Arbeitsgemeinschaft

Quantum field theory and stochastic PDEs

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14th January 2023

Quantum field theory (QFT) originated in the attempt to define a relativistic quantum mechanical theory for elementary particles in the 1940s and 1950s. Today, the term is used to describe the calculational framework for a wide range of phenomena is physics—from elementary particles to condensed matter physics—based on path integrals, which are measures on spaces of generalised functions. The mathematical construction and analysis of such measures is also known as Constructive QFT.

This Arbeitsgemeinschaft will begin by covering some background material and then explore some of the advances made in recent years that are based on the perspective of stochastic PDEs (SPDEs) for which the QFT measures are stationary measures.

The link between QFT and SPDE was first observed by the physicists Parisi and Wu [PW81], and the connection is known as Stochastic Quantisation. The study of solution theories and properties of solutions to these SPDEs derived from the Stochastic Quantisation procedure has stimulated substantial progress of the solution theory of singular SPDE, especially the invention of the theories of regularity structures [Hai14b] and paracontrolled distributions [GIP15] in the last decade. Moreover, Stochastic Quantisation allows us to bring in more tools including PDE and stochastic analysis to study QFT.

The focus of this Arbeitsgemeinschaft will be QFT models such as the Φ^4 and Yang–Mills models as examples to discuss stochastic quantisation and SPDE methods and their applications in these models. Other models such as Fermionic models, sine-Gordon, and exponential interaction will also be discussed to some extent. We will introduce the key ideas, results and applications of regularity structure and paracontrolled distributions, construction of local solutions and global solutions of SPDEs corresponding to these models, and use the PDE method to study some qualitative behaviors of these QFTs, and connections with the corresponding lattice or statistical physical models. We will also discuss some other topics of QFT, such as Wilsonian renormalisation group, log-Sobolev inequalities and their implications, and various connections between these topics and SPDEs.

Format: The Arbeitsgemeinschaft will have 8 blocks: Monday morning, Monday afternoon, Tuesday morning, Tuesday afternoon, Wednesday morning, Thursday morning, Thursday afternoon, Friday morning.

General information: 'Arbeitsgemeinschaft' means 'study group'. All lectures at the Arbeitsgemeinschaft are taught by participants with the goal of shared learning through active participation. All applicants must volunteer to give lectures when applying to the Arbeitsgemeinschaft (see below). Usually there will be more attendees than lectures.

Information on how to apply can be found at this website.

Below we outline 8 subjects to be discussed in the 8 time blocks. Each subject is divided into several smaller topics. We may not be able to cover all topics, so the final list will depend on participant interest.

1 Introduction to Euclidean QFT

Goal: Introduction to Euclidean quantum field theory, with some examples, including the Gaussian free field and the Φ^4_2 model in finite volume.

1.1 Quantum mechanics to path integrals: Feynman–Kac formula

Brief introduction to the quantum mechanical harmonic oscillator, and the ground state transformation that takes the harmonic oscillator to the Ornstein–Uhlenbeck semigroup. Discuss the Feynman–Kac formula with the goal to arrive at [GJ87, Theorem 3.4.1].

References: [GJ87, Chapters 1-3]

1.2 Reflection positivity and the Osterwalder–Schrader axioms

Osterwalder–Schrader axioms for QFT, for example as presented in [GJ87, Chapter 6], with focus on the reflection positivity property. In particular, the construction of the Hamiltonian [GJ87, Theorem 6.1.3]. Example that lattice QFT is reflection positive, and the analogy between the transfer matrix for lattice models and the quantum mechanical time evolution, see [GJ87, p. 96].

Reference: [GJ87, Chapter 6]. Further reference: [Sim74, Chapter 2]

1.3 The massive Gaussian free field

Generalities about Gaussian measures in infinite dimensions, including the integration by parts formula and the Wick theorem for Gaussian fields. Fock space representation of the canonical commutation relations and relation to chaos decomposition of the Gaussian L^2 space.

Define Gaussian free field on $\mathbf{R}^d$ with mass $m > 0$, which is a model for free (i.e. non-interacting) scalar boson. Sketch the validity of the Osterwalder–Schrader axioms.

References: [Sim74, Chapter 3], [Hai16a]. Further references: [NN18, Chapter 4], [She07].

1.4 The Φ^4_2 measure in finite volume

Construction of the Φ^4_2 measure in finite volume via Nelson's argument, following for instance [Hai16a, Section 8]. This includes: showing that as one removes the ultraviolet cutoff, i.e., the small scale regularisation, the Wick renormalised quartic interaction has a limit in L^p (see also [Sim74, Chapter 5]); the introduction of hypercontractivity of the Ornstein–Uhlenbeck semigroup; Nelson's argument, i.e., the density of the Φ^4_2 with respect to the Gaussian free field measure is integrable.

References: [Hai16a, Section 8]. Further references: [Sim74, Chapter 5], [Dim11, Section 13.4]

2 SPDEs in the Da Prato–Debussche regime

Goal: Introduction of the formal procedure of Parisi–Wu stochastic quantisation, which is a stochastic gradient flow (SPDE) with a Euclidean QFT model as its stationary measure. The Da Prato–Debussche argument [DPD03] for local solutions of the stochastic quantisation of Φ^4_2.

2.1 Functional setting and white noise

Introduction of the basic functional setting for SPDEs, such as Hölder–Besov spaces for distributions, Young's theorem for multiplying distributions. Define space-time white noise, and discuss its scaling and regularity properties.

References: [CW17, Section 1 and 2], [Hai14a, Section 2]

2.2 Linear stochastic heat equation

Discussion of the linear stochastic heat equation with additive space-time white noise, including the classical Schauder estimates, existence and uniqueness of initial value problem, stationary solution, regularity (and singularity), and Gaussianity of its solution. In particular, show that if the (Euclidean) space dimension is $d \geq 2$, then the solution to the linear stochastic heat equation exists only in the sense of distributions, and why this causes problem in interpreting the meaning of the solution to nonlinear SPDEs such as the dynamical Φ^4_2 model.

Introduce nonlinear perturbations to stochastic heat equation, and discuss scaling, and meanings of subcritical, critical and supercritical regimes.

References: [CW17, Section 2]

2.3 The Da Prato–Debussche argument

The Da Prato–Debussche argument [DPD03] for local solutions of the stochastic quantisation of Φ^4_2. In particular, discussion of Wick renormalised powers of the Gaussian solution to the linear stochastic heat equation, including moment calculations and convergence in suitable Hölder–Besov spaces. Then combine these with classical Schauder estimates and Young's theorem to construct local-in-time solution, by a fixed point argument.

Finally, discuss why the Da Prato–Debussche argument does not apply to stochastic quantisation of Φ^4_3.

References: [DPD03], [CW17, Section 2 and 3], [Hai14a, Section 3]

3 Global solution to stochastic quantisation of Φ^4_2

Goal: Global-in-time solutions for the dynamical Φ^4_2 model on the torus and the "coming down from infinity" property. Extension to infinite space $\mathbf{R}^2$ using weighted norms. Constructions of the Φ^4_2 measure via SPDEs.

3.1 Global solutions and coming down from infinity, I.

Following [MW17b, Section 3], recall standard results on embeddings and interpolations for Besov spaces and recall multiplication results for Besov spaces. Discuss L^p a priori estimates for stochastic quantisation of Φ^4_2 on torus following [MW17b, Section 6], including discussion on the proof of L^p energy identity, and bounds on various other terms making use of the $-\Phi^3$ term. Briefly discuss the extension to entire space $\mathbf{R}^2$ using weighted norms.

References: [MW17b], [GH19].

3.2 Global solutions and coming down from infinity, II.

Discuss the approach by Gubinelli and Hofmanová [GH19] on global solutions, focusing on $d = 2$. Recall Littlewood–Paley blocks, and introduce localization operators and their bounds. Introduce the decomposition of solution into singular and regular parts. Prove uniform bounds on solutions to regularised equations driven by ξ_ε which is regularised white noise, following [GH19, Appendix A]. Pass the estimates to limit as $\varepsilon \to 0$, using compactness argument, following [GH19, Section 4 and 6]. Prove the "coming down from infinity" property for Φ^4_2 following [GH19, Section 9].

References: [MW17b], [GH19]

3.3 Tightness via energy method

Discuss another approach based on tightness to construct the Φ^4_2 measure following [GH21]. Introduce the lattice approximation of the Φ^4_2 measure, and the corresponding lattice dynamic with stationary solution. Point out the relation of lattice approximation with reflection positivity. Then introduce the L^2 energy method. Decompose the solution into singular and regular parts, and demonstrate the uniform estimates in [GH21, Section 4] by simplifying the arguments for $d = 3$ there to $d = 2$.

References: [GH21]

3.4 Integrability of Φ^4_2

Discuss sub-Gaussian tail of Φ^4_2 via the Hairer–Steele argument [HS22]. One particular goal of stochastic quantisation is to obtain properties of the measure from the associated stochastic dynamics. Integrability of certain distributional norms where already studied in [MW17b] and [GH21] but is only with the paper of Hairer–Steele [HS22] that the optimal result with better-than Gaussian bounds have been established.

Reference: [HS22].

4 More specialized topics in $d = 2$

<u>**Goal:**</u> Extension of the discussion on Φ^4_2 and other topics and models in two Euclidean space dimensions.

4.1 More applications of the Da Prato–Debussche argument

Discuss additional examples where the Da Prato–Debussche argument is applicable, such as the stochastic quantisation of sine-Gordon model in 2D in the regime $\beta^2 < 4\pi$ following [HS16] assuming the regularity of imaginary Gaussian multiplicative chaos, the parabolic Anderson model in 2D following [HL15], as well as a recent simple construction of solution to dynamical Φ^4_3 equation by [JP21] using a multiplicative transformation (you may only discuss local solution here).

References: [HS16], [HL15], [JP21].

4.2 Kac–Ising model

Renormalized singular SPDEs describe certain non-linear fluctuations of microscopic statistical mechanics systems. In this context the (infinite) renormalization constants have a precise motivation as effects living at a different scale. As an example, discuss the Glauber dynamics of Kac-Ising model following the work of Mourrat and Weber [MW17a], and derivation of the Φ^4_2 equation in the limit. Explain the interpretation of renormalisation constant in this physical context.

References: [MW17a, Section 2], Lectures by H. Weber.

4.3 Perturbation theory of Φ^4_2

Discuss properties of the Φ^4_2 measure via the SPDE approach. Osterwalder–Schrader axioms. Integration by parts formula and the hierarchy of Dyson–Schwinger equations for correlation functions. Discuss bounds on perturbation theory errors following [SZZ21].

References: [GH21, SZZ21]

4.4 Elliptic stochastic quantisation

Discuss elliptic stochastic quantisation, following mainly [ADVG20, ADVG21]. This is a variant of the parabolic situation considered so far in this AG. Interestingly the proof of the correspondence between the measure and the SPDE requires here the use of arguments involving supersymmetry and superspaces (i.e. spaces with non-commuting coordinates). The key argument was first discovered by Parisi–Sourlas in the '80.

References: [ADVG20, ADVG21]. Further references: [GH19, BDV21]

4.5 Hyperbolic stochastic quantisation

Besides parabolic and elliptic stochastic quantisation (discussed in Sect. 4.4) another variant is the one provided by certain hyperbolic Hamiltonian equation. This observation also connects with the vast literature on Hamiltonian PDE with random initial data and in particular initial data distributed as the QFT measures [BDNY22, Section 1]. Discuss global solutions of the hyperbolic stochastic quantisation equation in the case of the ϕ^4_2 QFT following [GKOT22].

References: [GKOT22]. Further references: [BDNY22, Section 1], [GKO18]

4.6 Gaussian multiplicative chaos and the Liouville model

Review basic geometric notions such as metrics, Gauss curvature, conformal transformation and conformal factor. Define Gaussian multiplicative chaos for Gaussian free fields, state the main result on its convergence and basic properties such as shifting and negative moments.

Introduce the Liouville action functional on the Riemann sphere, vertex operator insertions. Under Seiberg assumptions, prove convergence of correlation functions of Liouville conformal field theory (following [DKRV16, Theorem 3.2 and Lemma 3.3]).

References: [DKRV16, Section 2 and 3], [LRV15]

4.7 Grassmann or Fermionic models

The Euclidean rotation of theories with Fermions involves algebras of Grassmann fields, as first observed by Osterwalder–Schrader [OS73]. Therefore in order to stochastically quantise such theories one need to dispose of a stochastic analysis of Grassmann random variables. Discuss Grassmann stochastic analysis, some Fermionic QFT models and related stochastic differential equations following [ABDVG22]. With time limitation, we only discuss these on lattices.

In particular, recall algebraic probability space, Grassmann algebras, Grassmann random variables. Define Dirac operator, and write down the action functionals for several Fermionic QFT models such as Yukawa model and Gross-Neveu model. Write down their stochastic quantisation equation on lattice.

Reference: [ABDVG22]. Further reference: [OS73]

5 Introduction to gauge theory

<u>Goal:</u> Introduction to lattice and continuum Yang–Mills theories and their formal relation. Wilson loop observables, the meaning of mass gap and confinement, etc. Langevin dynamics of lattice gauge theories.

5.1 Definition of Lattice Yang–Mills theory

Definition of Lattice Yang–Mills model on finite lattice, as a (well-defined) probability measure on product Lie groups. Discussion of different possible choices of lattice action, such as the Wilson and Villain actions (heat kernel action). Gauge transformations and gauge invariance of the model. The meaning of strong and weak coupling regimes. Definition of lattice Wilson loops and verification that they are gauge invariant observables. Introduce the concepts of mass gap, confinement, area law, large N factorization.

References: [Sei82], [Cha19b]. Further references: [Cha19a, Cha21]

5.2 Continuum Yang–Mills action

Definition of the continuum Yang–Mills action, including connection 1-forms, curvature 2-forms, and discuss its gauge invariance. Use the Baker–Campbell–Hausdorff formula to show that the lattice Yang–Mills action approximates the continuum Yang–Mills action. Define Wilson loops both on lattice and in continuum (for smooth connections).

Introduce the SPDE from stochastic quantisation or Langevin dynamic for 2D Yang–Mills, in terms of gauge covariant derivatives as well as in the coordinate form. Discuss the gauge covariance of the SPDE on the formal level. Explain the source of non-parabolicity. Introduce the DeTurck term which leads to a parabolic SPDE.

References: [Sei82], [CCHS22, Section 1]

5.3 Lattice Langevin dynamic of the Yang–Mills model

For gauge group $SO(N)$, introduce the Langevin dynamic of the Lattice Yang–Mills model. This includes finding the gradient on Lie groups and writing Lie group valued Brownian motions in terms of Lie algebra valued Brownian motions, and showing gauge covariance of the lattice Langevin dynamic. Time permitting, derivation of the Makeenko–Migdal / Dyson–Schwinger / master loop equations for lattice Wilson loops.

References: [SSZ22]

5.4 Applications of lattice Langevin dynamic

Discuss further topics of the lattice Langevin dynamic: the log-Sobolev inequality and Poincaré inequalities at strong coupling and its consequences, such as uniqueness of infinite volume lattice Yang–Mills measure, large N factorization / deterministic limit, and mass gap.

References: [SZZ22]

6 Regularity structures

Goal: Introduction to the theory of regularity structures and construction of local solution to stochastic quantisation of Φ^4_3 via regularity structures.

6.1 Basic concepts and reconstruction theorem

Introduce basic concepts in regularity structures, including regularity structures, models, modelled distributions. Perhaps give an example of polynomial regularity structure. Statement and proof of the reconstruction theorem.

References: [FH14] (2nd edition, download here). Further references: [Hai15, Hai16b]

6.2 Fixed point problem in the space of modelled distributions

Discussion how to formulate and solve a fixed point problem in the space of modelled distributions. Multiplication theorem. State the Schauder theorem in the space of modelled distributions which are important for solving the fixed point problems. Admissible models.

References: [FH14, Hai15]

6.3 Stochastic quantisation of Φ^4_3

Demonstrate the application of regularity structures using the example of the local solution theory of the stochastic quantisation of Φ^4_3. Start by stating the main result, including the introduction of renormalisation constants of orders $\frac{1}{\epsilon}$ and $\log(\epsilon)$.

Explain how to associate a regularity structure to such a given SPDE. Then describe the renormalisation group. Discuss the modelled distribution expansion for the solution, and derive the renormalised equation.

References: [Hai15, Hai16b]

6.4 Convergence of the renormalised models

We then prove convergence of the renormalised models. This includes recalling Wiener chaos, equivalence of moments, and introducing necessary diagrammatic tools. Demonstrate how the renormalisation constants introduced into the SPDE help cancel the divergences.

References: [Hai15, Hai16b]

7 Dynamics of $d = 2$ gauge theory

Goal: Discuss the construction of Langevin dynamic for 2D Yang–Mills based on the paper [CCHS22]. Understand the construction of state space, local solution theory, and idea of proving gauge covariance.

7.1 Construction of the state space

Discuss the construction of the state space. In particular, explain the problem with standard Besov–Hölder spaces. Then we start from "simple objects" which are functionals on line segments (in particular Hölder continuous functions that can be integrated along line segments); impose suitable norms on these functionals; and then take completion under these norms. Explain why the completion can be embedded in the standard Besov–Hölder spaces with negative regularities. Briefly mention that gauge transformations can be defined on this state space, and that the quotient space is completely metrizable.

References: [CCHS22, Section 3 and 4], [Che22]. Further Reference: Hairer's lectures

7.2 Local solution of Yang–Mills SPDE in 2D

Discuss Kolmogorov theorem in this state space, and the fact that Gaussian free field and solution to Stochastic heat equation belong to this state space – this is important for eventually showing that the solution to the Yang–Mills SPDE will belong to this state space.

Discuss local solution theory of the Yang–Mills SPDE, as an application of regularity structures. In particular, demonstrate the relevant trees associated to the SPDE, and derive the renormalised equation. One consequence from application of regularity structures is that the solution is the distributional solution to the stochastic heat equation plus an almost Lipschitz part; in particular this solution lies in the state space.

References: [CCHS22, Section 6 and 7], [Che22]

7.3 Gauge covariance of solution in 2D and general discussion on 3D

Discuss why a finite shift of the renormalisation constant will lead to gauge covariant limit, and how to exploit symmetries of the equation in this argument.

Make some general discussion on the challenges in 3D, such as increasing number of trees, necessity of using discrete symmetries to rule out certain a priori possible renormalisation terms, and the motivation for the construction of a new nonlinear state space.

References: [CCHS22, Section 6 and 7], [Che22]. Further reference: Hairer's lectures

8 Wilsonian renormalization group

Goal: Some applications of the Wilsonian renormalization group approach to QFT and SPDE. Wilsonian renormalization group keeps track of how QFT or SPDE changes under variation of scales.

8.1 Polchinski renormalization group equation

Consider QFT models which are Gaussian free fields perturbed by potentials V, and introduce scale-dependent potentials V_t. Derive the Polchinski renormalization (semi)group equation for the density $\exp(V_t)$ and for the potential V_t.

Apply Polchinski's RG equation either to Φ^4 model or to sine-Gordon model. In the case of Φ^4 model, write the potential as a formal power series expansion and derive the Polchinski flow (system of) equations for the components in this expansion. In the case of sine-Gordon model, introduce the Fourier expansion (or Mayer expansion) following [BK87] or [BB21], and write the flow equation in terms of this expansion.

References: [Pol84, BK87, BW88]

8.2 Stochastic control approach

Discuss the variational representation of the renormalised potential in terms of the Boué–Dupuis formula. Show tightness of the Φ^4_2 measure in finite volume using this approach. Possibly also discuss the sine-Gordon model below 4π.

References: [BG20, Bar22]

8.3 Log-Sobolev inequality

Introduce the Log-Sobolev inequality. Discuss the classical Bakry–Émery criterion, and why it fails to apply to sine-Gordon, Φ^4_2 and Φ^4_3. State and prove the multiscale Bakry-Émery criterion [BB21, Theorem 1.2].

Apply the multiscale Bakry–Émery criterion to the sine-Gordon model in the regime $\beta < 6\pi$ as in [BB21]. This includes introducing the Fourier representation of the scale-dependent potential V and bounding the components in this representation, at least for $\beta < 4\pi$. Possibly mention the extension of these results to the Φ^4 model [BD22].

References: [BB21, BD22]. Further reference: Bauerschmidt's lectures

8.4 Flow equations for dynamics

Discuss the Wilsonian renormalisation group flow approach to stochastic PDEs with additive noise and polynomial non-linearity, following [Duc22] [Duc21, Section 1, 2, 3]. This is a new approach to this class of singular SPDEs using the Wilsonian renormalization group and the Polchinski flow equation.

In particular, introduce the microscopic and the rescaled, i.e. macroscopic equations. including relevant / irrelevant coefficients. Introduce regularisation and the fixed point problem, decomposition of kernel and solution, "effective force functional" which depends on the scales, and the formal power series for the effective force functional. Explain that the effective force satisfies a flow equation (with "time" being the running scale), and how the effective force coefficients in the power series expansion are constructed recursively using the flow equation.

Discuss the "renormalization problem", namely, uniform bounds on the effective force coefficients and existence of their limits. Outline the ideas of the proof to these uniform bounds.

Reference: [Duc22]. Further references: [Duc21], [Kup16]

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