Oberwolfach seminar: Classification of C^*-algebras and C^*-dynamical systems

Objective

The objective of this proposed seminar is to give an introduction to the state of the art in the structure and classification of C^*-algebras and C^*-dynamical systems. This is an area which has been completely reinvented through repeated major breakthroughs over the last decade to provide full counterparts of celebrated results in the theory of von Neumann algebras from the 1970s and '80s. This Oberwolfach seminar aims to bring early-career researchers up to speed with the latest cutting-edge results, so that they can use both the classification theorems and underlying techniques in their own research. We will focus on instructive special cases and examples rather than working in complete generality and ensure that we expose the strong parallels and connections with von Neumann algebraic theory.

Thematic focus of lectures

Structure of simple nuclear C^*-algebras (White) These lectures will focus on the role of the Jiang–Su algebra and $\mathcal{Z}$-stability in the structure theory of simple nuclear C^*-algebras. A major goal will be to explain the new technique of *complemented partitions of unity* for combining results which hold for each trace on a C^*-algebra into results which hold uniformly over all traces. This course will also discuss abstract tools going back to work of Matui and Sato for verifying $\mathcal{Z}$-stability, which will be used in Geffen's course.

Abstract Classification (Schafhauser) These lectures will focus on the abstract approach to classification. They will focus on the role of of the trace-kernel extension and the Elliott–Kucerovsky absorption theorem allowing for the use of extension theoretic methods in the classification programme. The lectures will connect with those of Geffen and White through the use of strict comparison and $\mathcal{Z}$-stability.

Dynamical classification (Szabó) This course aims to give a comprehensive introduction to the fundamental techniques underpinning the classification of both C^*-algebras and group actions on them, some of which will be utilized in Schafhauser's course. These methods will be combined with more advanced techniques to demonstrate the core ideas behind the recent dynamical version of the Kirchberg-Phillips theorem in feasible special cases.

Examples (Geffen) The main goal of these talks will be to illustrate how to prove $\mathcal{Z}$-stability in concrete examples of C^*-algebras attached to dynamical systems. We will make use of two concepts which are currently at the heart of C^*-dynamical classification: *amenability of actions* and *dynamical comparison*. We will end with discussing some open problems.

The lectures will facilitate considerable interaction and connections between the different courses.

- Strong self-absorption will be a recurring theme. We will devote attention to the role of the Jiang–Su algebra $\mathcal{Z}$ and its purely infinite counterpart $\mathcal{O}_\infty$. This will be integrated across all 4 courses — it is a central objective that participants leave feeling the Jiang–Su algebra is no longer a mysterious object, but an algebra which they can work with.
- Intertwining is an essential tool in classification, and our courses will illustrate the parallels between the classical Elliott intertwining technology used in the classification of algebras with the very new cocycle category intertwining arguments introduced by Szabó.

- The structure theory developed in White's course will provide the 'von Neumann like' classification results which are the starting point for Schafhauser's course.
- Throughout, we will put the spotlight on important examples and model actions in order to strengthen the participants' intuition and to connect the courses by Geffen and Szabó.

Provisional programme

Each course will consist of three 90 minute lectures (which will probably be split in to two 40 min slots and a break). The provisional programme is below. We will encourage participants to give an optional lightening talk in the evening on either Monday or Tuesday evening, as an opportunity to introduce their research to the other participants.

We expect the programme will take roughly the following form.

	Monday	Tuesday	Wednesday	Thursday	Friday
8:00	Breakfast	Breakfast	Breakfast	Breakfast	Breakfast
9:00	Lecture	Lecture	Lecture	Lecture	Lecture
10:30	Break	Break	Break	Break	Break
11:00	Lecture	Lecture	Lecture	Lecture	Lecture
12:30	Lunch	Lunch	Lunch	Lunch	Lunch
2:30	Lecture	Break	Free afternoon	Break	Discussion
4:00	Break	Break	Free afternoon	Discussion session	
4:30	Lecture	Discussion session	Free afternoon	Discussion session	
6:00	Dinner	Dinner	Dinner	Dinner	Dinner
8:00	Early Career talks	Early Career talks			

Recommended reading

As core background, we expect participants will be familiar with:

- the basics of C^*-algebras including nuclearity and exactness, tensor products and crossed products (for example [2, Section 2, Section 3.1-3.6, Section 4.1-4.4]);
- K-theory (for example from [9]);
- the basics of II_1 factors (and finite von Neumann algebras), in particular the fact that projections in a II_1 factor are determined by their trace.

In terms of general background to the classification programme, we suggest participants look at the first 3 sections of the survey [10] and [3, Sections 1.1-1.2].

For more specific background, we recommend some familiarity with the topics below, though we do not expect that participants will have all of the following material at their finger tips.

- Elliott's intertwining argument in his classification of AF-algebras (for example from [9, Section 7]), and the intertwining arguments in [10, Section 2.3].
- Some familiarity with strongly self absorbing C^*-algebras ([13, Section 1]) and the Jiang-Su algebra $\mathcal{Z}$ and its key properties (including Rørdam and Winter's presentation from [11]). The background we need is summerised in [3, Section 4.1], and while the fact that $\mathcal{Z}$ is strongly self absorbing will be crucial; we will not need the details of this, nor will we need the proofs that the original presentation and the Rørdam-Winter presentation agree.
- Ultrapowers and ultraproducts of C^*-algebras and tracial von Neumann algebras. Appendix E of [2] gives a very brief account in the von Neumann setting for those who've not

encountered this before. Sequence algebras for C^*-algebras, and intertwining via reparameterisations ([6, Theorem 4.3]). The ultrapower / sequence algebra formulation of approximate intertwinings in [10, Section 7.2]

- Definition of and basic knowledge about Cuntz algebras $\mathcal{O}_n$ for $n \in \mathbb{N}^{\geq 2} \cup \{\infty\}$, e.g., as treated in [4]. Specifically regarding $\mathcal{O}_2$ and $\mathcal{O}_\infty$, some familiarity with the Kirchberg–Phillips absorption theorems [8] can be useful at the level of a black box.
- Voiculescu's theorem is very useful background. We recommend Arveson's seminal paper [1].
- Some familarity with KK-theory and extension theory. For KK-theory familiarity with the statements from [10, Section 2.4] or [12, Section 2] would be useful, but we don't expect any familiarity with proofs. Likewise it would be useful to be familiar with what it means for a C^*-algebra to satisfy the UCT.
- Familiarity with Cuntz comparison of positive elements, and the condition of *strict comparison* [7, Sections 2,3,8]. We will not need the abstract framework and the category **Cu** of Cuntz semigroups.
- Background on the Toms-Winter conjecture, for example from the surveys [5] and [14, Section 5].

References

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[4] K. R. Davidson. C^*-algebras by example, volume 6 of *Fields Institute Monographs.* American Mathematical Society, 1996.

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