ARBEITSGEMEINSCHAFTEN

1. Introduction

The Hitchin system [25], introduced in 1987, has turned out to be a versatile construction of algebraic completely integrable Hamiltonian systems, which has been related to most known integrable systems in algebraic geometry and mathematical physics see e.g. [14]. It has two outstanding applications in such distant fields as the quantum field theory explanation of the geometric Langlands program in the work of Kapustin-Witten's [28], and Ngô's proof [37] of the fundamental lemma in the Langlands program in number theory.

The Hitchin system is a proper map

$$h := \mathcal{M}^n_{Dol} \to \mathcal{A} \tag{1}$$

from the moduli space of certain Higgs bundles - pairs (E, Φ) of a rank n vector bundle E and a Higgs field $\Phi : E \to E \otimes K_C$ on a smooth complex projective curve C- to the Hitchin base, the affine space

$$\mathcal{A} := H^0(C; K) \times \cdots \times H^0(C; K^n).$$

The total space $\mathcal{M}^n_{Dol}$ of the Hitchin system carries a natural complex algebraic symplectic two form, with respect to which the components of h Poisson commute; additionally $\dim(\mathcal{A}) = \dim(\mathcal{M}^n_{Dol})/2$, which together imply complete integrability.

In fact, the holomorphic symplectic form is part of a hyperkähler structure on $\mathcal{M}^n_{Dol}$, with $\mathcal{M}^n_{Dol}$ representing the complex structure I. While in complex structure J there is a different moduli space interpretation, namely $\mathcal{M}^n_B$ the character variety of representations of the fundamental group of C into GL_n. These various moduli space interpretations go under the name of non-Abelian Hodge theory which attaches, in particular, the Dolbeault and Betti GL_n-moduli spaces to C, see [24, 42]. From the perspective of non-abelian Hodge theory we have a canonical diffeomorphism

$$\mathcal{M}^n_{Dol} \cong \mathcal{M}^n_B, \tag{2}$$

which underlies the change of complex structures in the hyperkähler metric on $\mathcal{M}^n_{Dol}$. By introducing certain twistings we can make sure that $\mathcal{M}^n_{Dol}$ and $\mathcal{M}^n_B$ are both smooth of dimension d_n. The latter is an affine variety, but the former is only quasi-projective, with large projective subvarieties (in the fibers of the proper Hitchin map h). By (2) they share their cohomology rings

$$H^*(\mathcal{M}^n_{Dol}; \mathbb{Q}) \cong H^*(\mathcal{M}^n_B; \mathbb{Q}). \tag{3}$$

Deligne's [12] mixed Hodge structure on their cohomology however does not agree. While the weight filtration on $H^*(\mathcal{M}^n_{Dol};\mathbb{Q})$ is pure, on $H^*(\mathcal{M}^n_B;\mathbb{Q})$ it is not.

In fact, the weight filtration

$$W_0 \subset \cdots \subset W_i \subset \cdots \subset W_{2k} = H^k(\mathcal{M}^n_B;\mathbb{Q}) \tag{4}$$

on $H^*(\mathcal{M}^n_B;\mathbb{Q})$ can be studied by arithmetic means [17] and was conjectured in *loc.cit.* to satisfy a "Curious Hard Lefschetz" theorem, that the Lefschetz map defined by capping with a certain power of the Higgs-Kähler class $\alpha \in H^2(\mathcal{M}^n_B;\mathbb{Q})$ of (curious) weight 4

$$\begin{array}{rccc} L^l: & Gr^W_{d_n-2l}H^{i-l}(\mathcal{M}^n_B;\mathbb{Q}) & \stackrel{\cong}{\to} & Gr^W_{d_n+2l}H^{i+l}(\mathcal{M}^n_B;\mathbb{Q}) \\ & x & \mapsto & x \cup \alpha^l \end{array} \tag{5}$$

is an isomorphism, which is not at all expected from a smooth affine variety.

There is a filtration on the cohomology of any proper map which satisfies a relative Hard Lefschetz theorem just like (5). This is the perverse Leray filtration, originating in the decomposition theorem of Beilinson, Bernstein, Deligne and Gabber [4], and carefully studied in [7], which in the case of the Hitchin map looks like (after certain shifts):

$$P_0 \subset \cdots \subset P_i \subset \cdots \subset P_k = H^k(\mathcal{M}^n_{Dol};\mathbb{Q}) \tag{6}$$

Motivated by explaining the Curious Hard Lefschetz theorem, as the relative Hard Lefschetz theorem satisfied by the perverse filtration, the following conjecture was made in 2012 in [8]:

Conjecture 1.1. *After suitably reindexing the filtrations, we have the "$P = W$" conjecture*

$$P_*(H^*(\mathcal{M}^n_{Dol},\mathbb{Q})) = W_*(H^*(\mathcal{M}^n_B;\mathbb{Q}))$$

under the isomorphism (3) *induced by the non-abelian Hodge theorem* (2).

The original paper [8] proved this for $n = 2$ using the detailed knowledge on the cohomology ring of $H^*(\mathcal{M}^2_{Dol};\mathbb{Q})$. Mellit proved the curious Hard Lefschetz theorem (5) for every n in [34] in 2019. More recently in 2022 using techniques of algebraic geometry of compact hyperkähler varieties [10] proved the conjecture for every n for genus 2 curves C.

Finally we list a few ramifications of the $P = W$ conjecture from the literature. For extensions of $P = W$ conjecture to parabolic cases and Hilbert schemes see e.g.[39]. For applications of wild analogues of the $P = W$ conjecture and the perverse filtration on $H^*(\mathcal{M}^n_{Dol};\mathbb{Q})$ to the study of torus link invariant and certain BPS invariants of Calabi-Yau 3-folds see [13] and the references therein. Extensions of $P = W$ conjectures for compact hyperkähler varieties were studied in [38]. Analogues of $P = W$ for mirror symmetry appeared in [29].

Most recently two complete proofs of the $P = W$ Conjecture 1.1 appeared in 2022. One by Maulik–Shen [32] and one by Hausel–Mellit–Minets–Schiffmann [21]. The aim of the Arbeitsgemeinschaft is to understand the $P = W$ Conjecture 1.1 and these two recent proofs.

2. Day 1: Character varieties, Higgs moduli spaces and the $P = W$ conjecture

2.1. **Lecture 1: Recollections on Hodge theory, examples.** The aim of this first lecture is to serve as a reminder of the basics of the Hodge theory of complex algebraic varieties (Hodge decomposition and filtrations, weight theory, mixed Hodge polynomials, (standard) Hard Lefschetz theorem for smooth projective varieties, ...). One standard reference is [45]. As examples, one could (for instance) treat in details the case of varieties of the form $(\mathbb{C}^*)^n \times \mathbb{C}^m$.

2.2. **Lecture 2: Recollections on perverse filtrations, examples.** The aim of the second lecture is to recall the definition and properties of the perverse filtration $P_\bullet H^*(X, \mathbb{C})$ induced on the cohomology of a smooth quasiprojective variety X by a proper morphism $f : X \to Y$ (in the cases at hand, Y will actually be an affine space). As a preliminary step, recall briefly the notions of perverse sheaves and the decomposition theorem of Beilinson-Bernstein-Deligne-Gabber, as in [4]. Recall also the notion of a Lefschetz structure, its relation to finite-dimensional $\mathfrak{sl}_2$-representations (as in [21, Section 8.1]), and the relative Hard Lefschetz theorem (as in [4, Thm. 5.4.10] or [7, Thm. 2.1.1, 2.1.4, 2.3.3, Sections 4.5, 4.6]), which is used the proof of $P = W$ in [21].

Introduce and give the basic properties of the sheaf-theoretic variant of the perverse degree of a cohomology class $c \in H^*(X, \mathbb{C})$ –the *strong perversity*– which plays a crucial role in the approach of Maulik-Shen (see [32, Section 1]).

2.3. **Lecture 3: The Higgs moduli space, the Hitchin map and the perverse filtration.** Introduce the moduli space $M^D_{r,d}$ of stable Higgs bundles of rank r and degree d on a smooth projective complex curve C, which is a smooth quasi-projective variety. Give the examples in the cases $r = 1$ or $g = 1$. Describe $M_{r,d}$, via the Beauville-Narasimhan-Ramanan correspondence, as a moduli space of one-dimensional sheaves on the surface T^*C, see e.g. [3]. Define the Hitchin morphism $\mu : M^D_{r,d} \to \mathcal{A} = \bigoplus_{i=1}^r H^0(C, \Omega_C^{\otimes i})$, state its basic properties (such as being proper, being an abelian fibration over a suitable open subset of $\mathcal{A}$) and explain why it can be viewed as a partial compactification of the cotangent $T^*N_{r,d}$ of the moduli space of stable vector bundles of rank r and degree d. The reference [16] could be useful here.

2.4. **Lecture 4: The character variety and its mixed Hodge structure.** Introduce the (twisted) character variety M^B_r of a genus g surface Σ_g associated to the group GL_r and a primitive rth root of unity. Prove that it is a smooth affine variety, and gives examples when $r = 1$ or $g = 1$, giving its mixed Hodge polynomial in both cases. Mention the SL_r and PGL_r-variants. The paper [17] can serve as a guide for all of this. If time, mention the parabolic variants, following [18].

State the non-abelian Hodge theorem of Corlette and Simpson (see [6], [42]) and observe it for $r = 1$ or $g = 1$. Give the statement of the $P = W$ conjecture and check it in the simple cases $r = 1$ or $g = 1$. Explain that the $P = W$ conjecture implies a 'curious' form of Poincaré duality (and also a curious form of Hard Lefschetz) for the mixed Hodge polynomial of the character variety.

3. Day 2: Tautological generators for the moduli of Higgs bundles

3.1. **Lecture 1: Tautological classes for the moduli of Higgs bundles.** We focus on the case where the rank and the degree are coprime. The main result to explain is that the cohomology of the moduli of stable Higgs bundles on a curve is generated by *tautological classes*, *i.e.*, the classes given by the Künneth components of a universal family, following Markman [31].

The proof in [31] is via moduli of stable sheaves on a surface, and the intersection theory argument may be technical. For the presentation, it may be a good idea to start with Beauville's proof [2] of a classical theorem of Atiyah–Bott [1] concerning an analogous statement for the cohomology of the moduli space of stable bundles on a curve where all the details can be explained. Then give an outline of the proof for the Higgs case [31] following the 2 steps: (1) explain [31, Theorem 1] which concerns $K3$ surfaces, (2) explain how a modification for the $K3$ case yields the Higgs case [31, Theorem 0.7] via a compactification.

3.2. **Lecture 2: Weights for the (Betti) tautological classes.** The purpose of this lecture is two-fold. For the first part, present the construction of the tautological classes for the character variety, and explain that they are matched with the generators on the Dolbeault (*i.e.* Higgs) side via non-abelian Hodge theory. A reference for this is [23] (before Section 6). For the second part, present Shende's result on the cohomology of a (twisted) character variety: the cohomology is of Hodge–Tate type and the weight is given by the Chern grading of the tautological class.

At the end, state that now the $P = W$ conjecture for GL_n is equivalent to "perverse = Chern" for tautological classes on the Dolbeault side.

If there is extra time, review the mixed Hodge structure associated with a simplicial scheme, focusing on the case of BG. This was used in Shende's argument.

3.3. **Lecture 3: Curious Hard Lefschetz.** The aim of the lecture is to give an overview of Mellit's proof the Curious Hard Lefschetz (CHL) property of the cohomology of character varieties, see [34]. The proof is long and relies on many different techniques hence the only realistic aim is to sketch the different steps of the argument, which *roughly* consist in :

(1) introducing a parabolic version $M_r^{B,par}$ of the character variety, which involves choosing one or more points on C along with generic semisimple conjugacy classes at each point,
(2) constructing a stratification of that parabolic character variety into 'cells' of the form $(\mathbb{C}^*)^a \times \mathbb{C}^b$, which allows one to deduce a version of the CHL property for $M_r^{B,par}$
(3) relating the cohomologies of $M_r^{B,par}$ and M_r^B to deduce the CHL property for M_r^B.

The second step hinges on the theory of braid varieties (see e.g. [41]) and Seifert surfaces while the last step uses the (finite) Springer action of the symmetric $\mathcal{S}_r$ on the cohomology of $M_r^{B,par}$, along with ideas coming from [19].

4. Day 3: Variants of Hitchin moduli spaces

4.1. **Lecture 1: Moduli of twisted Higgs bundles.** The purpose of the two lectures today is to introduce two types of Hitchin moduli spaces: (A) moduli of twisted (or meromorphic) Higgs bundles, (B) moduli of parabolic Higgs bundles. The definition of these moduli spaces is quite straight forward, so we will focus on why these spaces are useful.

In the first lecture, introduce the moduli space of Higgs bundles twisted by an effective divisor. Then explain two features of this moduli space:

(1) The decomposition theorem for the twisted Hitchin system is much easier than the usual case — every simple summand has full support. This is the support theorem of Chaudouard–Laumon [11], built on the seminal work of Ngô [37]. It would be good to compare it with [9] which shows that in the untwisted case at least every Levi subgroup of GL_n contributes a support. It would be helpful to explain the idea of the proof using the case of GL_2.
(2) The decomposition theorem of the twisted case can recover the decomposition theorem for the untwisted case, using the vanishing cycle functor. The reference is [33, Section 4]; for the presentation please feel free to focus only on the $\mathfrak{sl}_2$-case.

If there is time, explain at the end very briefly, following [33, Section 0.6], that (1) and (2) yield immediately a proof of the topological mirror symmetry conjecture of Hausel–Thaddeus [22]. This conjecture was first proven by Groechenig–Wyss–Ziegler by p-adic integration [15].

4.2. **Lecture 2: Moduli of parabolic Higgs bundles.** We discuss everything under the assumption $D = p$; that is, the boundary of the Riemann surface only consists of one point.

Following [21, Section 8.4], discuss the connection between the Poison variety $\overline{M}_{n,d}$ (the moduli of parabolic Higgs bundles without fixing the residue at the punctures) and the usual Higgs moduli space $M_{r,d}$. First introduce these spaces, and then present the details of Propositions 8.14, 8.15, 8.16, *i.e.* the trick of taking generic residue at the puncture. The reference for Proposition 8.14 above is [20, Corollary 1.3.3].

If there is time, mention that the restriction map preserves the perverse filtrations.

5. Day 4: Geometric representation theory techniques

5.1. **Lecture 1: Finite and affine Springer theory.** The aim of this talk is to pave the way for the next one (on Yun's global Springer theory). There are two settings : finite Springer theory (for a complex reductive group) and affine Springer theory (for the loop group of a complex reductive group).

In the finite case, present both the Springer and the Grothendieck-Springer resolutions as morphisms of stacks $\widetilde{\mathcal{N}}/G \to \mathcal{N}/G$ and $\widetilde{\mathfrak{g}}/G \to \mathfrak{g}/G$, and use the smallness of the latter map to define a Weyl group action on the Springer sheaf. Observe the decomposition of the pullback of the tautological bundle on the base into a successive extension of line bundles. This is classical and may be found, e.g. in [5].

In the affine case, introduce the affine Springer fiber (over regular semisimple and topologically nilpotent element is enough) and describe its main geometric properties. Describe the action of the affine Weyl group on its Borel-Moore homology, following the paper [30], Section 5.

5.2. **Lecture 2: Global Springer theory.** Review Yun's global Springer theory [43, 44]. One may follow the summary of [32, Section 3]; in particular (A,B,C) of Section 3.4. Note that the vanishing of (B) was not explained in details in [32], which was treated in [44, Lemma 3.2.3]: a toy model for thinking about it is that the first Chern class of the normalized Poincaré line bundle on $C \times J_C$ (for a smooth curve C) does not have nontrivial class $H^2(J_C)$ on its Jacobian side.

5.3. **Lecture 3: Cohomological Hall algebras and Hecke operators on surfaces.** Give the definition of cohomological Hall algebras on surfaces (one may restrict to zero-dimensional sheaves to avoid dealing with stacks of infinite type), and their action on Hecke patterns, following [27] or [35] [the construction involves the notion of derived stacks, but it is sufficient to use it as a black box]. Give the examples of A^2 and the Hilbert scheme of points (see e.g. [35, Section 5.]); State (and prove) Negut's Lemma for length one correspondences (see [36, Prop. 2.19], or [26, Section 8.]), and deduce the explicit form of length one Hecke operators; state the main theorem in [35], which is a description of this COHA as a $W_{1+\infty}$-algebra modeled on the homology of the surface.

5.4. **Lecture 4: Construction of the action of $\mathfrak{sl}_2$ and $\mathcal{H}_2$ on the cohomology of the Higgs moduli spaces.** Building from the results of the previous lecture (and sticking to the case of Higgs bundles rather than an arbitrary surface for simplicity), define an action of a suitable degenerate COHA, and then of $\mathfrak{sl}_2$ and of the Lie algebra $\mathcal{H}_2$ of Hamiltonian vector fields on the plane on the homology of the moduli spaces or stacks of (semi)stable Higgs bundles over the elliptic locus of the Hitchin base. This is explained in [21, Sections 6,7]. The parabolic case plays an important role in the proof of $P = W$, but the technical details pertaining to that case may safely be skipped.

6. Day 5: Proofs of the P=W conjecture

6.1. **Lecture 1: Maulik–Shen's proof.** Present an outline of the proof following the 4 steps listed in [32, Section 0.2], and recall the ingredients that have been discussed in the previous lectures. In particular, it would be great to fill some more details of certain steps when there was not much time to discuss them in the previous lectures.

6.2. **Lecture 2: Hausel–Minets–Mellit–Schiffmann's proof, (I).** Previous lectures explained the construction of Hecke operators on the cohomology of moduli spaces of Higgs bundles. The aim of this lecture is to explain the compatibility between these and the perverse filtration associated to the Hitchin fibration. In particular, this yields the compatibility between the $\mathfrak{sl}_2$-action and the perverse filtration (over the elliptic locus of the Hitchin base), see [21, Prop. 8.12].

6.3. **Lecture 3: Hausel–Minets–Mellit–Schiffmann's proof, (II).** Conclude the argument of the proof of $P = W$ following [21, Section 8] : one proves $P = W$ successively for (twisted) parabolic Higgs bundles over the elliptic locus, twisted parabolic bundles, twisted nilpotent parabolic bundles and finally (!) usual Higgs bundles. This also provides an independent proof of the curious Hard Lefschetz property.

References

[1] M. Atiyah and R. Bott, *The Yang–Mills equations over Riemann surfaces,* Phil. Trans. R. Soc. London A 308 (1982), 523–615.

[2] A. Beauville, *Sur la cohomologie de certains espaces de modules de fibrés vectoriels,* Tata Inst. Fund. Res., Geometry and analysis (Bombay, 1992), 37–40.

[3] A. Beauville, M. S. Narasimhan, M. S., S. Ramanan, *Spectral curves and the generalised theta divisor.*, J. Reine Angew. Math.398(1989), 169–179.

[4] A. Beilison, J. Bernstein, P. Deligne and O. Gabber, *Faisceaux pervers*, Astérisque 100 (1981).

[5] W. Borho and R. Macpherson, *Partial resolutions of nilpotent varieties.*, Analysis and topology on singular spaces, II, III (Luminy, 1981), 23–74. Astérisque, 101-102

[6] K. Corlette. *Flat G-bundles with canonical metrics.*, J. Differential Geom. 28 (1988), no. 3, 361–382. 25

[7] M.A. de Cataldo and L. Migliorini, *The Hodge Theory of Algebraic maps*, Ann. Scient. Ec. Norm. Sup., 4e série, t. 38, (2005), 693-750.

[8] M.A. de Cataldo , T. Hausel and L. Migliorini, "Topology of Hitchin systems and Hodge theory of character varieties: the case A_1." Annals of Mathematics (2012): 1329-1407.

[9] M.A. de Cataldo, J. Heinloth, and L. Migliorini, *A support theorem for the Hitchin fibration: the case of* GL_n *and* K_C*,* J. Reine Angew. Math., 780 (2021): 41–77.

[10] M.A. de Cataldo, D. Maulik and J. Shen, "Hitchin fibrations, abelian surfaces, and the P=W conjecture", J. Amer. Math. Soc. 35 (2022), no. 3, 911–953.

[11] P.-H. Chaudouard, G. Laumon, *Un théorème du support pour la fibration de Hitchin,* Ann. Inst. Fourier, Grenoble, Tome 66, no 2 (2016), p. 711–727.

[12] P. Deligne:Théorie de Hodge II. *Inst. hautes Etudes Sci. Publ. Math.* **40** (1971), 5–47.

[13] D.E. Diaconescu; R. Donagi and T. Pantev, BPS states, torus links and wild character varieties. Comm. Math. Phys.359(2018), no.3, 1027–1078.

[14] R.Donagi, E.Markman, "Spectral covers, algebraically completely integrable Hamiltonian systems and moduli of bundles" in *Integrable Systems and Quantum groups"* LNM 1620, 1-119.

[15] M. Groechenig, D. Wyss, and P. Ziegler, *Mirror symmetry for moduli spaces of Higgs bundles via p-adic integration,* Invent. Math. 221. (2020) 505–596.

[16] T. Hausel, *Global topology of the Hitchin system.*(Handbook of moduli. Vol. II, 29–69. Adv. Lect. Math. (ALM), 25 International Press, Somerville, MA, 2013.

[17] T. Hausel, F. Rodriguez-Viillegas, *Mixed Hodge polynomials of character varieties*, Invent. Math. 174 (2008), no. 3, 555–624.

[18] T. Hausel, E. Letellier, F. Rodriguez-Villegas, *Arithmetic harmonic analysis on character and quiver varieties.*,Duke Math. J.160(2011), no.2, 323–400.

[19] T. Hausel, E. Letellier, F. Rodriguez-Villegas, *Positivity for Kac polynomials and DT-invariants of quivers*, Ann. of Math. (2) 177 (2013), no. 3, 1147–1168.

[20] T. Hausel, F. Rodriguez-Villegas *Cohomology of large semiprojective hyperkähler varieties,* Astérisque (2015), no. 370, 113–156.

[21] T. Hausel, A. Mellit, A. Minets, and O. Schiffmann, *$P = W$ via H_2,* arXiv:2209.05429.

[22] T. Hausel, M. Thaddeus, *Mirror symmetry, Langlands duality, and the Hitchin system,* Invent. Math. 153 (2003), no. 1, 197–229.

[23] T. Hausel and M. Thaddeus, *Generators for the cohomology ring of the moduli space of rank 2 Higgs bundles,* Proc. London Math. Soc. (3) 88 (2004), no. 3, 632–658.

[24] N. J. Hitchin. *The self-duality equations on a Riemann surface.* Proc. London Math. Soc. (3), 55(1):59–126, 1987.

[25] ———, *Stable bundles and integrable systems*, Duke Math. J. **54** (1987), no. 1, 91–114.

[26] Q. Jiang, *Derived projectivizations of complexes*, arXiv:2202.11636.

[27] M. Kapranov and E. Vasserot, *The cohomological Hall algebra of a surface and factorization cohomology*, arXiv:1901.07641.

[28] A. Kapustin and E. Witten. Electric-magnetic duality and the geometric Langlands program. *Commun. Number Theory Phys.* 1(1):1–236, 2007. arXiv:`hep-th/0604151`

[29] L. Katzarkov, L.; V.V. Przyjalkowski and A. Harder $P = W$ Phenomena.(Russian. Russian summary)Mat. Zametki108(2020), no.1, 33–46; translation in Math. Notes108(2020), no.1-2, 39–49.

[30] G. Lusztig, *Affine Weyl groups and conjugacy classes in Weyl groups.*, Transform. Groups (1996), no.1-2, 83–97.

[31] E. Markman, *Generators of the cohomology ring of moduli spaces of sheaves on symplectic surfaces,* J. Reine Angew. Math. 544 (2002), 61–82.

[32] D. Maulik and J. Shen, *The $P = W$ conjecture for* GL_n*,* arXiv:2209.02568.

[33] D. Maulik and J. Shen, *Endoscopic decompositions and the Hausel–Thaddeus conjecture,* Forum Math., Pi, 9, (2021), No. e8, 49 pp.

[34] A. Mellit, *Cell decomposition of character varieties*, arXiv:1905.10685.

[35] A. Mellit, A. Minets, O. Schiffmann and E. Vasserot, *Cohomological Hall algebra of zero-dimensional sheaves on surfaces*, preprint.

[36] A. Negut, *Hecke correspondences for smooth moduli spaces of sheaves.*, Publ. Math. Inst. Hautes Études Sci.135(2022), 337–418.

[37] B.C. Ngô, *Le lemme fondamental pour les algèbres de Lie,* Publ. Math. IHES. 111 (2010) 1–169.

[38] J. Shen and Q. Yin: *Topology of Lagrangian fibrations and Hodge theory of hyper-Kähler manifolds*, Duke Math. J. 171 (2022), no. 1, 209–241.

[39] J. Shen and Z. Zhang, *Perverse filtrations, Hilbert schemes, and the P=W conjecture for parabolic Higgs bundles*. Algebr. Geom.8(2021), no.4, 465–489.

[40] V. Shende, *The weights of the tautological classes of character varieties,* Int. Math. Res. Not. 2017, no. 22, 6832–6840.

[41] V. Shende, D. Treumann and E. Zaslow, *Legendrian knots and constructible sheaves*, Invent. Math. 207 (2017), no. 3, 1031–1133.

[42] C.T. Simpson, *Higgs bundles and local systems.*, Publ. Math. I.H.E.S. 75 (1992) 5–95

[43] Z. Yun, *Global Springer Theory,* Adv. in Math. 228 (2011), 266–328.

[44] Z. Yun, *Langlands duality and global Springer theory,* Compositio. Math. 148 (2012), 835–867.
[45] C. Voisin, *Hodge theory*