ARBEITSGEMEINSCHAFT: TWISTOR D-MODULES AND THE DECOMPOSITION THEOREM

APRIL 2-8, 2023

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1. Introduction

The notion of twistor structure has been introduced by Simpson in [Sim97], following a letter to him by Deligne, in order to include the objects related by the Kobayashi-Hitchin correspondence on a compact Kähler manifold, namely stable Higgs bundles with vanishing characteristic classes and simple flat holomorphic bundles, into a larger family, so that to equip the enlarged moduli space of a hyperkähler structure, extending thereby the original construction of Hitchin [Hit87]. A more in-depth elaboration of this fundamental construction has recently been developed by Simpson [Sim08, Sim21]. The talks in this session will however not pursue in this direction, but the participants may consult these recent papers to better understand the backgrounds of the twistor approach.

Already in 1997, Simpson envisioned the following "meta-theorem":

Meta-theorem (Simpson, [Sim97]). *If the words "mixed Hodge structure" (resp. "variation of mixed Hodge structure") are replaced by the words "mixed twistor structure" (resp. "variation of mixed twistor structure") in the hypotheses and conclusions of any theorem in Hodge theory, then one obtains a true statement. The proof of the new statement will be analogous to the proof of the old statement.*

The aim of this session is to illustrate this meta-theorem with the proof of a conjecture made by Kashiwara in various talks around 1996 [Kas98], that is, the *decomposition theorem for semi-simple holonomic $\mathcal{D}$-modules.* One can summarize the twistor approach by saying that twistor theory provides with a theory of weights a category of objects that do not naturally exhibit a weight structure.

The decomposition theorem was first proved in [BBDG82] for pure perverse ℓ-adic sheaves on varieties over a finite field of characteristic $p \neq \ell$. It asserts that the push-forward by a proper morphism of such an object decomposes, in the derived category, into its perverse cohomology sheaves and each such is semi-simple. Furthermore, by a technique of reduction to characteristic p, Beilinson, Bernstein, Deligne and Gabber were able to extend it to semi-simple perverse sheaves on smooth complex projective varieties which are of geometric origin. M. Saito [Sai88, Sai90b] developed at the end of the eighties a completely new strategy to extend this result on complex varieties to any (semi-)simple perverse sheaf whose associated local system underlies a polarizable variation of $\mathbb{Q}$-Hodge structure, not necessarily of geometric origin. The theory of *mixed Hodge modules* is now widely used in Algebraic geometry.

The decomposition theorem for semi-simple perverse sheaves on smooth complex projective varieties has now two proofs. One is by Drinfeld [Dri01], extending the proof of [BBDG82] by reduction to characteristic p, relaxing the assumption of geometric origin by relying on a conjecture of de Jong, later proved by Böckle-Khare [BK06] and Gaitsgory [Gai07]. The other one, which will be the main topic of this session, applies the meta-theorem of Simpson to the strategy of M. Saito by introducing the category of polarizable twistor $\mathcal{D}$-modules. The starting point were the papers [Moc02, Sab05], and the proof was achieved in [Moc07]. One key point in M. Saito's theory is the use of Schmid's norm estimates and orbit theorems [Sch73] through the Hodge-Zucker theorem [Zuc79] yielding the Hodge theorem for the intersection complex of a polarizable variation of Hodge structure on a punctured compact Riemann surface. The "twistor analogues" of these results were provided by Simpson [Sim90].

Furthermore, Simpson also raised in [Sim90] the following:

"*A question is whether one could set up a correspondence in which some nontame harmonic bundles correspond to systems of equations with irregular singularities.*"

The strength of the twistor approach is that it enables to enlarge the scope of Hodge theory not only to arbitrary semi-simple perverse sheaves, equivalently semi-simple regular holonomic $\mathcal{D}$-modules via the Riemann-Hilbert correspondence, on smooth complex projective varieties, but also to possibly irregular semi-simple holonomic $\mathcal{D}$-modules. In such a way, the analogy with the arithmetic theory of pure ℓ-adic perverse sheaves on varieties over finite fields is made stronger, as the latter does not restrict to tame objects, contrary to M. Saito's Hodge modules, whose associated $\mathcal{D}$-modules are known to be regular holonomic. For example, the analogue of the Katz-Laumon ℓ-adic Fourier transformation exists in the theory of mixed twistor $\mathcal{D}$-modules.

Simpson's insight has first been confirmed in dimension 1 [Sab99, BB04] and, after a first step in [Sab09], the full development of the theory of wild twistor $\mathcal{D}$-modules, both in the pure and the mixed case, has been achieved by T. Mochizuki in the sequence of works [Moc11, Moc15], extending [Moc07]. In particular, the monographs [Moc07, Moc11] provide the complete proof in the complex analytic setting of the conjecture of Kashiwara for semi-simple holonomic $\mathcal{D}$-modules (note that a wild analogue of Drinfeld's proof for the regular case still not exists). An overview of this work is provided in [Moc14] (see also [Moc15, Chap. 1], and [Sab13] for a focus on the decomposition theorem). Let us also mention that the decomposition theorem in the Kähler setting, for regular holonomic $\mathcal{D}$-modules underlying a polarizable pure twistor $\mathcal{D}$-module, has recently been proved by T. Mochizuki [Moc22] (see also [Sai90a, Sai22] for the case of $\mathcal{O}_X$).

Let us end this introduction by emphasizing that Hodge module theory or twistor $\mathcal{D}$-module theory is not the only way to the decomposition theorem in complex algebraic geometry. For the case of regular holonomic $\mathcal{D}$-modules (or perverse sheaves) of geometric origin, so that Hodge theory is involved, we mention the work of de Cataldo and Migliorini [dCM02, dCM05, dCM09] (see also [Wil17]). A similar idea has been developed in [WY21] for proving the decomposition theorem in the case of a semi-simple local system on a smooth projective variety, relative to a morphism to another variety.

We suggest to have a look at the introductory chapters of [Sab05], [Moc07] and [Moc11] to understand how the various arguments fit together, leading to the proof of Kashiwara's conjecture.

2. Description of the talks

Monday: Introduction to twistor theory.

Lecture 1: Pure and mixed twistor structures, comparison with Hodge structures. This lecture does not contain much material, but allows the audience to become familiar with the notion at a quiet path.

Start with [Sim21, §1.1] in order to explain the word "twistor structure".

(a) Introduce the category of pure and mixed twistor structure, Tate objects, half-Tate objects $\mathcal{O}(p\{0\} + q\{\infty\})$. Two main two results to be proved in this lecture:

- the category of mixed twistor structure is abelian,
- faithfulness and exactness of some functors such as Gr^W, Ξ_{DR}, Ξ_{Dol}.

Define complex (pure or mixed) Hodge structures, explain the Rees construction, and interpret the w-oppositeness property in terms of twistor structures. Characterize Hodge structures among twistor structures by the existence of a $\mathbb{C}^*$-action, making the category of Hodge structures a subcategory of that of twistor structures compatible with the weight filtration.

Reduction of pure twistor structures to weight zero by half-Tate twists.

References: Mainly, [Sim97, §1], and for the Tate object refer to [Moc07, §3.3] and [Moc15, §2.1.8], [Moc22, §§7.1.1, 7.1.2, 7.1.6].

Further reading: [Sab18, §§1.1.a, 1.1.b].

Lecture 2: Twistor structure on a complex manifold, polarization.

(a) Variations of twistor structures on complex manifolds (simply called twistor structures on complex manifolds), λ-connections.

References: [Sim97, §3].

Further reading: [Moc07, §§3.1, 3.2] and [Moc22, §§7.2.1–7.2.3].

(b) Anti-holomorphic involution on a twistor structure, polarization of a pure twistor structure (on a point).

Equivalence "pure polarized twistor structure of weight 0 $\iff$ Complex vector space with a Hermitian metric".

Important example: polarization of the Tate objects.

Recall the definition of polarization of a complex Hodge structure and check that it is a polarization in the sense of twistor structures.

Polarization of a pure twistor structure on a complex manifold.

Main theorems to be proved:

- Any pure twistor substructure of a polarized pure twistor structure of the same weight is a direct summand, and the polarization induces a polarization of the substructure.
- The category of polarizable pure twistor structures of a given weight is semi-simple.

References: [Sim97, §§2, 3], [Moc07, §3.5], [Moc22, §7.2.4]

***Lecture 3: Harmonic bundles and equivalence with smooth polarized twistor structures of weight* 0.** Definition and basic examples of harmonic bundles on a complex manifold, example of variations of polarized Hodge structure.

Basic example: Make explicit the case of rank-one harmonic bundles on the punctured disc ([Moc07, §6.1], [Sab13, Ex. 1.2]). More basic examples ([Moc02, §3.2], [Moc07, §6.2]).

Equivalence with smooth polarized twistor structures of weight 0 (Lemma 3.1 in [Sim97], omitting the semi-simplicity property in the global case which is considered in the next lecture).

References: [Sim97, §3]

Lecture 4: The non-abelian Hodge correspondence and the Hodge-Simpson theorem. This lecture focuses on the global properties of harmonic bundles on a projective or Kähler complex manifold, or with respect to a projective morphism: these are the Hodge and semi-simplicity properties of harmonic bundles.

State the Corlette-Simpson theorem and give the proof of the "easy" direction of it: *on a smooth projective variety, harmonic bundles have semi-simple associated flat bundles.*

For the proof, two possible approaches depending on the speaker's taste:

- [MHMP, §4.3.b] which is a mixture of the original proof of Corlette [Cor88, §§2.2, 2.3] and arguments of Simpson [Sim92, §1].
- For another approach more related to the techniques used in the theory of harmonic bundles: [Moc09b, §2.3] (especially Lemma 2.34, which is the analogue for the Higgs case is explained in [Sim88, Lem. 3.2]).

State some consequences: A semi-simple local system on a smooth projective variety remains semi-simple after pullback.

What about projective pushforward? State the conjecture of Kashiwara. Reference: [Kas98].

First evidence for this conjecture: the Hodge-Simpson theorem for harmonic bundles on compact Kähler manifolds.

Give a statement of it following Simpson's meta-theorem: *The k-th cohomology of a pure polarizable twistor structure of weight* 0 *is a pure polarizable twistor structure of weight k and Hard Lefschetz holds in this context.* Emphasize the *strictness property*, corresponding to the degeneration at E_1 of the spectral sequence Hodge $\Rightarrow$ de Rham.

Sketch a proof of this theorem. Reference: [Sim92, §1], [Sim97, Th. 4.1].

Tuesday: Regular twistor $\mathcal{D}$-modules and the decomposition theorem for them. The talks will introduce the ambient category where regular twistor $\mathcal{D}$-modules live, and related functors. After having become more familiar with this category and its subcategory of regular twistor $\mathcal{D}$-modules, the talks will state and sketch some of the proofs of the main theorems.

An $\mathcal{R}$-triple has to be thought as a generalization to holonomic $\mathcal{D}$-modules of a C^∞ (or real analytic) bundle with flat connection and two filtrations, one being holomorphic and the other one anti-holomorphic with respect to the holomorphic structure induced by the $(0,1)$ component of the flat connection. The presentation as an $\mathcal{R}$-triple is intended to avoid passing to real-analytic $\mathcal{D}$-modules in order to keep control on coherence properties.

Lecture 5: The notion of $\mathcal{R}$-triple and various functors. Introduce the space $\mathcal{X} = X \times \mathbb{C}_\lambda$ and the ring $\mathcal{R}_\mathcal{X}$. Emphasize similarities and differences with the ring $\mathcal{D}_X$. Category of $\mathcal{R}_\mathcal{X}$-modules. Notion of strictness. Definition of coherence, characteristic variety, holonomicity.

Bundles $(\mathcal{E}, \mathbb{D})$ with flat λ-connections are $\mathcal{R}_X$-modules. $\mathcal{R}_X$-modules associated to filtered $\mathcal{D}_X$-modules. Functors to the category of $\mathcal{D}_X$-modules or Higgs sheaves by fixing the value of λ.

Motivations for considering sesquilinear pairings (see e.g. [MHMP, §12.9] for motivations). Category of $\mathcal{R}$-triples. Yoga of the category of $\mathcal{R}$-triples and basic operations: Hermitian duality, proper push-forward, (half) Tate twists.

Rephrase the smooth case of twistor theory in the language of $\mathcal{R}$-triples.

Basic example: Make the example of a harmonic bundle on the *punctured* disc (Lecture 3) an object of $\mathcal{R}$-triple on the *whole* disc (see e.g. the various incarnations of Example 1.2 in [Sab13]).

References: [Sab05, Chap. 1, 2], [Moc07, §§3.10 & 14.1], [Moc15, §2.1].

Lecture 6: Strict specializabilty, S-decomposability and canonical extensions.

Quick review without proofs of the theory of specialization of holonomic $\mathcal{D}$-modules: V-filtration with respect to a hypersurface and nearby/vanishing cycle constructions for holonomic $\mathcal{D}$-modules, monodromy, morphisms can and var, minimal (aka intermediate) extension along a hypersurface, decomposability with respect to the support (aka S-decomposability). State the theorem asserting that nearby/vanishing cycles of holonomic $\mathcal{D}$-modules commute with proper pushforward.

The goal of the lecture is to explain how this theory can be extended, by following the same path, to $\mathcal{R}$-modules and $\mathcal{R}$-triples (i.e., taking into account the sesquilinear pairing). It should only emphasize the analogies and differences of such constructions for $\mathcal{R}$-modules and their extensions to $\mathcal{R}$-triples. The main word that makes everything work is *strictness*. All notions should be illustrated with the *Basic example*.

- For $\mathcal{R}$-modules, introduce the Bernstein relation (justify the choice of the kind of Bernstein relation with the *Basic example*) and emphasize that the roots of the Bernstein polynomial are functions of λ. Definition of strict specializability and of the local (w.r.t. λ) existence and uniqueness of the canonical V-filtration.
- Sketch the proof of the the global (w.r.t. λ) existence of nearby and vanishing cycles, and the diagram can var for $\mathcal{R}$-modules.
- Definition of the nearby cycle functor of a sesquilinear pairing by means of Mellin transformation (the vanishing cycle functor and compatibility with can and var should only be mentioned).
- Conclude with the notion of minimal extension and strict S-decomposability of $\mathcal{R}$-triples.
- State the compatibility theorem between taking nearby/vanishing cycles and taking proper pushforward (emphasize the strictness condition). Give some indication on the proof.

If time permits, mention another approach to the theory, by means of the localization and dual localization functors, as well as Beilinson's maximal extension, following [Moc15, Chap. 3, 4].

References: [Sab05, Chap. 3], [Moc11, Chap. 22], [Moc07, §§14.2–14.4] (see also, for an easier setting, [MHMP, Chap. 9, 11, 12]).

Lecture 7: Polarizable pure twistor $\mathcal{D}$-modules. Notion of sesquilinear duality of some weight of an $\mathcal{R}_X$-triple. When X is a point, definition of a polarized pure twistor structure of some weight w in terms of $\mathcal{R}_X$-triples equipped with a sesquilinear duality of weight w.

Inductive definition of polarizable regular twistor $\mathcal{D}$-modules. Emphasize the purely imaginary case for further use in the proof of the conjecture of Kashiwara.

First properties (Kashiwara's equivalence, stability by direct summand, etc.). References: [Sab05, §4.1.b–4.1.e].

The word 'regular' is omitted below.

The following statements should be given, however without indication on the proofs. Details in Lecture 8.a for Statement 1. Statement 2 can be decomposed as the conjunction of a local one and a global one, by considering tame harmonic bundles as intermediate objects. It will be taken up in Lecture 11, after tame harmonic bundles have been introduced in great generality. In between, a proof of the "easy" direction of Statement 2 only relying on the work of Simpson about tame harmonic bundles on punctured Riemann surfaces will be given in Lecture 8.b. References: [Moc07, §1.4].

Statement 1 (decomposition theorem with respect to a projective morphism): The decomposition theorem for polarizable pure twistor $\mathcal{D}$-modules (Hard Lefschetz theorem, Riemann bilinear relations).

Statement 2: On a smooth projective variety, semi-simple perverse sheaves are in one-to-one correspondence with polarizable purely imaginary twistor $\mathcal{D}$-modules of weight 0 (generalization of Corlette-Simpson).

Lecture 8: Sketch of proofs of Statements 1 and "easy" direction of Statement 2.

This lecture could be divided in two sub-lectures, each of 45mn.

Lecture 8.a: Sketch of proof of Statement 1. The main goal of sub-Lecture 8.a is to explain how one can reduce the proof of Statement 1 to the case of regular twistor $\mathcal{D}$-modules on curves. The case of curves is assumed to hold and details will be given in Lectures 9–10.

Sketch of proof of the decomposition theorem for polarizable pure twistor $\mathcal{D}$-modules. References: [Sai88, §5.3] as the original reference for the Hodge case (see also [MHMP, §14.3]); for the twistor case, use [Sab05, Chap. 6], [Moc07, §§14.5, 14.6], and [Moc11, Chap. 18] by restricting to the regular case, especially §18.4 which corrects an error in [Sab05, §6.4(2)].

Lecture 8.b: Sketch of proof of the "easy" direction of Statement 2. The goal of sub-Lecture 8.b is to show that the perverse sheaf associated to a polarizable pure twistor $\mathcal{D}$-module on a projective manifold is semi-simple. It is obtained by reduction to the case of curves by a Zariski-Lefschetz type argument.

First step: Give the definition of a *tame* harmonic bundle on a punctured Riemann surface: on a disc centered at a puncture, the eigenvalues of the Higgs field have a pole of order at most one, [Sim90, Synopsis]; equivalently, the coefficients of the characteristic polynomial of the Higgs field are holomorphic on the disc.

State the main theorem of [Sim90, Synopsis] in the purely imaginary case:

One-to-one correspondence between purely imaginary tame harmonic bundles on a compact Riemann surface and semi-simple local systems on the complement of punctures.

Reference: [Sim90, p. 718], and [Sab05, §5.2] for the relation between stability and semi-simplicity in the purely imaginary case (called Deligne type in loc. cit.).

Second step: Give the proof that polarizable pure twistor $\mathcal{D}$-modules give rise to semi-simple perverse sheaves on a projective manifold, assuming that the theorem of Simpson on Riemann surfaces holds, and relying on Statement 1.

Note: Another proof will be sketched in Lecture 11.

Reference: [Sab05, Th. 4.2.12] with the caveat that one should use a corrected argument explained in (11) of the Erratum to [Sab05] or in [Moc07, §19.2.2].

Wednesday: Tame harmonic bundles. The lectures will focus on the analytic aspects of regular twistor $\mathcal{D}$-modules, namely tame harmonic bundles. They are the intermediate objects between perverse sheaves and polarized pure twistor $\mathcal{D}$-modules. Lectures 9 and 10 explain how to obtain Statements 1 and 2 on Riemann surfaces from the results of Simpson [Sim90].

Lecture 9: Tame harmonic bundles on curves, local properties. Content of this lecture: From a tame harmonic bundles on a punctured Riemann surface, one explains how to construct a polarized pure regular twistor $\mathcal{D}$-module of weight 0. In this lecture, the Riemann surface is a disc Δ centered at the origin, and the puncture is the origin, with complement Δ^*.

(1) Recall the definition of tameness for a harmonic bundle on Δ^*(Lecture 8.b). State Theorem 1 of [Sim90] for curvature bounds and mention that it is proved by means of Simpson's main estimate.
(2) State and sketch the proof of Theorem 2 in [Sim90]. Explain the analogous result for $(\mathcal{E}^\lambda, \mathbb{D}^\lambda)$ for each fixed $\lambda \in \mathbb{C}$ ([Moc07, §7.2.1]).
(3) State the main theorem of this lecture as follows: *the variation of polarized pure twistor structure of weight* 0 *attached to a harmonic bundle on* Δ^* *which is tame at the origin extends in a unique way as a polarized pure twistor* $\mathcal{D}$*-module on* Δ *whose underlying* $\mathcal{D}$*-module is the intermediate extension of the flat bundle underlying the variation.*

    Reference: [Moc07, §20.1] (see also [Sab05, §5.3] for a different approach to a similar global result on a compact Riemann surface).
(4) Illustrate the proof on a basic model. Reference: [Sab05, §5.1].
(5) The converse assertion: *a polarized twistor structure of weight* 0 *on* Δ^* *which is the restriction of a polarized pure twistor* $\mathcal{D}$*-module on* Δ *has an associated harmonic bundle which is tame at the origin* $t = 0$,
    is much simpler (it has already been used in Lecture 8.b, 2nd step):
    - the restriction of a polarized regular twistor $\mathcal{D}$-module of weight zero to the punctured Riemann surface corresponds to a harmonic bundle;

- by the regularity assumption, each negative step of the V-filtration is $\mathcal{O}$-locally free in the neighbourhood of $\lambda = 0$; since it is stable by the action of $t\eth_t$, it follows that the Higgs field, obtained by setting $\lambda = 0$ in the action of $t\eth_t$, satisfies the tameness assumption of [Sim90, §2].

Lecture 10: Tame harmonic bundles on curves, global properties. Content of this lecture: Interpretation (and adaptation) of the results of Simpson in terms of twistor $\mathcal{D}$-modules on curves. Sketch of proofs of Statements 1 and 2 of Lecture 7 in the case of curves. More precisely, for Statement 1, we consider the constant map from a compact Riemann surface to a point. In such a case, it amounts to the purity and the polarizability of the global de Rham cohomology of a polarized pure twistor $\mathcal{D}$-modules, since Hard Lefschetz is tautological. The proof follows the same lines as that of Zucker [Zuc79] for variations of polarized Hodge structures.

(1) Proof of Statement 2 in the case of curves: put together the theorems in Lectures 8.b and 9.
(2) Proof of Statement 1, local arguments: Comparison between the holomorphic L^2-complex and the L^2-complex for a tame harmonic bundle on a punctured Riemann surface.
    - Introduce the holomorphic L^2-complex and the L^2-complex, and state that the natural inclusion is a quasi-isomorphism (first with λ fixed, and then with parameters in $\mathbb{C}_\lambda$). Reference: [Sab05, §§6.2.a, 6.2.b], [Moc07, §§20.2.1, 20.2.2].
    - Explain the Dolbeault lemma for a singular Hermitian line bundle [Zuc79].
    - Sketch the proof of the quasi-isomorphism property by means of the norm estimates. Reference: [Moc07, §20.2.2] (see also the regular case of [Moc11, §§5.1, 5.2], and also [Sab05, §§6.2.b–6.2.f] for a proof of a different nature).
(3) Proof of Statement 1, global arguments: Comparison between the cohomology of a polarized pure twistor $\mathcal{D}$-module and the space of L^2-harmonic forms. References: [Zuc79], [Sab05, §6.2.g], [Moc02, §20.2], [Moc11, §18.2].

Lecture 11: Tame harmonic bundles in arbitrary dimension. This lecture is necessarily very sketchy. It aims at giving the main definitions [Moc07, §19.1]. Explain the statements of the following two theorems, and a few steps of their proofs:

(1) [Moc07, Th. 19.6]: Tame harmonic bundles are in one-to-one correspondence with polarized pure twistor $\mathcal{D}$-modules of weight 0. Make clear the statement with support on a closed irreducible analytic subset Z of the complex manifold X. Survey [Moc07, §§19.2-19.5] and insist on [Moc07, §19.6].
(2) Semi-simple local systems on the complement of a normal crossing divisor D in a projective manifold X are in one-to-one correspondence with purely imaginary tame harmonic bundles.
    For the proof, two possible approaches depending on the speaker's taste:
    - Follow [Moc07, §§22.1, 22.3] for the "easy" direction, and [Moc07, §25.5] for the other direction, relying on a method of Jost-Zuo [JZ97],
    - or give indications on the proof explained in [Moc09b], that will be taken up in the wild case in Lecture 17.

Thursday: Wild twistor $\mathcal{D}$-modules and the decomposition theorem for them. The lectures aim at explaining how twistor $\mathcal{D}$-module theory is able to overcome the drawback of Hodge theory of being restricted to the tame setting, in order to treat possibly irregular holonomic $\mathcal{D}$-modules. Compared to the tame setting, the new phenomenon is the Stokes phenomenon.

Lecture 12: Polarizable wild twistor $\mathcal{D}$-modules. This lecture is the wild analogue of Lecture 7.

First, it aims at explaining the modification, obtained by adapting an idea of Deligne, that is necessary to extend the definition of regular twistor $\mathcal{D}$-module to the wild (i.e., irregular) case. One strengthens the notion of strict specializability with respect to a holomorphic function by applying it to all possible locally twists by pullbacks of meromorphic connections in dimension one. The first part of the lecture explains this construction for $\mathcal{R}$-modules and $\mathcal{R}$-triples.

References: [Sab09, §§2.1–2.4] and [Moc11, Chap. 22].

One can then define the notion of polarizable wild twistor $\mathcal{D}$-module, and mention the basic properties in a way similar to what is done for regular twistor $\mathcal{D}$-modules.

References: [Sab09, §3] and [Moc11, §17.1].

State the following:

Wild Statement 1 (decomposition theorem with respect to a projective morphism): The decomposition theorem for polarizable pure wild twistor $\mathcal{D}$-modules (Hard Lefschetz theorem, Riemann bilinear relations).

Indicate that the reduction steps to the case of curves are similar to those of the tame case (Lecture 8.a). The case of curves needs further refinements (Lectures 14 and 15). Reference: [Moc11, Chap. 18].

Lecture 13: Irregular singularities in dimension one. This lecture starts by recalling the formal and asymptotic theory of meromorphic flat bundles on the disc with pole at the origin. It introduces the notions of irregular value, formal monodromy, associated nilpotent endomorphism, and Stokes structure, and states the Riemann-Hilbert-Birkhoff correspondence.

Reference: [Mal91, §§IV.1–3], in particular, Theorems (2.2) and (2.3).

Show how the RHB correspondence enables one to construct deformations of flat bundles by multiplying the irregular values by the same positive number T and otherwise keeping the Stokes structure fixed.

Reference: [Moc11, §4.5.2] in the particular case where the function T is a fixed positive number.

In order to develop a similar theory with the parameter λ, the strict specializability property with ramification and exponential twist considered in Lecture 12 proves important. One can state and possibly sketch a proof of [Sab09, Prop. 4.5.4].

State the Riemann-Hilbert-Birkhoff correspondence with the parameter λ.

References: [Moc11, §4.3] but only consider the one-variable case.

Lecture 14: Wild harmonic bundles on curves. This lecture focuses on a few fundamental results needed for passing from the tame case to the wild case in the proof of the decomposition theorem in dimension one, that will be done in the next lecture.

(1) Extend the definition of a tame harmonic bundle on a curve (Lecture 8.b) to that of a wild harmonic bundle (Definition 7.1.4 in [Moc11] in the case of curves).
(2) State the norm estimate of [Moc11, Prop. 8.1.1] and sketch the proof.
(3) Show rapid decay of harmonic forms [Moc11, §§8.1, 8.2].
(4) State without proof Wild Statement 2 for curves (see the statement and details in the Friday lectures), needed in the next lecture.

Reference: [Moc11, Chap. 8].

Lecture 15: Proof of Wild Statement 1 in the case of curves. This lecture ends the proof of Wild Statement 1 by considering the case of the constant map on a smooth compact Riemann surface. The arguments are given in [Moc11, §18.2], where one will follow and try to explain the references therein, in particular concerning the extension $\mathcal{Q}_\bullet$.

When needed, one will take for granted Wild Statement 2, as analyzed in the lectures on Friday, in the case of curves.

Friday: The Kobayashi-Hitchin correspondence for meromorphic flat bundles. The aim of the lectures is to explain the proof of

Wild Statement 2: On a smooth projective variety, semi-simple holonomic $\mathcal{D}$-modules are in one-to-one correspondence with polarizable purely imaginary wild twistor $\mathcal{D}$-modules of weight 0 (Kobayashi-Hitchin correspondence).

The combination of Wild Statements 1 and 2 gives a solution to Kashiwara's conjecture.

Lecture 16: Deligne-Malgrange lattices and purely imaginary wild harmonic bundles. We expect this lecture to be an exposition of the paper [Moc09c]. Explain the notion of (good) Deligne-Malgrange lattice. State the characterization of semi-simple meromorphic flat bundles in terms of the existence of a pluri-harmonic metric [Moc09c, Th. 3.4]. Details for the latter will be given in Lecture 17.

In higher dimensions, [Moc11], starting from [Moc09a], solves jointly both a conjecture of Sabbah in the projective setting and the conjecture of Kashiwara for possibly irregular holonomic $\mathcal{D}$-modules. In this lecture, one will omit the solution of the conjecture of Sabbah by relying on the independent proof given by Kedlaya [Ked10, Ked11], which allows to focus only on the analytic and twistor aspects of the Stokes phenomenon. Therefore, instead of the outline of the proof in the introduction, refer to [Ked11, §8.2] for Theorem 2.12 in [Moc09c].

On the other hand, if time permits, explain the approach of Mochizuki as outlined in the introduction of [Moc09c].

Lecture 17: Wild harmonic bundles and semi-simple meromorphic flat bundles. This lecture gives details on the Kobayashi-Hitchin correspondence for meromorphic flat bundles, that is, on the proof of the theorem stated in Lecture 16, which corresponds to [Moc11, Th. 16.2.4]. This is the wild analogue of Point (2) in Lecture 11.

The aim of this lecture is thus to explain the content of [Moc11, Chap. 16]. As indicated in Lecture 16, one will take the solution of the conjecture of Sabbah (and its generalization to Deligne-Malgrange lattices) for granted, and thus omit the proof of [Moc11, Th. 16.2.1].

Further reading: [Moc09b], [Moc21].

Lecture 18: Wild harmonic bundles and wild pure twistor $\mathcal{D}$-modules. This lecture is the wild analogues of Point (1) in Lecture 11. It is divided in the two sub-lectures 18.a and 18.b, each of 45mn.

Lecture 18.a: Essential surjectivity. The main result of the lecture is the essential surjectivity of the functor. State the correspondence of [Moc11, §19.1]. One should omit the refinement $\mathcal{A}$-wild during the lecture.

Reference: [Moc11, §19.2].

Lecture 18.b: Full faithfulness. The main result of the lecture is the full faithfulness of the functor considered in Lecture 18.a, and the end of the proof of the conjecture of Kashiwara's conjecture.

Reference: [Moc11, §§19.3–4].

References

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