Oberwolfach Seminar:
Long-time behavior in Fluids

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1. Introduction

The seminar will tie together two active research directions: complex fluids and long time behavior of Euler, Navier-Stokes and related equations. The time is ripe for such a union, with recent advances being made in both directions [**1, 2, 6, 7, 8, 9, 10, 11, 19**] and [**3, 4, 5, 12, 14, 15, 16, 17, 18, 21**].

Some overlap has already been established. For example, the work [**12**] tackles the problem of magnetic reconnection (topological change of magnetic field lines) and asymptotic approach to equilibrium in the context of Voigt regularization introduced to model complex fluids with polymeric interactions (Oskolkov, 1973), while the works [**1, 2, 9**] have extended ideas developed for understanding global attractors and large time behavior for Navier-Stokes to complex fluid models for electroconvection.

In what follows, we give a brief background and review some recent progress and directions which form the basis of mini-courses and lectures, and finish by briefly describing some specific themes of the lectures.

2. Long Time Behavior

Many geophysical and astrophysical systems, such as oceanic currents, large-scale weather patterns and planetary atmospheres are described, to good approximation, by two-dimensional ("2d") fluid equations since the vertical extent of these systems is typically much smaller than the horizontal. Understanding the long term dynamics of 2d fluids is thus fundamental to weather prediction, climate science, and astrophysics. The prototype of all such systems, the 2d incompressible Navier-Stokes ($\nu > 0$) and Euler ($\nu = 0$) equations for the fluid velocity field $u = (u_1, u_2)$, read

$$\partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u + f, \qquad \nabla \cdot u = 0$$

where p is the pressure, ν is the viscosity and f is an external forcing. We are particularly interested in the physically relevant regime of long time $t \gg 1$ and weak dissipation $\nu \ll 1$, in various orders of limits. Some rather surprising and mysterious features is observed in these systems are

- If $f = 0$ and $\nu \to 0$ first and then $t \to \infty$, the fluid tends towards order,
- If f is generic, $t \to \infty$ first and then $\nu \to 0$, the fluid tends towards disorder.

By disorder/order, we mean a measure of diversity or lack thereof of solutions in their phase space. This is all the more surprising since when $\nu = 0$, there is formally no dissipation mechanism in the Euler equations which govern the motion of a perfect (inviscid and incompressible) fluid. In fact, Euler is a time reversible, infinite dimensional Hamiltonian system. In finite dimensions, such systems have no mechanism (e.g. dissipation) by which to remove information and thus the complexity of a solution cannot reduce over time. However, Euler's infinite dimensional nature (possessing infinitely many conserved quantities) affords a hidden *inviscid friction* (namely, mixing), by which information can exit and order restored at long times. Here some key issues are

(1) Understand generic mixing of vorticity as $t \to \infty$,
(2) Understand emergence and persistence (stability) or large scale objects

Recent work has shed new light on some of these issues and, perhaps more importantly, opened the door to new lines of inquiry. The paper [**16**] reviews a variety of these advances, including

[**3, 4, 5, 15, 18**] and poses a number of open questions (approximately 30), many of which are suitably specific and tractable to be attacked by PhD students. A subsequent recent advance, [**17**], has introduced new dynamical and topological arguments aimed towards understanding the aforementioned generic mixing in the Euler equations. These recent works, among others, are the start of a new attempt to use tools from a range of mathematical fields (PDE, dynamical systems, geometry, topology) to understand classical issues.

With non-zero viscosity $\nu > 0$, most of the conservation laws of Euler are broken by the force and friction. Nevertheless, the friction is strong enough to order the motion at infinite time. Indeed, it is well known that the 2d Navier-Stokes equations possess a finite dimensional global attractor that governs all possible motions as $t \to \infty$. However, for generic forcing f, the attractor dimension is conjectured by Kolmogorov to unboundedly increase as $\nu \to 0$ in accord with observations of turbulent motion. The route towards this ultimate disorder is called the *transition to turbulence*. Some key issues are

(1) Understand instability of steady states and their bifurcations,
(2) Understand generic growth of attractor dimensions as $\nu \to 0$.

There are some very basic open issues here. Early work on instability, in the context of unidirectional shear flows, is due to Meshalkin and Sinai (1961), and it was leveraged to prove the emergence of secondary flows by Yudovich (1967). Beyond shear flows, the questions of instability are largely open but of great fundamental importance. In fact, Suri, Tithof, Grigoriev, and Schatz (2017) have shown on the basis of numerical simulation and physical experiment, that in a certain canonical setup, the issue of growth of attractor dimension is closely linked with the emergence of increasingly many unstable steady states (necessarily non-shear) in the inviscid limit.

3. Complex Fluids

A good testing ground for the methods developed in the study of long-time behavior and singularity formation in the Euler and Navier-Stokes equations is the study of complex fluids. A class of models of complex fluids consists of equations which are similar to classical fluid equations, except that the rate of strain tensor is modified by the an added strain, due to the added physical processes in the fluid.

Mathematically, this corresponds to considering the full Navier-Stokes system with an extra non-negative and symmetric strain tensor σ :

$$\partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u + \mathrm{div}(\sigma). \tag{1}$$

There are numerous closure models that allow us to determine the tensor σ by additional equations. One of the most natural couplings is the Oldroyd B system which, in simplified form has σ solving

$$\partial_t \sigma + u \cdot \nabla \sigma = (\nabla u)\sigma + \sigma(\nabla u)^t. \tag{2}$$

Replacing the Navier-Stokes equation (1) by a steady Stokes equation forced by the divergence of σ,

$$-\Delta u + \nabla p = \mathrm{div}(\sigma), \quad \nabla \cdot u = 0 \tag{3}$$

we arrive at an important and interesting model, studied in [**13**]. A a particular case is one in which $\sigma = \nabla^\perp \theta \otimes \nabla^\perp \theta$ and θ is transported,

$$\partial_t \theta + u \cdot \nabla \theta = 0. \tag{4}$$

We now describe two strands of research on singularity formation and long-time behavior.

3.1. Long-time Behavior. When considering the various works on the long-time behavior of the 2d Euler system or other active scalars, such as the SQG system, we find in (3)- (4) both extra structure and a lack of theory to accommodate that structure. One conceptual difference between (3)- (4) and the previously studied active scalar systems is that (3)- (4) enjoys a uniform H^1 bound on solutions (in addition to the conservation of all Casimirs). Such control is not available in the 2d Euler system, for example, where actually it is predicted that generic solutions become unbounded in H^1. At the same time, the nonlinearity of the coupling between the velocity and the unknown θ leads to numerous difficulties not present in the study of other active scalars. It is expected that studying this problem should require new tools and uncover new phenomena as compared to the previous works [**1, 2, 4, 7, 11, 12, 13, 17, 21**].

3.2. Singularity formation. The Oldroyd B system with (1) replaced by the Stokes system can be easily compared to the vorticity equation in the 3d Euler equation. It has a similar structure in that the unknown, σ, is being advected and stretched by a velocity field u that is related to σ by a -1 order operator (similar to the Biot-Savart law). Whether a finite-time singularity can form in this system is an outstanding open problem. A simpler model proposed in [**13**] that still retains many of the difficulties in studying the singularity problem is:

$$\partial_t \sigma = \sigma A^2(\sigma),$$

where A is a zeroth order and anti-symmetric operator. While singularity formation in this system is outstanding, the techniques of [**20**] can be effectively applied to study the singularity problem on the model and, likely, even for the full system (1)-(2). At the same time, the dissipative structure of the system should lead to interesting new phenomena not present in the previous works on singularity formation.

4. Lectures

We envision three pedagogical mini-courses aimed at exposing students to tractable problems in the areas described above, namely on long time behavior and inviscid limit, on complex fluids and on singularity formation. Problem sessions will accompany these courses, during which specific open problems will be stated and discussed. Specific lectures will describe methods of nonlinear and nonlocal analysis with emphasis on broad applicability.

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