Programm der Arbeitsgemeinschaft in Oberwolfach vom 29.3.-4.4.1998 über den Beweis der Milnor-Vermutung nach Voevodsky

Organizers: B. Kahn (Paris), M. Spieß (Heidelberg)

The aim of this Arbeitsgemeinschaft is to understand the proof of the following theorem, due to Vladimir Voevodsky:

Theorem. *Let k be a field of characteristic $\neq 2$ and m, n be positive integers. Then the norm residue homomorphism*

$$K_n^M(k)/2^m \longrightarrow H^n_{\text{ét}}(k, \mu_{2^m}^{\otimes n})$$

is an isomorphism.

This had been conjectured by Milnor. More generally one expects:

Conjecture (Kato).*For any positive integers m, n with m prime to* $\mathrm{char}(k)$ *the norm residue homomorphism*

$$K_n^M(k)/m \longrightarrow H^n_{\text{ét}}(k, \mu_m^{\otimes n})$$

is an isomorphism.

As in the previous cases where this conjecture could be established (Merkurjev, Suslin, Rost), the proof is based on the computation of certain cohomology groups attached to norm varieties. However, contrary to the former approaches, Voevodsky works with motivic cohomology rather than algebraic K-theory: for this reason, talks 1 to 12 will be devoted to the construction of this motivic cohomology. Two further key points in the proof are the existence of a pure motive associated by Markus Rost to certain quadrics, and Voevodsky's observation that these quadrics are related to the theory of complex cobordism (their **C**-valued points are generators of the mod 2 complex cobordism ring). In order to exploit this connection he developed (partly together with F. Morel) analogues of (stable) homotopy theory, Steenrod algebra, Thom spaces and the cobordism spectrum in the context of algebraic varieties. Talks 13–14 will cover the Rost motive and talks 15 to 19 will cover the homotopy part.

We shall try to cover as much of the general theory as possible (including the construction and properties of Voevodsky's triangulated categories of motives over a field of characteristic 0). Unfortunately, not all material for the homotopy part is available in preprint form yet, and the situation is similar for the Rost motive; so the preparation of these talks could turn out to be difficult. Fabien Morel and Markus Rost have agreed to help the lecturers here. Their e-mail addresses are:

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Each participant should be ready to give one of the 19 lectures. The lecturers of those talks which are grouped to pairs are encouraged to work together.

Lectures

1. Introduction
This talk should be an overview of the subject: Milnor's K-theory, the Milnor conjecture ($p = 2$), the (Bloch)-Kato conjecture (p arbitrary). Connection with algebraic K-theory: the conjectural Beilinson spectral sequence from truncated étale cohomology to algebraic K-theory with finite coefficients, the conjectural Beilinson and Lichtenbaum motivic complexes, the integral version of the conjectural Lichtenbaum spectral sequence, the Beilinson-Lichtenbaum conjecture. Attempts to construct the complexes: Bloch's complexes (higher Chow groups). Rationally they coincide with Adams eigenspaces on algebraic K-theory for smooth varieties over a field (Bloch, Levine). The Bloch-Lichtenbaum spectral sequence. Voevodsky's construction of motivic cohomology: it coincides with higher Chow groups for smooth schemes in characteristic 0. The Suslin-Voevodsky theorem (Kato $\Rightarrow$ Beilinson-Lichtenbaum in char. 0). Voevodsky's proof of the Milnor conjecture.

References: [27], [17], [2], [33], [1], [22], [3], [21], [4], [40], [47], [48].

2. Localization of categories; triangulated categories.
This talk should recall basic facts on these two topics.

Localization: this is completely general except for set-theoretic problems (see [11, I.1.1]). This is made nicer when calculus of fractions is possible (ibid., I.2). A nice set-up for localization with calculus of fractions (derived from topology) is Quillen's homotopical algebra. Definition of a model category, a closed model category. The associated homotopy category. Examples: the two model category structures on an abelian category; the category of simplicial sets and/or topological spaces and/or CW complexes. Derived functors as Kan extensions (e.g. [7, §8]).

Triangulated categories. Examples: the homotopy and derived categories of an abelian category (and their bounded variants). One word about derived functors. The stable homotopy category. Two adjoint functors between the stable homotopy category and the derived category of abelian groups (one is "chain complex", the other is "Eilenberg-Mac Lane spectrum"); after tensoring with $\mathbf{Q}$, they become equivalences of categories.

References: [11], [23], [32], [7], [41], [42], [13], [12].

3. + 4. Pretheories.
These talks should be devoted to [46] (see also [43]). The talks should include in particular the results of [46, Subsection 4.4], an introduction to the Nisnevich topology and the results of *ibid.*, Subsections 5.1 and 5.2. A good guideline for some of the main results is [45, Theorem 3.1.11]: the lecturers could concentrate on its proof (this includes comparing the notions of pretheory and presheaf with transfers).

References: [39], [46], [43], [31], [45].

5. The triangulated category of geometrical motives.
This talk should be devoted to [45, Section 2.1]. In particular, Proposition 2.1.4 should be proven. (One should admit [9, Theorem 7.1], a special case of [9, Theorem 7.4] which will be the topic of talk number 8. The latter requires resolution of singularities, while the former does not.) If time remains, an overview of the main results of the paper [45] should be given (as in section 2.2).

References: [45], [9].

6. Triangulated category of effective motivic complexes.
This talk should be devoted to [45, Section 3]. In particular: tensor structures on $D^-(Shv_{Nis}(SmCor(k)))$ and on $DM_-^{eff}(k)$, 3.2.6, 3.2.7 and the results of 3.5. Special emphasis should be given to the functors $\underline{C}_*$ and RC (compare [11, I.1.3]). It would be nice to trace where resolution of singularities is needed and where not in this part.

References: [45] and its references.

7. cdh topology.
In this talk cdh topology should be introduced and [9, Theorem 5.5] should be proved. It would be nice to also compare the idea of the cdh topology with Deligne's simplicial resolutions of singular varieties [36], [6].

References: [9, Sections 3 and 5], [46, Section 5], [45], [36], [6].

8. Moving lemma and duality.
This talk should devoted to (a sketch of) a proof the following theorem:

Let k be a field of characteristic zero, X a scheme of finite type over k and U a smooth quasi-projective equidimensional scheme of dimension n. Then the canonical map (of Nisnevich sheaves with transfers)

$$\mathcal{D} : z_{equi}(U, X, r) \to z_{equi}(X \times U, r + n)$$

induces an quasi-isomorphism of complexes of Nisnevich sheaves

$$\underline{C}_*(z_{equi}(U, X, r)) \to \underline{C}_*(z_{equi}(X \times U, r + n))$$

This is slightly more general than [9, Theorem 7.4] but as far as we can see, it can be proved by the same techniques. This result is everything one needs from [9] to prove the main results in [45, Section 4].

References: [9, Sections 2, 6 and 7], [8], [39].

9. Main properties of the triangulated category of motives.
This should include: motives for singular varieties, motives with compact support, quasi-invertibility of the Tate-object, duality [45, Theorem 4.3.7]. The relationship with Bloch's higher Chow groups should be indicated.

References: [45, Section 4].

10. + 11. The Beilinson-Lichtenbaum conjecture and the (Bloch-)Kato conjecture.
These two talks should give a fairly detailed proof of [40, Theorem 5.9]. Direct ancestors of the strategy (in a rather different setting) are [20] and [19]. Older ancestors are [18] and [26].

References: [40], [20], [19], [18], [26], [10] (for theorem 6.5), [14].

12. Norm varieties

This talk should cover [48, Section 2] (see also [16, Sections 1.2, 3 and 4]). Specifically: splitting varieties, examples (Severi-Brauer varieties, Pfister or Pfister neighbour quadrics, optionally splitting varieties for mod 3 symbols of length 3) and reduction to the proof of the Bloch-Kato conjecture, by induction on the degree, to [48, theorem 2.25] (see also [16, prop. 4.4 and th. 4.9]). Section 2.2 of [48] could just be sketched as it is essentially already in the Merkurjev-Suslin papers [24] and [25], to concentrate on the cohomology of $\check{C}(X)$, X a norm variety.

References: [47], [48], [16], [30].

13. + 14. The Rost motive.

These two talks should cover the following:

- Injectivity of $A_0(X, K_1) \xrightarrow{N} k^*$ for X the quadric associated to a Pfister neighbour over k, see [34]. A simpler proof (also due to Rost) is sketched in [16, 8.2].
- Construction of the Rost motive [35]. There are no details of the construction in *loc. cit.*, and not enough in [16, 8.1], but Rost offered to help with preparing this part.
- Voevodsky's translation in terms of triangulated motives [48, th. 4.4 and 4.8] (see also [47, Section 4], where the statements and arguments are somewhat more explicit). The main result is [48, Corollary 4.9].

References: [48], [47], [34], [35], [16].

15. + 16. Homotopy theory of schemes.

These two talks should cover the following:

- Construction of the homotopy category of schemes over a base [29]; a construction in the same spirit is given in [28]. See also [48, 3.2] over a field (which is the only case of interest here).
- The stable homotopy category of schemes. The only written reference is [48, 3.2]. The speakers can inspire themselves from [15] and [5].
- Cohomology and homology theories. The main result here is [48, Th. 3.12] and the two talks should aim towards this result. A proof is sketched in [16, proof of th. 5.6].

Fabien Morel agreed to help in the preparation of these talks. The unfinished manuscript [29] can be obtained on request.

References: [29], [28], [48], [16], [30], [5], [15].

17. Steenrod operations.
In view of the state of the published material, this talk can do little more than recall the classical Steenrod operations and their properties, give their definition in the motivic case and state the properties announced by Voevodsky in [48, 3.3]. The talk should also give the definition of the motivic Margolis cohomology groups $HM_n^{p,q}$ [48, comments after theorem 3.17], state the following theorem:

Let X be a $(2^{n-1}-1)$-dimensional quadric over F and let $\tilde{\mathcal{X}}$ be defined as in [48, th. 3.25]*. Then*

$$\tilde{H}M_i^{p,q}(\tilde{\mathcal{X}}) = 0$$

for any $i \leq n-1$ and $p, q \in \mathbf{Z}$.

and, finally, derive the proof of the Milnor conjecture from this assumed theorem and [48, cor. 4.9].

Fabien Morel agreed to help in the preparation of this talk.

References: [37], [48, 3.3], [16, §6], [30].

18. and 19. Thom spaces and algebraic cobordism.
These two talks should cover the following:

- Definition of the Thom space associated to an algebraic vector bundle [48, 3.5], [16, 7.2], [29]; statement of the homotopy purity theorem [29], [16, Theorem 7.1]. The T-spectrum $\mathbf{MU}_{*,*}$.
- The topological realisation functor [48, 3.4], [29], [16, 7.2].
- Definition of a (v_n, l)-variety [48, Definition 3.5]. Examples: Severi-Brauer varieties of dimension $l-1$ ($n=1$), quadrics of dimension 2^n-1 ($l=2$).
- Statement of [48, Theorem 3.25] and sketch of a proof. This relies on [48, Theorem 3.22 and Proposition 3.24]; a sketch of a proof of the first one is given in [16, proof of Theorem 7.8].

Fabien Morel agreed to help in the preparation of these talks.

References: [48], [16], [30].

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Applications should be sent, preferably *via* e-mail, to M. Spieß and/or B. Kahn until 21.2.1998.

Please send this application form also to Mathematisches Forschungsinstitut, Lorenzenhof, 77709 Oberwolfach, Fax: 07834/97955.

I would like to participate in the Arbeitsgemeinschaft on "The Milnor conjecture". I am willing to give a lecture:

- preferably Nr. ,
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