335.

[4] S. Hofmann, S. Mayboroda and A. McIntosh: *Second order elliptic operators with complex bounded measurable coefficients in L^p, Sobolev and Hardy spaces*, Ann. Sci. Éc. Norm. Supér. (4) 44 (2011), no. 5, 723–800.

[5] A. McIntosh: *Operators which have an H_∞ functional calculus*, Proc. Centre Math. Anal. Austral. Nat. Univ. 14 (1986), 210–231.

Recommended Reading for Frey's lectures

[1] A. Córdoba and C. Fefferman: *Wave packets and Fourier integral operators*, Comm. Partial Differential Equations 3 (1978), no. 11, 979–1005.

[2] D. Frey and P. Portal: *L^p estimates for wave equations with specific $C^{0,1}$ coefficients*, preprint available online at `https://arxiv.org/abs/2010.08326`

[3] A. Hassell, P. Portal, and J. Rozendaal: *Off-singularity bounds and Hardy spaces for Fourier integral operators*, Trans. Amer. Math. Soc. 373 (2020), no. 8, 5773–5832.

[4] A. Hassell and J. Rozendaal: *L^p and H^p_{FIO} regularity for wave equations with rough coefficients, Part I*, preprint available online at `https://arxiv.org/abs/2010.13761`

[5] H. F. Smith: *A Hardy space for Fourier integral operators*, J. Geom. Anal. 8 (1998), no. 4, 629–653.