Oberwolfach Seminar 2448a

“Exponential motives”

Organizers

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Summary

Exponential motives are meant to be a universal cohomology theory for pairs (X, f) consisting of an algebraic variety X over some field and a regular function f on X. These objects have attracted considerable attention in recent years, especially in connection with the following three topics:

- *Exponential periods.* From an elementary point of view, periods are convergent integrals of the form $\int_\sigma g(x_1, \dots, x_n)dx_1 \cdots dx_n$, where g is an algebraic function and $\sigma \subset \mathbf{C}^n$ is a semi-algebraic domain, both defined by polynomials with algebraic coefficients. Examples include $2\pi i$, logarithms of algebraic numbers, elliptic integrals and multiple zeta values. A more modern approach consists in interpreting the integrand as a differential form on an algebraic variety X defined over $\overline{\mathbf{Q}}$, the integration domain as a singular chain on the topological space $X(\mathbf{C})$, and the integral itself as a pairing between algebraic de Rham cohomology and singular cohomology of X, possibly relative to a subvariety. It is through this perspective that motives become a powerful tool to study periods, namely to predict all algebraic relations among them. One of the starting points of the theory of exponential motives is the observation that numbers such as e, the square root of π, Euler's γ constant or the special values of the gamma and the Bessel functions are not expected to be periods in that usual sense. Nevertheless, in all these examples there is an integral representation of the form $\int_\sigma e^{-f(x_1,\dots,x_n)}g(x_1, \dots, x_n)dx_1 \cdots dx_n$, where g and σ are as above and the extra datum is that of an algebraic function with algebraic coefficients f. This suggests treating the integrand as a class in de Rham cohomology, except that the natural differential is now $d - df$ instead of the exterior derivative. Besides, the convergence of the integral is only ensured if the function e^{-f} decays rapidly along the boundary of σ. This leads to the definition of two cohomology theories for pairs (X, f), twisted de Rham cohomology and rapid decay cohomology, as well as a canonical comparison isomorphism between them.
- *Exponential sums over finite fields.* When estimating various quantities in analytic number theory (the number of solutions of Diophantine equations, the size of Fourier coefficients of automorphic forms, etc.), one regularly encounters exponential sums of the form $\sum_{x\in X(\mathbf{F}_p)} \psi(f(x))$, where $\psi \colon \mathbf{F}_p \to \mathbf{C}^\times$ is an additive character of the finite field $\mathbf{F}_p$ (e.g., $x \mapsto \exp(2\pi i x/p)$) and X is an algebraic variety over $\mathbf{F}_p$ endowed with a regular function f. Kloosterman sums are, for

instance, of this shape. Proving optimal bounds for their absolute values relies crucially on the interpretation, through the so-called function-sheaf dictionary, of these exponential sums as traces of the Frobenius operator acting on the étale cohomology of an ℓ-adic local system $\mathcal{L}_{\psi(f)}$ built out of the Artin–Schreier covering of the affine line. By a now classic analogy, these local systems are the counterparts over finite fields of the exponential differential equations $d - df$, and one would like to have a systematic framework to navigate between characteristic p and characteristic zero at disposal. Among other things, this could allow one to use variants of Hodge theory to obtain bounds for the p-adic valuations of Frobenius eigenvalues, in the spirit of Mazur's celebrated "Newton above Hodge" theorem, or to better understand a theorem by Katz comparing the monodromy groups of the local systems $\mathcal{L}_{\psi(f)}$ with the differential Galois groups of $d - df$.

- *Mirror symmetry and the irregular Hodge filtration.* One of the resounding successes of the dialogue between mathematicians and physicists is the idea that Calabi-Yau varieties (for instance, degree $d+1$ hypersurfaces in $\mathbb{P}^d$) should come in "mirror pairs" reflecting a duality between complex and symplectic structures. For Fano varieties (such as projective spaces, Grassmannians or hypersurfaces of degree at most d in $\mathbb{P}^d$), it is not expected that the mirror is still a Fano variety but a Landau-Ginzburg model. From a mathematical point of view, this is once again a pair (X, f) consisting of a variety X and a regular function f. For example, the mirror of $\mathbb{P}^1$ is the torus $\mathbb{G}_m$ endowed with the function $x + 1/x$, which gives rises to special values of the Bessel function in the setting of exponential periods and to Kloosterman sums in the setting of exponential sums. Katzarkov, Kontsevich and Pantev conjecture that certain algebro-geometric invariants of Fano varieties, such as their Hodge numbers, should be read off the twisted de Rham cohomology of their associated Landau-Ginzburg models by means of a so-called irregular Hodge filtration first introduced by Deligne. In the previous example, the non-zero Hodge numbers of $\mathbb{P}^1$ are $h^{0,0} = h^{1,1} = 1$ and the twisted de Rham cohomology of $(\mathbb{G}_m, x + 1/x)$ has, as expected, irregular Hodge numbers equal to $h^{0,1} = h^{1,0} = 1$. In general, the irregular Hodge filtration jumps at rational numbers, as is suggested by certain algebraic relations between exponential periods and classical periods or by the formulas for the p-adic valuations of Gauss sums.

At first sight unrelated, these three topics enjoy rich connections with each other. Guided by the philosophy of exponential motives, we can sometimes turn into theorems what had earlier been inspiring analogies. Two recent breakthroughs in this directions are the answer to a 1929 question by Siegel on the existence of E-functions that are not polynomial expressions in hypergeometric functions [1], and the proof that a family of global L-functions built out of moments of Kloosterman sums over finite fields have meromorphic continuation and satisfy a functional equation [3].

Tentative program

During the first three or four days of the seminar, the morning lectures by the organisers will cover the foundational material from their monograph [1], namely:

- Rapid decay cohomology, twisted de Rham cohomology and the comparison isomorphism.
- The category of perverse sheaves with vanishing cohomology on the affine line.
- The construction of the tannakian category of exponential motives over a subfield of the complex numbers and various realization functors.
- The characterisation of classical motives within the category of exponential motives.
- The exponential period conjecture.

For each of these topics, we will provide a list of significant examples to be worked out by the participants during the afternoon sessions, as well as open problems of various levels of difficulty (running from something that could be presented towards the end of a week of focused work to we-don't-have-the-slighlest-clue-how-to-solve questions). The second part of the morning lectures will be devoted to the applications to E-functions and Kloosterman sums. Besides the philosophy of exponential motives, they rely on techniques such as the differential Galois theory of $\mathscr{D}$-modules on the affine line or the irregular Hodge filtration that we plan to introduce through examples and workshop sessions in the afternoon.

References

[1] J. Fresán and P. Jossen, *A non-hypergeometric E-function*, Ann. of Math.**194** (2021), no. 3, 903–942.
[2] J. Fresán and P. Jossen, *Exponential motives*, `http://javier.fresan.perso.math.cnrs.fr/expmot.pdf`.
[3] J. Fresán, C. Sabbah, and J-D. Yu, *Hodge theory of Kloosterman connections*, Duke Math. J. **171** (2022), no. 8, 1649-1747.