OBERWOLFACH ARBEITSGEMEINSCHAFT: CLUSTER ALGEBRAS

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Introduction

Background. Cluster algebras, invented [FZ02] by Sergey Fomin and Andrei Zelevinsky around the year 2000, are commutative algebras whose generators, the cluster variables, are constructed in a recursive manner. Among these algebras, there are the algebras of homogeneous coordinates on Grassmannians, on flag varieties and on many other varieties which play an important role in geometry and representation theory. Fomin and Zelevinsky's main aim was to set up a combinatorial framework for the study of the so-called dual canonical bases which these algebras possess [Lus90, Kas90] and which are closely related to the notion of total positivity [Lus98, Fom10] for the associated varieties.

Since Fomin–Zelevinsky's invention, cluster algebras have had strikingly fruitful interactions with an ever-growing array of other subjects including

- Poisson geometry [GSV03, GSV05, GSV08, GSV10, BZ05] ...;
- discrete dynamical systems [FZ03c, DFK09, IIK+10, Ked08, Kel10, Kel13, KNS11] ...;
- Teichmüller spaces and higher Teichmüller spaces [FST08, FT18, Sch10, Mus11, MSW11a, MSW13, FP14, FLL19, FG06, FG07, FG09a, FG09b, FG09c] ...;
- combinatorics and in particular the study of polyhedra like the Stasheff associahedra [Cha05, CFZ02, FR05, FST08, IT09, Kra06] ...;
- commutative and non commutative algebraic geometry and in particular the study of stability conditions in the sense of Bridgeland [Bri07], Calabi-Yau algebras [Gin, IR08] , Donaldson-Thomas invariants in geometry [JS12, KS, KS11, Rei11, Nag13, Sze08] ... and in string theory [ACC+13, ACC+14, CCV, CNV, CV13, GMN10, GMN13a, GMN13c] ...;
- the representation theory of quivers and finite-dimensional algebras, cf. for example the survey articles [BBM06, Rin07, GLS08, Kel10, Lec10, Rei10, Rei, Ami11, Kel12, Pla18] ...;
- the representation theory of quantum affine algebras and of quiver Hecke algebras [HL10, HL13, HL16a, HL16b, Qin17, KKKO18, Kas18, Her19, KK19, HO19, KKOP20, HL21, FH22, FHOO22], ... , as well as the representation theory of p-adic groups [LM18, LM20, LM], ... ;
- mirror symmetry [GHK15, CGM+17, Man17, GHKK18, CM20, CKM, DM21, BA] ... ;
- Exact WKB analysis and supersymmetric quantum field theory [GMN13b, GMN14, All19, IN14, IN14]... ;
- Scattering amplitudes in quantum field theory, the amplituhedron ... [AHT14, GPSV14, GGS+14, AHBC+16, GMSV19, PSBW22, Wil] ... .

They have also appeared in the study of KP solitons [KW11], hyperbolic 3-manifolds [NTY19],... . We refer to the book in progress [FWZc, FWZd, FWZa, FWZb], to the introductory articles [Fom10, FZ03b, Zel, Zel02, Zel05, Zel07, Kel12, KD20] and to the cluster algebras portal [Fom] for more information on cluster algebras and their links with other subjects in mathematics (and physics).

This Arbeitsgemeinschaft. In this Arbeitsgemeinschaft, we have chosen to focus on three main subjects:

A. the basic theory of cluster algebras,
B. the most important classical examples of cluster structures on varieties and
C. the recent interaction between cluster algebras and symplectic topology and its application to the construction of cluster structures on braid varieties.

Part C focuses on developments in symplectic geometry that have either used cluster algebras or been used to study them. In particular, this last series of lectures aims at developing the intuitions and techniques from symplectic geometry [Cas22, CW, PT20, GSW] and the microlocal theory of sheaves [KS13, GKS12, STZ17] to complement the more algebraic and combinatorial methods often used to study cluster algebras. On the one hand, these lectures will explain new results in the study of Lagrangian surfaces, including the detection of infinitely many Lagrangian fillings, via techniques from cluster algebras [CG22, CW]. On the other, the combinatorics of weaves [CL22, CZ22] will also be presented from their original symplectic geometric viewpoint and then applied to prove new results in the study of cluster algebras. To wit, the lectures will show that the coordinate rings of braid varieties, which arise as certain moduli of Lagrangian fillings and generalize Richardson varieties, are indeed cluster algebras [CGGS, CGG+22]. Note that [GLSBS] also provides an alternative, more combinatorial, construction of such cluster structures on braid varieties.

The talks. In sections A, B and C below, we give the list of proposed topics for talks. Here is the provisional schedule for the week:

	Mon	Tue	Wed	Thu	Fri
09:30–10:30	A1	A4	B2	B3	B5
11:15–12:15	A2	A5	C3	B4	C6
16:00–17:00	C1	C2		C4	
17:30–18:30	A3	B1		C5	

Software for working with cluster algebras. In addition to mathematical talks, we will also have one or two sessions (to be held in the afternoon or evening) that explain the various pieces of software that are useful for working with cluster algebras. These include Keller's quiver mutation applet, Musiker–Stump's Sage code for cluster algebras, Bourjaily's Mathematica code for the combinatorics of positroids, Galashin's applet for plabic graphs/ tilings, Weng's seed calculator, ...

Prerequisites. For parts A and B, the participants should have a working knowledge of basic algebraic geometry (regular functions, affine varieties, projective varieties, irreducibility, ...) cf. for example sections I.1–I.5 of [Har77]. They should be familiar with examples like affine spaces, projective spaces and Grassmannians. For part C, familiarity with basic notions of differential topology is desirable, cf. [GP74, Mil65], see Section C for details.

What is an Arbeitsgemeinschaft. An Arbeitsgemeinschaft (study group) is mainly meant for non-specialists who want to broaden their outlook on mathematics and for junior researchers who wish to enter a field for future research. Experts are also welcome. The idea is to 'learn by doing'. Participants have to volunteer for one of the lectures described in the program of the Arbeitsgemeinschaft. After the deadline for application, the organizers choose the actual speakers to give them enough time to understand the subject and to prepare for their lectures. We refer to

https://www.mfo.de/scientific-program/meetings/arbeitsgemeinschaft

for information on how to apply. The application deadline for this Arbeitsgemeinschaft is

June 16, 2023.

A. Cluster algebra basics (5 talks)

A.1. Cluster algebras: Definition and first examples.

A.1.1. *Goal of the talk.* The goals are

(1) to introduce the cluster algebra and the upper cluster algebra associated with an ice quiver (and, more generally, with a valued ice quiver),
(2) to state the Laurent phenomenon and
(3) to illustrate everything on classes of simple examples: cluster algebras rank two (without coefficients, skew-symmetric and skew-symmetrizable), the Markov cluster algebra and the homogeneous coordinate algebra of the Grassmannian of planes in $(n+3)$-space.

A.1.2. *Suggested plan.* Begin with quiver mutation and explain its link to matrix mutation. Illustrate on examples (using for example [Kel]). Mention the more general case of rectangular integer matrices with *skew-symmetrizable* principal part (corresponding to valued ice quivers, cf. section 3 of [Kel12]). Define seed mutation and cluster variables. Illustrate on examples given by: the A_2-quiver, the Kronecker quiver, the Markov quiver, the valued B_2-quiver. Define the cluster algebra, state the Laurent phenomenon, define the upper cluster algebra, illustrate the difference on the example of the Markov quiver. Extend the definitions to ice quivers (quivers some of whose vertices are frozen). Define what a cluster structure on a (homogeneous) coordinate algebra is. Illustrate it on the example of the Grassmannian of planes in $(n+3)$-space (present the parametrization of cluster variables resp. clusters by arcs resp. triangulations of the $(n+3)$-gon following Fomin–Zelevinsky).

A.1.3. *Remarks and references.* The historical reference is [FZ02] but we will use the definitions of [FZ07]. Full information can be found in [FWZc]. A concise definition in the restricted generality needed for this talk is given in sections 1.2–1.6 of [GLS13] (assume that all (or no) coefficients are invertible) as well as sections 2–4 of [Kel12], which contains many further references (the conjectures mentioned there are now for the most part proved thanks to [GHKK18, Qin17, KKKO18, CL20]).

A.2. Classification of cluster-finite cluster algebras.

A.2.1. *Goal of the talk.* State the classification of cluster-finite[1] cluster algebras (those with only finitely many cluster variables) and, in the acyclic case, the parametrization of their non initial cluster variables by the positive roots. Present the method for constructing the cluster variables of acyclic cluster-finite (valued) quivers using generalized friezes.

[1]They are traditionally called 'cluster algebras of finite type' which is unfortunate because there are many cluster algebras which are commutative algebras of finite type but have infinitely many cluster variables.

A.2.2. *Suggested plan:* Start with a reminder on (finite) root systems (illustrated on examples of rank 2 and 3). State the classification and the parametrization of cluster variables by almost positive roots (following Ch. 5 of [FWZd] or [FZ03a]). Present the algorithm on examples. Sketch why cluster-finite cluster algebras must come from finite root systems following [FWZd] or [FZ03a]. You might conclude by mentioning that for arbitrary acyclic quivers, the non-initial cluster variables are similarly parametrized by the real Schur roots of the quiver, cf. [CK06].

A.2.3. *Remarks and references:* The historical reference is [FZ03a]. Exhaustive information is given in [FWZd]. The construction of cluster variables using generalized friezes (knitting algorithm) is presented in section 1.2 of [Kel10] and implemented in [Kel]. The fact that (valued) Dynkin quivers have only finitely many cluster variables can be deduced from Gabriel's theorem (resp. Dlab–Ringel's theorem) in quiver representation theory using the results of [CC06] but this will not be part of the talk.

A.3. **Cluster structures on coordinate algebras, example of the Grassmannian.**

A.3.1. *Goal of the talk.* The goal is to give a first example of how to show that a coordinate ring has a cluster algebra structure.

A.3.2. *Suggested plan.* State and give a proof (or proof sketch) of the Starfish Lemma, which is a tool for showing that each cluster variable is a regular function [FWZa, Section 6.4]. Then use this to prove Scott's result [Sco06] that the coordinate ring of the affine cone over the Grassmannian has a cluster structure [FWZa, Section 6.7].

A.3.3. *Remarks and references.* Scott's original proof of the Grassmannian's cluster structure [Sco06] used the combinatorics of *alternating strand diagrams* (which are equivalent to *plabic graphs*), and the fact that there exists a special seed Σ in which every cluster variable is a Plücker coordinate *and* all cluster variables obtained by mutating one step away from Σ are also Plücker coordinates. Scott also used the fact that every Plücker coordinate appears as a face label of some alternating strand diagram. However, it will probably be more instructive to present the proof given in [FWZa, Section 6.7] as this does not require any intricate combinatorial machinery.

A.4. **More cluster combinatorics: g-vectors, c-vectors ..., maximal green sequences.**

A.4.1. *Goal of the talk.* The goals are

(1) to present the recursive definitions of g-vectors, c-vectors and F-polynomials and the formulas they yield for cluster variables respectively y-variables (in the terminology of Fomin–Zelevinsky) following Nakanishi–Zelevinsky [NZ12],
(2) to present Nakanishi–Zelevinsky duality,
(3) to present the bijection between g-vectors and cluster variables,
(4) if time permits[2]: to present the notions of green/red mutation, to introduce maximal green and green-to-red sequences and mention some of their uses.

[2] If there is not enough time, this topic could also be treated in a software demonstration session.

A.4.2. *Remarks and references.* The g-vectors were originally defined in [FZ07] and c-vectors appear there implicitly. They appear explicitly in [NZ12]. The g-vectors can also be interpreted as tropical points of the cluster Poisson variety (X-variety) given by the quiver, cf. the talk devoted to cluster ensembles. The sign coherence conjecture mentioned in [NZ12] is now proved in full generality in [GHKK18]. The fact that g-vectors are in bijection with cluster variables was shown in [CIKLFP13] for skew-symmetric cluster algebras. The skew-symmetrizable case can be deduced from the results of [GHKK18] using the 'proper Laurent polynomial property' of [CILF12]. Information on maximal green sequences and their uses is given in the survey [KD20].

A.5. **Cluster algebras from surfaces.**

A.5.1. *Goal of the talk.* The goal is to explain how to associate a cluster algebra to a bordered orientable surface together with a choice of marked points; roughly speaking, cluster variables correspond to arcs, clusters correspond to triangulations, and exchange relations are special kinds of *skein relations.* [FST08, FT18] are good references for main results plus proofs. [MSW11b] includes a quick overview of the basics; [MW13] discuss skein relations (inspired by earlier work of Fock-Goncharov).

A.5.2. *Suggested plan.* Introduce bordered marked surfaces, arcs, triangulations, skein relations; for simplicity, it may be best to focus on surfaces without punctures. Explain (without proof) the geometric interpretation of cluster variables as *lambda lengths* (also called *Penner coordinates*) for points in the corresponding decorated Teichmüller space. If time permits, could mention combinatorics of Laurent expansions as in [MSW11b].

B. Cluster algebras in geometry

B.1. **Techniques for constructing cluster structures on varieties.**

B.1.1. *Goal of the talk.* The goal of the talk is to explain how one might go about showing that the coordinate ring of a variety is a cluster algebra. There are several steps here: constructing an initial seed, showing that each cluster variable is a regular function, and showing that the cluster variables generate the entire coordinate ring. Or alternatively: showing that the algebra of regular functions is an upper cluster algebra, then showing that the upper cluster algebra equals the cluster algebra.

B.1.2. *Suggested plan.* Recall the Starfish Lemma (from talk A.3) as a tool for showing that each cluster variable is a regular function [FWZa, Section 6.4]. Introduce the notions of *locally acyclic* [Mul13] cluster algebras, and maybe also *Louise* cluster algebras [MS16] and/or *sink-recurrent* cluster algebras [GLSBS, Definition 5.4]; such properties imply that a cluster algebra equals its upper cluster algebra. Possibly mention localization techniques as in [CGG$^+$22, Section 5.3]. Fraser's work on quasi-homomorphisms of cluster algebras [Fra16] is also useful. In particular, if one is interested in showing that a cluster algebra $\mathcal{A}$ equals its upper cluster algebra U, sometimes the following lemma is useful: if all elements of a generating set for U are either in $\mathcal{A}$ or in a cluster algebra quasi-equivalent to $\mathcal{A}$, then $\mathcal{A} = U$. Note that the twist map is an example of a quasi-homomorphism of cluster algebras.

B.2. **Combinatorics of plabic graphs.**

B.2.1. *Goal of the talk.* To get some familiarity with the Postnikov's plabic graphs and related combinatorics (esp decorated permutations), which encode positroid cells, positroid varieties and their associated cluster structures.

B.2.2. *Suggested plan.* Define plabic graphs and reduced plabic graphs and how they generalize combinatorial objects like triangulations of a polygon, wiring (and double) wiring diagrams (cf [FWZb, Section 7.2]). Another important example is that of *plabic fences* which come from (not necessarily reduced) expressions in simple reflections (cf. [FPST, Section 12] and [CW, Section 2.5]). Explain the *fundamental theorem of reduced plabic graphs*, which says that two reduced plabic graphs are move-equivalent if and only if the associated trip permutations are the same, see [Pos] and [FWZb, Theorem 7.4.25]. Explain some criteria for checking that a graph is reduced (e.g. [FWZb, Theorem 7.11.5 or Theorem 7.8.6]; explain how to label faces of a plabic graph to get a *weakly separated collection*. Explain how these combinatorial objects can be used to define a positroid cell in the positive Grassmannian (as in [Pos]) and a positroid variety in the Grassmannian (as in [KLS13]).

B.3. **Webs and Gr(3,n).**

B.3.1. *Goal of the talk.* Explain how trivalent tensor diagrams encode elements of the coordinate ring $\mathbb{C}[\widehat{Gr_{3,n}}]$ of the Grassmannian of 3-planes in $\mathbb{C}^n$, and explain the Fomin-Pylyavskyy conjectures. [FP16] is a main reference but to get a quick introduction, see [Fra20, Section 9.1].

B.3.2. *Suggested plan.* Explain what is a tensor diagram, and how to use one to associate an element of $\mathbb{C}[\widehat{Gr_{3,n}}]$. Two ways to do this: either as a repeated contraction of the volume form and dual volume form [FP16, Section 4] (see also [FLL19]), and via the application of *skein relations* (as in [Fra20, Section 9.1]). Explain main results and conjectures of [FP16] and [FP14].

B.4. **Double Bruhat cells and generalizations.**

B.4.1. *Goal of the talk.* The goal is to present the cluster structures on double Bruhat cells [BFZ05] and on double Bott–Samelson cells [SW21]. For simplicity, the emphasis is on type A.

B.4.2. *Suggested plan.* Present the definition of double Bruhat cells following section 2 of [BFZ05] with emphasis on the case of $G = SL_{r+1}(\mathbb{C})$ (consider using a bigger running example than [BFZ05]). Present the construction of the quiver (illustrating it on examples [Kel][3]) and the initial cluster (do not enter into the details of the construction of generalized minors). Present special cases (for example $G^{c,c^{-1}}$, $G^{c^3,c^{-2}}$, G^{e,w_0}, G^{w_0,w_0}). Present the definition of double Bott–Samelson cells following section 2 of [SW21] concentrating on type A. Present the definition of the cluster structure (=cluster K_2 structure) following section 3.2 of [SW21].

B.4.3. *Remarks and references.* For double Bruhat cells, you might want to give a more informal description of the quivers in the spirit of section II.4 of [BBIRS09] (which gives the quiver for a one-sided Bruhat cell without the frozen part). For this talk, familiarity with linear algebraic groups is useful, cf. for example [Bor91, Hum75, Kum02]. However, it will mostly serve to specialize the contents of the original papers to type A.

[3]Please contact B. Keller for this undocumented feature

B.5. **Cluster ensembles (A-variety, X-variety).**

B.5.1. *Goal of the talk.* The first goal of the talk is to present Fock–Goncharov's geometric vision of cluster theory encoded in pairs of 'varieties', now called Poisson cluster variety (=X-variety) and cluster K_2 variety (=A-variety), linked by a map $p : A \to X$. The second goal is to illustrate this setting on the example of double Bruhat and/or double Bott–Samelson cells using the results presented in talk B.4.

B.5.2. *Suggested plan.* Introduce the cluster ensembles and the map $p : A \to X$ following [FG09a] (cf. below for useful additional references). Show the link between tropical points of the cluster Poisson variety and the g-vectors of talk A.4. Present the example of the cluster ensemble enriching a double Bott–Samelson variety (introduced in talk B.4) following [SW21].

B.5.3. *Remarks and references.* The historical reference for cluster ensembles is [FG09a]. A very concise summary is in appendix A of [SW20] and a more detailed one (in a refined setting) in Appendix A.2 of [SW21].

C. Cluster algebras and symplectic topology (6 talks)

The main goal of this series of lectures is to present a family of cluster algebras that arise naturally in symplectic topology. These cluster algebras are the coordinate rings of algebraic varieties, known as braid varieties, that parametrize certain configurations of flags. These lectures use the combinatorics of weaves, first introduced to manipulate Legendrian and Lagrangian surfaces in contact and symplectic manifolds, to show that these coordinate rings are indeed cluster algebras. These weaves allow for a quick and diagrammatic translation of symplectic geometric constructions, such as Lagrangian disk surgeries and microlocal holonomies, into algebraic ones, such as cluster mutations and cluster variables.

The lectures build the necessary symplectic background from the ground up, including preliminaries on wavefronts and microlocal sheaf theory. Once the necessary constructions from symplectic geometry have been introduced, they are applied to the study of cluster algebras. Applications in the other direction, proving new results in symplectic geometry by studying cluster algebras, will be also highlighted.

Background and introductory references: For lectures C.1 and C.2, familiarity with differential geometry or topology can be helpful, such as a graduate course in differential topology, e.g. following [Bre93, GP74] or [Mil65]. Introductory material on contact and symplectic topology can be found in the earlier chapters of [CdS01], [AdG01] [Gei08] or [MS98]. For lectures C.3 and C.4 a background in (classical) sheaf theory or homological algebra, following graduate level materials [Bre97, Dim04, Hat02], is likely useful. For lectures C.5 and C.6, familiarity with the basics of cluster algebras, at the level of [FWZc], is recommended. As with lectures in series A and B, some basic knowledge of complex algebraic varieties [Har77, Sha13a, Sha13b], particularly in the form of explicit examples (especially Grassmannians and flag varieties), is recommended.

C.1. **Introduction to Lagrangian fillings.**

C.1.1. *Goal of the talk.* The main goal is introducing the geometric problem of studying embedded exact Lagrangian fillings in the symplectic Darboux ball $(T^*\mathbb{R}^2, \lambda_{st})$ with boundary condition a Legendrian link in a contact Darboux ball $(\mathbb{R}^3, \xi_{st}) \subset (T^*_\infty\mathbb{R}^2, \ker \lambda_{st})$.

C.1.2. *Suggested plan.*

(1) Definition of contact and symplectic structures, proceeding from the examples of cotangent bundles. (See [Gei08, Sections 1.1,1.4], [Ad89, Appendix 4], [Ad90, Chapter 1] or [AdG01].)
(2) Motivation and salient properties: Darboux theorem [Ad89, Appendix 4.H], Donaldson's divisor decomposition [Gir02] and Lagrangian skeleta of Weinstein manifolds [CE12].
(3) Definition of exact Lagrangian submanifolds and Legendrian submanifolds. Example of conormals in cotangent bundles, e.g. [KS13, Section 6.2]. (See [Gei08] and [MS98, Section 3.3].)
(4) Explanation of the problem: classification of embedded exact Lagrangian fillings of a given Legendrian link up to isotopy. Intuitively explain Lagrangian disk surgery and briefly survey the status of the problem. Present layout of lectures C2 through C6. (See [Cas22, Section 5].)

C.1.3. *Simplifying assumptions.* It suffices to focus on the 4-dimensional symplectic Darboux space $(T^*\mathbb{R}^2, \lambda_{st})$, and the contact manifolds $(\mathbb{R}^3, \xi_{st})$, (J^1S^1, ξ_{st}), $(\mathbb{R}^5, \xi_{st})$. Focus on embedded exact Lagrangian surfaces and Legendrian links and surfaces on these manifolds.

C.1.4. *Remarks and additional references.* Item (4) above is the core content of the talk. Subsequent talks aim at studying algebraic spaces that act as moduli of Lagrangian fillings. In general, the two introductory textbooks [Gei08] and [MS98] provide details on the basic contact and symplectic background, respectively. The sources [AdG01, Ad90, Ad89] are written in a survey manner, which can also be of help.

C.2. **Fronts and Lagrangian fillings of Legendrian links.**

C.2.1. *Goal of the talk.* The main goal is presenting Legendrian fronts, through examples and computations, and explaining how they are used to construct embedded exact Lagrangian fillings of Legendrian links.

C.2.2. *Suggested plan.*

(1) Introduce fronts for Legendrian links in $(\mathbb{R}^3, \xi_{st})$, present examples and state the Legendrian Reidemeister Theorem. (See [Gei08, Chapter 3] and [Etn05, Section 2], also [Ad90, Chapter 3].)
(2) Classical invariants [Etn05, Section 2] and statement of non-simplicity [Etn05, Section 4]. Geometric intuition towards modern invariants [GKS12, KS13].
(3) Legendrian lift of an exact Lagrangian surface. Study of Lagrangian fillings of Legendrian links via fronts of Legendrian surfaces ([CZ22, Section 7].)
(4) Definition of Legendrian weaves and construction of Lagrangian fillings via weaves. (See [CZ22, Sections 2,4].) Y-trees and 1-cycles of the Legendrian surface associated to the weave.

C.2.3. *Simplifying assumptions.* For surface fronts restrict to discusss the A_1^2, A_1^3 and D_4^- singularities, which are the ones used for weaves. (Note that the latter is not generic and generic singularities, such as A_3, do not need to be covered.)

C.2.4. *Remarks and additional references.* It is important to emphasize that each front, of a Legendrian link or a Legendrian surface, provides a stratification of the target space, $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively in the cases at hand. Fronts and their perestroikas for surfaces are discussed in detail in [Ad90, Chapter 3].

C.3. **A short introduction to constructible sheaves.**

C.3.1. *Goal of the talk.* The main goal is introducing constructible sheaves, through explicit examples and six-functor computations, and discuss the $R\Gamma$ functor and their derived (or dg) category.

C.3.2. *Suggested plan.*

(1) Definition of sheaves (of complexes of $\mathbb{C}$-vector spaces), with motivation from local systems. Examples of sheaves in $\mathbb{R}^2$ and $\mathbb{R}^3$ via inclusions i, j of open sets U, Z and functors $i_*, i_!, j_*$. (See [KS13, Chapter II].)
(2) Definition of constructible sheaves. Examples based on stratifications given by planar and spatial fronts of Legendrians. (See [KS13, Chapter VIII].)
(3) Explanation of the derived sections functor $R\Gamma$. Mention of the $\mathcal{H}om$ functor and the construction of the dg-category of constructible sheaves. (See [KS13] and [CL22, Appendix A].)

C.3.3. *Simplifying assumptions.* All stratifications that will be used are either of $\mathbb{R}^2$ or of $\mathbb{R}^3$. Assume that the stratifications are Whitney and, if it helps, restrict to those stratifications coming from fronts with the singularities discussed in Lecture C.2.

C.4. **Microsupport and Legendrian fronts.**

C.4.1. *Goal of the talk.* The main goal is to introduce the notion of singular support and compute moduli of sheaves singularly supported on a front of a Legendrian link and on a weave.

C.4.2. *Suggested plan.*

(1) Definition of microsupport [KS13, Section 5.1] and the statement of the involutivity theorem [KS13, Theorem 6.5.4]. (See also [KS13, Chapter VIII] for the constructible case.)
(2) Categories of sheaves on $\mathbb{R}^2$ with microsupport on a Legendrian link [GKS12, KS13]. (See also [CL22, CW, STZ17].) The Legendrian invariance theorem [GKS12, Theorem 3.7].
(3) Computations for local models in the Legendrian links case [STZ17, Section 3.3] and the case of rainbow closures of positive braids [STZ17, Section 6.2]. (See also [CW, Section 4.1].) Computation for weaves [CZ22, Section 5].

C.4.3. *Simplifying assumptions.* Focus on sheaves on $\mathbb{R}^2$ and $\mathbb{R}^3$ that are constructed via $i_*, i_!$, where i is the inclusion of an open (or closed) set. Always suppose that the Legendrian admits a binary Maslov index and thus it suffices to work with sheaves (instead of complexes of sheaves). The moduli of (pseudoperfect) objects is the most important space in the context of these series. In particular, ignore *hom* and μhom matters and focus on describing objects. In the setup of the subsequent lectures, this moduli space is an affine algebraic variety.

C.5. **Microlocal holonomies and the Bott-Samelson case.**

C.5.1. *Goal of the talk.* The main goal is to describe microlocal holonomies and show that the coordinate ring of regular functions of a Bott-Samelson variety is a cluster algebra.

C.5.2. *Suggested plan.*

1. Microlocalization functor [Gui] and [CL22, Sections 5.1.1, Appendix B.2]. Microlocal holonomies: monodromies and merodromies [CW, Sections 4.4,4.6].
2. Proof via weaves that the coordinate ring of a Bott-Samelson variety is a cluster algebra [CW, Theorem 1.1]. (The key step in the proof is [CW, Section 4.9].)

C.5.3. *Simplifying assumptions.* Suppose that all Lagrangian fillings are orientable, spin, and there exists a binary Maslov index for all Legendrian submanifolds being discussed. For the microlocalization functor, adopt the direct combinatorial approach (stalks computed as cones) and describe it as a functor to local systems on the Legendrian, bypassing the Kashiwara-Schapira stack, as in [CL22]. (If that helps, ignore signs.) For the Bott-Samelson varieties, restrict to $G = GL_n(\mathbb{C})$.

C.6. **Cluster structures on braid varieties.**

C.6.1. *Goal of the talk.* The main goal of the talk is to show that the coordinate ring of regular functions of any braid variety is a cluster algebra.

C.6.2. *Suggested plan.*

1. Lusztig cycles in weaves, associated quivers and (candidate) cluster variables [CGG$^+$22, Section 4].
2. Proof that the coordinate ring of regular functions of a braid variety is an upper cluster algebra [CGG$^+$22, Section 5.3].
3. Proof that the upper cluster algebra coincides with the cluster algebra [CGG$^+$22, Section 5.5].
4. If time permits, additional results on the geometry of braid varieties: DT-transformations as contact transformations and holomorphic symplectic structures.

C.6.3. *Simplifying assumptions.* Focus on the $G = GL_n(\mathbb{C})$ case.

References

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