Oberwolfach Seminar: Recent topics on the Navier-Stokes equations
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The seminar is addressed to the recent development of research on the Navier-Stokes equations. The mathematical analysis of the Navier-Stokes equations was founded by Leray in 1993 and in 1994 for the stationary and the non-stationary cases, respectively. In the non-stationary case, Leray constructed a global weak solution in $\mathbb{R}^3 \times (0, \infty)$ with finite energy and dissipation for arbitrary L^2-initial data. It is still a big open question whether such a weak solution is unique and regular. So far as the problem on local well-poseness, almost optimal result has been obtained since the space $BMO^{-1} = \dot{F}^{-1}_{\infty,2}$ is admissible, but $\dot{B}^{-1}_{\infty,q}$ yields ill-posedness for all $1 \le q \le \infty$. Notice that $\dot{B}^{-1+\frac{n}{p}}_{p,\infty}$ is admissible to well-posedness for all $1 \le p < \infty$.

Concerning the problem on global well-posedness, in recent years, Buckmaster-Vicol [1] had proved non-uniqueness of weak solutions in the class finite energy. Many of us believe that Leray's weak solution becomes, in fact, unique and regular. However, such a faith may not be true so that it seems a challenging problem whether the same non-uniqueness does hold if the weak solutions has both finite energy and dissipation, i.e., the Leray-Hopf class.

In the stationary case, there remains several fundamental questions in 3D exterior domains Ω. For instance, Leray constructed a weak solution u with the finite Dirichlet integral $\int_\Omega |\nabla u(x)|^2 dx < \infty$. The uniqueness question of such weak solutions is still open. Furthermore, there remains a fundamental problem whether the solution does exist for the prescribed inhomogeneous data on the boundary $\partial\Omega = \cup_{j=1}^N \Gamma_j$ with the multiple connected components $\Gamma_1, \cdots, \Gamma_L$ in $\mathbb{R}^3$. Recently, the second and third organizers with their colleagues [3], [4] established an L^r-Helmholtz-Weyl decomposition of vector fields in 3D exterior domains, and applied to the exterior problem on the stationary Navier-Stokes equations.

Besides these interesting problems, the free boundary problem of two phase flow is also fundamental in the fluid mechanics, and the approach in terms of maximal L^p-regularity theorem has been fully developed by the third organizer [5], [6]. Based on this background, we bring a focus onto the following three topics.

(i) Onsager conjecture and non-uniqueness of the weak solutions to the Navier-Stokes equations.
(ii) L^r-Helmholtz-Weyl decomposition in 3D exterior domains and its application to the stationary Navier-Stokes equations
(iii) Free boundary problem on incompressible two-phase flows with phase transitions

More precisely, we have the following plan;

Lecture 1 given by T. Buckmaster:
Onsager conjecture and non-uniqueness of the weak solutions to the Navier-Stokes equations
This lecture will discuss the use of convex integration to construct wild weak solutions in the context of the Euler and Navier-Stokes equations. In particular, an outline of the resolution of Onsager's conjecture as well as the recent proof of non-uniqueness of weak solutions to the Navier-Stokes equations will be shown. Onsager's conjecture is linked to the phenomena of anomalous dissipation in turbulent cascades, which has been called the zeroth law of turbulence. Non-uniqueness of weak solutions to the Navier-Stokes equations has an important consequence for a well known strategy for resolving the famous Millennium Prize problem. Specifically, one may hope to prove the existence of global smooth solutions to the Navier-Stokes equations by first constructing global weak solutions and then proving that such solutions are smooth. The recent non-uniqueness result of Buckmaster and Vicol demonstrates that such a strategy is doomed to fail, at least for the class of weak solutions considered. It remains open whether Leray-Hopf weak solutions are non-unique which is the subject of the famous Ladyzhenskaja conjecture.

Lecture 2 given by H. Kozono:
L^r-Helmholtz-Weyl decomposition in 3D exterior domains and its application to the stationary Navier-Stokes equations

This lecture is addressed to the Helmholtz-Weyl decomposition of L^r-vector fields on exterior domains with compact smooth surfaces in $\mathbb{R}^3$. In bounded domains, an L^r-Helmholtz decomposition has been established by Solonnikov and Fujiwara-Morimoto. Later on Kozono-Yanagisawa gave a concrete characterization of 3D harmonic vector fields from a viewpoint of topological invariance called Betti number. It is not a trivial problem whether the corresponding L^r-decomposition does hold in an exterior domain $\Omega \subset \mathbb{R}^3$ with smooth compact surface $\partial\Omega$. In this lecture, we show that in accordance with the boundary condition $\boldsymbol{u} \cdot \boldsymbol{\nu}|_{\partial\Omega} = 0$ or $\boldsymbol{u} \times \boldsymbol{\nu}|_{\partial\Omega} = \mathbf{0}$ ($\boldsymbol{\nu}$:unit outer normal to $\partial\Omega$), the space of L^r-harmonic vector fields has a different aspect. For instance, there is a threshold exponent $r = 3/2$ in the sense that the first Betti number is changed. The L^r-Helmholtz-Weyl decomposition states that every $\boldsymbol{u} \in L^r(\Omega)$ is uniquely expressed in such a way that

$$\boldsymbol{u} = \boldsymbol{h} + \mathrm{rot}\ \boldsymbol{w} + \nabla p.$$

In the lecture, we prove how to determine the vector potential $\boldsymbol{w}$ and the scalar potential p according to $1 < r < \infty$. As an application, we prove the existence theorem on D-solutions to the stationary Navier-Stokes equations in 3D exterior domains with inhomogeneous boundary condition.

Lecture 3 given by S. Shimizu:

Free boundary problem on incompressible two-phase flows with phase transitions

There is a large literature on isothermal incompressible Newtonian two-phase flows without phase transitions, and also on the two-phase Stefan problem with surface tension modeling temperature driven phase transitions. On the other hand, mathematical work on two-phase flow problems including phase transitions are few. In this lecture we will explain the modeling of one-component two-phase flows with moving boundary. The models are derived from first principles, and some of the main structural properties are presented. In particular, we show that the models are thermodynamically consistent, identify the equilibria, and discuss their thermodynamic stability. In addition, several analytical results in the incompressible case with phase transition are presented, which include topics like short-time well-posedness, stability of equilibria, and long time existence and convergence to equilibria.

References

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