OBERWOLFACH SEMINAR 2322A: ANALYSIS OF AUTOMORPHIC FORMS AND L-FUNCTIONS IN HIGHER RANK 28 MAY - 03 JUN 2023

FARRELL BRUMLEY, ANDRE REZNIKOV, PAUL D. NELSON, AKSHAY VENKATESH

L-functions are generalizations of the Riemann zeta function that are considered fundamental in modern number theory. For example, the distribution of their zeros controls the behavior of prime numbers in several refined senses, going far beyond the classical prime number theorem. The problem of estimating L-functions, on average or individually, has been of focused interest over the past few decades. One source of motivation is that such estimates have turned out to be at the heart of solutions to seemingly unrelated problems concerning, for instance, quadratic forms and quantum chaos (see for instance [4, 5]). L-functions are often studied through the lens of period integrals of automorphic forms, whose estimation is of independent interest.

The orbit method is a tool for reducing problems in the representation theory of Lie groups to simpler problems in symplectic geometry, invariant theory and linear algebra. The week-long "Oberwolfach Seminar 2322a" will introduce to young researchers how this tool may be applied to analytic problems in the theory of automorphic forms and L-functions, highlighting recent progress on the estimation of the latter [9, 7, 8, 3]. Many further challenges remain, and the transformative potential of these techniques has not yet been realized. The target audience is PhD students or post-doctoral researchers wishing to be quickly immersed in a modern, active research area. The number of participants is limited to 25; priority will be given to young, motivated researchers.

We now discuss in more detail the intended thematic focus, noting that this may evolve as we flesh out the lectures. At the end, we give some more precise references within the primary ones.

(1) **Basics on integral representations of automorphic L-functions.** Our hope is that our seminar will be accessible to young researchers of varied background, including those coming from analytic number theory, representation theory and automorphic forms. For this reason, we plan to spend a couple lectures summarizing some basic background concerning integral representations of L-functions and how such representations may be applied as a tool for estimation, following [1]. As a general reference, we mention [2].

(2) **Microlocal analysis on Lie group representations.** The main references here are parts of [9, 7, 8]. The general theme is to study analytic problems involving representations π of a Lie group G by approximately decomposing the action of group elements close to the identity, whose actions

approximately commute. The approximate eigenvectors for such decomposition are called *localized vectors* and form the building blocks of this approach. The corresponding systems of eigenvalues are described by parameters τ lying in the dual Lie algebra $\mathfrak{g}^*$. In the case of an irreducible unitary representation π, the eigenvalues that arise are often described by the coadjoint orbit $\mathcal{O}_\pi \subseteq \mathfrak{g}^*$, which in turn describes the character of π; this is the content of the Kirillov character formula. In the metaphor of the correspondence principle of quantum mechanics, one understands $\mathcal{O}_\pi$ as a phase space underlying the Hilbert space π. One can organize the theory via an assignment Op from functions $a : \mathfrak{g}^* \to \mathbb{C}$ defined on the dual Lie algebra to operators $\mathrm{Op}(a)$ on π, generalizing the classical pseudodifferential calculus. Localized vectors can be constructed or otherwise seen to exist in many ways: directly via analysis of a suitable model (induced, Whittaker, ...), indirectly by estimating the trace of $\mathrm{Op}(a)$ for suitable a via the Kirillov formula. For analytic purposes, one significant feature of localized vectors is that they admit approximate reproducing kernels defined using functions on G supported close to the identity element. These kernels can be chosen so that their image under π is small outside a fairly narrow window of π, which makes them useful for studying averages of small families of π. Another useful property of localized vectors is that their matrix coefficients concentrate along the centralizers of the corresponding parameters τ. This often renders feasible the analysis of expressions defined using such matrix coefficients, such as the local integrals occurring in the Ichino–Ikeda formula.

(3) **Invariant theory of Gan–Gross–Prasad pairs.** The main tool for studying automorphic L-functions is the theory of integral representations. Starting with an automorphic form φ defined on the adelic quotient $[G]$ of a group G, one consider its integral over the adelic quotient $[H]$ of a subgroup $H \leq G$. In favorable circumstances, such integrals are given up to "local" constants by some L-function. Given an automorphic representation π of G, one can study localized vectors $\varphi \in \pi$, with corresponding parameters $\tau \in \mathfrak{g}^*$, and study the period integrals $\int_{[H]} \varphi$. One general theme we hope to develop in the lectures is how analytic problems involving such integrals may be reduced to algebro-geometric or linear-algebraic problems concerning the action of H on $\mathfrak{g}^*$. This theme features repeatedly in [9, 7, 8]. In particular, the H-stabilizer and H-orbit of τ play an important role; when the former is trivial and the latter is closed, the element τ is called *stable*, following Mumford, and this condition turns out to play an important role in the analytic theory. For example, it is closely related to the notion of conductor dropping, and is central to the study of the matrix coefficient integrals arising in the Ichino–Ikeda conjecture.

(4) **Application to estimates for L-functions** The basic modules described above aim to introduce tools that should be of general use. We hope to motivate them by spending a few lectures explaining their application to the problem of estimating automorphic L-functions, as in [9, 7, 8].

We are working on creating some self-contained expositions on these topics in advance of the workshop. In the meantime, the following sections of the primary references may be useful for understanding some of the themes that we plan to highlight, even though they do not provide a complete or self-contained introduction to the topics.

- Localized vectors:
  - Basic notions: [9, §1], [7, §2.5], [8, §14], [6, §4].
  - Constructions: [9, Remark 6.6], [8, §1.3.11, Part 3].
  - Matrix coefficients: [9, §19.7].
  - Approximate reproducing kernels: [7, §2.5, §4, §14].
- Infinitesimal characters and the Kirillov formula: [9, §6, §9].
- Branching: [9, §19], [7, §12, §14], [8, §1.3.10, §3, §11].
- Invariant theory for GGP pairs: [9, §13], [7, §13], [9, §14, §17.3.1], [7, §11].
- Relative trace formula and amplification: [7, §1.5.3, §2.7, §6]
- Volume estimates: [7, §1.5.4–1.5.7, §2.8–2.10, §15-16], [8, §1.3.5]

References

[1] Joseph Bernstein and Andre Reznikov. Subconvexity bounds for triple L-functions and representation theory. *Ann. of Math. (2)*, 172(3):1679–1718, 2010.

[2] J. W. Cogdell. Notes on L-functions for GL_n. In *School on Automorphic Forms on* $\mathrm{GL}(n)$, volume 21 of *ICTP Lect. Notes*, pages 75–158. Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2008.

[3] Kevin Hartnett. Mathematicians clear hurdle in quest to decode primes. January, 2022, 2022.

[4] Henryk Iwaniec and Peter Sarnak. Perspectives on the analytic theory of L-functions. *Geom. Funct. Anal.*, (Special Volume, Part II):705–741, 2000. GAFA 2000 (Tel Aviv, 1999).

[5] Philippe Michel and Akshay Venkatesh. Equidistribution, L-functions and ergodic theory: on some problems of Yu. Linnik. In *International Congress of Mathematicians. Vol. II*, pages 421–457. Eur. Math. Soc., Zürich, 2006.

[6] Paul D. Nelson. Quantum variance on quaternion algebras, III. preprint, 2019.

[7] Paul D. Nelson. Spectral aspect subconvex bounds for $\mathrm{U}_{n+1} \times \mathrm{U}_n$. *arXiv e-prints*, page arXiv:2012.02187, December 2020.

[8] Paul D. Nelson. Bounds for standard L-functions. *arXiv e-prints*, page arXiv:2109.15230, September 2021.

[9] Paul D. Nelson and A. Venkatesh. The orbit method and analysis of automorphic forms. *Acta Math.*, 226(1):1–209, 2021.