Invitation to the Arbeitsgemeinschaft at Oberwolfach on Mirror Symmetry
October 5 - October 12, 1997
Program proposed by V.V. Batyrev and D. van Straten

Mirror symmetry is an unexpected duality between Calabi-Yau manifolds of arbitrary dimension. This phenomenon has been discovered by physicists and during last six years considerable progress has been made. However, it is fair to say that most of the progress consists of lucky guesses at a correct mathematical formulations of the mirror symmetry, mathematical conjectures about its properties and examples confirming these conjectures or consequences of them. For this reason mirror symmetry for Calabi-Yau manifolds still remains far from complete mathematical understanding. One of the main achievements of the last years is the rigorous mathematical foundation of *quantum cohomology*, which arose as an attemp to give a firm basis to the exciting calculation of Candelas et. al. of the numbers n_d of rational curves of degree d on a general quintic threefold, [18].
It turned out to be very difficult to organize the recently published material on mirror symmetry in a logically connected series of 18 lectures. So we decided to select 5 topics (one for each day) and try to keep logical connections between lectures mainly within one day.

Monday

Gromov-Witten invariants, quantum cohomology and WDVV-equations

The purpose of this day is to give an overview of the rich mathematical material concerning quantum cohomology and its application to enumerative geometry. Our general references are [22] and the lecture notes of P. Aluffi taken at the Mittag Leffler Institute, [1].

Lecture 1. *Spaces of stable maps.*
(Comparable to Lecture 2 in [1], Part 1, p.3-4). Definition of a stable map ([22], (1.1), [35]). Existence of the spaces $\overline{M}_{g,n}(X,\beta)$ of stable maps ([22], thm.1). Basic properties for convex manifolds ([22], 0.4, thm.2, thm.3). Examples ($\overline{M}_{0,n}(\mathbf{P}^r, d)$ for small n and d, [22], p.5-6)

Lecture 2.*Gromov-Witten Invariants*
(Comparable to Lecture 3 in [1], Part 3, p.5-9)
Evalution maps, definition of Gromov-Witten (G-W) invariants, ([22], §7). First properties of G-W classes ([22], p.33-34). Case of homogeneous space ([22], lemma 13). G-W invariants for $\mathbf{P}^r$. Axioms for G-W invariants, (elements from Lecture 7/8 in [1], Part III, [35], [13]).

Lecture 3. *Quantum cohomology*
(Comparable to Lecture 4 in [1], Part I)
The quantum product ([22], §8, p.35). The potential Φ, ([22],p.36, lemma 14). Associativity of the quantum product,([22], thm. 4).

Lecture 4. *Applications to Enumerative Geometry*
(Comparable to Lecture 5 in [1], Part I)
Associativity and WDVV equations. The case of $\mathbf{P}^2$, Kontsevich recursion formula, ([35], §5, [22], §9).

Tuesday

Rational curves on the quintic threefold.

The purpose of this day is to present ideas around the mathematical verification of the "numbers of rational curves" on quintic threefolds.

Lecture 5. *The computation of Candelas c.s., [18]*
The number of lines, conics and twisted cubics on a general quintic threefold, Clemens conjecture (Beginning of the Lecture 5 in [1], Part III). The formula of Candelas c.s. ([12] (2.1)) for the computation of the n_d. The hypergeometric series as period integral of the quintic mirror family ([8], Exam. 4.5). Duality of the Hodge numbers [40]. The n_d as G-W invariants and the issue of counting of the multiple coverings of a rational curve, ([4], [40], [45]).

Lecture 6. *Enumeration of rational curves via torus action*
The idea of using Bott's formula ([21], §1). Counting of lines. Sketch of an application to counting twisted cubics (citeE-S, §3). The fixed points of $(\mathbf{C}^*)^{n+1}$ on $\overline{M}_{0,k}(\mathbf{P}^n, d)$, a review of the Kontsevich's computation [36]. (Comparable to a part of the Lecture 10 in [1], Part III)

Lecture 7. *Quantum Differential Systems*
Frobenius manifolds of Dubrovin, flatness of the quantum connection and

WDVV-equation, ([38], (2.3), see also Lecture 10, [1], part II, [27], §7). Example of $\mathbf{P^n}$ and hypergeometric solution, ([38] (2.3.5), Lecture 4 in [1], Part II, p.50, [26]). Givental's general cohomology valued solution to the Quantum Differential System ([38] (2.3.1)). Restriction to a hypersurface, convex vector bundles, Euler class and modified product ([38] (2.4), [27] §4).

Lecture 8. *Equivariant Quantum Cohomology*
Equivariant version of Quantum Cohomology (Lecture 4 in [1], Part II, [38] (2.2.3), [27]). Idea of Givental's proof [38] (2.5). Recursion relation ([38], §4), Kim's quantum hyperplane theorem. [38]. See also [37].

Wednesday

Mirror symmetry in physics

During this day we suggest to discuss the origin of mirror symmetry from the representation theory of $N = 2$ superconformal algebra. As a general reference for this topic we suggest the book of B. Greene [29].

Lecture 9. $N = 2$ *superconformal field theories.*
Definition of $N = 2$ superconformal algebra [29](3.1). Representation theory of $N = 2$ superconformal algebra, chiral primary fields, spectral flow and $U(1)$-projection [29](3.2,3.3,3.4). Examples [29](3.5). Minimal models and its patition functions [28](§4), [29](Exam.4 on p. 53 and 5.1).

Lecture 10 *Greene-Plesser mirror construction for Fermat hypersurfaces, [30].*
Conjectural correspondence between tensor products of minimal models Calabi-Yau hypersurfaces of Fermat type [28](5.1), [29](5.3). Arguments supporting this correspondence [28](5.2), [29](5.3), [30] The mirror construction for Fermat hypersurfaces and its examples [29](6.1,6.4,6.5,6.6), [28](6.3). The calculations of the partition functions supporting the mirror construction [28](6.2), also [29](6.2,6.3).

Thursday

Mirror symmetry and toric geometry

The purpose of this day is to explain the mirror symmetry constructions and to check its consequnces for Calabi-Yau hypersurfaces and complete intersections in Gorenstein toric Fano varieties. As a general review of this topic we suggest [2].

Lecture 11. *Mirror construction for hypersurfaces in toric varieties.*
Reflexive polyhedra [7](4.1.5) and their properties [7](4.1.6, 4.1.7, 4.1.9). Mirror construction [7](§5). Reflexive simplices and Greene-Plesser mirror construction [7](5.1,5.4,5.5) Resolutions of singularities of Calabi-Yau hypersurfaces [7](4.2). The computations of the Hodge numbers $h^{1,1}$ and $h^{n-1,1}$ for minimal desingularizations of generic Calabi-Yau hypersurfaces in toric varieties [7](4.3.7, 4.4.2). The monomial-divisor mirror map [3](Definitions in §§2,3, Theorem in §and Conjecture 3 in §5).

Lecture 12 *Periods of Calabi-Yau hypersurfaces.*
The main period of a Calabi-Yau hypersurface and the GKZ-hypergeometric system [8](§14), [34](§3). Compactifications of moduli spaces of toric hypersurfaces [8](§10), [34](§4). Maximal unipotent monodromy point [34](§5). Calculation of the prepotential and the instanton numbers of rational curves on 3-dimensional Calabi-Yau hypersurfaces [34](§6).

Lecture 13 *Mirror construction for Calabi-Yau complete intersections in toric varieties.*
Nef-partitions and associated polyhedra [17](§2). The combinatorial duality for nef-partitions [17](§3). Reflexive Gorenstein cones and their relations to nef-partitions [9] (Prop. 3.6). The duality for reflexive Gorenstein cones and the duality for nef-partitions [9] (Th. 4.6). Application to mirrors of rigid Calabi-Yau manifolds [9] (§5).

Lecture 14 *Topological test for mirror symmetry and string-theoretic Hodge numbers.*
Definition of string-theoretic Hodge numbers for varieties with at worst Gorenstein toroidal or quotient singularities [11] (6.7). Some properties [11] (6.9, 6.12). Mirror duality for string-theoretic Hodge numbers of Calabi-Yau complete intersections in toric varieties [10] (4.15).

Friday

Mirror symmetry "is" T-duality

The purpose of this day is to give a presentation of the mathematical results related to the conjectural mirror construction due to Strominger-Yau-Zaslov

[43] and its recent confirmation in the work of Gross-Wilson [32] for $K3$-surfaces and some Calabi-Yau 3-folds. As a general review of this material we suggest [23].

Lecture 15. *Mirror symmetry for lattice polarized K3 surfaces.*
The notion of lattice polarised $K3$-surfaces and m-admissble vector ([32], p.4, [19], §1, §5). Moduli spaces of marked M-polarized $K3$'s and the period domains D_M ([32], p.4, [19], §3). The tube domain T_M ([32], p.5, prop 1.1, [19], §4, §6). Mirror symmetric families ([32], p.6, [19], §6). Examples ([19], (1.2), (7.1), (7.9), etc.). Relation to toric mirror construction ([19],(8.1), (8.2), (8,6), [36], §4, (4.3.3)). (Relation to Arnolds Strange duality, [19], [20], [36].)
See also: [5], [31].

Lecture 16. *Special Lagrangian Submanifolds*
The notion of special Lagrangian submanifold ([32], p.1, [23], p.7, [33], III). Examples and properties ([33], (3.1), [39], p.1-2.) Their deformation theory ([32], p.1-2, [23], p.7-8, [39]). Special Lagrangian n-torus fibrations ([32], p.2, [23], p.9-10). The Strominger-Yau-Zaslow construction of the mirror ([32], p.2, [23], p.5, Conjecture 2, [43], p.1-2). Quantum mirror symmetry ([23], §1).

Lecture 17 *Mirror symmetric families and special Lagrangian fibrations on $K3$-surfaces.*
Special Lagrangian fibration on a $K3$ surface, [32], p.6-9, prop.1.2-1.4,[23], §6). The moduli space of simple sheaves on a $K3$ surface, ([23], [41], [42]). Mukai vector, pairing and Fourier-Mukai transform ([6], §4). Fourier-Mukai functor as mirror symmetry ([6], §5).

Lecture 18 *T-duality for Borcea-Voisin mirror pairs of Calabi-Yau 3-folds.*
$K3$ surfaces with involutions, the numbers N and N', the dual involution ([32], §2, [44], (1.1), (2.6), [32], thm (2.1)). Definition and Hodge numbers of the Borcea-Voisin threefolds ([32], §3, [44], (1.2), (1.8)). Special Lagrangian 3-torus fibration in the examples. Analysis of the topology ([32], p. 18-22).

References

[1] P. Aluffi, *Quantum cohomology at the Mittag Leffler Institute*,available at `http://www.math.fsu.edu/ aluffi/eprint.archive.html` (FSU97-01)

[2] P. Aspinwall, B. R. Greene, D.R. Morrison, *Calabi-Yau Moduli Space, Mirror Manifolds and Spacetime Topology Change in String Theory*, In: Essays on Mirror Manifolds II, Greene, Yau (eds), Int. Press 1996, 213-279. hep-th/9309097.

[3] P. Aspinwall, B. R. Greene, D.R. Morrison, *Monomial-divisor mirror map*, Internat. Math. Res. Notices (1993), 319-337.

[4] P. S. Aspinwall and D. R. Morrison, *Topological field theory and rational curves*, Comm. Math. Phys. **151** (1993), 245–262, `hep-th/9110048`.

[5] P. Aspinwall, *$K3$ Surfaces and String Duality*, hep-th/9611137.

[6] C. Bartocci, U. Bruzzo, D. Hernandez Ruiperez, J. Munoz Porras, *Mirror symmetry on elliptic K3 surfaces via Fourier-Mukai transform*, alg-geom/9704023.

[7] V.V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties*, J. Alg. Geom., (1994), 493-535.

[8] V.V. Batyrev, *Variations of the mixed Godge structure of affine hypersurfaces in algebraic tori*, Duke Math. J. **69** (1993), 349-409.

[9] V.V. Batyrev, L.A. Borisov, *Dual Cones and Mirror Symmetry for Generalized Calabi-Yau Manifolds*, In: Essays on Mirror Manifolds II, Green, Yau (eds), Int. Press 1996, 71-86. alg-geom/9402002.

[10] V.V. Batyrev, L.A. Borisov, *Mirror duality and string-theoretic Hodge numbers*, Invent. Math. **126** (1996), 183-203.

[11] V.V. Batyrev, D.I. Dais, *Strong Mckay correspondence, string-theoretic Hodge numbers and mirror symmetry*, Topology **35** (1996), 901-929.

[12] V.V. Batyrev, D. van Straten, *Generalized Hypergeometric Functions and Rational Curves on Calabi-Yau Complete Intersections in Toric Varieties*, alg-geom/9307010.

[13] K. Behrend, *Gromov-Witten invariants in algebraic geometry*, Invent. Math. **127** (1997), 601-617.

[14] K. Behrend and B. Fantechi, *The intrinsic normal cone*, Invent. Math. **128** (1997), 45–88, `alg-geom/9601010`.

[15] K. Behrend and Yu. Manin, *Stacks of stable maps and Gromov–Witten invariants*, Duke Math. J. **85** (1996), 1–60, `alg-geom/9506023`.

[16] C. Borcea, *$K3$ surfaces with involution and Mirror Pairs of Calabi-Yau Manifolds*, In: Essays on Mirror Manifolds II, Green, Yau (eds), Int. Press 1996.

[17] L.A. Borisov, *Towards the mirror symmetry for Calabi-Yau complete intersections in Gorenstein toric Fano varieties*, alg-geom/9310003.

[18] P. Candelas, X.C de la Ossa, P.S. Green, and P.S Parkes, *A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory*, Nuclear Phys. B **359** (1991), 21-74.

[19] I. V. Dolgachev, *Mirror Symmetry for Lattice Polarized K3-surfaces*, alg-geom/9502005.

[20] W. Ebeling, *Strange duality, mirror symmetry, and the Leech lattice*, alg-geom/9612010.

[21] G. Ellingrud, S.A. Stromme, *Botts formula and enumertative geometry*, alg-geom/9411005.

[22] W. Fulton, R. Pandharipande, *Notes on stable maps and quantum cohomology*, Preprint (1996), alg-geom/9608011.

[23] D. Morrison, *The Geometry Underlying Mirror Symmetry*, alg-geom/9608006.

[24] D. Morrison, *Mathematical Aspects of Mirror Symmetry*, alg-geom/9609021.

[25] A. Givental. *Homological geometry and mirror symmetry.* In: Proceedings of the International Congress of Mathematicians, Zürich, 1994, Birkhäuser, 1995, v. 1, pp. 472 – 480.

[26] A. Givental. *Homological geometry I: projective hypersurfaces.* Selecta Math., New Series, **1** (1995), No. 2, 325 – 345.

[27] A. B. Givental, *Equivariant Gromov–Witten invariants*, Internat. Math. Res. Notices (1996), 613–663, `alg-geom/9603021`.

[28] B. R. Greene, *Constructing Mirror manifolds*, In: Essays on Mirror Manifolds II, Green, Yau (eds), Int. Press 1996, 29-69.

[29] B. R. Greene, *String Theory on Calabi-Yau Manifolds*, hep-th/9702155.

[30] B. R. Greene and M. R. Plesser, *Duality in Calabi–Yau moduli space*, Nuclear Phys. B **338** (1990), 15–37.

[31] V. Grisenko, V. Nikulin, *K3 surfaces, Lorentzian Kac–Moody algebras and Mirror Symmetry*, alg-geom/9510008.

[32] M. Gross, P. Wilson, *Mirror Symmetry via* 3*-tori for a class of Calabi-Yau Threefolds*, alg-geom/9608004.

[33] R. Harvey and H. B. Lawson, Jr., *Calibrated geometries*, Acta Math. **148** (1982), 47–157.

[34] S. Hosono, *GKZ Systems, Gröbner Fans and Moduli Spaces of Calabi-Yau hypersurfaces*, alg-geom/9707003.

[35] M. Kontsevich, Yu. Manin, *Gromov-Witten classes, quantum cohomology and enumerative geometry*, Commun. math. Phys. **164** (1994), 525-562.

[36] M. Kontsevich, *Enumeration of rational curves via torus actions*, The Moduli Space of Curves (R. Dijkgraaf, C. Faber, and G. van der Geer, eds.), Progress in Math., vol. 129, Birkhäuser, 1995, pp. 335–368, `hep-th/9405035`.

[37] B. Kim. *Notes on Givental's paper "Equivariant Gromov-Witten Invariants*, available on request from organizers.

[38] B. Kim, *The Quantum Hyperplane Theorem*, available on request from organizers.

[39] R. C. McLean, *Deformations of calibrated submanifolds*, Comm. Anal. Geom., to appear, preprint available at `http://www.math.duke.edu/preprints/1996.html`.

[40] D.R. Morrison, *Mirror symmetry and rational curves on quintic three-folds: A guide for mathematicians*, J. Amer. Math. Soc. **6** (1993), 223-247.

[41] S. Mukai, *Symplectic structure of the moduli space of sheaves on an abelian or $K3$ surface*, Inv. Math. **77**(1984), 101-116.

[42] S. Mukai, *On the moduli space of bundles on $K3$ surfaces*, In: Vector bundles on Algebraic Varieties, Bombay 1984, Oxford Univ. Press (1987), 341-413.

[43] A. Strominger, S.-T. Yau, E. Zaslow, *Mirror Symmetry is T-duality*, hep-th/9606040.

[44] C. Voisin, *Miroirs et involutions sur les surfaces $K3$*, In: Astérisque **218**, 273-322 (1993).

[45] C. Voisin, *A mathematical proof of a formula of Aspinwall and Morrison*, Compositio Math. **104** (1996), 135–151.

Speakers are asked to observe that talks last **60** *minutes*

For preprints or further information, please contact the organizers:

V.V Batyrev
Mathematisches Institut
Eberhard-Karls-Universität Tübingen
72076 Tübingen.
tel: 07071-297 3241, 07071-791605 (private).
e-mail: batyrev@bastau.mathematik.uni-tuebingen.de

D. van Straten
FB17, Mathematik
Johannes Gutenberg-Universität Mainz
55099 Mainz.
tel: 06131 39 2435, 06131-831830 (private).
e-mail: straten@math.uni-mainz.de

Applications should be send to V. Batyrev and/or D. van Straten until the end of August.